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NEW NONPARAMETRIC CLASSES OF DISTRIBUTIONS IN TERMS OF MEAN TIME TO FAILURE IN AGE REPLACEMENT

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Abstract

The mean time to failure (MTTF) function in age replacement is used to evaluate the performance and effectiveness of the age replacement policy. In this paper, based on the MTTF function, we introduce two new nonparametric classes of lifetime distributions with nonmonotonic mean time to failure in age replacement; increasing then decreasing MTTF (IDMTTF) and decreasing then increasing MTTF (DIMTTF). The implications between these classes of distributions and some existing classes of nonmonotonic ageing classes are studied. The characterizations of IDMTTF and DIMTTF in terms of the scaled total time on test transform are also obtained.

Keywords: BFR distribution; convex and concave functions; nonmonotonic ageing class; NWBUE distribution; total time on test transform

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1. Introduction

Failure of units during operation might sometimes be costly or dangerous. Age replacement policy is the most common maintenance policy employed to prevent a unit from failure during its operation. In age replacement policy, the unit is replaced either at failure time or at the prespecified time t if it is alive at time t (known as planned replacement age). Let $X_{[t]}$ denote the time to the first in-service failure of an item under the age replacement policy with the planned replacement age t. Assuming that F is the lifetime distribution of a new item, the survival function of $X_{[t]}$ (denoted by S_t) is

$$S_t(x) = [\overline{F}(t)]^n \overline{F}(x - nt), \qquad nt \le x < (n + 1)t, \quad n = 0, 1, \dots,$$

where $\overline{F} = 1 - F$; see [3]. The mean of $X_{[t]}$, denoted by $M_F(t)$, is called the mean time to failure (MTTF) in age replacement. It was introduced by Barlow and Proschan [3] to evaluate the performance and effectiveness of the age replacement policy. Suppose that the replacement of items are continued even after the first failure is observed. Let X_i, $i = 1, 2, \dots,$ be the lifetime of the ith item used in the age replacement policy with distribution F. Define $W_i = \min\{X_i, t\}$, $i = 1, 2, \dots,$ and N as the total number of items used until we observe the

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first W_i which is strictly less than t. Then $X_{[t]} = \sum_{i=1}^{N} W_i$. Now,

$$\begin{aligned}
\mathbb{E}[X_{[t]} \mid N = n] &= \mathbb{E}\bigg[\sum_{i=1}^{n} W_i \,\bigg|\, N = n\bigg] \\
&= \mathbb{E}\bigg[\sum_{i=1}^{n-1} W_i \,\bigg|\, N = n\bigg] + \mathbb{E}[W_n \mid N = n] \\
&= (n-1)t + \mathbb{E}[X_n \mid X_n < t] \\
&= (n-1)t + \frac{-t\overline{F}(t) + \int_0^t \overline{F}(x)\,\mathrm{d}x}{F(t)}.
\end{aligned}$$

Thus, from the fact that the random variable N has a geometric distribution with parameter $F(t)$, it follows that

$$M_F(t) = \bigg(\frac{1}{F(t)} - 1\bigg)t + \frac{-t\overline{F}(t) + \int_0^t \overline{F}(x)\,\mathrm{d}x}{F(t)} = \frac{\int_0^t \overline{F}(x)\,\mathrm{d}x}{F(t)}.$$

Kayid *et al.* [15] obtained $M_F(t)$ by a different argument. The distribution F is said to be decreasing (increasing) mean time to failure (DMTTF (IMTTF)) in age replacement if $M_F(t)$ is decreasing (increasing) on $[0, \infty)$. DMTTF means a type of 'deterioration' and IMTTF means 'nondeterioration' or improvement in some senses. The relationship between DMTTF and IMTTF and some well-known ageing classes of distributions, which we review next, were investigated by Klefsjö [16].

A lifetime distribution F is said to be an increasing (decreasing) failure rate (IFR (DFR)) if $R(x) = -\log[\overline{F}(x)]$ is convex (concave) on $[0, \infty)$ when finite; see [4]. Equivalently, F is IFR (DFR) if and only if the failure rate function of F, $r_F(x) = f(x)/\overline{F}(x)$, is increasing (decreasing) on $[0, \infty)$, provided that the lifetime density f exists. The distribution F is said to be an increasing (decreasing) failure rate average (IFRA (DFRA)) if $R(x)/x$ is increasing (decreasing) on $[0, \infty)$ when finite or, equivalently, if the failure rate average function of F, $\tilde{r}_F(t) = \int_0^t r_F(x)\,\mathrm{d}x/t$, is increasing (decreasing) on $[0, \infty)$. The distribution F is said to be new better (worse) than used in expectation (NBUE (NWUE)) if $\mu_F (\geq [\leq]) \int_t^\infty \overline{F}(x)\,\mathrm{d}x/\overline{F}(t)$ for all $t \geq 0$, where μ_F is the finite mean of F.

