AN ITERATIVITY CONDITION FOR THE MEAN-VALUE PRINCIPLE UNDER CUMULATIVE PROSPECT THEORY

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ABSTRACT

In this paper, we present the full characterization of the iterativity condition for the mean-value principle under the cumulative prospect theory. It turns out that the premium principle is iterative for exactly six pairs of probability distortion functions. Some of the corresponding premium principles are the classical mean-value principle, essential infimum or essential supremum of the random loss. Moreover, from the proof of the main theorem of this paper, it follows that the iterativity of the mean-value principle is equivalent to the iterativity of the generalized Choquet integral.

KEYWORDS

Cumulative Prospect Theory, Generalized Choquet Integral, iterativity, premium principles, mean-value principle, distorted probability

1. INTRODUCTION

The concept of iterativity dates back at least to Bühlmann (1970), who explains the difference between risk (individual) and collective premium. In order to calculate an individual premium, an insurer takes into account all the features of decision maker's risks. If the parameter y of the aforementioned risk is known, then $H(X|y)$ is the premium for risk X whose characteristic is y. However, this specific feature y is usually a realization of some random variable Y. Therefore, the collective premium cannot be determined in a similar straightforward way, but it should be calculated in two steps. First, an insurance company should determine $H(X|Y)$, which is a random variable dependent on Y. Then, a risk structure Y should be compensated by evaluating $H(H(X|Y))$. Since the premium $H(X)$ is in most cases different from $H(H(X|Y))$, there appears a problem to find under which circumstances these two values are the same. Bühlmann (1970) and Gerber (1974) also note an analogy between iterativity and the method of evaluating the credibility premium.

Gerber (1974) proves that the premium principle which satisfies a continuity condition is iterative if and only if it is mean-value principle, i.e. it is the

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solution of $v(H(X)) = Ev(X)$, where v is a strictly increasing, convex and twice differentiable function. A generalization of the result by Gerber is given by Goovaerts and de Vylder (1979). They prove that the Swiss principle is iterative if and only if it reduces to the mean-value principle or the zero-utility principle with a linear or exponential utility function. Gerber (1979) also notes that if $S = X_1 + \cdots + X_N$ is a random sum and premium principle $H(X)$ is both additive and iterative, then $H(S) = H(H(S|N)) = H(H(X)N)$. Moreover, Goovaerts *et al.* (2010) conclude that if the premium principle is a mixture of exponential functions, then it is iterative if and only if the mixture function is degenerate.

Kupper and Schachermayer (2009) note that there is a connection between iterative premium principles and dynamic time-consistent risk measures. They show that the only law invariant, time consistent and relevant dynamic risk measure is the entropic one. More results on dynamic time-consistent risk measures are given by Acciaio and Penner (2011) and Föllmer and Schied (2011).

In the rank-dependent utility model, it is assumed that probabilities are distorted by some increasing function $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$, called probability distortion function (e.g. Segal, 1989; Denneberg, 1994). Let $\mathcal{G}$ denote the class of all probability distortion functions. For a fixed $g \in \mathcal{G}$ and non-negative random variable X, the Choquet integral is defined by

$$E_g X := \int_0^\infty g\left(P\left(X > t\right)\right) dt.$$

Furthermore, we assume that all random variables are defined on some probability space $(\Omega, \mathcal{A}, P)$. If X takes finite number of values $x_1 < x_2 < \cdots < x_n$ with probabilities $P(X = x_i) = p_i > 0$, then $E_g X = x_1 + \sum_{i=1}^{n-1} g(q_i)(x_{i+1} - x_i)$, where $q_i = \sum_{k=i+1}^{n} p_k$; in particular, for $n = 2$ we have $E_g X = x_1(1 - g(p_2)) + g(p_2)x_2$.

For $g, h \in \mathcal{G}$ and an arbitrary random variable X, the generalized Choquet integral is defined as

$$E_{gh} X = E_g X_+ - E_h\left(-X\right)_+,$$

provided that both integrals are finite. Here and subsequently, $X_+ = \max\{0, X\}$. The generalized Choquet integral is introduced by Tversky and Kahneman (1992) for discrete random variables and is used to describe mathematical foundations of cumulative prospect theory. In numerous experiments, Tversky and Kahneman note that probabilities of losses are distorted in a different way than probabilities of gains. They suggest replacing the utility function with a value function that depends on a relative payoff. In contrast to expected utility theory, the value function measures losses and gains but not absolute wealth. Under cumulative prospect theory, both value function and probability distortion function do not have to be differentiable.

