BOUNDED COMPLETENESS AND SCHAUDER'S BASIS FOR $C[0, 1]$

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A basis $\{x_i\}_{i=1}^{\infty}$ for a Banach space X is said to be *boundedly complete* [**4**, p. 284] if whenever $\{a_i\}_{i=1}^{\infty}$ is a sequence of scalars for which $\sup_n \left\| \sum_{i=1}^{n} a_i x_i \right\| < +\infty$, then $\sum_{i=1}^{\infty} a_i x_i$ converges. It is well-known [**2**, p. 70] that if $\{x_i\}_{i=1}^{\infty}$ is a boundedly complete basis for X then X is isometric to a conjugate space; in fact, $X = [f_i]^*$, where $\{f_i\}_{i=1}^{\infty} \subseteq X^*$ is the sequence of coefficient functionals associated with the basis $\{x_i\}_{i=1}^{\infty}$. It follows that no basis for $C[0, 1]$ can be boundedly complete since no separable conjugate space contains c_0[**1**], yet $C[0, 1]$ is a separable space which contains c_0.

In fact, a considerably stronger result of the same general nature is true.

THEOREM. *There is no semi-normalized basis $\{x_i\}_{i=1}^{\infty}$ for $C[0, 1]$ with the property that whenever $\{a_i\} \in c_0$ and $\sup_n \left\| \sum_{i=1}^{n} a_i x_i \right\| < +\infty$, then $\sum_{i=1}^{\infty} a_i x_i$ converges in $C[0, 1]$.*

Proof. Suppose $\{x_i\}_{i=1}^{\infty}$ is a semi-normalized basis for $C[0, 1]$ with the property that whenever $\{a_i\}_{i=1}^{\infty} \in c_0$ and $\sup_n \left\| \sum_{i=1}^{n} a_i x_i \right\| < +\infty$, then $\sum_{i=1}^{\infty} a_i x_i$ converges in $C[0, 1]$. Then since $0 < \inf_i \|x_i\| \leq \sup_i \|x_i\| < +\infty$ it follows that any semi-normalized block basic sequence $\{b_k\}_{k=1}^{\infty} = \left\{ \sum_{i=N_k}^{N_{k+1}-1} c_i x_i \right\}$ taken with respect to the basis $\{x_i\}_{i=1}^{\infty}$ in $C[0, 1]$ has the same property. We show this cannot be.

Let λ denote the symmetric sequence space defined by

$$\lambda = \left\{ (c_i) \in c_0 \,\middle|\, \lim_{n \to \infty} \frac{\sum_{i=1}^{n} c_i^*}{\sum_{i=1}^{n} \frac{1}{i}} = 0 \right\},$$

where

$$\|(c_i)\|_\lambda = \sup_n \frac{\sum_{i=1}^{n} c_i^*}{\sum_{i=1}^{n} \frac{1}{i}}$$

and $\{c_i^*\}_{i=1}^{\infty}$ denotes the arrangement of the sequence $\{|c_i|\}_{i=1}^{\infty}$ into one which decreases to zero. It is well-known that with the indicated norm λ is a Banach space in which the sequence $\{e_i\}_{i=1}^{\infty}$ defined by $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, etc., is a basis which is equivalent to each of its subbases (i.e. $\sum_i b_i e_i$ converges in $\lambda \Leftrightarrow \sum_i b_i e_{n_i}$ converges in λ, for any subsequence $\{n_i\}_{i=1}^{\infty}$ of the positive integers). Moreover the basis $\{e_i\}_{i=1}^{\infty}$ converges weakly to zero in λ since $\lambda^* = \left\{(d_i) \in c_0 \;\middle|\; \sum_{i=1}^{\infty} \frac{|d_i^*|}{i} < \infty\right\}$ (see [**3**, p. 139–150] for a discussion of these matters). It is also easy to see that the basis $\{e_i\}_{i=1}^{\infty}$ in λ does not have the property mentioned in the theorem since if $a_i = \frac{1}{i}$ for all i then $(a_i) \in c_0$ and $\sup \left\|\sum_{i=1}^{n} a_i e_i\right\| = 1$, but yet $\sum_{i=1}^{\infty} a_i e_i = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ is not in λ.

Now λ is separable so it can be isometrically embedded in $C[0, 1]$, and since the basis $\{e_i\}_{i=1}^{\infty}$ for λ converges weakly to zero it follows that a subsequence $\{e_{i_k}\}_{k=1}^{\infty}$ is equivalent to a semi-normalized block basis sequence $\{b_k\}_{k=1}^{\infty} = \left\{\sum_{i=N_k}^{N_{k+1}-1} c_i x_i\right\}_{k=1}^{\infty}$ in $C[0, 1]$ [**1**]. Since $\{e_{i_k}\}_{k=1}^{\infty}$ is equivalent to $\{e_i\}_{i=1}^{\infty}$ and $\{e_i\}_{i=1}^{\infty}$ fails to have the property in question it follows that $\{b_k\}_{k=1}^{\infty}$ also fails to have it, a contradiction to our previous assumption. Since $\{x_i\}_{i=1}^{\infty}$ was an arbitrary semi-normalized basis for $C[0, 1]$ the theorem follows.

