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ISOMORPHISM OF RELATIVE HOLOMORPHS AND MATRIX SIMILARITY

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Abstract

Let V be a finite dimensional vector space over the field with p elements, where p is a prime number. Given arbitrary $\alpha, \beta \in \mathrm{GL}(V)$, we consider the semidirect products $V \rtimes \langle\alpha\rangle$ and $V \rtimes \langle\beta\rangle$, and show that if $V \rtimes \langle\alpha\rangle$ and $V \rtimes \langle\beta\rangle$ are isomorphic, then α must be similar to a power of β that generates the same subgroup as β; that is, if H and K are cyclic subgroups of $\mathrm{GL}(V)$ such that $V \rtimes H \cong V \rtimes K$, then H and K must be conjugate subgroups of $\mathrm{GL}(V)$. If we remove the cyclic condition, there exist examples of *nonisomorphic*, let alone nonconjugate, subgroups H and K of $\mathrm{GL}(V)$ such that $V \rtimes H \cong V \rtimes K$. Even if we require that noncyclic subgroups H and K of $\mathrm{GL}(V)$ be abelian, we may still have $V \rtimes H \cong V \rtimes K$ with H and K nonconjugate in $\mathrm{GL}(V)$, but in this case, H and K must at least be isomorphic. If we replace V by a free module U over $\mathbb{Z}/p^m\mathbb{Z}$ of finite rank, with $m > 1$, it may happen that $U \rtimes H \cong U \rtimes K$ for nonconjugate cyclic subgroups of $\mathrm{GL}(U)$. If we completely abandon our requirements on V, a sufficient criterion is given for a finite group G to admit nonconjugate cyclic subgroups H and K of $\mathrm{Aut}(G)$ such that $G \rtimes H \cong G \rtimes K$. This criterion is satisfied by many groups.

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1. Introduction

We fix throughout a prime number p and write F for the field with p elements and V for an F-vector space of finite dimension $n > 0$. Given an automorphism α of V, we may consider the semidirect product $G_\alpha = V \rtimes \langle\alpha\rangle$, where

$$\alpha v \alpha^{-1} = \alpha(v), \quad v \in V.$$

Likewise, given $\beta \in \mathrm{GL}(V)$, we have the semidirect product $G_\beta = V \rtimes \langle\beta\rangle$. It is well known that if $\langle\alpha\rangle$ and $\langle\beta\rangle$ are conjugate subgroups of $\mathrm{GL}(V)$, then G_α is isomorphic to G_β. In Theorem 4.1, we prove the converse: if $G_\alpha \cong G_\beta$, then $\langle\alpha\rangle$ and $\langle\beta\rangle$ must be conjugate in $\mathrm{GL}(V)$. Here $\langle\alpha\rangle$ and $\langle\beta\rangle$ are conjugate if and only if α is similar to β^i for some integer i coprime to the order of β, that is, such that $\langle\beta^i\rangle = \langle\beta\rangle$. The proof of

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Theorem 4.1 is somewhat subtle. A more transparent argument is given in Theorem 5.2, provided α and β are unipotent, in which case, α and β must themselves be similar. Theorem 4.1 seems to be exceptional in the realm of group theory, in the sense that any changes to the given linear algebra setting will tend to make it fail, as explained below.

Given an arbitrary group G, its holomorph is $\mathrm{Hol}(G) = G \rtimes \mathrm{Aut}(G)$, where

$$\alpha g \alpha^{-1} = \alpha(g), \quad \alpha \in \mathrm{Aut}(G), \quad g \in G.$$

For a subgroup H of $\mathrm{Aut}(G)$, we have the relative holomorph $\mathrm{Hol}(G, H) = G \rtimes H$, viewed as a subgroup of $\mathrm{Hol}(G)$. If K is also a subgroup of $\mathrm{Aut}(G)$, it is well known that if H and K are conjugate in $\mathrm{Aut}(G)$, then $\mathrm{Hol}(G, H)$ and $\mathrm{Hol}(G, K)$ are conjugate in $\mathrm{Hol}(G)$, and hence isomorphic. The converse is false in general. In Example 6.4, we exhibit *nonisomorphic* noncyclic subgroups H and K of $\mathrm{GL}(V)$ such that $\mathrm{Hol}(V, H) \cong \mathrm{Hol}(V, K)$, provided p is odd and $n \geq 2$. The case $p = 2$ and $n \geq 4$ is dealt with in Example 6.5. Moreover, Example 6.3 gives nonconjugate, noncyclic, abelian subgroups H and K of $\mathrm{GL}(V)$ such that $\mathrm{Hol}(V, H) \cong \mathrm{Hol}(V, K)$ for any $n \geq 6$. However, in Corollary 3.4, we show that if H and K are abelian subgroups of $\mathrm{GL}(V)$ such that $\mathrm{Hol}(V, H) \cong \mathrm{Hol}(V, K)$ and the sum of all subspaces $(h - 1)V$, with $h \in H$, is equal to V, then H is necessarily conjugate to K. Furthermore, according to Proposition 5.3, $\mathrm{Hol}(V, H) \cong \mathrm{Hol}(V, K)$ always forces $H \cong K$ when H and K are abelian. Example 6.2 shows that Theorem 4.1 fails, in general, if V is replaced by a free module U over $\mathbb{Z}/p^m\mathbb{Z}$ of finite rank n. Indeed, when $p = 2, m = 3, n = 4$, we exhibit an explicit automorphism α of U such that $\mathrm{Hol}(U, \alpha) \cong \mathrm{Hol}(U, \beta)$ for exactly two conjugacy classes of cyclic subgroups $\langle\beta\rangle$ of $\mathrm{GL}(U)$. When used in conjunction with Lemma 3.3, Examples 6.2 and 6.3 reveal instances when $\mathrm{Hol}(G, H) \cong \mathrm{Hol}(G, K)$ but no isomorphism between them sends G back into itself.

If we completely abandon our requirements on V, Example 6.6 gives a sufficient criterion for a group G to admit cyclic subgroups H and K such that $\mathrm{Hol}(G, H) \cong \mathrm{Hol}(G, K)$, but H and K are not conjugate in $\mathrm{Aut}(G)$. This is illustrated with various instances of such G, H and K. Finally, Example 6.7 lists a few groups G such that the existence of an isomorphism between $\mathrm{Hol}(G, H)$ and $\mathrm{Hol}(G, K)$, for arbitrary subgroups H and K of $\mathrm{Aut}(G)$, forces H to be similar to K.

The holomorph of a group appeared early in the literature to produce examples of complete groups, that is, groups having trivial centre and only inner automorphisms. In 1908, Miller [10] showed that the holomorph of any finite abelian group of odd order is complete, based on prior work of Burnside [2]. Miller's result was extended to show that the holomorph, or a relative holomorph, of other classes of abelian groups, not necessarily finite, is also complete (see [1, 4, 6, 13]). A long-standing problem in this regard was the existence of complete groups of odd order, settled positively by Dark [5] in 1975.