From Klefsjö [16], we know that

$$\text{IFR(DFR)} \implies \text{IFRA(DFRA)} \implies \text{DMTTF(IMTTF)} \implies \text{NBUE(NWUE)}.$$

The classes DMTTF and IMTTF of distributions have received extensive attention in the literature; see [1], [12]–[15], [18], [19], and [22].

The ageing patterns in the above classes are monotone. However, in practical situations, it is often seen that the ageing pattern is nonmonotonic; see [5], [9], and [26]. The various nonmonotonic ageing classes have been introduced in the literature to model such situations. The classes of distributions with bathtub failure rate (BFR) and upside-down bathtub failure rate (UBFR) are well-known nonmonotonic ageing classes and have been extensively studied in the context of reliability; see [20]. Guess *et al.* [11] introduced the nonmonotonic ageing classes of distributions with initially increasing (decreasing) and then decreasing (increasing) mean residual life. The class of new worse (better) then better (worse) than used in the expectation (NWBUE (NBWUE)) is defined in [17]. Deshpande and Suresh [9] introduced the

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nonmonotonic ageing class of increasing (decreasing) initially and then decreasing (increasing) residual life. Belzunce *et al.* [5], based on the Laplace transform of the residual lifetime, introduced a new concept of nonmonotonic ageing to model some situations in insurance. The relations between the nonmonotonic ageing classes of distributions have also been studied in the literature; see, for example, [5], [8], [9], and [24]–[26].

Motivated by the definition of the above nonmonotonic ageing classes, using the MTTF function, we propose two new nonparametric classes of distributions as follows.

Definition 1. A lifetime distribution F is said to be the initially increasing then decreasing mean time to failure (IDMTTF) if there exists a change point $\tau \geq 0$ such that $M_F(t)$ is increasing on $[0, \tau)$ and decreasing on $[\tau, \infty)$.

The dual class of 'decreasing initially then increasing mean time to failure' (DIMTTF) distributions is defined similarly by changing the order of the monotonicity.

The class of IDMTTF distributions can be used to model a situation in which the effect of age replacement is initially beneficial and then adverse, and the dual class of DIMTTF distributions models the case that the effect of age replacement is initially adverse and then beneficial. The change point of $M_F(t)$ is important in IDMTTF distributions. Clearly, $M_F(t)$ is maximum at this point, so it may be taken as a possible optimal age replacement. Thus, the IDMTTF property of the F distribution is of great interest in connection with the age replacement optimization.

There is also a close relationship between the function $M_F(t)$ and the expected cost rate which makes the MTTF change point more interesting. Let c_1 be the cost of replacing each failed unit that includes all costs resulting from a failure and its replacement and let c_2 $(< c_1)$ be the cost of exchanging each nonfailed unit. One of the most familiar criteria to determine the optimal replacement time is minimizing the expected cost rate (see [27, p. 72]) which is given by

$$C_F(t) = \frac{c_1 F(t) + c_2 \overline{F}(t)}{\int_0^t \overline{F}(x)\,\mathrm{d}x} = \frac{c_1 + c_2(\overline{F}(t)/F(t))}{M_F(t)}.$$

Now, let F be an IDMTTF distribution with the change point τ and let T be the optimal replacement time that minimizes $C_F(t)$. From the fact that $\overline{F}(t)/F(t)$ is decreasing on $(0, \infty)$, it follows that $C_F(t)$ is decreasing on $(0, \tau)$ which, in turn, implies that τ is a lower bound for T. Furthermore, usually c_2/c_1 is relatively small, and for large t, $\overline{F}(t)/F(t) \simeq 0$ from which it follows that for large τ, $T \simeq \tau$. We demonstrate this point in the following examples.