Now, we remind a premium principle which is a modification of the mean-value principle adjusted to cumulative prospect theory. Let X be an arbitrary random variable which does not have to be non-negative. Then X should be regarded as a total claim made by the insured, decreased by the possible gain earned from investment. This allows us to consider insurance products involving some investment options such as investment-linked life insurance or variable annuity. In the case of non-life insurance, it is plausible to study non-negative random variables. Consider a decision maker whose reference point is $w \geq 0$ (e.g. initial wealth) and who wants to purchase an insurance policy paying out the monetary equivalent of the random loss X. Furthermore, we call $(X - w)_+$ losses (or catastrophic losses) and $(w - X)_+$ gains (or non-catastrophic losses). Assume that $u_1, u_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are some strictly increasing value functions, where u_1 measures gains and u_2 measures losses. Let g and h be probability distortion functions of gains and losses, respectively. Kaluszka and Krzeszowiec (2012) introduce the premium $H(X)$ for insuring X as the solution of

$$u_1((w-H(X))_+)-u_2((H(X)-w)_+) = E_g u_1((w-X)_+)-E_h u_2((X-w)_+). \quad (1)$$

Note that (1) can be rewritten as

$$u\,(w - H\,(X)) = E_{gh} u\,(w - X) \quad (2)$$

with strictly increasing function $u(x) = u_1(x_+) - u_2((-x)_+)$ for $x \in \mathbb{R}$, where

$$E_{gh}u(w - X) = E_g[[u(w - X)]_+] - E_h[[-u(w - X)]_+].$$

Gerber (1979) considers a similar equation for premium $H(X)$ under the assumptions that the value function u is convex and probabilities are not distorted, i.e. $g(p) = h(p) = p$. In a more general model, Luan (2001) assumes that $h = \overline{g}$, g is concave and the value function is convex, where $\overline{g}(x) = 1 - g(1 - x)$. Van der Hoek and Sherris (2001) analyze a functional with different probability distortion functions for gains and losses. However, they study only the case when the value functions are linear. Goovaerts *et al.* (2010) consider a risk measure obtained by applying the equivalent utility principle in rank-dependent utility and analyze when such a defined measure is additive. Their result states that the probability distortion functions for gains and losses must be identities. Wang and Young (1998) use distorted probabilities and study properties of the risk-adjusted credibility premium. Al-Nowaihi *et al.* (2008), by solving functional equations, state necessary and sufficient conditions describing preference-homogeneity and risk-aversion under cumulative prospect theory. In particular, they prove that probability distortion functions for gains and losses are identical. Kaluszka and Krzeszowiec (2012) analyze the mean-value principle under cumulative prospect theory and study its properties. Some of them are satisfied only if probabilities for gains and losses are distorted in the same way (or, in particular, they are not distorted). The aim of this paper is to give the complete characterization of the iterativity condition for the mean-value principle under cumulative prospect theory.

2. Main result

In the actuarial literature, we can meet different approaches for defining the conditional Choquet integral (e.g. Wang and Young, 1998; Chateauneuf *et al.*, 2001; Lehrer, 2005; Kast *et al.*, 2008). We define the conditional generalized Choquet integral as

$$E_{gh}(X|Y) = \int_0^\infty g(P(X_+ > s|Y))ds - \int_0^\infty h(P((-X)_+ > s|Y))ds,$$

if both integrals are finite. Then, $H(X|Y)$ is introduced as the solution of

$$u(w - H(X|Y)) = E_{gh}[u(w - X)\,|\,Y].$$

A premium principle $H(X)$ is said to be iterative, if for all X, Y

$$H(X) = H(H(X|Y)),$$

provided that both $H(X)$ and $H(H(X|Y))$ exist.

In order to characterize the property of iterativity of the premium principle which is the solution of (2), we need an auxiliary lemma. Furthermore, we denote $\sup X = \inf\{x : P(X > x) = 0\}$ and $\inf X = -\sup(-X)$.

