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DIVISIBILITY OF THE PARTITION FUNCTION $\mathrm{PDO}_t(n)$ BY POWERS OF 2 AND 3

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Abstract

Lin introduced the partition function $\mathrm{PDO}_t(n)$, which counts the total number of tagged parts over all the partitions of n with designated summands in which all parts are odd. Lin also proved some congruences modulo 3 and 9 for $\mathrm{PDO}_t(n)$, and conjectured certain congruences modulo 3^{k+2} for $k \geq 0$. He proved the conjecture for $k = 0$ and $k = 1$ ['The number of tagged parts over the partitions with designated summands', *J. Number Theory* **184** (2018), 216–234]. We prove the conjecture for $k = 2$. We also study the lacunarity of $\mathrm{PDO}_t(n)$ modulo arbitrary powers of 2 and 3. Using nilpotency of Hecke operators, we prove that there exists an infinite family of congruences modulo any power of 2 satisfied by $\mathrm{PDO}_t(n)$.

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1. Introduction and statement of results

A partition of a positive integer n is a nonincreasing sequence of positive integers, called parts, whose sum is n. In [1], Andrews, Lewis and Lovejoy investigated the partition function $\mathrm{PD}(n)$ which counts the number of partitions of n with designated summands. A partition of n with designated summands is obtained from an ordinary partition of n by tagging exactly one of each part size. For example, $\mathrm{PD}(4) = 10$ with the relevant partitions being $4', 3'+1', 2'+2, 2+2', 2'+1'+1, 2'+1+1', 1'+1+1+1, 1+1'+1+1, 1+1+1'+1, 1+1+1+1'$. They also studied another partition function $\mathrm{PDO}(n)$ which counts the number of partitions of n with designated summands in which all parts are odd. From the above example, $\mathrm{PDO}(4) = 5$. Later, many authors have studied these two partition functions (see for example [2–4, 17]).

Recently, Lin [9] introduced two new partition functions $\mathrm{PD}_t(n)$ and $\mathrm{PDO}_t(n)$ related to partitions with designated summands. The partition function $\mathrm{PD}_t(n)$ counts the total number of tagged parts over all the partitions of n with designated summands.

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For instance, $\mathrm{PD}_t(4) = 13$. The other partition function $\mathrm{PDO}_t(n)$ counts the total number of tagged parts over all the partitions of n with designated summands in which all parts are odd. For example, $\mathrm{PDO}_t(4) = 6$. Lin found the generating functions of $\mathrm{PD}_t(n)$ and $\mathrm{PDO}_t(n)$. The generating function of $\mathrm{PDO}_t(n)$ is given by

$$G(q) := \sum_{n=0}^{\infty} \mathrm{PDO}_t(n)q^n = \frac{qf_2 f_3^2 f_{12}^2}{f_1^2 f_6}, \tag{1.1}$$

where $f_k = (q^k; q^k)_\infty$ and $(a; q)_\infty := \prod_{j=0}^{\infty}(1 - aq^j)$.

Lin also established many congruences modulo small powers of 3 satisfied by $\mathrm{PD}_t(n)$ and $\mathrm{PDO}_t(n)$. For example, he proved the following Ramanujan-type congruences modulo 9 and 27 satisfied by $\mathrm{PDO}_t(n)$: for $n \geq 0$,

$$\mathrm{PDO}_t(8n) \equiv \mathrm{PDO}_t(12n) \equiv \mathrm{PDO}_t(12n + 8) \equiv 0 \pmod 9,$$
$$\mathrm{PDO}_t(24n) \equiv \mathrm{PDO}_t(36n) \equiv \mathrm{PDO}_t(36n + 24) \equiv 0 \pmod{27}.$$

He further conjectured the following congruences.

CONJECTURE 1.1 [9, Conjecture 6.1]. For $k, n \geq 0$,

$$\mathrm{PDO}_t(8 \cdot 3^k n) \equiv 0 \pmod{3^{k+2}},$$
$$\mathrm{PDO}_t(12 \cdot 3^k n) \equiv 0 \pmod{3^{k+2}}.$$

Lin proved Conjecture 1.1 for $k = 0, 1$ using basic q-series techniques. We prove the following theorem which establishes Conjecture 1.1 for $k = 2$.

