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ON THE TREEWIDTH OF RANDOM GEOMETRIC GRAPHS AND PERCOLATED GRIDS

ANSHUI LI,* *Hangzhou Normal University*
TOBIAS MÜLLER,** *Utrecht University*

Abstract

In this paper we study the treewidth of the random geometric graph, obtained by dropping n points onto the square $[0, \sqrt{n}]^2$ and connecting pairs of points by an edge if their distance is at most $r = r(n)$. We prove a conjecture of Mitsche and Perarnau (2014) stating that, with probability going to 1 as $n \to \infty$, the treewidth of the random geometric graph is $\Theta(r\sqrt{n})$ when $\liminf r > r_c$, where r_c is the critical radius for the appearance of the giant component. The proof makes use of a comparison to standard bond percolation and with a little bit of extra work we are also able to show that, with probability tending to 1 as $k \to \infty$, the treewidth of the graph we obtain by retaining each edge of the $k \times k$ grid with probability p is $\Theta(k)$ if $p > \frac{1}{2}$ and $\Theta(\sqrt{\log k})$ if $p < \frac{1}{2}$.

Keywords: Random geometric graph; treewidth

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1. Introduction and main results

The *random geometric graph* $\mathcal{G}(n, r)$ is the random graph obtained by taking n points $X_1, \dots, X_n$ independent and identically distributed (i.i.d.) uniformly at random from the square $[0, \sqrt{n}]^2$, and joining X_i and X_j by an edge if their Euclidean distance is at most r. Here $r = r(n)$ may and often does depend on n. To avoid having to deal with annoying trivial special cases we assume that $r \le \sqrt{2n}$ throughout the paper. The study of random geometric graphs essentially goes back to Gilbert [7] who defined a very similar model in 1961. For this reason random geometric graphs are often also called the *Gilbert model* of random graphs. Random geometric graphs have been the subject of a considerable research effort in the last two decades. As a result, detailed information is now known on various aspects such as (k-)connectivity [22], [23], the largest component [24], the chromatic number and clique number [17], [20], the (non-)existence of Hamilton cycles [2], [21], monotone graph properties in general [8], and the simple random walk on the graph [4]. One of the most well-known phenomena in random geometric graphs is the 'sudden emergence of a giant component'. By this we mean that there exists a critical value r_c such that if $\limsup r < r_c$ then, a.a.s., every component of $G(n, r)$ has $O(\log n)$ vertices, whereas if $\liminf r > r_c$ then, a.a.s., there exists a 'giant' component with $\Omega(n)$ vertices. Here and in the rest of the paper, we say that a sequence of events $(E_n)_n$ holds *asymptotically almost surely* (a.a.s.) if $\lim_{n\to\infty} \mathbb{P}(E_n) = 1$. The exact value of r_c is not known at this time, but simulations suggest that the exact value is approximately 1.2 (see [24]).

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* Postal address: Department of Mathematics, Hangzhou Normal University, Hangzhou, 310000, P.R. China.
Email address: anshuili@hznu.edu.cn
** Postal address: Mathematical Institute, Utrecht University, Utrecht, 3508TA, The Netherlands.
Email address: t.muller@uu.nl

For more details, and proofs, on the giant component phenomenon and background on random geometric graphs in general we refer the reader to [24].

In this paper we consider the *treewidth* of random geometric graphs. The treewidth of a graph was introduced by Halin [10] and independently, but later, by Robertson and Seymour [25]. It is a graph parameter that in a sense measures how similar a given graph is to a tree. (We postpone the precise—and technical—definition of treewidth until Section 2.) Treewidth plays an important role in modern algorithmic graph theory. Many NP-hard algorithmic decision problems have, for instance, been shown to be polynomially solvable when restricted to the class of instances with a bounded tree-width. In fact, a striking result of Courcelle [5] states that any algorithmic decision problem that can be expressed in monadic second-order logic can be solved in linear time for the class of graphs with bounded treewidth. An example of a decision problem that is NP-hard, in general, and can be expressed in monadic second-order is k-colorability (for any fixed k). As random geometric graphs have been used extensively as models for modeling communication networks, this motivated Mitsche and Perarnau [19] to consider the treewidth (tw) of random geometric graphs. They proved that if $r \in (0, r_c)$ is fixed then, a.a.s., $\mathrm{tw}(G(n, r)) = \Theta(\log n/\log\log n)$, while if $r > C$, where C is a large constant, then, a.a.s., $\mathrm{tw}(G(n, r)) = \Theta(r\sqrt{n})$. Mitsche and Perarnau [19] also conjectured that the second result should extend all the way to the critical value. Here we will prove their conjecture.

Theorem 1.1. *If $r = r(n)$ is such that* $\liminf r > r_c$, *where r_c is the critical value for the emergence of the giant component, then, a.a.s. as $n \to \infty$,* $\mathrm{tw}(G(n, r)) = \Theta(r\sqrt{n})$.