Lemma 1. *Let* X, Y *be arbitrary random variables. Then* $\sup(\sup(X|Y)) = \sup X$.

Proof of Lemma 1. Let $\sup(X|Y) = \inf\{x : P(X > x|Y) = 0\}$. Assume that $P(X > 0) > 0$. Since $X \le \sup X$ a.s., thus $X_+^k \le (\sup X)^k$ a.s. for all $k \in \mathbb{N}$, where $X_+^k = (X_+)^k$. Hence, $[E(X_+^k|Y)]^{1/k} \le \sup X$ for all $k \in \mathbb{N}$ and

$$\sup(X|Y) = \sup_{k\in\mathbb{N}}\left[E\left(X_+^k|Y\right)\right]^{1/k} \le \sup X \text{ a.s.}$$

Thus, $\sup(\sup(X|Y)) \le \sup X$. As for any random variable Z, we have $\sup Z = \sup_{k\in\mathbb{N}} (EZ_+^k)^{1/k}$ (see Aliprantis and Border, 2006, p. 462); hence, for all $k \in \mathbb{N}$ we have

$$\begin{aligned}\sup(\sup(X|Y)) &= \sup_{k\in\mathbb{N}}\left(E(\sup(X|Y))_+^k\right)^{1/k} \ge \left(E(\sup(X|Y))_+^k\right)^{1/k}\\ &\ge \left[E\left(\left(E\left(X_+^k|Y\right)\right)^{1/k}\right)_+^k\right]^{1/k} = \left(EX_+^k\right)^{1/k}. \qquad (3)\end{aligned}$$

From (3), it follows that $\sup(\sup(X|Y)) \ge \sup X$. Finally, $\sup(\sup(X|Y)) = \sup X$. If $P(X \le 0) = 1$, then we may add some c such that $P(X + c > 0) > 0$ and use the fact that $\sup(X + c) = \sup X + c$. ■

Let $H(X)$ be the premium principle determined from (2). Consider the following cases:

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(i) If $g(x) = h(x) = x$ for $0 \le x \le 1$, then $E_{gh}X = EX$ and $H(X) = w - u^{-1}(Eu(w - X))$. Thus, $H(X)$ is the mean-value principle, which is iterative (see Gerber, 1979; Goovaerts *et al.*, 1984).

(ii) If $g(x) = \mathbf{1}_{\{1\}}(x)$ and $h(x) = \overline{g}(x) = \mathbf{1}_{(0,1]}(x)$, then $E_{gh}X = \inf X$ and $H(X) = \sup X$. From Lemma 1, it follows that $H(X)$ is iterative.

(iii) If $g(x) = \mathbf{1}_{(0,1]}(x)$ and $h(x) = \overline{g}(x) = \mathbf{1}_{\{1\}}(x)$, then $E_{gh}X = \sup X$ and $H(X) = \inf X$. As $\inf X = -\sup(-X)$, from (ii) it follows that $\inf(\inf(X|Y)) = \inf X$.

(iv) If $g(x) = h(x) = \mathbf{1}_{\{1\}}(x)$ for $0 \le x \le 1$, then $E_{gh}X = (\inf X)_+ - (-\sup X)_+$ and

$$H(X) = \begin{cases} \sup X & \text{if } X \le w \text{ a.s.,} \\ \inf X & \text{if } X \ge w \text{ a.s.,} \\ w & \text{if } \inf X \le w \le \sup X. \end{cases}$$

It is clear that if $H(X) = w$, then $H(X)$ is iterative. From this, (ii) and (iii), it follows that $H(X)$ is iterative.

(v) If $g(x) = x$ and $h(x) = \mathbf{1}_{\{1\}}(x)$ for $0 \le x \le 1$, then $E_{gh}X = EX_+ - (-\sup X)_+$ and

$$H(X) = \begin{cases} w - u^{-1}(Eu(w - X)) & \text{if } X \le w \text{ a.s.,} \\ \inf X & \text{if } X \ge w \text{ a.s.,} \\ w - u^{-1}\left(E[u(w - X)]_+\right) & \text{if } \inf X \le w \le \sup X. \end{cases}$$

Note that $E[E(X_+|Y)]_+ = E[E(X_+|Y)] = EX_+$. From this, (i) and (iii), it follows that $H(X)$ is iterative.