We are concerned here with a specific case of the general problem of finding necessary and sufficient conditions for two semidirect products $A \rtimes_f B$ and $C \rtimes_g D$ to be isomorphic, where $f : B \to \mathrm{Aut}(A)$ and $g : D \to \mathrm{Aut}(C)$ are homomorphisms. No general answer is known. When $A = C$ and $B = D$, this problem was investigated by Taunt [14] and Kuzennyi [9]. If $B = \mathrm{Aut}(A)$, $D = \mathrm{Aut}(C)$, and f and g are the identity maps, Mills showed in [11, 12] respectively that $\mathrm{Hol}(A) \cong \mathrm{Hol}(C)$ forces $A \cong C$ when both A and C are finitely generated abelian groups and when A or C are finite abelian groups. Mills [11] also pointed out that when $n \geq 3$, the nonisomorphic dihedral and generalised quaternion groups of order $4n$ have isomorphic holomorphs. A proof can be found in Kohl's paper [7, Proposition 3.10]. In Miller's work [10], the holomorph of a group G is viewed as the normaliser in the symmetric group $S(G)$ of the left (or right) regular representation of G. Miller referred to the normaliser in $S(G)$ of $\mathrm{Hol}(G)$ as the multiple holomorph, say $M(G)$, of G. The quotient group $T(G) = M(G)/\mathrm{Hol}(G)$ has interesting properties and has received considerable attention recently. The structure of $T(G)$ was determined by Kohl [8] for dihedral and generalised quaternion groups, and by Caranti and Dalla Volta [3] for finite perfect groups with trivial centre.

The conjugacy of subgroups H and K of $\mathrm{Aut}(G)$ is an obvious sufficient condition for $\mathrm{Hol}(G, H)$ to be isomorphic to $\mathrm{Hol}(G, K)$, and in the present paper, we show that it is also necessary, provided H and K are cyclic and $G = V$. Our examples show that one cannot deviate much from the stated hypotheses for the necessity of conjugacy to hold. One may use our results in the classification, up to isomorphism, of the relative holomorphs of an elementary abelian group such as V. This is a difficult problem, as attested by one of its simplest cases [15], namely when $p = 2$ and $n = 4$, in which case, there are 138 relative holomorphs.

2. Background from linear algebra

Given $\alpha \in \mathrm{End}(V)$, we write $m_\alpha \in F[X]$ for the minimal polynomial of α. Given a group G and $g \in G$, we let $o(g)$ stand for the order of g.

LEMMA 2.1. *Let $\alpha \in \mathrm{GL}(V)$. Then $p \nmid o(\alpha)$ if and only if m_α is square-free.*

PROOF. Let $m = o(\alpha)$. Then m is the smallest natural number such that $m_\alpha \mid (X^m - 1)$. Now $X^m - 1$ and its formal derivative, namely mX^{m-1}, are relatively prime if and only if $p \nmid m$. □

LEMMA 2.2. *Let $\alpha \in \mathrm{GL}(V)$. Then α is similar to α^p if and only if $p \nmid o(\alpha)$.*

PROOF. If $p \mid o(\alpha)$, then α and α^p have different orders, whence they are not similar.

Suppose $p \nmid o(\alpha)$. Then by Lemma 2.1, the invariant factors of α are all square-free. Thus, the rational canonical form of α is the direct sum of companion matrices C_f to square-free monic polynomials $f \in F[X]$. Let $f \in F[X]$ be an invariant factor of α of degree d. Since every irreducible polynomial over F has distinct roots in any splitting field and f is square-free, it follows that f has distinct roots in a splitting field, say K. In particular, C_f is similar in $\mathrm{GL}_d(K)$ to a diagonal matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_d)$. Thus, C_f^p is similar in $\mathrm{GL}_d(K)$ to $\mathrm{diag}(\lambda_1^p, \ldots, \lambda_d^p)$. Now the map $\Omega : K \to K$ given by $\Omega(k) = k^p$, for $k \in K$, is an automorphism of K with fixed field F. We also

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write Ω for the associated automorphism $K[X] \to K[X]$. Since $f \in F[X]$, $f = \Omega(f) = (X - \lambda_1^p)\cdots(X - \lambda_d^p)$ is the minimal polynomial of C_f^p. Thus, C_f^p is similar to C_f in $\mathrm{GL}_d(F)$, whence α and α^p have the same rational canonical form. □

Given a field K, and K-vector spaces V_1 and V_2 with automorphisms α_1 and α_2, respectively, we say that α_1 and α_2 are similar if there is an isomorphism $f : V_1 \to V_2$ such that $f\alpha_1 f^{-1} = \alpha_2$.

LEMMA 2.3. *Let R be a principal ideal domain, M a nonzero finitely generated torsion R-module and $q \in R$ an irreducible element. Then the isomorphism type of M is completely determined by the composition length of M and the isomorphism type of $N = qM$. In particular, if K is a field, W is a nonzero finite dimensional K-vector space, $q \in K[X]$ is irreducible, and $u, v \in \mathrm{End}(W)$ are such that $u|_{q(u)W}$ and $v|_{q(v)W}$ are similar, then u and v are similar.*

PROOF. For any $r \in R$, set

$$M_r = \{x \in M \mid r^t x = 0 \text{ for some } t \geq 1\}, \quad N_r = \{x \in N \mid r^t x = 0 \text{ for some } t \geq 1\}.$$

Let $(q_1^{a_1} \cdots q_s^{a_s})$ be the annihilating ideal of M, where $q_1, \ldots, q_s$ are nonassociate irreducible elements of R and each $a_i \geq 1$. If q is nonassociate to every q_i, then $M = N$ and there is nothing to do. We assume henceforth that q is associate to some q_i, say q_1. Then $M_{q_j} = N_{q_j}$ for every $j > 1$. Moreover, $M = M_{q_1} \oplus \cdots \oplus M_{q_s}$.

It remains to show that the isomorphism type of $M_q = M_{q_1}$ is determined by that of N_q and the composition length of M. There exists a unique sequence of nonnegative integers $(e_1, e_2, \ldots)$, which is eventually 0, and such that $M_q = U_1 \oplus U_2 \oplus \cdots$, where each U_i is the direct sum of e_i cyclic submodules with annihilating ideal (q^i). The corresponding sequence for N_q is clearly $(e_2, e_3, \ldots)$. Thus, $e_2, e_3, \ldots$ are determined by the isomorphism type of N. Moreover, the composition length of M is equal to the sum of the composition lengths of the M_{q_j}, $j > 1$, plus $e_1 + 2e_2 + 3e_3 + \cdots$, so e_1 is determined by the composition length of M and the isomorphism type of N. This proves the first statement.

As for the second, take $R = K[X]$, and view $M = W$ and $N = q(u)W$ as R-modules via u. The similarity types of u and $u|_N$ are completely determined by the isomorphism types of M and N, respectively, as R-modules. Moreover, in the above notation, $e_2, e_3, \ldots$ are determined by the isomorphism type of N, and

$$\dim_K M = (\deg q)(e_1 + 2e_2 + 3e_3 + \cdots) + \dim_K M_{q_2} + \cdots + \dim_K M_{q_s},$$

so e_1 is determined by the fixed quantities $\dim_K M$ and $\deg q$, together with the isomorphism type of N as R-module. This proves the second statement. □

3. Background from group theory

Given a group G with subgroups H and K, we say that H and K are conjugate in G if there is $g \in G$ such that $gHg^{-1} = K$, in which case, we write $H \sim K$.