Our proof of Theorem 1.1 makes use of a comparison to bond percolation on $\mathbb{Z}^2$. Recall that this refers to the infinite random graph obtained by retaining each edge of the familiar integer lattice with probability p and discarding it with probability $1 - p$, independently of the choices for all other edges. We will denote by $\Gamma(k, p)$ the restriction of this process to the $k \times k$ integer grid. That is, $\Gamma(k, p)$ has vertex set $[k]^2$ and for every pair of points $u, v \in [k]^2$ with Euclidean distance equal to 1, we add an edge with probability p, independently of the choices for all other pairs. (Here and in the rest of the paper we use the notation $[k] := \{1, \ldots, k\}$.) For the proof of Theorem 1.1 we only need to consider the treewidth of $\Gamma(k, p)$ when p is very close to 1, but with a little bit of extra work we are able to obtain the following result in addition to Theorem 1.1.

Theorem 1.2. *If $p \in (0, 1)$ is fixed then, a.a.s. as $k \to \infty$,*

$$\mathrm{tw}(\Gamma(k, p)) = \begin{cases} \Theta(k) & \text{if } p > \frac{1}{2}, \\ \Theta(\sqrt{\log k}) & \text{if } p < \frac{1}{2}. \end{cases} \tag{1.1}$$

Note that k is the *square root* of the number of vertices of $\Gamma(k, p)$.

2. Notation and preliminaries

In this section we give some definitions and results which we will need in the sequel. We start with the precise definition of treewidth. For a graph $G = (V, E)$ on n vertices, we call $(T, \mathcal{W})$ a *tree decomposition* of G, where $\mathcal{W}$ is a set of vertex subsets $W_1, \ldots, W_s \subset V$, called *bags*, and T is a forest with vertices in $\mathcal{W}$, such that

- $\bigcup_{i=1}^{s} W_i = V$;
- for any $e = uv \in E$ there exists a set $W_i \in \mathcal{W}$ such that $u, v \in W_i$;
- for any $v \in V$, the subgraph induced by the $W_i \ni v$ is connected as a subgraph of T.

The *width* of a tree-decomposition is $w(T, \mathcal{W}) = \max_{1 \le i \le s} |W_i| - 1$, and the *treewidth* of a graph G can be defined as

$$\mathrm{tw}(G) := \min_{(T,\mathcal{W})} w(T, \mathcal{W}),$$

where the minimum is taken over all tree decompositions $(T, \mathcal{W})$ of G. From the definition of treewidth, one can see that the treewidth of a tree is one, while the treewidth of a k-clique is $k - 1$. We also observe that if H is a subgraph of G, then $\mathrm{tw}(H) \le \mathrm{tw}(G)$, and if G is a graph with connected components $G_1, \ldots, G_m$, then $\mathrm{tw}(G) = \max_{1 \le i \le m} \mathrm{tw}(G_i)$.

Given an edge xy of graph G, the graph G/xy is obtained from G by *contracting* the edge xy. That is, to obtain G/xy, we identify the vertices x and y and remove all resulting loops and duplicate edges. A graph H is a *minor* of G if it is a subgraph of the graph obtained from G by a sequence of edge-contractions. Again, one can see from the definitions that if H is a *minor* of G, $\mathrm{tw}(H) \le \mathrm{tw}(G)$.

Alon *et al.* [1] proved the following powerful result, bounding the treewidth of graphs without a given minor.

Theorem 2.1. (See [1].) *If G does not have H as a minor, then* $\mathrm{tw}(G) \le |V(H)|^{3/2}\sqrt{|V(G)|}$.

In this paper we will make use of the following immediate corollary.

Corollary 2.1. *There exists a constant $C > 0$ such that every planar graph G satisfies* $\mathrm{tw}(G) \le C\sqrt{|V(G)|}$.

Throughout the paper we will denote by $\Gamma(k)$ $(:= \Gamma(k, 1))$ the $k \times k$ grid. The next observation appears as Exercise 16 in [6, Chapter 12].

Lemma 2.1. *We have* $\mathrm{tw}(\Gamma(k)) = k$.

For one of our lower bounds on the treewidth, we will need the following lemma which links the treewidth of a graph and the existence of a partition of its vertex set with special properties. A vertex partition $V = \{A, S, B\}$ is a *balanced k-partition* if $|S| = k + 1$, there are no edges in G between a vertex in A and a vertex in B, and $\frac{1}{3}(n - k - 1) \le |A|, |B| \le \frac{2}{3}(n - k - 1)$. In this case, S is called a *balanced separator*. The following result connecting balanced partitions and treewidth is due to Kloks [14], which provides a necessary condition for a graph to have a treewidth of certain size.

Lemma 2.2. (See [14].) *Let G be a graph and suppose that* $\mathrm{tw}(G) \le k \le |V(G)| - 1$. *Then G has a balanced k-partition.*

We say that $A \subseteq \{0, 1\}^n$ is an up-set if whenever we take a point of A and we change one of its coordinates into a one, then the resulting point is still in A. We will use the following lemma later on.

Lemma 2.3. (Harris' lemma, [11].) *Let $A, B \subseteq \{0, 1\}^n$ be up-sets and let $X = (X_1, \ldots, X_n)$ be a vector of independent Bernoulli random variables. Then* $\mathbb{P}(X \in A \cap B) \ge \mathbb{P}(X \in A)\mathbb{P}(X \in B)$.