(vi) If $g(x) = \mathbf{1}_{\{1\}}(x)$ and $h(x) = x$ for $0 \le x \le 1$, then $E_{gh}X = (\inf X)_+ - E(-X)_+$ and

$$H(X) = \begin{cases} \sup X & \text{if } X \le w \text{ a.s.,} \\ w - u^{-1}(Eu(w - X)) & \text{if } X \ge w \text{ a.s.,} \\ w - u^{-1}\left(E[-u(w - X)]_+\right) & \text{if } \inf X \le w \le \sup X. \end{cases}$$

From (i), (ii) and (v), it follows that $H(X)$ is iterative.

The main theorem of this paper is the following.

Theorem 2. *Let $w \ge 0$ be fixed. Assume that u is strictly increasing, continuous, $u(0) = 0$ and $g, h \in \mathcal{G}$. Then, $H(X)$ which is the solution of* (2) *is iterative if and only if $H(X)$ is defined by one of the formulas from* (i) *to* (vi).

Proof. For a fixed w, let $v(x) := u(w - x)$. Then $H(X) = v^{-1}(E_{gh}v(X))$. Moreover,

$$H(X|Y) = v^{-1}(E_{gh}(v(X)|Y)) \tag{4}$$

and

$$v(H(H(X|Y))) = E_{gh}v(H(X|Y)). \tag{5}$$

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From (4) and (5), the condition $H(X) = H(H(X|Y))$ is equivalent to

$$v^{-1}(E_{gh}v(X)) = v^{-1}(E_{gh}v(H(X|Y))) = v^{-1}(E_{gh}(E_{gh}(v(X)|Y))),$$

which means that the premium principle is iterative if and only if

$$E_{gh}Z = E_{gh}(E_{gh}(Z|Y)), \tag{6}$$

where $Z = v(X)$. The first implication of this theorem follows from (i) to (vi). Now, we prove the inverse implication.

Let (Z, Y) be the random vector with distribution $P(Z=0, Y=1) = 1/4$, $P(Z=s, Y=1) = 1/4$, $P(Z=s, Y=2) = 1/4$, $P(Z=1, Y=2) = 1/4$, where $0 < s < 1$ is arbitrary. Then

$$\begin{aligned} E_{gh}Z &= s\,(g\,(3/4) - g\,(1/4)) + g\,(1/4)\,,\\ E_{gh}\,(Z|\,Y=1) &= sg\,(1/2)\,,\\ E_{gh}\,(Z|\,Y=2) &= s\,(1-g\,(1/2)) + g\,(1/2)\,. \end{aligned} \tag{7}$$

Since $E_{gh}\,(Z|\,Y=1) \le E_{gh}\,(Z|\,Y=2)$, we have

$$E_{gh}(E_{gh}(Z|Y)) = s(2g(1/2) - 2(g(1/2))^2) + (g(1/2))^2. \tag{8}$$

From (6), (7) and (8), it follows that the premium principle $H(X)$ is iterative if

$$s(g\,(3/4) - g\,(1/4)) + g\,(1/4) = s(2g\,(1/2) - 2\,(g\,(1/2))^2) + (g\,(1/2))^2$$

for $0 < s < 1$. We obtained the equality of two polynomials of variable s. Comparing their coefficients yields

$$\begin{cases} g\,(3/4) - g\,(1/4) = 2g\,(1/2) - 2\,(g\,(1/2))^2\,,\\ g\,(1/4) = (g\,(1/2))^2\,. \end{cases} \tag{9}$$

Now, let random vector (Z, Y) has the distribution $P(Z=0, Y=1) = 1/4$, $P(Z=1, Y=1) = 1/4$, $P(Z=g\,(1/2)\,, Y=2) = 1/2$. Then

$$\begin{aligned} E_{gh}Z &= g\,(1/2)\,(g\,(3/4) - g\,(1/4)) + g\,(1/4)\,,\\ E_{gh}\,(Z|\,Y=1) &= E_{gh}\,(Z|\,Y=2) = g\,(1/2) \end{aligned} \tag{10}$$

and since $E_{gh}c = c$ for $c \in \mathbb{R}$ (see Kaluszka and Krzeszowiec, 2012), thus

$$E_{gh}(E_{gh}(Z|Y)) = g\,(1/2)\,. \tag{11}$$

From (6), (10) and (11), it follows that $H(X)$ is iterative if $g\,(1/2)\,(g\,(3/4) - g\,(1/4)) + g\,(1/4) = g\,(1/2)$. From this and (9), we have that $2\,(g\,(1/2))^3 - 3\,(g\,(1/2))^2 + g\,(1/2) = 0$. Hence, $g\,(1/2) = 0$, or $g\,(1/2) = 1/2$ or $g\,(1/2) = 1$.