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LEMMA 3.1. *Let G be a group and suppose that H_1, H_2 are conjugate subgroups of* $\mathrm{Aut}(G)$. *Then* $\mathrm{Hol}(G, H_1)$ *and* $\mathrm{Hol}(G, H_2)$ *are conjugate subgroups of* $\mathrm{Hol}(G)$, *and are therefore isomorphic.*

PROOF. By assumption, there is $\gamma \in \mathrm{Aut}(G)$ such that $\gamma H_1 \gamma^{-1} = H_2$. Then $\gamma \in \mathrm{Hol}(G)$ and we have $\gamma G \gamma^{-1} = \gamma(G) = G$ inside $\mathrm{Hol}(G)$. Therefore,

$$\gamma \mathrm{Hol}(G, H_1)\gamma^{-1} = \gamma(G \rtimes H_1)\gamma^{-1} = \gamma(G) \rtimes \gamma H_1 \gamma^{-1} = G \rtimes H_2 = \mathrm{Hol}(G, H_2). \qquad \square$$

LEMMA 3.2. *Let G be a finite group having a normal subgroup N with* $\gcd(|G/N|, |N|) = 1$. *Then any subgroup K of G such that $|K|$ is a factor of $|N|$ must be included in N. In particular, if $x \in G$ is such that $o(x)$ is a factor of $|N|$, then $x \in N$.*

PROOF. As $KN/N \cong K/(K \cap N)$, we have $|KN/N| \mid |K|$. However, KN/N is a subgroup of G/N, so $|KN/N| \mid |G/N|$. Since $\gcd(|G/N|, |N|) = 1$, we infer that KN/N is trivial, so $K \subseteq N$. □

LEMMA 3.3. *Let A be an abelian group, and let H and K be subgroups of* $\mathrm{Aut}(A)$. *Suppose that $f : \mathrm{Hol}(A, H) \to \mathrm{Hol}(A, K)$ is an isomorphism such that $f(A) = A$. Then $H \sim K$. In particular, if A is finite and abelian, and* $\gcd(|A|, |H|) = 1$, *then* $\mathrm{Hol}(A, H) \cong \mathrm{Hol}(A, K)$ *forces $H \sim K$.*

PROOF. By hypothesis, f restricts to an automorphism u of A and induces an isomorphism $g : \mathrm{Hol}(A, H)/A \to \mathrm{Hol}(A, K)/A$. Moreover, there are isomorphisms $i : H \to \mathrm{Hol}(A, H)/A$ and $j : K \to \mathrm{Hol}(A, K)/A$. Let $v : H \to K$ be the isomorphism given by $v = j^{-1}gi$.

Given any $h \in H$, we have $f(h) = bk$ for unique $b \in A$ and $k \in K$ and, by definition, $v(h) = k$. Since A is abelian, conjugation by $k = v(h)$ and $bk = f(h)$ agree on A. Thus, for any $a \in A$,

$$u((h)(a)) = u(hah^{-1}) = f(hah^{-1}) = f(h)f(a)f(h)^{-1} = v(h)u(a)v(h)^{-1} = v(h)(u(a)). \tag{3.1}$$

Therefore, $uhu^{-1} = v(h)$ for all $h \in H$. As v is an isomorphism, we infer $uHu^{-1} = K$, which proves the first part. As for the second, suppose that A is finite and abelian, $\gcd(|A|, |H|) = 1$, and that $s : \mathrm{Hol}(A, H) \to \mathrm{Hol}(A, K)$ is an isomorphism. Then $s(A)$ is a normal subgroup of $\mathrm{Hol}(A, K)$ of order $|A|$, where $|A|$ is relatively prime to $|H| = |K| = |\mathrm{Hol}(A, K)/A|$. It follows from Lemma 3.2 that $s(A) = A$, whence $H \sim K$ by the first part. □

The condition that A be abelian is not necessary for the second part of Lemma 3.3, although we will not require this more powerful result. The condition that $f(A) = A$ cannot be removed with impunity from the first part of Lemma 3.3, as Examples 6.2 and 6.3 show.

COROLLARY 3.4. *Let A be a finite (additive) abelian group and let H and K be abelian subgroups of* $\mathrm{Aut}(A)$. *Suppose that* $\mathrm{Hol}(A, H) \cong \mathrm{Hol}(A, K)$ *and that the sum of all subgroups* $(h - 1)(A)$, *as h runs through H, is equal to A. Then $H \sim K$.*

PROOF. Let $f : \mathrm{Hol}(A, H) \to \mathrm{Hol}(A, K)$ be an isomorphism. The stated hypotheses imply that the derived subgroup of $\mathrm{Hol}(A, H)$ (respectively $\mathrm{Hol}(A, K)$) is A (respectively a subgroup of A). Thus, f maps A inside of A. However A is finite, so $f(A) = A$ and Lemma 3.3 applies. □

Examples 6.2 and 6.3 show that Corollary 3.4 fails if the condition on H is removed.

4. Isomorphism of relative holomorphs forces conjugacy of the complements

We are ready to prove our main result.

THEOREM 4.1. *If $\alpha, \beta \in \mathrm{GL}(V)$, then $\mathrm{Hol}(V, \alpha) \cong \mathrm{Hol}(V, \beta)$ if and only if $\langle\alpha\rangle \sim \langle\beta\rangle$.*

PROOF. For $\pi \in \mathrm{GL}(V)$, set $G_\pi = \mathrm{Hol}(V, \pi)$ and let $V_\pi = [G_\pi, G_\pi]$ be the derived subgroup of G_π. We readily see that $V_\pi = (\pi - 1)(V)$.

If $\langle\alpha\rangle \sim \langle\beta\rangle$, then $G_\alpha \cong G_\beta$ by Lemma 3.1. Suppose next $G_\alpha \cong G_\beta$. If $\gcd(p, o(\alpha)) = 1$, then $\langle\alpha\rangle \sim \langle\beta\rangle$ by Lemma 3.3. We suppose henceforth that $p \mid o(\alpha)$.

There is a group isomorphism, say $f : G_\alpha \to G_\beta$, sending V_α onto V_β. We have $f(\alpha) = w\gamma$ for unique $w \in V$ and $\gamma \in \langle\beta\rangle$. It is clear that the subspaces V_α and V_β are invariant under α and γ, respectively. A calculation analogous to (3.1) shows that $\alpha|_{V_\alpha}$ is similar to $\gamma|_{V_\beta}$. Indeed, let $u : V_\alpha \to V_\beta$ be the isomorphism of F-vector spaces induced by f. Then for any $v \in V_\alpha$,

$$u(\alpha(v)) = f(\alpha(v)) = f(\alpha v\alpha^{-1}) = f(\alpha)f(v)f(\alpha)^{-1} = \gamma f(v)\gamma^{-1} = \gamma(f(v)) = \gamma(u(v)),$$

so that $u\alpha|_{V_\alpha}u^{-1} = \gamma|_{V_\beta}$, as claimed.