By a slight abuse of notation, throughout this paper we will denote the graph with vertex set $\mathbb{Z}^2$ and an edge $vw \in E(\mathbb{Z}^2)$ if and only if $\|v - w\| = 1$ also by $\mathbb{Z}^2$. Recall that bond percolation on $\mathbb{Z}^2$ refers to the random process where we keep each edge of $\mathbb{Z}^2$ with probability p and discard it with probability $1 - p$, independently of all other edges. The edges that are kept are referred to as *open* and the discarded edges as *closed*. If $R := \{a, \ldots, b\} \times \{c, \ldots, d\}$ is an axis-parallel rectangle, then we say that R has a *horizontal crossing* if there is an open path

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that stays inside R and connects the left side $\{a\} \times \{c, \dots, d\}$ to the right side $\{b\} \times \{c, \dots, d\}$. A vertical crossing is defined similarly. We denote by $H(R)$ the event that there is a horizontal crossing of R, and $V(R)$ the event that there is a vertical crossing of R. In the sequel, we will use the following well-known result on bond percolation on $\mathbb{Z}^2$ with $p > \frac{1}{2}$. A proof can, for instance, be found in [3, Lemma 8, p. 64].

Lemma 2.4. *If* $p > \frac{1}{2}$ *then* $\lim_{k\to\infty} \mathbb{P}(H([3k] \times [k])) = 1$.

In words, when $p > \frac{1}{2}$ then the probability of crossing a $3k \times k$ rectangle in the long direction can be made arbitrarily close to 1 by making k large.

Formally speaking, we can describe bond percolation on $\mathbb{Z}^2$ as a random vector X taking values in $\{0, 1\}^{E(\mathbb{Z}^2)}$. Here $X_e = 1$ if e is open, and $X_e = 0$ otherwise. In the standard setup, the coordinates X_e are i.i.d. Bernoulli random variables. One can also consider more general bond percolation models in which the coordinates are not independent. We say that such a bond percolation model Y is 1-*independent* if, for every pair of sets $S, T \subseteq E(\mathbb{Z}^2)$ with the property that no edge in S shares an endpoint with any edge in T, the random vectors $(Y_e)_{e\in S}$ and $(Y_e)_{e\in T}$ are independent. Recall that a coupling of two random objects X, Y is a joint probability space on which both are defined (and have the correct marginal distributions). The following result is a reformulation of a special case of a result by Liggett *et al.* [15].

Theorem 2.2. (See [15].) *There exists a function* $\pi : [0, 1] \to [0, 1]$ *such that,* $\lim_{p\uparrow 1} \pi(p) = 1$, *and the following holds. Suppose that* Y *follows a* 1*-independent bond percolation model on* $\mathbb{Z}^2$ *and set* $p := \inf_{e\in E(\mathbb{Z}^2)} \mathbb{P}(Y_e = 1)$*. Then there exists a coupling of* Y *with standard (i.e. independent) bond percolation* X *with* $\mathbb{P}(X_e = 1) = \pi(p)$*, such that, almost surely,* $X_e \le Y_e$ *for all* $e \in E(\mathbb{Z}^2)$.

In words, this last theorem says that every 1-independent bond percolation model contains the edges of an independent bond percolation model, and the edge probability $\pi(p)$ of this independent bond percolation approaches 1 as $p := \inf_{e\in E(\mathbb{Z}^2)} \mathbb{P}(Y_e = 1)$ approaches 1.

When working with random geometric graphs, it is often useful to consider a *Poissonized* version of the random geometric graph. We define $G_{\text{Po}}(n, r)$ analogously to $G(n, r)$ except that now we take the points of a Poisson process of intensity 1 on $[0, \sqrt{n}]^2$ and then build our graph on that as before. Equivalently, we can say that we throw $N_n \overset{\text{D}}{=} \text{Po}(n)$ i.i.d. uniform points onto $[0, \sqrt{n}]^2$ and then build the graph on those as before where, '$\overset{\text{D}}{=}$' denotes equality in distribution. Working with the Poissonized version is often useful in proofs because of the convenient independence properties of the Poisson process. Recall that if $N_n \overset{\text{D}}{=} \text{Po}(n)$ then $\mathbb{P}(N_n > (1 + \varepsilon)n) = o(1)$, as can for instance be seen by Chebyschev's inequality. Using a straightforward coupling and rescaling, this gives the following observation.

Corollary 2.2. *There is a coupling such that for every* $r = r(n)$*, a.a.s.,* $G_{\text{Po}}((1-\varepsilon)n, r\sqrt{1-\varepsilon})$ *is a subgraph of* $G(n, r)$.

It, of course, also makes sense to simply consider the random geometric graph built on a Poisson process $\mathcal{P}$ of intensity 1 on all of the plane $\mathbb{R}^2$. This is the well-known *continuum percolation* model defined originally by Gilbert [7]. We remark that Gilbert and several other sources in the literature fix $r = 1$ and allow the intensity of the Poisson process to vary, but it is easily seen that this is equivalent to the setting where we vary r and the intensity of the Poisson process is fixed to be 1. Note that $G_{\text{Po}}(n, r)$ is just the restriction of continuum percolation to the square $[0, \sqrt{n}]^2$. We also remark that the critical r_{c} for the 'emergence of a giant component' in $G(n, r)$ is the same as the critical value for the existence of an infinite

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component in continuum percolation (see [24, Chapters 9 and 10]). Similarly to the case of bond percolation on $\mathbb{Z}^2$, we can define crossing events for continuum percolation. Our definition follows Meester and Roy [18]. For $R = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ an axis-parallel rectangle, we say that $H(R)$ holds (i.e. there is a horizontal crossing of R) if it is possible to draw a continuous curve between the right and left side that stays inside R and is completely covered by the balls of radius $r/2$ centered on the points of $\mathcal{P}$. Note that this in particular implies that there is a path between a vertex that is within $r/2$ of the left side of R, and a vertex within $r/2$ of the right side of R such that all other vertices of the path are either inside R or within distance $r/2$ of R. We have the following analogue of Lemma 2.4.