In particular the classical Schauder basis $\{\varphi_i\}_{i=1}^{\infty}$ for $C[0, 1]$ defined by $\varphi_0(t) \equiv 1$, $\varphi_1(t) = t$, and

$$\varphi_{2^n+l}(t) = \begin{cases} 0 \text{ if } t \notin \left(\dfrac{2l-2}{2^{n+1}}, \dfrac{2l}{2^{n+1}}\right) \\ 1 \text{ if } t = \dfrac{2l-1}{2^{n+1}} \\ \text{linear otherwise on } [0, 1] \end{cases},$$

where $n = 0, 1, 2, \ldots$ and $l = 1, 2, \ldots, 2^n$ fails to have the given property. The purpose of this note is to observe that, in contrast, the Schauder system $\{\varphi_i\}_{i=1}^{\infty}$ *does* have a weaker (yet closely related) property to which we now give a name.

DEFINITION. The semi-normalized basis $\{x_i\}_{i=1}^{\infty}$ for the Banach space X is said to be *monotonically boundedly complete* if whenever $\{a_i\}_{i=1}^{\infty}$ is a sequence of scalars which decreases monotonically to zero and for which $\sup_n \left\|\sum_{i=1}^{n} a_i x_i\right\| < +\infty$, then $\sum_{i=1}^{\infty} a_i x_i$ converges.

The fact that Schauder's basis $\{\varphi_i\}_{i=1}^{\infty}$ (along with certain other non-boundedly complete bases) is monotonically boundedly complete is a consequence of the following general result.

THEOREM. *Let $\{x_i\}_{i=1}^{\infty}$ be a semi-normalized basis for a Banach space X satisfying the following conditions:*

(i) *There exists a strictly increasing sequence $\{N_k\}_{k=1}^{\infty}$ of positive integers and a constant $m_0 > 0$ for which $N_1 = 1$ and for which $\left\|\sum_{i=N_k}^{N_{k+1}-1} c_i x_i\right\| \le m_0\left(\sup_{N_k \le i < N_{k+1}} |c_i|\right)$ for all $k = 1, 2, 3, \ldots$ and for all scalars $\{c_i\}_{i=1}^{\infty}$.*

(ii) *There exists a co stant $P > 0$ such that given any $k = 1, 2, \ldots$ there is $F_k \in X^*$ for which $\|F_k\| = 1$, $\langle F_k, x_i\rangle \geq 0$ for all i satisfying $1 \leq i \leq N_k$, and $\langle F_k, x_{N_i}\rangle \geq P$ for $i = 1, 2, \ldots, k$.*

Then $\{x_i\}_{i=1}^{\infty}$ is a monotonically boundedly complete basis for X.

Proof. Let $\{a_i\}_{i=1}^{\infty}$ decrease monotonically to zero and suppose $\sup_n \left\|\sum_{i=1}^{n} a_i x_i\right\| = M < +\infty$. If, for each $k = 1, 2, \ldots$, we let $y_k = \sum_{i=N_k}^{N_{k+1}-1} a_i x_i$ then $\|y_1 + y_2 + \ldots + y_k\| \leq M$ for all k. Hence if $\{F_k\}_{k=1}^{\infty} \subseteq X^*$ is as in (ii) above, then $\langle F_k, y_1 + \ldots + y_k\rangle \leq M$ for all $k = 1, 2, \ldots$. That is, for every $k = 1, 2, \ldots$,

$$\langle F_k, a_1 x_1 + \ldots + a_{N_2-1}x_{N_2-1}\rangle + \langle F_k, a_{N_2}x_{N_2} + \ldots + a_{N_3-1}x_{N_3-1}\rangle + \ldots + \langle F_k, a_{N_k}x_{N_k} + \ldots + a_{N_{k+1}-1}x_{N_{k+1}-1}\rangle \leq M.$$

But since $\{a_i\}_{i=1}^{\infty} \downarrow 0$ and F_k satisfies (ii) above, this says that $a_{N_2} . P + \ldots + a_{N_k+1} . P \leq M$ for all k, so $\sum_{i=1}^{\infty} a_{N_i}$ converges.