Consider the random vector (Z, Y) with distribution $P(Z=s, Y=1) = 1/4 + c$, $P(Z=1, Y=1) = 1/4 - c$, $P(Z=s, Y=2) = 1/4 + d$,

$P(Z=1,\ Y=2) = 1/4 - d$, where $0 < s < 1$ is arbitrary and $0 \leq c, d \leq 1/4$. Then

$$\begin{aligned} E_{gh}Z &= s(1-g(1/2-c-d)) + g(1/2-c-d), \\ E_{gh}(Z|Y=1) &= s(1-g(1/2-2c)) + g(1/2-2c), \\ E_{gh}(Z|Y=2) &= s(1-g(1/2-2d)) + g(1/2-2d). \end{aligned} \tag{12}$$

Moreover, $E_{gh}(Z|Y=1) \leq E_{gh}(Z|Y=2)$ if and only if $c \geq d$. Thus,

$$\begin{aligned} E_{gh}(E_{gh}(Z|Y)) = {} & [s(1-g(1/2-2c)) + g(1/2-2c)](1-g(1/2)) \\ & + [s(1-g(1/2-2d)) + g(1/2-2d)]\,g(1/2) \end{aligned} \tag{13}$$

if $0 \leq c \leq d \leq 1/4$ and

$$\begin{aligned} E_{gh}(E_{gh}(Z|Y)) = {} & [s(1-g(1/2-2d)) + g(1/2-2d)](1-g(1/2)) \\ & + [s(1-g(1/2-2c)) + g(1/2-2c)]\,g(1/2) \end{aligned} \tag{14}$$

if $0 \leq d \leq c \leq 1/4$. From (6) and (12)–(14), it follows that $H(X)$ is iterative if

$$\begin{aligned} & s(1-g(1/2-c-d)) + g(1/2-c-d) \\ & \quad = [s(1-g(1/2-2c)) + g(1/2-2c)](1-g(1/2)) + [s(1-g(1/2-2d)) \\ & \qquad + g(1/2-2d)]\,g(1/2) \end{aligned} \tag{15}$$

for $0 < s < 1$ if $0 \leq c \leq d \leq 1/4$ and

$$\begin{aligned} & s(1-g(1/2-c-d)) + g(1/2-c-d) \\ & \quad = [s(1-g(1/2-2d)) + g(1/2-2d)](1-g(1/2)) + [s(1-g(1/2-2c)) \\ & \qquad + g(1/2-2c)]\,g(1/2) \end{aligned} \tag{16}$$

for $0 < s < 1$ if $0 \leq d \leq c \leq 1/4$. Consider the following cases:

(a) $g(1/2) = 0$. Then clearly $g(x) = 0$ for $0 \leq x \leq 1/2$.
(b) $g(1/2) = 1/2$. Then from (15) and (16), we have

$$\begin{aligned} & 2s(1-g(1/2-c-d)) + 2g(1/2-c-d) \\ & \quad = s(1-g(1/2-2c)) + g(1/2-2c) + s(1-g(1/2-2d)) + g(1/2-2d) \end{aligned}$$

for $0 < s < 1$ and $0 \leq c, d \leq 1/4$. Comparing coefficients of the above polynomials gives

$$2g(1/2-c-d) = g(1/2-2c) + g(1/2-2d) \tag{17}$$

for $0 \leq c, d \leq 1/4$. Let $f(x) = g(1/2-2x)$. Then, (17) can be rewritten as $2f((c+d)/2) = f(c) + f(d)$ for $0 \leq c, d \leq 1/4$, which is the Jensen functional equation. Since f is monotonic, thus measurable, it follows that f is linear (see Kuczma, 2009, p. 354). As $f(0) = g(1/2) = 1/2$ and $f(1/4) = g(0) = 0$, hence $f(x) = -2x + 1/2$ for $0 \leq x \leq 1/4$. Finally, $g(x) = x$ for $0 \leq x \leq 1/2$.