Lemma 2.5. (See [18, Corollary 4.1].) *If* $r > r_c$ *then* $\lim_{a\to\infty} \mathbb{P}(H([0, 3a] \times [0, a])) = 1$.

We say that an event A defined with respect to the Poisson process $\mathcal{P}$ is *increasing* if whenever A holds for some set of points $X = \{x_1, x_2, \dots\} \subseteq \mathbb{R}^2$ (i.e. some realization of $\mathcal{P}$), then A also holds for any set X' that contains X. We have the following analogue of Lemma 2.3 above.

Lemma 2.6. (See [18, Theorem 2.2].) *If* A, B *are increasing events (with respect to* $\mathcal{P}$*) then* $\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$.

3. The treewidth of the percolated grid $\Gamma(k, p)$

3.1. When p is large

Instead of proving the $p > \frac{1}{2}$ part of Theorem 1.2 directly, we first prove the following weaker version.

Proposition 3.1. *There exist constants* $c > 0$ *and* $p < 1$ *such that* $\operatorname{tw}(\Gamma(k, p)) \geq ck$ *a.a.s.*

If $A \subseteq \mathbb{Z}^2$ is finite and connected (as a subgraph of $\mathbb{Z}^2$) then there is a well defined 'surrounding cycle' $\operatorname{surr}(A)$ in the dual lattice $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ (that separates A from ∞, and every other cycle in $(\mathbb{Z}^2)^*$ that separates A from ∞ contains $\operatorname{surr}(A)$ in its interior). For $A \subseteq [k]^2$ connected, we define $\operatorname{outer}(A)$ to be the set of edges of $\Gamma(k)$ that cross $\operatorname{surr}(A)$. (See Figure 1 for a depiction.)

We will make use of the following straightforward observation. We include a proof for completeness.

Lemma 3.1. *Suppose that* $A \subseteq [k]^2$ *is connected (as a subgraph of* $\Gamma(k)$*) and does not contain a horizontal crossing of* $[k]^2$. *Then* $|\operatorname{outer}(A)| \geq \max(\sqrt{|A|}, |\operatorname{surr}(A)|/4)$.

Proof. First suppose that A contains a vertical crossing. Since A does not contain a horizontal crossing, $\operatorname{surr}(A)$ must contain a (dual) path that separates the left edge of $[k]^2$ from its right edge. This implies that $\operatorname{outer}(A)$ contains at least k edges. Hence, $\operatorname{outer}(A) \geq \sqrt{|A|}$. Note that, since A can intersect at most one of the vertical sides of $[k]^2$, we have that the number of edges of $\operatorname{surr}(A)$ that do not intersect edges of $\Gamma(k)$ is at most $|\operatorname{surr}(A)| - |\operatorname{outer}(A)| \leq 3k$. This shows $|\operatorname{outer}(A)| \geq |\operatorname{surr}(A)|/4$.

Let us then assume that A contains neither a horizontal nor a vertical crossing. Let $a := |\pi_x(A)|, b := |\pi_y(A)|$, where π_x, respectively π_y, denote the projection onto the x-axis, respectively the y-axis. Clearly, we have $|A| \leq ab$ and $|\operatorname{outer}(A)| \geq a + b$. Thus,

$$\sqrt{|A|} \leq \max(a, b) \leq a + b \leq |\operatorname{outer}(A)|.$$

Also note that $|\operatorname{surr}(A)| - |\operatorname{outer}(A)| \leq a + b$. So certainly $|\operatorname{outer}(A)| \geq |\operatorname{surr}(A)|/4$. □

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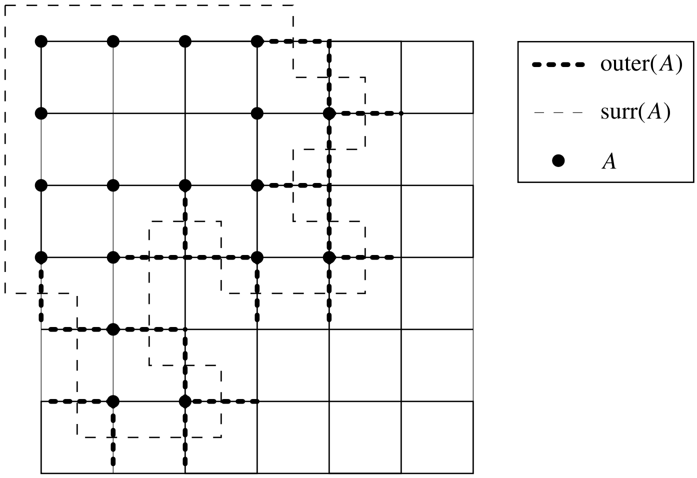

FIGURE 1: Depiction of surr(A) and outer(A) for a set $A \subseteq [7]^2$.