Since $f(V_\alpha) = V_\beta$, we see that f also induces an isomorphism $g : G_\alpha/V_\alpha \to G_\beta/V_\beta$. As $V \cap \langle\alpha\rangle$ is trivial, we have $o(\beta) = o(\alpha) = o(V_\alpha\alpha) = o(g(V_\alpha\alpha)) = o(V_\beta w\gamma)$. Let m denote this common number. Since $\gamma \in \langle\beta\rangle$, we see that $o(\gamma) \mid m$. Set

$$h = 1 + X + \cdots + X^{p-1} = \frac{X^p - 1}{X - 1} = \frac{(X - 1)^p}{X - 1} = (X - 1)^{p-1} \in F[X].$$

Then,

$$(w\gamma)^p = h(\gamma)(w)\gamma^p \quad \text{and} \quad h(\gamma)(w) \in (\gamma - 1)(V) \subseteq V_\beta. \tag{4.1}$$

Suppose first that $p \mid o(\gamma)$. Then (4.1) implies $(V_\beta w\gamma)^{o(\gamma)} = V_\beta$. Thus, $m \mid o(\gamma)$ and therefore $m = o(\gamma)$, whence $\langle\gamma\rangle = \langle\beta\rangle$. It follows that $G_\beta = G_\gamma$ and $V_\beta = V_\gamma$. Thus, $\alpha|_{V_\alpha}$ is similar to $\gamma|_{V_\gamma}$, which implies that α is similar to γ by Lemma 2.3.

Suppose next that $p \nmid o(\gamma)$. Then $p \nmid o(\gamma|_{V_\beta})$. As $\alpha|_{V_\alpha}$ is similar to $\gamma|_{V_\beta}$, we deduce that $p \nmid o(\alpha|_{V_\alpha})$.

The map $f^{-1} : G_\beta \to G_\alpha$ is also an isomorphism, and $f^{-1}(\beta) = z\delta$ for unique $z \in V$ and $\delta \in \langle\alpha\rangle$. If $p \mid o(\delta)$, we may deduce, as above, that β is similar to δ, where $\langle\delta\rangle = \langle\alpha\rangle$. Suppose next that $p \nmid o(\delta)$, which implies, as above, that $p \nmid o(\beta|_{V_\beta})$.

Now (4.1) implies $m \mid o(\gamma)p$. Since $p \mid o(\alpha)$, then p and $o(\gamma)$ are relatively prime factors of m, so $o(\gamma)p \mid m$ and therefore $m = o(\gamma)p$. Thus, $\gamma = \beta^{pj}$, where $j \in \mathbb{Z}$ and $\gcd(pj, m) = p$. We may thus write $pj = p^s k$, where $s \geq 1$ and $k \in \mathbb{Z}$ is relatively

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prime to m. Therefore, $\langle\beta\rangle = \langle\beta^k\rangle$, $V_\beta = V_{\beta^k}$ and $o(\beta|_{V_\beta}) = o(\beta^k|_{V_\beta})$. Since $\gamma = (\beta^k)^{p^s}$ and $p \nmid o(\beta^k|_{V_\beta})$, Lemma 2.2 implies that $\beta^k|_{V_\beta}$ is similar to $\gamma|_{V_\beta}$. However, $\gamma|_{V_\beta}$ is similar to $\alpha|_{V_\alpha}$, so $\beta^k|_{V_{\beta^k}}$ is similar to $\alpha|_{V_\alpha}$. By Lemma 2.3, α is similar to β^k. □

5. The unipotent case

For a group G, its lower central series $G^1, G^2, \ldots$ is inductively defined by $G^1 = G$ and $G^{i+1} = [G, G^i]$, $i \geq 1$. If $G^{i+1} = 1$ for some $i \geq 1$, we say that G is nilpotent, and the smallest such i is called the nilpotency class of G. It is well known that every finite p-group is nilpotent.

Given an additive abelian group A and an endomorphism α of A, we say that α is unipotent if $\alpha = 1 + \beta$, with $\beta \in \mathrm{End}(A)$ nilpotent.

LEMMA 5.1. *Let A be a nontrivial finite abelian p-group. Let $\alpha \in \mathrm{Aut}(A)$ and set $G_\alpha = \mathrm{Hol}(A, \alpha)$. Then G_α is nilpotent $\Leftrightarrow$ α is unipotent $\Leftrightarrow$ the order of α is a power of p.*

PROOF. It is readily seen that the proper terms of the lower central series of G_α are $(\alpha - 1)^i A$, $i \geq 1$, so G_α is nilpotent $\Leftrightarrow$ α is unipotent. If the order of α is a power of p, then G_α is a p-group, and hence nilpotent. Suppose α is unipotent, so $\alpha = 1 + \beta$, with $\beta \in \mathrm{End}(A)$ nilpotent, and therefore $\beta^{p^m} = 0$ for some $m \geq 0$. We show by induction on m that this implies that the order of $1 + \beta$ is a power of p. The case $m = 0$ is trivial. Suppose $m > 0$ and the result is true for $m - 1$. We have $p^\ell A = 0$ for some $\ell \geq 1$. Since $p^\ell \mid \binom{p^\ell}{i}$ for $0 < i < p$, we have $(1 + \beta)^{p^\ell} = 1 + \beta^p \gamma$ for some $\gamma \in \mathbb{Z}[\beta]$. As $(\beta^p \gamma)^{p^{m-1}} = 0$, the order of $(1 + \beta)^{p^\ell}$ is a power of p, and hence so is that of $1 + \beta$. □

THEOREM 5.2. *Suppose $\alpha, \beta \in \mathrm{GL}(V)$ are unipotent and $\mathrm{Hol}(V, \alpha) \cong \mathrm{Hol}(V, \beta)$. Then α is similar to β.*

PROOF. Set $G_\alpha = \mathrm{Hol}(V, \alpha)$ and $G_\beta = \mathrm{Hol}(V, \beta)$. Then G_α and G_β are nilpotent by Lemma 5.1, and we let m be the common nilpotency class of G_α and G_β.

As indicated earlier, the proper terms of the lower central series of G_α (respectively G_β) are $(\alpha - 1)^i V$ (respectively $(\beta - 1)^i V$), $i \geq 1$.

There is basis of V relative to which the matrix of α (respectively β) is the direct sum of $e_1, \ldots, e_m$ (respectively $f_1, \ldots, f_m$) Jordan blocks with eigenvalue 1 of sizes $1, \ldots, m$, respectively. Thus,

$$e_1 + 2e_2 + \cdots + me_m = \dim V = f_1 + 2f_2 + \cdots + mf_m.$$

This yields a basis of $(\alpha - 1)V$ (respectively $(\beta - 1)V$) relative to which the matrix of α (respectively β) is the direct sum of $e_2, \ldots, e_m$ (respectively $f_2, \ldots, f_m$) Jordan blocks with eigenvalue 1 of sizes $1, \ldots, m - 1$. Since $[G_\alpha, G_\alpha] = (\alpha - 1)V$ (respectively $[G_\beta, G_\beta] = (\beta - 1)V$), it follows that

$$e_2 + 2e_3 + \cdots + (m - 1)e_m = \dim(\alpha - 1)V = \dim(\beta - 1)V = f_2 + 2f_3 + \cdots + (m - 1)f_m.$$

Continuing in this way, we see that the column vector with entries $e_1 - f_1, \ldots, e_m - f_m$ is annihilated by the upper triangular matrix with equal entries along each

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superdiagonal, these entries being $1, 2, \ldots, m$, in each successive superdiagonal. As this matrix is invertible, we see that $e_1 = f_1, \ldots, e_m = f_m$, whence α is similar to β. □

PROPOSITION 5.3. *Suppose H and K are abelian subgroups of* $\mathrm{GL}(V)$ *such that* $\mathrm{Hol}(V, H) \cong \mathrm{Hol}(V, K)$. *Then* $H \cong K$.