Example 1. Let F be a lifetime distribution with survival function

$$\overline{F}(x) = \begin{cases} \mathrm{e}^{-x}, & 0 < x < 1, \\ \frac{16}{15}\mathrm{e}^{-1}\big(1 - \frac{1}{16}x^3\big), & 1 \leq x < 2, \\ \frac{8}{15}\mathrm{e}^{-x+1}, & x \geq 2. \end{cases} \tag{1}$$

The corresponding $M_F(t)$ is

$$M_F(t) = \begin{cases} 1, & 0 < t < 1, \\ \big[1 - \frac{123}{60}\mathrm{e}^{-1} + \frac{16}{15}\mathrm{e}^{-1}t - \frac{1}{60}\mathrm{e}^{-1}t^4\big]\big[1 - \frac{16}{15}\mathrm{e}^{-1}\big(1 - \frac{1}{16}t^3\big)\big]^{-1}, & 1 \leq t < 2, \\ \big[1 + \frac{21}{60}\mathrm{e}^{-1} - \frac{8}{15}\mathrm{e}^{-t+1}\big]\big[1 - \frac{8}{15}\mathrm{e}^{-t+1}\big]^{-1}, & t \geq 2. \end{cases}$$

It is possible to show that F is IDMTTF with the change point $\tau = 1.7389$; see Figure 1. In Figure 2 we present plots of the expected cost rate function for $c_1 = 1000$ and some small

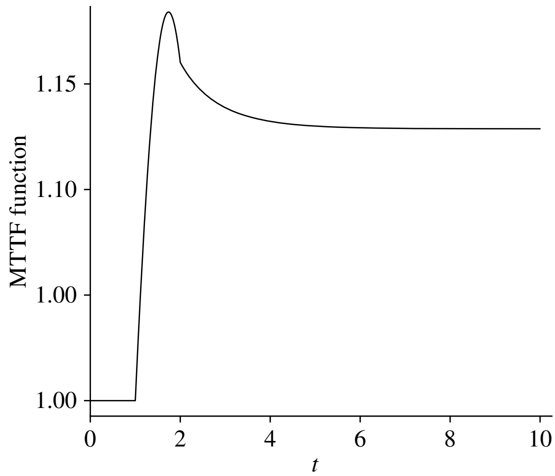

FIGURE 1: The MTTF function of distribution given in (1).

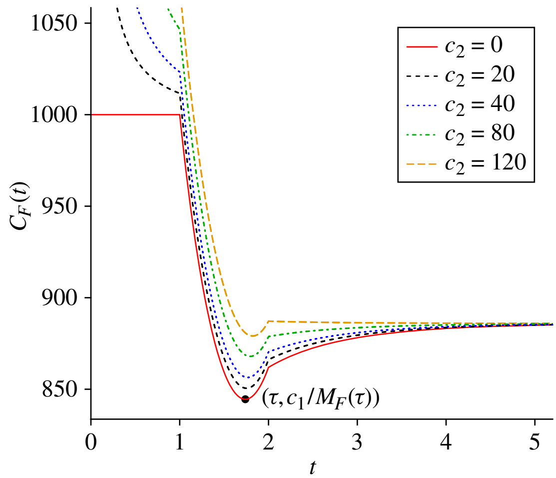

FIGURE 2: Plots of the expected cost rate function $C_F(t)$ of the distribution given in (1) for $c_1 = 1000$ and $c_2 = 0, 20, 40, 80, 120$.

values of c_2. From the figure we see the justification that $C_F(t)$ is close to $c_1/M_F(t)$ for large t. In Table 1 we present the optimal replacement time T for the given values of c_1 and c_2. We see that as c_2/c_1 decreases, T approaches $\tau = 1.7389$ from above.

Example 2. The modified Weibull (MW) model introduced in [21] has the survival function

$$\overline{F}(t) = \exp(-at^{\alpha}\mathrm{e}^{\lambda t}), \qquad t > 0,$$

with parameters $a > 0$, $\alpha > 0$, and $\lambda > 0$. Lai *et al.* [21] showed that the MW model is BFR whenever $0 < \alpha < 1$. From Theorem 4, the MW model is also IDMTTF. The change point of $M_F(t)$ is $\tau = 2.007$ for the parameter values $a = 0.8$, $\alpha = 0.5$, and $\lambda = 0.3$.