THEOREM 1.2. *For all $n \geq 0$,*

$$\mathrm{PDO}_t(72n) \equiv 0 \pmod{81}, \tag{1.2}$$

$$\mathrm{PDO}_t(108n) \equiv 0 \pmod{81}. \tag{1.3}$$

In addition to the study of Ramanujan-type congruences, it is an interesting problem to study the distribution of the partition function modulo positive integers M. To be precise, given an integral power series $F(q) := \sum_{n=0}^{\infty} a(n)q^n$ and $0 \leq r < M$, we define

$$\delta_r(F, M; X) := \frac{\#\{n \leq X : a(n) \equiv r \pmod M\}}{X}.$$

An integral power series F is called *lacunary modulo M* if

$$\lim_{X\to\infty} \delta_0(F, M; X) = 1,$$

that is, almost all of the coefficients of F are divisible by M.

It is a well-known fact that modular forms with integer Fourier coefficients are lacunary modulo any positive integer. Recently, in [5, Theorem 1.1], Cotron *et al.* extended this fact to integral weight eta-quotients modulo arbitrary powers of primes under certain strong conditions. In [9], Lin remarked that the generating function of $\mathrm{PDO}_t(n)$ is a modular form. However, this observation is not quite correct because

$\mathrm{PDO}_t(n)$ is not holomorphic at the cusp 1. Also, the generating function of $\mathrm{PDO}_t(n)$ does not satisfy the conditions of [5, Theorem 1.1]. Therefore, it is an interesting problem to study the lacunarity of $G(q) = \sum_{n=0}^{\infty} \mathrm{PDO}_t(n)q^n$ modulo arbitrary powers of primes. In the following theorem, we prove that $G(q)$ is lacunary modulo arbitrary powers of 2 and 3.

THEOREM 1.3. *For any positive integer k,*

$$\lim_{X\to\infty} \delta_0(G, 2^k; X) = 1, \tag{1.4}$$

$$\lim_{X\to\infty} \delta_0(G, 3^k; X) = 1. \tag{1.5}$$

Serre observed and Tate proved that the action of Hecke algebras on spaces of modular forms of level 1 modulo 2 is locally nilpotent (see for example [14–16]). Ono and Taguchi [13] showed that this phenomenon generalises to higher levels. We observe that, for any positive integer k, the generating function of $\mathrm{PDO}_t(n)$ is congruent to an eta-quotient modulo 2^k, and the eta-quotient is a modular form whose level is in Ono and Taguchi's list. This allows us to use a result of Ono and Taguchi to prove the following congruence for $\mathrm{PDO}_t(n)$.

THEOREM 1.4. *Let n be a nonnegative integer. Then there is an integer $s \geq 0$ such that for every $u \geq 1$ and distinct primes $p_1, \ldots, p_{s+u}$ coprime to 6,*

$$\mathrm{PDO}_t(p_1 \cdots p_{s+u} \cdot n) \equiv 0 \pmod{2^u}$$

whenever n is coprime to $p_1, \ldots, p_{s+u}$.

The rest of this paper is organised as follows. In Section 2, we recall some basic properties of modular forms and η-quotients. In Section 3, we prove Theorem 1.2 using standard dissection of q-series. The proofs of Theorems 1.3 and 1.4 rely on properties of modular forms and we prove these theorems in Sections 4 and 5, respectively. Finding a proof of Conjecture 1.1 for $k \geq 3$ using standard dissection of q-series looks difficult. However, it might be possible to prove Conjecture 1.1 using modular forms.

2. Preliminaries

In this section, we recall some definitions and basic facts on modular forms and eta-quotients. For more details, see for example [8, 12].

2.1. Spaces of modular forms. We first define the matrix groups

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\},$$

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and

$$\Gamma(N) := \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N} \text{ and } b \equiv c \equiv 0 \pmod{N}\right\},$$

where N is a positive integer. A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subseteq \Gamma$ for some N. The smallest N such that $\Gamma(N) \subseteq \Gamma$ is called the level of Γ. For example, $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level N.

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half of the complex plane. The group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0\right\}$$

acts on $\mathbb{H}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d}.$$

We identify ∞ with $1/0$ and define

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{r}{s} = \frac{ar+bs}{cr+ds},$$

where $r/s \in \mathbb{Q} \cup \{\infty\}$. This gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended upper half-plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A cusp of Γ is an equivalence class in $\mathbb{P}^1 = \mathbb{Q} \cup \{\infty\}$ under the action of Γ.