We say a set $A \subseteq [k]^2$ is *dirty* (with respect to $\Gamma(k, p)$) if

- A is connected (as a subgraph of $\Gamma(k)$),
- A intersects at most three sides of $[k]^2$, and
- at most half of the edges of outer(A) are open in $\Gamma(k, p)$.

We say that a vertex $v \in [k]^2$ is dirty if it is contained in some dirty set.

Lemma 3.2. *There exists a $p_0 < 1$ such that whenever $p \geq p_0$ then, a.a.s., there are at most $k^2/10^{10}$ dirty vertices.*

Proof. Let Y denote the number of vertices contained in some dirty set A with $|\text{surr}(A)| \geq k^{0.01}$ and let Z denote the number of vertices contained in some dirty set A with $|\text{surr}(A)| < k^{0.01}$. It is easy to see that the number of cycles in $(\mathbb{Z}^2)^*$ that have length ℓ and that surround a given vertex $v \in \mathbb{Z}^2$ is at most $\ell 4^\ell$. This allows us to bound the expectation of Y as

$$\begin{aligned}\mathbb{E}(Y) &\leq k^2 \sum_{\ell \geq k^{0.01}} 4^\ell \binom{\ell}{\ell/8}(1-p)^{\ell/8} \\ &\leq k^2 \sum_{\ell \geq k^{0.01}} 8^\ell (1-p)^{\ell/8} \\ &= \frac{k^2(8(1-p)^{1/8})^{k^{0.01}}}{1-8(1-p)^{1/8}} \\ &= o(1),\end{aligned}$$

where we have used the fact that $|\text{outer}(A)| \geq |\text{surr}(A)|/8$ in the first line, that $\binom{\ell}{\ell/8} \leq 2^\ell$ in the second line, and where the last equality holds provided p_0 was chosen sufficiently close to 1 (and $p \geq p_0$). In particular, for p_0 sufficiently close to 1 and $p > p_0$, we have $Y = 0$ a.a.s.

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Next, we consider Z. For $v \in [k]^2$, we denote by E_v the event that v is contained in a dirty A with $|\text{surr}(A)| < k^{0.01}$. We have

$$\begin{aligned}
\mathbb{P}(E_v) &\leq \sum_{\ell \leq k^{0.01}} \ell 4^\ell \binom{\ell}{\ell/8} (1-p)^{\ell/8} \\
&\leq \sum_{\ell \leq k^{0.01}} \ell (8(1-p)^{1/8})^\ell \\
&\leq \frac{8(1-p)^{1/8}}{(1-8(1-p)^{1/8})^2} \\
&\leq 10^{-11},
\end{aligned}$$

where the last inequality holds provided p_0 is chosen sufficiently close to 1 (and $p \geq p_0$).

On the other hand, we clearly have

$$\mathbb{P}(E_v) \geq (1-p)^4.$$

Hence, we have $k^2(1-p)^4 \leq \mathbb{E}Z = \sum_v \mathbb{P}(E_v) \leq k^2/10^{11}$. In particular, $\mathbb{E}Z = \Theta(k^2)$.

Next we consider the second moment of Z. Observe that if $|u-v|_\infty \geq 3k^{0.01}$, then E_u and E_v are independent. (Here $|(x,y)|_\infty = \max(|x|,|y|)$ denotes the familiar L_∞-norm.) This allows us to write

$$\begin{aligned}
\mathbb{E}Z^2 &= \sum_{u,v} \mathbb{P}(E_u \cap E_v) \\
&= \sum_v \mathbb{P}(E_v) \sum_{|u-v|_\infty < 3k^{0.01}} \mathbb{P}(E_u \mid E_v) + \sum_v \mathbb{P}(E_v) \sum_{|u-v|_\infty \geq 3k^{0.01}} \mathbb{P}(E_u \mid E_v) \\
&\leq \sum_v \mathbb{P}(E_v) 36k^{0.02} + \sum_{u,v} \mathbb{P}(E_v)\mathbb{P}(E_u) \\
&= \mathbb{E}(Z) \cdot o(k^2) + (\mathbb{E}Z)^2 \\
&= (1+o(1))(\mathbb{E}Z)^2.
\end{aligned}$$

This shows that $\text{var}(Z) = o((\mathbb{E}Z)^2)$. An application of Chebyschev's inequality shows that

$$\mathbb{P}\left(Z > \frac{k^2}{10^{10}}\right) \leq \mathbb{P}\left(|Z - \mathbb{E}Z| \geq \frac{9}{10}\mathbb{E}Z\right) \leq \left(\frac{10}{9}\right)^2 \frac{\text{var}(Z)}{(\mathbb{E}Z)^2} = o(1).$$

In conclusion, we have seen that, when p_0 is sufficiently close to 1 and $p_0 < p \leq 1$, then $Y = 0$ a.a.s. and $Z \leq k^2/10^{10}$ a.a.s., which obviously implies the lemma. □

Proof of Proposition 3.1. Let us pick $1 > p > p_0$ with p_0 as provided by Lemma 3.2. Then, a.a.s., $\Gamma(k,p)$ has no more than $k^2/10^{10}$ dirty vertices. In the remainder of the proof we therefore assume we are given a subgraph $G \subseteq \Gamma(k)$ for which there are at most $k^2/10^{10}$ dirty vertices, but which is otherwise arbitrary. We will show that any such G satisfies $\text{tw}(G) \geq k/1000$.