PROOF. Set $V_H = [\mathrm{Hol}(V, H), \mathrm{Hol}(V, H)]$ and $V_K = [\mathrm{Hol}(V, K), \mathrm{Hol}(V, K)]$, which are subgroups of V. Let $f : \mathrm{Hol}(V, H) \to \mathrm{Hol}(V, K)$ be an isomorphism. Then f restricts to an isomorphism between V_H and V_K. This yields an isomorphism between $\mathrm{Hol}(V, H)/V_H$ and $\mathrm{Hol}(V, K)/V_K$. However,

$$\mathrm{Hol}(V, H)/V_H \cong (V/V_H) \times H, \;\; \mathrm{Hol}(V, K)/V_K \cong (V/V_K) \times K.$$

Here $V/V_H \cong V/V_K$, as both are F-vector spaces of the same dimension, so the uniqueness part of the fundamental theorem of finite abelian groups yields $H \cong K$. □

6. Examples

Given a group G and $x, y \in G$, we set ${}^x y = xyx^{-1}$. If H and N are groups such that H acts on N by automorphisms via a homomorphism $T : H \to \mathrm{Aut}(N)$, we may consider the semidirect product $N \rtimes_T H$, where ${}^h x = hxh^{-1} = T(h)(x)$, $h \in H, x \in N$.

There is a slight generalisation of Theorem 4.1 to a semidirect product $V \rtimes_T \langle \alpha \rangle$, where the action of $\langle \alpha \rangle$ on V is not necessarily faithful. We omit the details of the proof, which is similar to that of Theorem 4.1 but notationally more complicated.

We let $U_n(p)$ stand for the subgroup of $\mathrm{GL}_n(p)$ consisting of all upper triangular matrices with 1 values along the main diagonal. For later reference, we observe that $U_3(p) = \mathrm{Heis}(p)$, the Heisenberg group over F.

Throughout this section, we set $R = \mathbb{Z}/p^m\mathbb{Z}$, where $m \geq 1$, and let U be a free R-module of finite rank $n > 0$. Given a subgroup H of $\mathrm{GL}_n(R)$, taking $U = R^n$, we may consider the semidirect product $U \rtimes_T H$, where T is the homomorphism $H \hookrightarrow \mathrm{GL}_n(R) \to \mathrm{GL}(U)$, and $\mathrm{GL}_n(R) \to \mathrm{GL}(U)$ is the isomorphism associated to the canonical basis of U, so that ${}^h v = h \cdot v$ is the product of the $n \times n$ matrix h by the vector v of length n. It is clear that $U \rtimes_T H \cong \mathrm{Hol}(U, T(H))$, and we will write $\mathrm{Hol}(U, H) = U \rtimes_T H$ from now on.

LEMMA 6.1. *Let H be a subgroup of* $\mathrm{GL}_n(R)$ *and set $U = R^n$. Suppose there are subgroups W and K of* $\mathrm{Hol}(U, H)$ *such that W is normal, $W \cong U$,* $\mathrm{Hol}(U, H) = WK$ *and $W \cap K$ is trivial. Assume that the homomorphism $K \to$* $\mathrm{GL}(W)$ *arising from the conjugation action of K on W is faithful. Fix an R-basis $B = \{w_1, \ldots, w_n\}$ of W, and let L be the image of the corresponding homomorphism $v : K \to$* $\mathrm{GL}(W) \to \mathrm{GL}_n(R)$. *Then* $\mathrm{Hol}(U, H) \cong \mathrm{Hol}(U, L)$.

PROOF. For $w \in W$, let $[w]_B \in U$ be the coordinates of w relative to B. Thus, the map $u : W \to U$, given by $w \mapsto [w]_B$, is an isomorphism. By assumption, the map $K \to L$, given by $k \mapsto v(k)$ is an isomorphism. Consider the map $f : W \rtimes K \to \mathrm{Hol}(U, L)$ given

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by $wk \mapsto u(w)v(k)$. We claim that f is an isomorphism. It suffices to verify that f maps ${}^k w$ into ${}^{v(k)}u(w)$, that is, $[{}^k w]_B = v(k) \cdot [w]_B$. To see this, observe that by definition, if $k \in K$ and $1 \le j \le n$, then

$$ {}^k w_i = \sum_{1 \le j \le n} v(k)_{ji} w_j, $$

so that $[{}^k w_i]_B$ is the ith column of $v(k)$, that is, $[{}^k w_i]_B = v(k) \cdot [w_i]_B$. As K acts linearly on W, it follows that $[{}^k w]_B = v(k) \cdot [w]_B$. □

EXAMPLE 6.2. Take $p = 2$ and $m = 3$, so that $R = \mathbb{Z}/8\mathbb{Z}$, and further take $n = 4$ and $U = R^4$, whose elements are viewed as column vectors. Let

$$ A = \begin{pmatrix} 3 & -1 & 1 & -2 \\ 0 & 3 & -3 & 1 \\ 0 & 3 & 4 & 3 \\ 2 & 0 & -2 & 3 \end{pmatrix} \in \mathrm{GL}_4(R), $$

and set $G = \mathrm{Hol}(U, A) = U \rtimes \langle A \rangle$, where $AvA^{-1} = A \cdot v$, the product of A by the column vector $v \in U$. Then for $S \in \mathrm{GL}_4(R)$, we have $G \cong \mathrm{Hol}(U, S)$ if and only if $\langle S \rangle \sim \langle A \rangle$ or $\langle S \rangle \sim \langle B \rangle$, where $\langle B \rangle \nsim \langle A \rangle$ and

$$ B = \begin{pmatrix} -1 & -2 & 2 & 4 \\ 0 & 3 & -3 & 1 \\ 0 & 3 & 4 & 3 \\ 1 & 0 & -2 & 3 \end{pmatrix} \in \mathrm{GL}_4(R). \tag{6.1} $$

Indeed, for $M \in \mathrm{GL}_4(R)$, we denote by $\overline{M}$ the image of M under the canonical projection $\mathrm{GL}_4(R) \to \mathrm{GL}_4(F)$. The characteristic polynomial of $\overline{A}$ is $f(X) = (X+1)^2(X^2 + X + 1)$. Thus, the minimal polynomial of $\overline{A}$ is $f(X)$ or $g(X) = (X+1)(X^2 + X + 1) = X^3 + 1$. In the latter case, $\overline{A}$ has order 3, which is easily seen to be false. Thus, the minimal polynomial of $\overline{A}$ is $f(X)$. As the degree of $f(X)$ is the size of $\overline{A}$, it follows that $\overline{A}$ is similar to the companion matrix of $f(X)$, or the direct sum of the companion matrices of $f_1(X) = X^2 + 1$ and $f_2(X) = X^2 + X + 1$. Thus, the order of $\overline{A}$ is 6, whence the order of A is a multiple of 6. The same comments apply to B. In particular, $\overline{A}$ is similar to $\overline{B}$. However, $\langle A \rangle \nsim \langle B \rangle$, since the determinant of A is -1 and that of B is 3. Thus, every odd power of B has determinant 3, so no odd power of B is similar to A. Therefore, $\langle A \rangle$ and $\langle B \rangle$ are not conjugate in $\mathrm{GL}_4(R)$.