(c) $g(1/2) = 1$. Then from (15), we have

$$s(1 - g(1/2 - c - d)) + g(1/2 - c - d) = s(1 - g(1/2 - 2d)) + g(1/2 - 2d)$$

for $0 < s < 1$ and $0 \leq d \leq c \leq 1/4$. Comparing the coefficients of the above polynomials gives

$$g(1/2 - c - d) = g(1/2 - 2d) \tag{18}$$

for all $0 \leq d \leq c \leq 1/4$. Putting $d = 0$ yields $g(1/2 - c) = g(1/2) = 1$ for $0 \leq c \leq 1/4$. Thus, $g(x) = 1$ for $1/4 \leq x \leq 1/2$. Setting $c = 1/4$ in (18) gives $g(x) = g(2x)$ for $0 \leq x \leq 1/4$. Since $g(1/2) = 1$, we have $g(1/2) = g(2/4) = g(1/4) = g(2/8) = g(1/8)$ and so on. Hence, $g(x) = 1$ for $0 < x \leq 1/2$.

Consider the random vector (Z, Y) with distribution $P(Z = s, Y = 1) = 1/4 - c$, $P(Z = 1, Y = 1) = 1/4 + c$, $P(Z = s, Y = 2) = 1/4 - d$, $P(Z = 1, Y = 2) = 1/4 + d$, where $0 < s < 1$ is arbitrary and $0 \leq c, d \leq 1/4$. Then, an analogous argumentation as in the previous part of the proof shows that $H(X)$ is iterative if

$$\begin{aligned} s(1 - g(1/2 + c + d)) &+ g(1/2 + c + d) \\ &= [s(1 - g(1/2 + 2c)) + g(1/2 + 2c)](1 - g(1/2)) \\ &\quad + [s(1 - g(1/2 + 2d)) + g(1/2 + 2d)]\, g(1/2) \end{aligned} \tag{19}$$

for $0 < s < 1$ if $0 \leq c \leq d \leq 1/4$ and

$$\begin{aligned} s(1 - g(1/2 + c + d)) &+ g(1/2 + c + d) \\ &= [s(1 - g(1/2 + 2d)) + g(1/2 + 2d)](1 - g(1/2)) \\ &\quad + [s(1 - g(1/2 + 2c)) + g(1/2 + 2c)]\, g(1/2) \end{aligned} \tag{20}$$

for $0 < s < 1$ if $0 \leq d \leq c \leq 1/4$. Consider the following cases:

(a) $g(1/2) = 0$. From (19), we get

$$s(1 - g(1/2 + c + d)) + g(1/2 + c + d) = s(1 - g(1/2 + 2c)) + g(1/2 + 2c)$$

for $0 \leq c \leq d \leq 1/4$. Comparing coefficients of the above polynomials gives

$$g(1/2 + c + d) = g(1/2 + 2c) \tag{21}$$

for $0 \leq c \leq d \leq 1/4$. Putting $c = 0$ yields $g(1/2 + d) = g(1/2) = 0$ for $0 \leq d \leq 1/4$. Thus, $g(x) = 0$ for $1/2 \leq x \leq 3/4$. If we set $d = 1/4$ in (21), then we get

$$g(3/4 + c) = g(1/2 + 2c) \tag{22}$$

for $0 \leq c \leq 1/4$. Putting $c = 1/8$ in (22) gives $g(7/8) = g(3/4) = 0$. Hence, $g(x) = 0$ for $1/2 \leq x \leq 7/8$. Setting $c = 3/16$ in (22) gives $g(15/16) = g(7/8) = 0$. Thus, $g(x) = 0$ for $1/2 \leq x \leq 15/16$. An analogous reasoning shows that $g(x) = 0$ for $1/2 \leq x < 1$.