The group $\mathrm{GL}_2^+(\mathbb{R})$ also acts on functions $f : \mathbb{H} \to \mathbb{C}$. In particular, suppose that $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{R})$. If $f(z)$ is a meromorphic function on $\mathbb{H}$ and ℓ is an integer, then define the slash operator $|_\ell$ by

$$(f|_\ell\gamma)(z) := (\det \gamma)^{\ell/2}(cz+d)^{-\ell} f(\gamma z).$$

Definition 2.1. Let Γ be a congruence subgroup of level N. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form with integer weight ℓ on Γ if it satisfies the following conditions:

(1) $f((az+b)/(cz+d)) = (cz+d)^\ell f(z)$ for all $z \in \mathbb{H}$ and all $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$;
(2) if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_\ell\gamma)(z)$ has a Fourier expansion of the form

$$(f|_\ell\gamma)(z) = \sum_{n \geq 0} a_\gamma(n) q_N^n,$$

where $q_N := e^{2\pi i z/N}$.

For a positive integer ℓ, the complex vector space of modular forms of weight ℓ with respect to a congruence subgroup Γ is denoted by $M_\ell(\Gamma)$.

Definition 2.2 [12, Definition 1.15]. If χ is a Dirichlet character modulo N, then we say that a modular form $f \in M_\ell(\Gamma_1(N))$ has Nebentypus character χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

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for all $z \in \mathbb{H}$ and all $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_\ell(\Gamma_0(N), \chi)$.

2.2. Modularity of eta-quotients. The relevant modular forms in this paper are those that arise from eta-quotients. Recall that the Dedekind eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi iz}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta},$$

where N is a positive integer and r_δ is an integer.

We now recall two theorems from [12, page 18] on modularity of eta-quotients. We will use these two results to verify modularity of certain eta-quotients appearing in the proofs of our main results.

THEOREM 2.3 [12, Theorem 1.64]. *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient such that $\ell = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$,*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for every $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$. Here, the character χ is defined by $\chi(d) := \left(\frac{(-1)^\ell s}{d}\right)$, where $s := \prod_{\delta|N} \delta^{r_\delta}$.

Suppose that f is an eta-quotient satisfying the conditions of Theorem 2.3 and that the associated weight ℓ is a positive integer. If $f(z)$ is holomorphic at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_\ell(\Gamma_0(N), \chi)$. The following theorem gives the necessary criterion for determining orders of an eta-quotient at cusps.

THEOREM 2.4 [12, Theorem 1.65]. *Let c, d and N be positive integers with $d \mid N$ and* $\gcd(c, d) = 1$. *If f is an eta-quotient satisfying the conditions of Theorem 2.3 for N, then the order of vanishing of $f(z)$ at the cusp c/d is*

$$\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d, N/d)d\delta}.$$

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Finally, we recall the definition of Hecke operators. Let m be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi)$. Then, the action of the Hecke operator T_m on $f(z)$ is defined by

$$f(z)|T_m := \sum_{n=0}^{\infty} \bigg(\sum_{d|\gcd(n,m)} \chi(d) d^{\ell-1} a\Big(\frac{nm}{d^2}\Big)\bigg) q^n.$$

In particular, if $m = p$ is prime,

$$f(z)|T_p := \sum_{n=0}^{\infty} \bigg(a(pn) + \chi(p) p^{\ell-1} a\Big(\frac{n}{p}\Big)\bigg) q^n.$$

We adopt the convention that $a(n) = 0$ unless n is a nonnegative integer.

3. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following lemma.

LEMMA 3.1. *The following identities hold:*

$$\begin{aligned}
\frac{f_3}{f_1^3} &= \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \\
\frac{1}{f_1^4} &= \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}, \\
\frac{f_2}{f_1^2} &= \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6}, \\
f_1 f_2 &= \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}, \\
f_1^3 &= \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}.
\end{aligned}$$

PROOF. For the proof of first identity, see [10]. The second identity is proved in [6]. The remaining three identities of the lemma are proved in [7]. □

PROOF OF THEOREM 1.2. By (1.1), we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(n) q^n = q \frac{f_2 f_3^2 f_{12}^2}{f_1^2 f_6}.$$