We next show that $\mathrm{Hol}(U, A) \cong \mathrm{Hol}(U, B)$. For this purpose, note that

$$ A^2 = \begin{pmatrix} -3 & -3 & -2 & -2 \\ 2 & 0 & 1 & -3 \\ -2 & -3 & 1 & 0 \\ 4 & 0 & 4 & -1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 3 & 4 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & 4 & 3 & 4 \\ 2 & 0 & -2 & 1 \end{pmatrix}, \quad A^6 = \begin{pmatrix} 3 & 0 & -2 & 4 \\ 4 & 1 & 4 & 4 \\ 4 & 0 & -3 & -2 \\ 4 & 0 & 4 & -1 \end{pmatrix}, $$

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which confirms that the order of $\overline{A}$ is 6. Moreover,

$$A^{12} = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so the order of A is 24. Set $G = \mathrm{Hol}(U, A)$,

$$x = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in U, \quad y = (x, A^{12}) \in G,$$

where this notation will be used to avoid confusion when dealing with elements of G which are neither in U nor in $\langle A \rangle$. We further let W be the subgroup of G generated by $[G, G]$ and y. As W contains $[G, G]$, it is a normal subgroup of G. We claim that W is abelian and, in fact, isomorphic to U. To see this, note that $[G, G]$ is the subgroup of U generated by the columns of $A - 1$, say f_1, f_2, f_3, f_4. Thus, W is abelian if and only if A^{12} commutes with $[G, G]$, which means that A^{12} fixes the columns of $A - 1$, that is, $(A^{12} - 1)(A - 1) = 0$, which is true as the fourth row of $A - 1$ is annihilated by 4. Row reducing $A - 1$, we find that $\langle f_1, f_2, f_3, f_4 \rangle = \langle f_1 \rangle \oplus \langle f_2 \rangle \oplus \langle f_3 \rangle \oplus \langle f_4 \rangle$, where f_2, f_3, f_4 have order 8 and f_1 has order 4. In particular, this yields $[G, G] \cong C_8^3 \times C_4$ and $G/[G, G] \cong C_2 \times C_{24}$. Moreover,

$$y^2 = (x, A^{12})^2 = x + A^{12} \cdot x = (1 + A^{12}) \cdot x = f_1,$$

whence $W \cong U$ and $W/[G, G] \cong C_2$. Note that W is complemented by $K = \langle A \rangle$ in G, since their intersection is trivial, so their product has the right order. Also, x, f_2, f_3, f_4 are the columns of an invertible matrix, say Q, so they generate U.

The conjugation action of $\langle A \rangle$ on W is faithful, for if A^i acts trivially on W, then $(A^i - 1)Q = 0$, whence $A^i = 1$. Let M be the matrix of the conjugation action of A on W relative to the R-basis $\{y, f_2, f_3, f_4\}$. We claim that $M = B$, as given in (6.1). Indeed, denoting by $C_i(P)$ the i-column of a matrix P,

$$\begin{aligned} {}^{A}f_i &= A \cdot f_i = A \cdot C_i(A - 1) = C_i(A(A - 1)) = (A - 1)C_i(A) \\ &= A_{1i} f_1 + A_{2i} f_2 + A_{3i} f_3 + A_{4i} f_4 \end{aligned}$$

for all $i \in \{2, 3, 4\}$. Recalling that $y^2 = f_1$, it follows that M and B share the last three columns. Let us verify that M and B share the first column. To see this, observe that

$${}^{A}y = (A \cdot x, A^{12}) = ((A - 1) \cdot x, 1)y.$$

Here, the definition of x gives $(A - 1)x = -f_1 + f_4$, where $f_1 = y^2$, so ${}^{A}y = y^{-2} f_4 y = y^{-1} f_4$, so the first columns of M and B are identical. It follows from Lemma 6.1 that $\mathrm{Hol}(U, A) \cong \mathrm{Hol}(U, B)$.

Let $S \in \mathrm{GL}_4(R)$ and suppose that $H = \mathrm{Hol}(U, S)$ is isomorphic to G. We proceed to show that $\langle S \rangle \sim \langle A \rangle$ or $\langle S \rangle \sim \langle B \rangle$. By hypothesis, we have an isomorphism $\Delta : H \to G$. Here, U is a normal subgroup of H containing $[H, H]$; U is complemented in H by the

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cyclic subgroup $\langle S\rangle$ of order 24; and relative to the canonical basis of U, the matrix of the action of S on U by conjugation is S.

Set $N = \Delta(U)$ and $T = \Delta(S)$. Then N is a normal subgroup of G isomorphic to U containing $[G, G]$; N is complemented in G by the cyclic subgroup $\langle T\rangle$ of order 24; and relative to some R-basis of N, the matrix of the action of T on N by conjugation is S.

As indicated above, $[G, G] \cong C_8^3 \times C_4$, so $N/[G, G] \cong C_2$. As $G/[G, G] \cong C_2 \times C_{24}$, it follows that there are exactly three normal subgroups of G containing $[G, G]$ as a subgroup of index 2, and N must be one of them. The first possibility is $N = U$, in which case, $\langle S\rangle \sim \langle A\rangle$ by Lemma 3.3. The second possibility is

$$N = [G, G] \times \langle A^{12}\rangle \cong C_8^3 \times C_4 \times C_2 \not\cong U,$$

which cannot be. It remains to analyse the third possibility, namely $N = W$. Since $G = W \rtimes \langle T\rangle$, the order of T modulo W is also 24. As $G = W \rtimes \langle A\rangle$, it follows that $T = wA^i$, where $w \in W$ and $i \in \mathbb{Z}$ is relatively prime to 24. Since $M = B$, the conjugation action of T on W relative to the R-basis $\{y, f_2, f_3, f_4\}$ of W is B^i. However, relative to some R-basis of W, the matrix of the conjugation action of T on W is S. Thus, S is similar to B^i with $\gcd(24, i) = 1$, as required.

EXAMPLE 6.3. Suppose that $n \geq 6$ and set $V = F^n$. Then there are abelian subgroups H and L of $U_n(p)$ such that $\mathrm{Hol}(V, H) \cong \mathrm{Hol}(V, L)$ but $H \not\sim L$ in $\mathrm{GL}_n(p)$.

Indeed, suppose first that $n = 6$ and let $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ be the canonical basis of V. For $A \in M_3(F)$, set

$$S_A = \begin{pmatrix} I_3 & A \\ 0 & I_3 \end{pmatrix} \in U_6(p),$$

and let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

T_H be the additive subgroup of $M_3(F)$ generated by A_1, A_2, A_3, and H the subgroup of $U_6(p)$ generated by $S_{A_1}, S_{A_2}, S_{A_3}$.