TABLE 1: The optimal replacement time T for $c_1 = 1000$ and some small values of c_2.

c_2	T	$C_F(T)$
20	1.7525	850.5263
40	1.7661	856.4259
80	1.7936	856.6161
120	1.8211	857.1896

In Figure 3 we present plots of the expected cost rate function for $c_1 = 1000$ and some small values of c_2. From the figure we see the justification that $C_F(t)$ is close to $c_1/M_F(t)$ for large t. In Table 2 we present the optimal replacement time T for the given values of c_1 and c_2. We see that as c_2/c_1 decreases, T approaches $\tau = 2.007$ from above.

In Section 2 we study the properties of IDMTTF and DIMTTF distributions and also investigate their relationships with other well-known nonmonotonic ageing notions. We prove that

$$\text{BFR} \implies \text{IDMTTF} \implies \text{NWBUE}$$

and

$$\text{UBFR} \implies \text{DIMTTF} \implies \text{NBWUE}.$$

We conclude this section by recalling some definitions and theorems which we use later in this paper. Throughout, the terms increasing and decreasing are used for nonincreasing and nondecreasing, respectively.

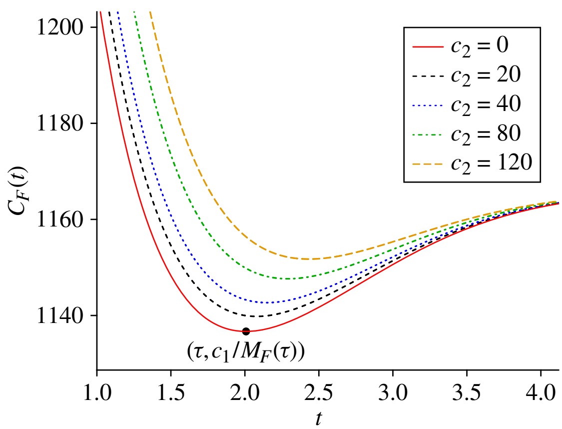

FIGURE 3: Plots of the expected cost rate function $C_F(t)$ of the MW model for $c_1 = 1000$ and $c_2 = 0, 20, 40, 80, 120$.

TABLE 2: The optimal replacement time T for $c_1 = 1000$ and some small values of c_2.

c_2	T	$C_F(T)$
20	2.0785	1139.826
40	2.1490	1142.683
80	2.2902	1147.652
120	2.4329	1151.749

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Let F be a lifetime distribution having support on $[0, \infty)$. We say that F is BFR (UBFR) if there exists a change point $x_0 \geq 0$ such that $R(x) = -\log[\overline{F}(x)]$ is concave (convex) on $[0, x_0)$ and convex (concave) on $[x_0, \infty)$. Equivalently, F is BFR (UBFR) if there exists a change point $x_0 \geq 0$ such that $r_F(x)$ is decreasing (increasing) in $[0, x_0)$ and increasing (decreasing) on $[x_0, \infty)$, provided that r_F exists. We say that F is NWBUE (NBWUE) if there exists a change point $x^* \geq 0$ such that $\int_x^\infty \overline{F}(t)\,\mathrm{d}t (\geq [\leq]) \mu_F \overline{F}(x)$ for $x < x^*$ and $\int_x^\infty \overline{F}(t)\,\mathrm{d}t (\leq [\geq]) \mu_F \overline{F}(x)$ for $x \geq x^*$.

The scaled total time on test (TTT) transform associated with F is

$$\varphi_F(u) = \frac{\int_0^{F^{-1}(u)} \bar{F}(x)\,\mathrm{d}x}{\mu_F}, \qquad u \in [0, 1],$$

where $F^{-1}(u) = \inf\{x,\ F(x) \geq u\}$. The TTT transform plays an important role in reliability theory because of its application in characterizing many ageing classes of lifetime distributions; see, for example, [2], [6], and [7]. The following characterizations will help us to prove some results in the next section.

Theorem 1. (Despande and Suresh [9].) *A lifetime distribution F is BFR (UBFR) if and only if there exist $u_0 \in [0, 1]$ such that $\varphi_F(u)$ is convex (concave) on $[0, u_0)$ and concave (convex) on $[u_0, 1]$.*

Theorem 2. (Klefsjö [17].) *A lifetime distribution F is NWBUE (NBWUE) if and only if there exist $u^* \in [0, 1]$ such that $\varphi_F(u) \leq [\geq] u$ for $u \in [0, u^*)$ and $\varphi_F(u) \geq [\leq] u$ for $u \in [u^*, 1)$.*

2. Characterization and implications

In this section we first obtain a characterization for the IDMTTF distribution in terms of the scaled TTT transform. Then we discuss the interrelationships between IDMTTF, BFR, and NWBUE.