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(b) $g(1/2) = 1/2$. Then from (19) and (20), we have

$$\begin{aligned} &2s(1 - g(1/2 + c + d)) + 2g(1/2 + c + d) \\ &\quad = s(1 - g(1/2 + 2c)) + g(1/2 + 2c) + s(1 - g(1/2 + 2d)) \\ &\qquad + g(1/2 + 2d) \end{aligned}$$

for $0 < s < 1$ and $0 \le c, d \le 1/4$. Comparing coefficients of the above polynomials gives

$$2g(1/2 + c + d) = g(1/2 + 2c) + g(1/2 + 2d) \tag{23}$$

for $0 \le c, d \le 1/4$. Let $f(x) = g(1/2 + 2x)$. Then (23) can be rewritten as $2f((c + d)/2) = f(c) + f(d)$ for $0 \le c, d \le 1/4$. Since f is measurable, thus it is linear (see Kuczma, 2009, p. 354). As $f(0) = g(1/2) = 1/2$ and $f(1/4) = g(1) = 1$; hence, $f(x) = 2x + 1/2$ for $0 \le x \le 1/4$. Finally, $g(x) = x$ for $1/2 \le x \le 1$.

(c) $g(1/2) = 1$. Then clearly $g(x) = 1$ for $1/2 \le x \le 1$.

Reassuming, so far we proved that if the premium principle $H(X)$ is iterative, then $g(x) = x$, or $g(x) = \mathbf{1}_{\{1\}}(x)$ or $g(x) = \mathbf{1}_{(0,1]}(x)$ for $0 \le x \le 1$. Furthermore, it suffices to note that $E_{gh}(-X) = -E_{hg}X$ in order to conclude that $h(x) = x$, or $h(x) = \mathbf{1}_{\{1\}}(x)$ or $h(x) = \mathbf{1}_{(0,1]}(x)$ for $0 \le x \le 1$ (see (6)). Thus, we obtained nine possible pairs of functions g and h. We will show that for three of these pairs the iterativity condition is not satisfied. Let random vector (Z, Y) has the distribution $P(Z = -1, Y = 1) = 1/4$, $P(Z = 1, Y = 1) = 1/2$, $P(Z = -1, Y = 2) = 1/4$. Then $E_{gh}Z = g(1/2) - h(1/2)$, $E_{gh}(Z|Y = 1) = g(2/3) - h(1/3)$ and $E_{gh}(Z|Y = 2) = -1$. As $E_{gh}(Z|Y = 1) \ge E_{gh}(Z|Y = 2)$, thus $E_{gh}(E_{gh}(Z|Y)) = (g(2/3) - h(1/3))g(3/4) - h(1/4)$ if $g(2/3) - h(1/3) \ge 0$ and $E_{gh}(E_{gh}(Z|Y)) = (g(2/3) - h(1/3))(1 - h(1/4)) - h(1/4)$ if $g(2/3) - h(1/3) \le 0$. From (6), it follows that if $H(X)$ is iterative, then

$$g(1/2) - h(1/2) = (g(2/3) - h(1/3))\, g(3/4) - h(1/4) \tag{24}$$

if $g(2/3) - h(1/3) \ge 0$ and

$$g(1/2) - h(1/2) = (g(2/3) - h(1/3))(1 - h(1/4)) - h(1/4) \tag{25}$$

if $g(2/3) - h(1/3) < 0$. From (24) and (25), it follows that $H(X)$ is not iterative if $g(x) = \mathbf{1}_{(0,1]}(x)$ and $h(x) = x$, or $g(x) = x$ and $h(x) = \mathbf{1}_{(0,1]}(x)$, or $g(x) = h(x) = \mathbf{1}_{(0,1]}(x)$. ■

3. Concluding remarks

In this paper, by solving functional equations, we analyze the mean-value principle under cumulative prospect theory. We give full characterization of the iterativity condition for this premium principle which turns out to be equivalent to the iterativity of the generalized Choquet integral. We prove that this property is satisfied not only when probabilities are not distorted (which corresponds to the classical mean-value principle), but also for five other cases. For all of them, the probability distortion functions are either identities or are degenerated to two points, producing the premium principles which are not commonly met in the actuarial literature. Regarding this result, in order to study possibly wide class of functionals under cumulative prospect theory, there is a justified necessity to put possibly weakest assumptions on value function and probability distortion functions.

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