Aiming for a contradiction, we assume that there exists some balanced partition $\{A, S, B\}$ of $V(G) = [k]^2$ with $|S| < k/1000$.

We first observe that we can assume, without loss of generality, that A does not contain a horizontal crossing. For, if it does then B cannot contain a vertical crossing (otherwise A, B

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would not be disjoint). Hence, by applying symmetry (switching the roles of A, B and rotating by 90 degrees) we can indeed assume A does not contain a horizontal crossing. Observe that

$$|A| \geq k^2 - |B| - |S| \geq k^2 - \tfrac{2}{3}k^2 - \tfrac{1}{1000}k \geq \tfrac{1}{10}k^2.$$

Let $A_1, \ldots, A_m$ denote the connected components of A (connected when considered as subgraphs of G). We set

$$\mathcal{I} := \{i : A_i \text{ is not dirty}\}, \qquad A' := \bigcup_{i \in \mathcal{I}} A_i.$$

Since the total number of dirty vertices is less that $k^2/10^{10}$, we have

$$|A'| \geq |A| - \frac{k^2}{10^{10}} \geq \frac{k^2}{100}.$$

Note that every edge (in G) between a vertex of A' and a vertex of $[k^2] \setminus A'$ must in fact connect a vertex of A' to a vertex of S. Hence, it follows that

$$4|S| \geq \tfrac{1}{2} \sum_{i \in \mathcal{I}} |\mathrm{outer}(A_i)| \geq \tfrac{1}{2} \sum_{i \in \mathcal{I}} \sqrt{|A_i|} \geq \tfrac{1}{2}\sqrt{|A'|} \geq \tfrac{1}{20}k.$$

(Here we have used Lemma 3.1 for the third expression and the concavity of the square root function for the fourth expression.) So it follows that $k/1000 \geq |S| \geq k/80$, a contradiction.

This shows that there is no balanced partition with $|S| < k/1000$, which implies that $\mathrm{tw}(G) \geq k/1000$ by Kloks' lemma (Lemma 2.2). □

3.2. When $p > \frac{1}{2}$

We are now ready to prove the first part of Theorem 1.2 with the help of Proposition 3.1.

Proof of Theorem 1.2. (The $p > \frac{1}{2}$ case.) Our proof is an application of a standard technique for comparing supercritical percolation to percolation with p close to 1, by means of Lemma 2.4 and Theorem 2.2. See, for instance, [3, pp. 74–75].

Let p_0 be as provided by Proposition 3.1, and let π be as provided by Theorem 2.2. We now pick p_1 such that $\pi(p_1) > p_0$. By Lemma 2.4, we can find an $a \in \mathbb{N}$ such that $\mathbb{P}(H([3a] \times [a])) > \sqrt[3]{p_1}$.

For R a $3a \times a$ rectangle, we define the event $E(R) := H(R) \cap V(R_{\mathrm{L}}) \cap V(R_{\mathrm{R}})$, where R_{L} denotes the leftmost $a \times a$ subrectangle, and R_{R} denotes the rightmost $a \times a$ rectangle (see Figure 13 of [3, p. 74] for a depiction). If R is a $a \times 3a$ rectangle then we define $E(R) := V(R) \cap H(R_{\mathrm{B}}) \cap H(R_{\mathrm{T}})$ with R_{B}, respectively R_{T}, the bottom, respectively top, $a \times a$ subrectangle of A. Note that, by choice of a and Harris' lemma, we have $\mathbb{P}(E(R)) > p_1$ for every $3a \times a$ or $a \times 3a$ rectangle R.

We now define a (dependent) bond percolation model Y on $\mathbb{Z}^2$ as follows. We declare the horizontal edge between (i, j) and $(i+1, j)$ open in Y if $E(\{2ai+1, \ldots, 2ai+3a\} \times \{2aj+1, \ldots, 2aj+a\})$ holds; similarly, the edge between (i, j) and $(i, j+1)$ is open in Y if $E(\{2ai+1, \ldots, 2ai+a\} \times \{2aj+1, \ldots, 2aj+3a\})$ holds. To clarify the construction, let us mention that one could think of the square $\{2ai+1, \ldots, 2ai+a\} \times \{2aj+1, \ldots, 2aj+a\}$ as representing the point (i, j) and in the $3a \times a$ rectangle R that represents the edge between (i, j) and $(i+1, j)$, we have that R_{L} represents (i, j) and R_{R} represents $(i+1, j)$. It is not difficult to see that Y is in fact 1-independent. Hence, by Theorem 2.2, $Y \geq X$, where X is standard (independent) percolation on $\mathbb{Z}^2$ with edge-probability $> p_0$.