Substituting the 3-dissection formula for f_2/f_1^2 from Lemma 3.1,

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(n) q^n = q \frac{f_3^2 f_{12}^2}{f_6} \bigg(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6} \bigg).$$

Extracting those terms of the form q^{3n} on both sides of this equation and replacing q^3 by q, we find that

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(3n)q^n = 4q\frac{f_2 f_4^2 f_6^3}{f_1^4}.$$

Substituting the 2-dissection formula for $1/f_1^4$ from Lemma 3.1 yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(3n)q^n = 4q f_2 f_4^2 f_6^3 \Big(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\Big).$$

Extracting those terms of the form q^{2n} on both sides of this equation and replacing q^2 by q,

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(6n)q^n = 16q f_2^4 f_4^4 \Big(\frac{f_3}{f_1^3}\Big)^3.$$

Substituting the 2-dissection formula for f_3/f_1^3 from Lemma 3.1 yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(6n)q^n = 16q\frac{f_4^{22} f_6^9}{f_2^{23} f_{12}^6} + 16\cdot 9q^2\frac{f_4^{18} f_6^7}{f_2^{21} f_{12}^2} + 16\cdot 27q^3\frac{f_4^{14} f_6^5 f_{12}^2}{f_2^{19}} + 16\cdot 27q^4\frac{f_4^{10} f_6^3 f_{12}^6}{f_2^{17}}.$$

Extracting those terms of the form q^{2n} on both sides of this equation and replacing q^2 by q,

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(12n)q^n = 16\cdot 9q\frac{f_2^{18} f_3^7}{f_1^{21} f_6^2} + 16\cdot 27q^2\frac{f_2^{10} f_3^3 f_6^6}{f_1^{17}}. \tag{3.1}$$

By the binomial theorem,

$$\frac{f_2^{10}}{f_1^{17}} \equiv \frac{f_2^9}{f_1^{18}} f_1 f_2 \equiv \frac{f_6^3}{f_3^6} f_1 f_2 \pmod 3.$$

Substituting the 3-dissection formula for $f_1 f_2$ from Lemma 3.1,

$$\frac{f_2^{10}}{f_1^{17}} \equiv \frac{f_6^4 f_9^4}{f_3^7 f_{18}^2} - q\frac{f_6^3 f_9 f_{18}}{f_3^6} - 2q^2\frac{f_6^2 f_{18}^4}{f_3^5 f_9^2} \pmod 3. \tag{3.2}$$

Again, using the binomial theorem,

$$\frac{f_2^{18}}{f_1^{21}} \equiv \frac{f_2^{18}}{f_1^{27}} f_1^6 \equiv \frac{f_6^6}{f_3^9}(f_1^3)^2 \pmod 9.$$

Substituting the 3-dissection formula for f_1^3 from Lemma 3.1,

$$\frac{f_2^{18}}{f_1^{21}} \equiv \frac{f_6^8 f_9^{12}}{f_3^{11} f_{18}^6} - 6q\frac{f_6^7 f_9^9}{f_3^{10} f_{18}^3} + 9q^2\frac{f_6^6 f_9^6}{f_3^9} + 8q^3\frac{f_6^5 f_9^3 f_{18}^3}{f_3^8} - 24q^4\frac{f_6^4 f_{18}^6}{f_3^7} + 16q^6\frac{f_6^2 f_{18}^{12}}{f_3^5 f_9^6} \pmod 9. \tag{3.3}$$

Substituting (3.2) and (3.3) in (3.1), and then extracting the terms of the form q^{3n} on both sides, we find that

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(36n)q^{3n} \equiv 16 \cdot 81q^3\frac{f_6^4 f_9^6}{f_3^2} - 16 \cdot 27q^3\frac{f_6^9 f_9 f_{18}}{f_3^3} \pmod{81}.$$

Replacing q^3 by q, and then substituting the 2-dissection formula for f_3/f_1^3 from Lemma 3.1,

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PDO}_t(36n)q^{n} &\equiv -16 \cdot 27q\frac{f_2^9 f_3 f_6}{f_1^3} \pmod{81} &(3.4)\\ &\equiv -16 \cdot 27q f_2^9 f_6\frac{f_3}{f_1^3} \pmod{81}\\ &\equiv -16 \cdot 27q\frac{f_4^6 f_6^4}{f_{12}^2} - 16 \cdot 81q^2 f_2^2 f_4^2 f_6^2 f_{12}^2 \pmod{81}\\ &\equiv -16 \cdot 27q\frac{f_4^6 f_6^4}{f_{12}^2} \pmod{81}. &(3.5)\end{aligned}$$

Extracting the terms of the form q^{2n} on both sides of (3.5),

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(72n)q^{2n} \equiv 0 \pmod{81}.$$

This completes the proof of (1.2).