To prove the following theorem, we use similar arguments to those of Klefsjö [16] to prove the characterizations of the DMTTF and IMTTF properties in terms of the scaled TTT transform.

Theorem 3. *A lifetime distribution F is IDMTTF if and only if there exist $\tilde{u} \in [0, 1]$ such that $\varphi_F(u)/u$ is increasing in $u \in [0, \tilde{u})$ and decreasing in $u \in [\tilde{u}, 1]$.*

Proof. If F is continuous and strictly increasing, the proof follows by using the substitution $u = F(t)$. We prove the theorem in the general case as follows.

Necessary. Suppose that F is IDMTTF with the change point τ. Then F is continuous on $(0, \tau)$ and strictly increasing on $[\tau, \infty)$. To show that F is continuous on $(0, \tau)$, let t be an arbitrary point in $(0, \tau)$. We can find a sequence $(t_n)_{n=1}^\infty$ that increases to t. Then, we have

$$\frac{\int_0^{t_n} \overline{F}(x)\,\mathrm{d}x}{F(t_n)} \leq \frac{\int_0^{t} \overline{F}(x)\,\mathrm{d}x}{F(t)}, \qquad n = 1, 2, \ldots.$$

Now, letting $n \to \infty$, it follows that $F(t^-) \geq F(t_0)$ which means that F is continuous in t. Now, to show that F is strictly increasing on $[\tau, \infty)$, we suppose that this does not hold, that is, there are $\tau \leq t_1 < t_2$ such that $F(t_1) = F(t_2)$. From the fact that $M_F(t)$ is decreasing on $[\tau, \infty)$, it follows that $\int_0^{t_1} \overline{F}(x)\,\mathrm{d}x \geq \int_0^{t_2} \overline{F}(x)\,\mathrm{d}x$ which is a contradiction.

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Now, to show that $\varphi(u)/u$ is increasing and then decreasing, let $u_1 < u_2 < \tilde{u} = F(\tau)$ and $t_i = F^{-1}(u_i)$, $i = 1, 2$. Then, $t_1 < t_2 < \tau$ and

$$\frac{\int_0^{t_1} \overline{F}(x)\,\mathrm{d}x}{F(t_1)} \leq \frac{\int_0^{t_2} \overline{F}(x)\,\mathrm{d}x}{F(t_2)}.$$

The continuity of F on $(0, \tau)$ implies that $\varphi_F(u_1)/u_1 \leq \varphi_F(u_2)/u_2$. Now, let $\tilde{u} \leq u_1 < u_2$ and, again, $t_i = F^{-1}(u_i)$, $i = 1, 2$. If $t_1 = t_2$ then $\varphi_F(u_1) = \varphi_F(u_2)$ which implies that $\varphi_F(u_1)/u_1 \geq \varphi_F(u_2)/u_2$. If $\tau \leq t_1 < t_2$ then we can find two sequences $(y_j)_{j=1}^{\infty}$ and $(z_k)_{k=1}^{\infty}$ such that $(y_j)_{j=1}^{\infty}$ increases to t_2 and $(z_k)_{k=1}^{\infty}$ decreases to t_1 and $z_k < y_j$ for $k, j = 1, 2, \ldots$. As $M_F(t)$ is decreasing on $[\tau, \infty)$, it follows that for $j, k = 1, 2, \ldots,$

$$\frac{\int_0^{z_k} \overline{F}(x)\,\mathrm{d}x}{F(z_k)} \geq \frac{\int_0^{y_j} \overline{F}(x)\,\mathrm{d}x}{F(y_j)}.$$

Now, letting j and k go to ∞ and from the fact that $\lim_{j\to\infty} F(y_j) \leq u_2$ and $\lim_{k\to\infty} F(z_k) \geq u_1$, we obtain $\varphi_F(u_1)/u_1 \geq \varphi_F(u_2)/u_2$.