Let W be the subgroup of $\mathrm{Hol}(V, H)$ generated by $v_1, v_2, v_3, S_{A_1}, S_{A_2}, S_{A_3}$. Then $W \cong V$. Moreover, W is a normal subgroup of $\mathrm{Hol}(V, H)$. Let K be the subgroup of $\mathrm{Hol}(V, H)$ generated by v_4, v_5, v_6. We see that $K \cong H$ and $\mathrm{Hol}(V, H) = W \rtimes K$. When K acts on W by conjugation, the matrices corresponding to the actions of v_4, v_5, v_6 relative to the basis $\{v_1, v_2, v_3, S_{A_1}, S_{A_2}, S_{A_3}\}$ are respectively equal to $S_{B_1}, S_{B_2}, S_{B_3}$, where B_i is the opposite of the matrix formed by the ith columns of A_1, A_2, A_3, in that order, for $i \in \{1, 2, 3\}$. Thus,

$$B_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

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so the action of K on W is faithful. Let L be the subgroup of $U_6(p)$ generated by $S_{B_1}, S_{B_2}, S_{B_3}$, so that $\mathrm{Hol}(V,H) \cong \mathrm{Hol}(V,L)$ by Lemma 6.1. Let T_L be the additive subgroup of $M_3(p)$ generated by B_1, B_2, B_3.

Suppose, if possible, that $H \sim L$ in $\mathrm{GL}_6(p)$. Then, there is $X \in \mathrm{GL}_6(p)$ such that $XHX^{-1} = L$. Thus, X gives rise to the isomorphism $f : \mathrm{Hol}(V,H) \to \mathrm{Hol}(V,L)$ given by $vh \mapsto (X \cdot v)(XhX^{-1})$. Then f must map the centre of $\mathrm{Hol}(V,H)$ onto the centre of $\mathrm{Hol}(V,L)$. However, both centres are equal to $\langle v_1, v_2, v_3\rangle$, so

$$X = \begin{pmatrix} Y & Q \\ 0 & Z \end{pmatrix},$$

where $Y, Z \in \mathrm{GL}_3(p)$ and $Q \in M_3(F)$. Then $XHX^{-1} = L$ gives $YT_HZ^{-1} = T_L$. However, all matrices in T_L have rank at most 2, whereas T_H has a matrix of rank 3. We deduce that $H \nsim L$ in $\mathrm{GL}_6(p)$.

In general, take $m = \lceil n/2 \rceil$, for $A \in M_m(F)$, set

$$S_A = \begin{pmatrix} I_m & A \\ 0 & I_{n-m} \end{pmatrix} \in U_n(p),$$

and let $A_1, \ldots, A_{n-m} \in M_{m,n-m}(p)$ be defined as follows: $A_1 = \mathrm{diag}(1, \ldots, 1)$, where the number of 1 values is $n-m$, $A_2 = E^{1,2}, \ldots, A_{n-m} = E^{1,n-m}$, where $E^{i,j}$ is the matrix having a 1 in position (i, j) and 0 elsewhere. We can then continue as above, noting that all matrices in T_L have rank at most 2, whereas T_H has a matrix of rank $n - m \geq 3$.

We will refer to a group G as admitting if there exist nonconjugate subgroups H and K of $\mathrm{Aut}(G)$ such that $\mathrm{Hol}(N,H) \cong \mathrm{Hol}(N,K)$. We will say that G is highly admitting if there exist nonisomorphic subgroups H and K of $\mathrm{Aut}(G)$ such that $\mathrm{Hol}(N,H) \cong \mathrm{Hol}(N,K)$.

EXAMPLE 6.4. Suppose that p is odd and $n \geq 2$. Then V is highly admitting.

Indeed, suppose first $n = 2$ and set $V = F^2$. Let $\{v_1, v_2\}$ be the canonical basis of V and let H be the subgroup of $\mathrm{GL}_2(p)$ generated by

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since $o(B) = p$, $o(A) = 2$ and ${}^AB = B^{-1}$, we see that H is the dihedral group of order $2p$.

Let W be the subgroup of $\mathrm{Hol}(V,H)$ generated by v_1, B. Then $W \cong V$. Moreover, W is a normal subgroup of $\mathrm{Hol}(V,H)$. Let K be the subgroup of $\mathrm{Hol}(V,H)$ generated by v_2, A. Clearly, $K \cong C_{2p}$ is not isomorphic to H, and $\mathrm{Hol}(V,H) = W \rtimes K$. When K acts on W by conjugation, the matrices corresponding to the actions of v_2 and A relative to the basis $\{v_1, B\}$ are respectively equal to

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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The subgroup of $\mathrm{GL}_2(p)$ generated by these matrices, say L, is isomorphic to C_{2p}, so the action of K on W is faithful. It follows from Lemma 6.1 that $\mathrm{Hol}(V, L) \cong \mathrm{Hol}(V, H)$, even though L is not isomorphic to H.

The general case when $\{v_1, \ldots, v_n\}$ is the canonical basis of V follows by extending A, B so that they fix $v_3, \ldots, v_n$.

In addition to proving that V is highly admitting when $n \geq 2$ and p is odd, this example shows that even though $\mathrm{Hol}(V, H) \cong \mathrm{Hol}(V, L)$ and the Sylow p-subgroup of H is conjugate to the Sylow p-subgroup of L, this conjugation cannot be extended to all of H and L.

We next provide an analogue of Example 6.4 when $p = 2$, provided $n \geq 4$.

EXAMPLE 6.5. Suppose that $n \geq 4$ and $p = 2$. Then V is highly admitting.

Indeed, suppose first $n = 4$ and set $V = F^4$. Let $\{v_1, v_2, v_3, v_4\}$ be the canonical basis of V and let H be the subgroup of $\mathrm{GL}(V)$ generated by

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $H \cong C_2^3$. Let W be the subgroup of $\mathrm{Hol}(V, H)$ generated by $v_1 + v_4, v_2 + v_3, X, Z$. Then $W \cong V$. Moreover, W is a normal subgroup of $\mathrm{Hol}(V, H)$. Let K be the subgroup of $\mathrm{Hol}(V, H)$ generated by v_1, Y. We see that $K \cong D_8$ and $\mathrm{Hol}(V, H) = W \rtimes K$. When K acts on W by conjugation, the matrices corresponding to the actions of v_1, Y relative to the basis $\{v_1 + v_4, v_2 + v_3, X, Z\}$ are respectively equal to

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The subgroup of $\mathrm{GL}_4(2)$ generated by these matrices, say L, is isomorphic to D_8, so the action of K on W is faithful. It follows from Lemma 6.1 that $\mathrm{Hol}(V, L) \cong \mathrm{Hol}(V, H)$ even though L is not isomorphic to H.

The general case when $\{v_1, \ldots, v_n\}$ is the canonical basis of V follows by extending X, Y, Z so that they fix $v_5, \ldots, v_n$.

EXAMPLE 6.6. Conditions (C1)–(C3) below ensure that a group G admits automorphisms α and β such that $\mathrm{Hol}(G, \alpha) \cong \mathrm{Hol}(G, \beta)$ but $\langle \alpha \rangle \nsim \langle \beta \rangle$.