We next prove (1.3). By the binomial theorem,

$$\frac{f_2^9}{f_1^3} \equiv \frac{f_6^3}{f_3} \pmod 3. \tag{3.6}$$

Combining (3.6) and (3.4),

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(36n)q^{n} \equiv -16 \cdot 27q f_6^4 \pmod{81}.$$

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Extracting the terms of the form q^{3n} on both sides yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(108n)q^{3n} \equiv 0 \pmod{81}.$$

This completes the proof of (1.3). □

4. Proof of Theorem 1.3

Given a prime p, let

$$A_p(z) = \prod_{n=1}^{\infty} \frac{(1-q^n)^p}{(1-q^{pn})} = \frac{\eta^p(z)}{\eta(pz)}.$$

Then, using the binomial theorem,

$$A_p^{p^k}(z) = \frac{\eta^{p^{k+1}}(z)}{\eta^{p^k}(pz)} \equiv 1 \pmod{p^{k+1}}.$$

Define $B_{p,k}(z)$ by

$$B_{p,k}(z) = \left(\frac{\eta(2z)\eta(3z)^2\eta(12z)^2}{\eta(z)^2\eta(6z)}\right)A_p^{p^k}(z). \tag{4.1}$$

Modulo p^{k+1},

$$B_{p,k}(z) \equiv \frac{\eta(2z)\eta(3z)^2\eta(12z)^2}{\eta(z)^2\eta(6z)} = \frac{qf_2f_3^2f_{12}^2}{f_1^2f_6}. \tag{4.2}$$

Combining (1.1) and (4.2),

$$B_{p,k}(z) \equiv \sum_{n=0}^{\infty} \mathrm{PDO}_t(n)q^n \pmod{p^{k+1}}. \tag{4.3}$$

PROOF OF THEOREM 1.3. We put $p = 2$ in (4.1) to obtain

$$B_{2,k}(z) = \left(\frac{\eta(2z)\eta(3z)^2\eta(12z)^2}{\eta(z)^2\eta(6z)}\right)A_2^{2^k}(z) = \frac{\eta(2z)^{1-2^k}\eta(3z)^2\eta(12z)^2\eta(z)^{2^{k+1}-2}}{\eta(6z)}.$$

Now, $B_{2,k}$ is an eta-quotient with level $N = 144$. The cusps of $\Gamma_0(144)$ are represented by fractions c/d, where $d \mid 144$ and $\gcd(c, d) = 1$ (see for example [11, page 5]). By Theorem 2.4, $B_{2,k}(z)$ is holomorphic at a cusp c/d if and only if

$$\begin{aligned} S := 12\frac{\gcd(d,1)^2}{\gcd(d,12)^2}(2^{k+1}-2) + 6\frac{\gcd(d,2)^2}{\gcd(d,12)^2}(1-2^k) + 8\frac{\gcd(d,3)^2}{\gcd(d,12)^2} \\ - 2\frac{\gcd(d,6)^2}{\gcd(d,12)^2} + 2 \geq 0. \end{aligned}$$

To find all the possible values of S, we prepared Table 1 using MATLAB. Using Table 1, we find that $S \geq 0$ for all $d \mid 144$. Hence, $B_{2,k}(z)$ is holomorphic at every

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TABLE 1. Data to find the values of S.

$d \mid 144$	$\frac{\gcd(d,1)^2}{\gcd(d,12)^2}$	$\frac{\gcd(d,2)^2}{\gcd(d,12)^2}$	$\frac{\gcd(d,6)^2}{\gcd(d,12)^2}$	$\frac{\gcd(d,3)^2}{\gcd(d,12)^2}$
1	1	1	1	1
2	0.2500	1	1	0.2500
3, 9	0.1111	0.1111	1	1
4, 8, 16	0.0625	0.2500	0.2500	0.0625
6, 18	0.0278	0.1111	1	0.2500
12, 24, 36, 48, 72, 144	0.0069	0.0278	0.2500	0.0625