Sufficiency. Suppose that $\varphi_F(u)/u$ is increasing and then decreasing with the change point $\tilde{u}$. Let $\tau = F^{-1}(\tilde{u})$. Similar to the necessary part, we show that F^{-1} is continuous on $(0, \tilde{u})$ and strictly increasing on $[\tilde{u}, 1)$ which is equivalent to F being strictly increasing on $(0, \tau)$ and continuous on $[\tau, \infty)$; see, for example, [10]. Let $t_1 < t_2 < \tau$ and $u_i = F(t_i)$, $i = 1, 2$. Then $u_1 < u_2 < \tilde{u}$ and from the fact that $\varphi_F(u)/u$ is increasing on $(0, \tilde{u})$ and the continuity of F^{-1}, it follows that $M_F(t_1) \leq M_F(t_2)$. Now, let $\tau \leq t_1 < t_2$. If $u_1 = u_2$ then $M_F(t_1) \leq M_F(t_2)$. If $\tilde{u} \leq u_1 < u_2$ then we have two sequences $(v_j)_{j=1}^{\infty}$ and $(w_k)_{k=1}^{\infty}$ such that $(v_j)_{j=1}^{\infty}$ increases to u_2 and $(w_k)_{k=1}^{\infty}$ decreases to u_1 and $w_k < v_j$ for $k, j = 1, 2, \ldots$. Since $\varphi_F(u)/u$ is decreasing on $(\tilde{u}, 1)$, it follows that for $j, k = 1, 2, \ldots,$

$$\frac{\int_0^{F^{-1}(w_k)} \overline{F}(x)\,\mathrm{d}x}{w_k} \geq \frac{\int_0^{F^{-1}(v_j)} \overline{F}(x)\,\mathrm{d}x}{v_j}.$$

Now, letting j and k go to ∞ and from the fact that

$$\lim_{j\to\infty} F^{-1}(v_j) \geq t_2 \quad \text{and} \quad \lim_{k\to\infty} F^{-1}(w_k) \leq t_1,$$

we obtain $M_F(t_1) \leq M_F(t_2)$ and the proof is complete. □

In Figure 4 we present the plot of the scaled TTT transform of an IDMTTF distribution. Geometrically, from Theorem 3, a distribution is IDMTTF if and only if the angle $\theta(u)$ is increasing and then decreasing on $(0, 1)$. It is obvious that in an IDMTTF distribution, $\tilde{u}$ is a point that maximizes the angle, that is, $\sup_{0<u<1} \theta(u) = \theta(\tilde{u})$.

Remark 1. Using similar arguments to those in the proof of Theorem 3, we can show that a lifetime distribution F is DIMTTF if and only if there exist $\tilde{u} \in [0, 1]$ such that $\varphi_F(u)/u$ is decreasing in $u \in [0, \tilde{u})$ and increasing in $u \in [\tilde{u}, 1]$.

We need to prove the following lemma to show that the BFR class of lifetime distributions is a subset of the IDMTTF class of distributions.

Lemma 1. *Let $f\colon [x^*, x^{**}] \to R^+$ be a positive and concave function on $[x^*, x^{**}]$ and define $\tilde{a} = \max\{a;\ f(x) - ax = 0$ has at least one root in $[x^*, x^{**}]\}$, $\tilde{x}_1 = \min\{x;\ \tilde{a}x = f(x)\}$ and $\tilde{x}_2 = \max\{x;\ \tilde{a}x = f(x)\}$.*

(i) *If $\tilde{x}_2 = x^{**}$ then $f(x)/x$ is increasing on $[x^*, x^{**}]$.*

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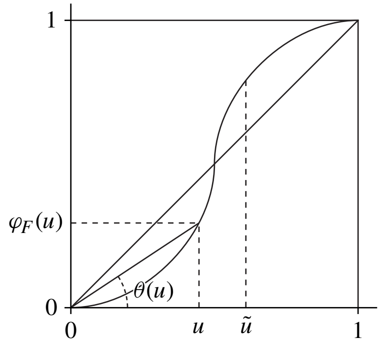

FIGURE 4: The scaled TTT transform function of an IDMTTF distribution.

(ii) *If* $\tilde{x}_1 = x^*$ *then* $f(x)/x$ *is decreasing on* $[x^*, x^{**}]$.

(iii) *If* $x^* < \tilde{x}_1 \le \tilde{x}_2 < x^{**}$ *then* $f(x)/x$ *is increasing on* $[x^*, \tilde{x}]$ *and decreasing on* $[\tilde{x}, x^{**}]$, *where* $\tilde{x}$ *is any value in* $[\tilde{x}_1, \tilde{x}_2]$.