Let G be a group having elements x and y such that:

(C1) $o(x) = o(y)$;

(C2) $o(x) = o(xZ(G))$ and $o(y) = o(yZ(G))$ (so that $\langle x \rangle \cap Z(G) = 1 = \langle y \rangle \cap Z(G)$);

(C3) there is no automorphism of G that sends x^i to yz for any i relatively prime to the order of x and any $z \in Z(G)$ (this means that when $\mathrm{Aut}(G)$ acts on $\mathrm{Inn}(G)$

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by conjugation, the subgroups generated by $i(x)$ and $i(y)$ are in different orbits, where $i(g) \in \mathrm{Inn}(G)$ is the inner automorphism associated to $g \in G$).

Let $\alpha = i(x)$ and $\beta = i(y)$. By condition (C2), $\mathrm{Hol}(G, \alpha) = G \times \langle u \rangle$, where $u = x^{-1}\alpha$ has the same order as x, and $\mathrm{Hol}(G,\beta) = G \times \langle v \rangle$, where $v = y^{-1}\beta$ has the same order as y. Thus, $\mathrm{Hol}(G, \alpha) \cong \mathrm{Hol}(G,\beta)$ by condition (C1). Also, $\langle \alpha \rangle$ and $\langle \beta \rangle$ are not conjugate subgroups of $\mathrm{Aut}(G)$ by condition (C3).

Many groups satisfy conditions (C1)–(C3). A centreless group G having cyclic subgroups of the same order that are not in the same $\mathrm{Aut}(G)$-orbit meets conditions (C1)–(C3). For instance: S_n, $n \geq 4$, taking $x = (1, 2)$ and $y = (1, 2)(3, 4)$, and looking at the size of their centralisers; a free group on $n \geq 2$ generators $x_1, \ldots, x_n$, taking $x = x_1$ and $y = [x_1, x_2]$. The general linear group $\mathrm{GL}_n(p)$, with $n \geq 4$ and p odd, is not centreless and satisfies conditions (C1)–(C3), taking $x = \mathrm{diag}(-1, 1, \ldots, 1)$ and $y = \mathrm{diag}(-1, -1, 1, \ldots, 1)$, and looking at the size of their centralisers.

As another example, assume $m \geq 2$ and let $G = \mathrm{Hol}(V, \alpha)$, where α acts on V, with respect to some basis, via the direct sum of m copies of the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, where $n = 2m$. Thus, the order of G is p^{2m+1}. As an abstract group,

$$G = \langle x_1, x_2, \ldots, x_{2m-1}, x_{2m}, y \mid [x_i, x_j] = 1, x_i^p = 1 = y^p, {}^{y}x_{2i-1} = x_{2i-1}, {}^{y}x_{2i} = x_{2i}x_{2i-1} \rangle. \tag{6.2}$$

The simplest example occurs when $m = 2$ and $p = 2$, in which case, $|G| = 32$. In this case, G can also be described as a Sylow 2-subgroup P of $\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$. Indeed, let N be the kernel of the canonical map $\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$, which consists of all 16 matrices of the form $1 + X$, where $X \in M_2(2\mathbb{Z}/4\mathbb{Z})$. Then $N \cong C_2^4$, and N is generated by

$$B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Defining $A \in P$ by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we see that

$${}^{A}B = BC, \quad {}^{A}C = C, \quad {}^{A}D = DE, \quad {}^{A}E = E.$$

Using the notation from (6.2), the centre of G is generated by all x_{2i-1}. Take $x = x_2$ and y as given. Then x and y have order p, which remains the same modulo $Z(G)$. The centraliser of x^i, when $p \nmid i$, is the subgroup T generated by all x_j. The centraliser of yz, when $z \in Z(G)$, is equal to $Z(G)\langle y \rangle$. Here, $|T| = p^{2m}$ and $|Z(G)\langle y \rangle| = p^{m+1}$. Since $m \geq 2$, it follows that T cannot be mapped into $Z(G)\langle y \rangle$ by any automorphism of G, so x and y meet the required conditions. Note that if $m = 1$, then $G = \mathrm{Heis}(p)$, in which case, G does not satisfy conditions (C1)–(C3). Indeed, if p is odd, then any two noncentral elements of G produce subgroups of inner automorphisms of order p that are in the

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same Aut(G)-orbit; if $p = 2$, then $G = D_8$, and D_8 is nonadmitting (see Example 6.7 below).

Our last example discusses instances of nonadmitting finite groups.

EXAMPLE 6.7. (a) Suppose that $n = 2$ and $p = 2$. Then V is nonadmitting.

Indeed, every proper subgroup of $\mathrm{GL}(V) \cong S_3$ is cyclic, and all cyclic subgroups of S_3 of the same order are conjugate in S_3.

(b) Suppose that $n = 3$ and $p = 2$. Then V is nonadmitting.

Indeed, set $V = F^3$ and let $\{v_1, v_2, v_3\}$ be the canonical basis of V. It is known that the only cases, when $\mathrm{GL}_3(2)$ has subgroups of the same order that are not conjugate, occur for orders 4, 12 and 24, and that there are three conjugacy classes of groups of order 4 and two conjugacy classes of groups of orders 12 and 24. In order 4, let H and K respectively consist of all matrices of the form

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $G_1 = \mathrm{Hol}(V, H)$ is not isomorphic to $G_2 = \mathrm{Hol}(V, K)$, since $[G_1, G_1] = \langle v_1, v_2\rangle$ and $[G_2, G_2] = \langle v_1\rangle$ (this gives a quick way to verify that $H \nsim K$). Next, let J be the subgroup generated by the upper triangular Jordan block with eigenvalue 1. Thus, J is cyclic of order 4. Set $G_3 = \mathrm{Hol}(V, J)$. Then $[G_3, G_3] = \langle v_1, v_2\rangle$, but G_3 is nilpotent of class 3 and G_1 is nilpotent of class 2. Thus, G_3 is not isomorphic to G_1 or to G_2.

In order 24, let H and K respectively consist of all matrices of the form

$$\begin{pmatrix} A & u \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & w \\ 0 & A \end{pmatrix},$$

where $A \in \mathrm{GL}_2(2)$, u is a column vector of length 2 and w is a row vector of length 2. Observe that $G_1 = \mathrm{Hol}(V, H)$ is not isomorphic to $G_2 = \mathrm{Hol}(V, K)$, since $[G_1, G_1] = \langle v_1, v_2\rangle \rtimes A_4$ and $[G_2, G_2] = V \rtimes A_4$ (this gives a quick way to verify that $H \nsim K$).

In order 12, the situation is as above, but with $A \in \langle C\rangle$, where C is the companion matrix of the polynomial $t^2 + t + 1 \in F[t]$. The outcome is the same.

(c) In addition to the group C_2^3 discussed above, every other group of order 8 is nonadmitting. These cases are easily verified and we omit the details.

(d) In addition to the group C_8 mentioned above, every cyclic group C_{p^n} is nonadmitting. The case when p is odd is trivial, and the case when $p = 2$ requires routine calculations that we omit.

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