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We can view $\Gamma(k, p)$ as the restriction of the (independent, edge-probability p) percolation process to the $k \times k$ grid $[k]^2$. We let Γ_X, respectively Γ_Y, denote the subgraph that X, respectively Y, defines on $[\ell]^2$, where $\ell := \lfloor k/2a \rfloor - 1$. As the reader can easily check, we have chosen ℓ so that each of the rectangles corresponding to the edges of Γ_Y is contained in $[k]^2$. Observe that by construction (and Proposition 3.1) we have a.a.s.,

$$\mathrm{tw}(\Gamma_Y) \geq \mathrm{tw}(\Gamma_X) = \Omega(\ell) = \Omega(k).$$

Next, we remark that Γ_Y is in fact a minor of $\Gamma(k, p)$ (under the natural coupling associated with the construction of Y). To see this, we can proceed as follows. If $E(R)$ holds with R a $3a \times a$ rectangle that corresponds to some edge of Γ_Y, then we perform a sequence of contractions that will identify all vertices of R_{L} that participate in (horizontal or vertical) crossings of R_{L} into a single vertex x, we produce a vertex y via contractions in R_{R} similarly, and then we contract the remaining edges of a long, horizontal crossing of R into a single edge that connects x and y. If we carry this out for each rectangle corresponding to an edge of Γ_Y and discard any unneeded vertices (making sure to keep exactly one vertex in each $a \times a$ square that corresponds to a vertex $(i, j) \in [\ell]^2$ that was not incident to any edge of Y), then we obtain a graph isomorphic to Γ_Y.

Since Γ_Y is a minor of $\Gamma(k, p)$, we have $\mathrm{tw}(\Gamma(k, p)) \geq \mathrm{tw}(\Gamma_Y) = \Omega(k)$, a.a.s., as required, completing the proof. □

3.3. When $p < \frac{1}{2}$

In this section we prove the upper and lower bound of the treewidth $\Gamma(k, p)$ for $p < \frac{1}{2}$. We need the following result from percolation theory, that is originally due to Kesten [12], [13].

Theorem 3.1. (See [12], [13].) *Consider bond percolation on $\mathbb{Z}^2$ and let C_0 denote the number of vertices in the cluster (component) of the origin. For each $p < \frac{1}{2}$ there exists $\lambda(p) > 0$ such that*

$$\mathbb{P}(|C_0| \geq n) \leq \mathrm{e}^{-n\lambda(p)} \quad \textit{for all } n \geq 0.$$

This has the following easy consequence.

Corollary 3.1. *If $0 < p < \frac{1}{2}$ then, a.a.s., all components of $\Gamma(k, p)$ have $O(\log k)$ vertices.*

Proof. Let us fix $0 < p < \frac{1}{2}$ and let $\lambda(p)$ be as provided by Theorem 3.1. Let $K := 100/\lambda(p)$. Observe that, for every $v \in [k]^2$ and $\ell \in \mathbb{N}$, the probability that it is in a component of order $\geq \ell$ in $\Gamma(k, p)$ is no more than the probability that $|C_0|$ exceeds ℓ. Thus, we can conclude that

$$\begin{aligned}\mathbb{P}(\Gamma(k, p) \text{ has a component of size } \geq K \log k) &\leq k^2 \exp[-100 \log k] \\ &= \exp[-98 \log k] \\ &= o(1).\end{aligned}$$

□

Since the treewidth of a graph is equal to the maximum of the treewidth of its components, and all components of $\Gamma(k, p)$ are planar, the required upper bound for $\mathrm{tw}(\Gamma(k, p))$ in the case when $p < \frac{1}{2}$ follows immediately using Corollary 2.1.

Corollary 3.2. *If $0 < p < \frac{1}{2}$ then, a.a.s.,* $\mathrm{tw}(\Gamma(k, p)) = O(\sqrt{\log k})$.

The following lemma now completes the proof of Theorem 1.2.

Lemma 3.3. *Fix $0 < p < \frac{1}{2}$ then, a.a.s.,* $\mathrm{tw}(\Gamma(k, p)) = \Omega(\sqrt{\log k})$.

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Proof. We fix a $\varepsilon = \varepsilon(p)$ (small, to be determined later), and we set $\ell := \lceil\sqrt{\varepsilon \log k}\rceil$. We now fix $N := \lfloor k/(\ell+1)\rfloor^2 = \Omega(k^2/\log k)$ (vertex-)disjoint $\ell \times \ell$-subgrids $G_1, \ldots, G_N$ in $[k]^2$. We will say that the subgrid G_i is *intact* if all of its edges are present in $\Gamma(k, p)$. By independence of the events that the G_i-s are intact, we have

$$\begin{aligned}\mathbb{P}(\text{at least one } G_i \text{ is intact}) &= 1 - (1 - p^{2\ell(\ell-1)})^N \\ &\geq 1 - \exp[-Np^{2\ell(\ell-1)}] \\ &\geq 1 - \exp[-Np^{\ell^2}] \\ &\geq 1 - \exp[-Np^{\varepsilon \log k}].\end{aligned}$$

Next, note that

$$Np^{\varepsilon \log k} = \Omega\left(\frac{k^2}{\log k}\exp[\varepsilon \log p \log k]\right) = \Omega(\exp[2\log k - \log\log k + \varepsilon \log p \log k]).$$

Hence, provided we choose $\varepsilon < -2/\log p$, we have $Np^{\varepsilon \log k} \to \infty$ and, hence, also

$$\mathbb{P}(\text{at least one } G_i \text{ is intact}) = 1 - o(1).$$

Hence, by Lemma 2.1, and since $\operatorname{tw}(H) \leq \operatorname{tw}(G)$ if $H \subseteq G$, it follows that $\operatorname{tw}(\Gamma(k, p)) \geq \ell = \Omega(\sqrt{\log k})$ a.a.s. □

Corollary 3.2 and Lemma 3.3 together give the $p < \frac{1}{2}$ part of Theorem 1.2.