Proof. (i) Suppose that $f(x)/x$ is not increasing on $[x^*, x^{**}]$, that is, there exist x_1 and x_2 such that $x^* \le x_1 < x_2 \le x^{**}$ and $a_1 = f(x_1)/x_1 > f(x_2)/x_2 = a_2$. It follows from the concavity of f that

$$\frac{f(\tilde{x}_2) - a_1 x_1}{\tilde{x}_2 - x_1} = \frac{f(\tilde{x}_2) - f(x_1)}{\tilde{x}_2 - x_1} \ge \frac{f(\tilde{x}_2) - f(x_2)}{\tilde{x}_2 - x_2} = \frac{f(\tilde{x}_2) - a_2 x_2}{\tilde{x}_2 - x_2} > \frac{f(\tilde{x}_2) - a_1 x_2}{\tilde{x}_2 - x_2}.$$

After some algebraic manipulation, we arrive at $f(\tilde{x}_2) < a_1 \tilde{x}_2$ or, equivalently, $\tilde{a} < a_1$, since $f(\tilde{x}_2) = \tilde{a}\tilde{x}_2$. This is a contradiction to the definition of $\tilde{a}$.

(ii) Similar to part (i), suppose that $f(x)/x$ is not decreasing on $[x^*, x^{**}]$, that is, there exist x_1 and x_2 such that $x^* \le x_1 < x_2 \le x^{**}$ and $a_1 = f(x_1)/x_1 < f(x_2)/x_2 = a_2$. Now, the concavity of f again implies that

$$\frac{f(\tilde{x}_1) - a_2 x_1}{\tilde{x}_1 - x_1} > \frac{f(\tilde{x}_1) - a_1 x_1}{\tilde{x}_1 - x_1} = \frac{f(\tilde{x}_1) - f(x_1)}{\tilde{x}_1 - x_1} \ge \frac{f(\tilde{x}_1) - f(x_2)}{\tilde{x}_1 - x_2} = \frac{f(\tilde{x}_1) - a_2 x_2}{\tilde{x}_1 - x_2},$$

from which it follows that $\tilde{a} = f(\tilde{x}_1)/\tilde{x}_1 < a_2$ and this is a contradiction to the definition of $\tilde{a}$.

(iii) Applying parts (i) and (ii) to $[x^*, \tilde{x}_1]$ and $[\tilde{x}_1, x^*]$ (or $[x^*, \tilde{x}_2]$ and $[\tilde{x}_2, x^*]$), respectively, we obtain the required result. □

Theorem 4. *If a lifetime distribution* F *is BFR with the change point* t_0, *then* F *is IDMTTF with a change point* $\tau > t_0$.

Proof. Since F is BFR, it follows from Theorem 3 that there exists $u_0 \in [0, 1]$ such that $\varphi_F(u)$ is convex on $[0, u_0)$. Using this observation, from [23, Proposition 21.A.11] and the fact that $\varphi_F(0) = 0$, we see that $\varphi_F(u)/u$ is increasing on $(0, u_0)$. Now, again from the BFR property of F, it follows from Theorem 3 that $\varphi_F(u)$ is concave on $[u_0, 1]$. Now, it follows from this and Lemma 1 that there exists $\tilde{u} \in [u_0, 1]$ such that $\varphi_F(u)/u$ is increasing on $[u_0, \tilde{u})$ and decreasing on $[\tilde{u}, 1]$, where $\tau = F^{-1}(\tilde{u}) \ge F^{-1}(u_0) = t_0$. □

In the following example we show that an IDMTTF lifetime distribution might not be BFR.

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Example 3. Let F be a lifetime distribution with survival function given in (1). From Example 1, we known that F is IDMTTF with the change point $\tau = 1.7389$. The hazard rate function of F is

$$r_F(t) = \begin{cases} 1, & 0 < t < 1, \\ \dfrac{3t^2 e^1}{16(1 - t^3/16)}, & 1 \leq t < 2, \\ 1, & t \geq 2. \end{cases}$$

Now, $r_F(1.5) = 1.45 > 1 = r_F(0.5) = r_F(2.5)$ which means that F is not BFR.

In the next theorem we establish the fact that the IDMTTF property implies the NWBUE property.