cusp c/d. From Theorem 2.3, the weight of $B_{2,k}(z)$ is $\ell = 2^{k-1} + 1$. Also, the associated character for $B_{2,k}(z)$ is given by

$$\chi_1 = \left(\frac{(-1)^{2^{k-1}+1}2^{4-2^k}3^3}{\bullet}\right).$$

Finally, by Theorem 2.3, $B_{2,k}(z) \in M_{2^{k-1}+1}(\Gamma_0(144), \chi_1)$ for $k \geq 1$. Given any positive integer m, by a deep theorem of Serre [12, page 43], if $f(z) \in M_\ell(\Gamma_0(N), \chi)$ has the Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} c(n)q^n \in \mathbb{Z}[[q]],$$

then there is a constant $\alpha > 0$ such that

$$\#\{n \leq X : c(n) \not\equiv 0 \pmod{m}\} = O\left(\frac{X}{(\log X)^\alpha}\right).$$

This yields

$$\lim_{X\to\infty} \delta_0(f, m; X) = \lim_{X\to\infty} \frac{\#\{n \leq X : c(n) \equiv 0 \pmod{m}\}}{X} = 1.$$

Since $B_{2,k}(z) \in M_{2^{k-1}+1}(\Gamma_0(144), \chi_1)$, the Fourier coefficients of $B_{2,k}(z)$ are almost always divisible by $m = 2^k$. Now, using (4.3) completes the proof of (1.4).

We next prove (1.5). We put $p = 3$ in (4.1) to obtain

$$B_{3,k}(z) = \left(\frac{\eta(2z)\eta(3z)^2\eta(12z)^2}{\eta(z)^2\eta(6z)}\right)A_3^{3^k}(z) = \frac{\eta(2z)\eta(3z)^{2-3^k}\eta(12z)^2\eta(z)^{3^{k+1}-2}}{\eta(6z)}.$$

Now, $B_{3,k}$ is an eta-quotient with $N = 144$. As before, the cusps of $\Gamma_0(144)$ are represented by fractions c/d, where $d \mid 144$ and $\gcd(c, d) = 1$. By Theorem 2.4, $B_{3,k}(z)$ is holomorphic at a cusp c/d if and only if

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$$L := 12\frac{\gcd(d,1)^2}{\gcd(d,12)^2}(3^{k+1}-2) + 4\frac{\gcd(d,3)^2}{\gcd(d,12)^2}(2-3^k) + 6\frac{\gcd(d,2)^2}{\gcd(d,12)^2} - 2\frac{\gcd(d,6)^2}{\gcd(d,12)^2} + 2 \geq 0.$$

From Table 1, $L \geq 0$ for all $d \mid 144$. By Theorem 2.3, $B_{3,k}(z) \in M_{3^k+1}(\Gamma_0(144), \chi_2)$, where χ_2 is the associated Nebentypus character. Using the same reasoning and (4.3), we find that $\mathrm{PDO}_t(n)$ is divisible by 3^k for almost all $n \geq 0$. This completes the proof of (1.5). □

5. Proof of Theorem 1.4

In this section, we prove Theorem 1.4 using nilpotency of Hecke operators. We apply a result of Ono and Taguchi [13] to the modular form $B_{2,k}(z)$ to deduce the infinite family of congruences.

Proof of Theorem 1.4. Taking $p = 2$ in (4.3), we have

$$B_{2,k}(z) \equiv \sum_{n=0}^{\infty} \mathrm{PDO}_t(n) q^n \pmod{2^{k+1}}.$$

Note that $B_{2,k}(z) \in M_{2^{k-1}+1}(\Gamma_0(144), \chi_1)$. By [13, Theorem 1.3(3)], there is an integer $s \geq 0$ such that for any $u \geq 1$,

$$B_{2,k}(z) | T_{p_1} | T_{p_2} | \cdots | T_{p_{s+u}} \equiv 0 \pmod{2^u}$$

whenever $p_1, \ldots, p_{s+u}$ are coprime to 6. It follows from the definition of the Hecke operators that if $p_1, \ldots, p_{s+u}$ are distinct primes and if n is coprime to $p_1 \cdots p_{s+u}$, then

$$\mathrm{PDO}_t(p_1 \cdots p_{s+u} \cdot n) \equiv 0 \pmod{2^u}.$$

This completes the proof of the theorem. □

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