4. Proof of Theorem 1.1

Since Mitsche and Perarnau [19] have already shown the result holds when $r = r(n)$ is larger than some fixed constant C, we only need to consider the case when $r_c < \liminf r \leq \limsup r \leq C$. Note that in this case $\Theta(r\sqrt{n})$ simplifies to $\Theta(\sqrt{n})$. Moreover, by monotonicity, we see that for any such sequence r, a.a.s., $\operatorname{tw}(G(n, r)) \leq \operatorname{tw}(G(n, C)) = O(\sqrt{n})$ by Mitsche and Perarnau's result. Hence, we only need to prove an a.a.s. lower bound for the treewidth of order $\Omega(\sqrt{n})$. Using Corollary 2.2 and monotonicity, Theorem 1.1 follows if we can establish the following lemma.

Lemma 4.1. *For every fixed $r > r_c$, we have $\operatorname{tw}(G_{Po}(n, r)) = \Omega(\sqrt{n})$ a.a.s.*

Proof. The proof is almost exactly the same as the proof of the $p > \frac{1}{2}$ case of Theorem 1.2 above. Again, we let p_0 be as provided by Proposition 3.1, we let π be as provided by Theorem 2.2, and we pick p_1 such that $\pi(p_1) > p_0$. Using Lemma 2.5, we find an a such that $\mathbb{P}(H([0, 3a] \times [0, a])) > \sqrt[3]{p_1}$. For R a $3a \times a$ or $a \times 3a$ rectangle we define $E(R)$ as in the proof of the $p > \frac{1}{2}$ case of Theorem 1.2 above. By choice of a and Lemma 2.6 we have $\mathbb{P}(E(R)) > p_1$ for any such rectangle.

We again define a 1-independent bond percolation model Y on $\mathbb{Z}^2$, by declaring the horizontal edge between (i, j) and $(i+1, j)$ open in Y if $E([2ai, 2ai + 3a] \times [2aj, 2aj + a])$ holds; and the edge between (i, j) and $(i, j+1)$ is open in Y if $E([2ai, 2ai + a] \times [2aj, 2aj + 3a])$ holds. (Note that 1-independence holds provided we chose a sufficiently large.) Again from Theorem 2.2 it follows that $Y \geq X$, where X is standard (independent) percolation on $\mathbb{Z}^2$ with edge probability $> p_0$.

We set $k := \lfloor\sqrt{n}/2a\rfloor - 1$, and we let Γ_X, respectively Γ_Y, be the restriction of X, respectively Y, to $[k]^2$. Arguing analogously to the way we did in the proof of the $p > \frac{1}{2}$

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case of Theorem 1.2, we see that Γ_Y is in fact a minor of $G_{\text{Po}}(n, r)$ (under the natural coupling we get from the construction of Y). Hence, using Proposition 3.1, we have, a.a.s.,

$$\text{tw}(G_{\text{Po}}(n, r)) \geq \text{tw}(\Gamma_Y) \geq \text{tw}(\Gamma_X) = \Omega(k).$$

Since $k = \Theta(\sqrt{n})$, this concludes the proof. □

5. Discussion and further work

Together with the work of Mitsche and Perarnau [19], our Theorem 1.1 provides an almost complete picture of the behavior of the treewidth of random geometric graphs, up to the order of the leading constants.

Corollary 5.1. *Asymptotically almost surely,*

$$\text{tw}(G(n, r)) = \begin{cases} \Theta\left(\dfrac{\log n}{\log\log n}\right) & \text{if } 0 < \liminf r \leq \limsup r < r_c, \\ \Theta\left(r\sqrt{n}\right) & \text{if } \liminf r > r_c. \end{cases}$$

Interestingly, by a result of McDiarmid [16], the clique number of random geometric graphs is a.a.s. equal to $(1 + o(1)) \log n/\log\log n$ when r is constant. This gives rise to the following natural questions.

Question 5.1. Suppose that $0 < \liminf r \leq \limsup r < r_c$.

- Is $\text{tw}(G(n, r)) = (1 + o(1)) \log n/\log\log n$ a.a.s.?
- Is $\text{tw}(G(n, r)) = \omega(G(n, r))$ a.a.s.?

Of course we would also be very interested to learn the precise leading constants for the supercritical case. With our methods and those of Mitsche and Perarnau [19], the following natural conjecture still seems out of reach.

Conjecture 5.1. *Suppose that $r > r_c$ is fixed. Then there exists a $c = c(r)$ such that $\text{tw}(G(n, r)) = (c + o(1))\sqrt{n}$ a.a.s.*

Another tantalizing question is what happens precisely at the critical point. Based on widely believed conjectures on the 'critical exponents' for two-dimensional percolation (see [9, Chapters 9 and 10]), we offer the following conjectures.

Conjecture 5.2. *Asymptotically almost surely,* $\text{tw}(G(n, r_c)) = n^{91/192+o(1)}$.

Conjecture 5.3. *Asymptotically almost surely,* $\text{tw}(\Gamma(k, \frac{1}{2})) = k^{91/96+o(1)}$.

We have made two separate conjectures and added some slack in the exponent so that there is a bit more hope that at least one of the conjectures will be solved.

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