Theorem 5. *If a lifetime distribution F is IDMTTF with change point τ, then F is NWBUE with the change point $t_0 < \tau$.*

Proof. From Theorem 3, there exist $0 < \tilde{u} < 1$ such that $\varphi_F(u)/u$ is increasing on $(0, \tilde{u})$ and decreasing on $[\tilde{u}, 1]$. Thus, for $u \in [\tilde{u}, 1]$, $\varphi_F(u)/u \geq \varphi(1)/1 = 1$, that is, $\varphi_F(u) \geq u$. It is obvious that $\varphi_F(u) - u$ has at most one sign change in $[0, \tilde{u}]$; if there is a sign change, it is in the order $-, +$. If there is no sign change, $\varphi_F(u) \geq u$ for $u \in [0, 1]$ then F is NBUE which is a NWBUE with change point $t_0 = 0$. If there is a sign change then there exists a $u_0 \in [0, \tilde{u}]$ such that $\varphi_F(u) \leq u$ for $u \in [0, u_0]$ and $\varphi_F(u) \geq u$ for $u \in [u_0, \tilde{u}]$. Hence, the proof is complete. □

In the next example we show that the converse of the above result does not hold in general.

Example 4. Let F be a lifetime distribution with survival function

$$\overline{F}(x) = \begin{cases} e^{-x}, & 0 < x < 1, \\ \dfrac{e^{-1}}{x^2}, & 1 \leq x < 2, \\ \frac{1}{8}x \exp\left(-\left(\frac{1}{8}x^2 + \frac{1}{2}\right)\right), & x \geq 2. \end{cases} \tag{2}$$

Mitra and Basu [26] showed that F is NWBUE with change point $x^* = 4$. The MTTF function of F is

$$M_F(t) = \begin{cases} 1, & 0 < t < 1, \\ \dfrac{t^2 - e^{-1}t}{t^2 - e^{-1}}, & 1 \leq t < 2, \\ \left[1 - \frac{1}{2}\exp\left(-\left(\frac{1}{8}t^2 + \frac{1}{2}\right)\right)\right]\left[1 - \frac{1}{8}t\exp\left(-\left(\frac{1}{8}t^2 + \frac{1}{2}\right)\right)\right]^{-1}, & t \geq 2; \end{cases}$$

see Figure 5, from which we see that F is not IDMTTF.

Remark 2. From Theorems 4 and 5, we have the following implications:

$$\text{BFR} \implies \text{IDMTTF} \implies \text{NWBUE}.$$

Remark 3. Using similar arguments used to prove the above implications, we obtain the following implications:

$$\text{UBFR} \implies \text{DIMTTF} \implies \text{NBWUE}.$$

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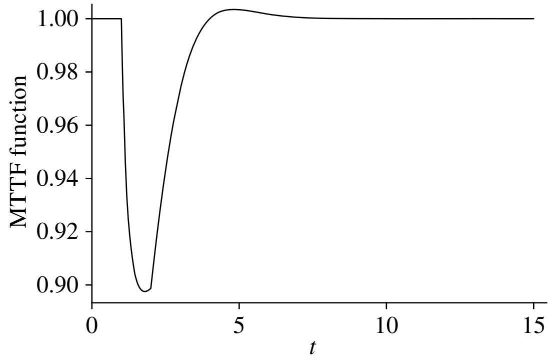

FIGURE 5: Plot of the MTTF function of the distribution given in (2).

3. Summary and future work

In this paper we introduced two new nonparametric classes of distributions with non-monotonic MTTF in age replacement. IDMTTF (DIMTTF) is a class of distribution with increasing (decreasing) and then decreasing (increasing) MTTF function. We characterized these classes of distribution in terms of the scaled TTT transform function. Using these characterizations, we obtained the following chain of implications between IDMTTF (DIMTTF) class of distributions with the well-known classes of BFR and NWBUE (UBFR and NBWUE) distributions:

$$\text{BFR} \implies \text{IDMTTF} \implies \text{NWBUE}$$

and

$$\text{UBFR} \implies \text{DIMTTF} \implies \text{NBWUE}.$$

A reasonable starting point in reliability analysis is to determine the ageing class of the underlying distribution F. Hence, it is of practical importance to investigate the problem of testing exponentiality against the IDMTTF property. If the IDMTTF property of the underlying distribution is characterized, in practice, the estimation of the change point of the MTTF function is of great interest. These problems form the basis of our future work.

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