Now let $\varepsilon > 0$ be given, m_0 the number given in (i), and r a positive integer for which $\sum_{i=r}^{\infty} a_{N_i} < \frac{\varepsilon}{3m_0}$ (note, then, that $0 \leq a_i < \frac{\varepsilon}{3m_0}$ for all $i \geq N_r$). If $N_r \leq m < n$ we then have

$$\begin{aligned}
&\|a_m x_m + a_{m+1}x_{m+1} + \ldots + a_n x_n\| \\
&\quad = \|(a_m x_m + \ldots + a_{N_j-1}x_{N_j-1}) + y_{N_j} + y_{N_j+1} + \ldots + y_{N_q} + (a_{N_q+1}x_{N_q+1} + \ldots + a_n x_n)\| \\
&\qquad\qquad \text{(for some } j \text{ and } q > r \text{ for which } m \geq N_j - 1 \text{ and } n \leq N_{q+1} - 1) \\
&\quad \leq \|a_m x_m + \ldots + a_{N_j-1}x_{N_j-1}\| + \sum_{i=j}^{q} \|y_{N_i}\| + \|a_{N_q+1}x_{N_q+1} + \ldots + a_n x_n\| \\
&\quad \leq m_0 . \sup_{m \leq i \leq N_j - 1} |a_i| + \sum_{i=j}^{q} m_0 . \sup_{N_i \leq s \leq N_{i+1}-1} |a_s| + m_0 . \sup_{N_q+1 \leq i \leq n} |a_i| \quad \text{(by (i))}.
\end{aligned}$$

Since $(a_i) \downarrow 0$ this last is

$$\begin{aligned}
&\leq m_0 . a_m + \sum_{i=j}^{q} m_0 . a_{N_i} + m_0 . a_{N_q+1} \\
&< m_0 . \frac{\varepsilon}{3m_0} + m_0 . \sum_{i=j}^{q} a_{N_i} + m_0 . \frac{\varepsilon}{3m_0} \\
&< m_0\left[\frac{\varepsilon}{3m_0} + \frac{\varepsilon}{3m_0} + \frac{\varepsilon}{3m_0}\right] = \varepsilon,
\end{aligned}$$

by choice of r. That is, if $N_r \leq m < n$ then $\left\|\sum_{i=m}^{n} a_i x_i\right\| < \varepsilon$, so $\sum_{i=1}^{\infty} a_i x_i$ converges in X and $\{x_i\}_{i=1}^{\infty}$ has been shown to be monotonically boundedly complete.

COROLLARY. *The Schauder basis $\{\varphi_i\}_{i=0}^{\infty}$ for $C[0, 1]$ is monotonically boundedly complete.*

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Proof. It is sufficient to show that the basic sequence $\{x_i\}_{i=1}^{\infty}=\{\varphi_j\}_{j=3}^{\infty}$ is monotonically boundedly complete. We show the conditions of the previous theorem are satisfied for the sequence $\{x_i\}_{i=1}^{\infty}$ in $C[0,1]$.

To do this we first define a sequence of dyadic rational numbers $\{t_k\}_{k=1}^{\infty}$ by: $t_1=\frac{1}{4}$, $t_2=\frac{3}{8}$, and $t_{n+2}=\frac{1}{2}(t_n+t_{n+1})$ for all $n\geq 1$. By the Nested Interval Theorem the sequence $\{t_k\}_{k=1}^{\infty}$ converges to some number $t_0\in[0,1]$. Moreover each t_k is the midpoint of the interval of support of a unique function in the set $\{\varphi_i\}_{i=3}^{\infty}$. If we denote this function by x_{N_k} then $x_{N_1}=\varphi_3$, $x_{N_2}=\varphi_6$, $x_{N_3}=\varphi_{11},\ldots$, and by construction of $\{t_k\}_{k=1}^{\infty}$ and the definition of the Schauder functions it is clear that $x_{N_k}(t_0)\geq\frac{1}{2}$ for all $k=1,2,\ldots$. If for each $k=1,2,\ldots$ we let $F_k=\delta_{t_0}\in C[0,1]^*$ then $\langle F_k,x_i\rangle=x_i(t_0)\geq 0$ for all $i=1,2,\ldots$, $\langle F_k,x_{N_k}\rangle=x_{N_k}(t_0)\geq\frac{1}{2}$ for $k=1,2,\ldots$, and condition (ii) of the previous theorem is satisfied with $P=\frac{1}{2}$.

To see that (i) is also satisfied, note that for any k, $x_{n_k}=\varphi_{2^k+l}$ for some $1\leq l\leq 2^k$, and hence each x_i for which $N_k\leq i<N_{k+1}$ is either of the form $x_i=\varphi_{2^k+r}$ for some $1\leq r\leq 2^k$ or of the form $x_i=\varphi_{2^{k+1}+s}$ for some $1\leq s\leq 2^{k+1}$. It follows, then, from the definition of the Schauder functions (and the fact that a linear combination of such functions is piecewise-linear with a relative maximum or minimum only at nodal points) that for any k and any scalars $\{c_i\}_{i=N_k}^{N_{k+1}-1}$ we have

$$\left\|\sum_{i=N_k}^{N_{k+1}-1} c_i x_i\right\|\leq\frac{3}{2}\sup_{N_k\leq i<N_{k+1}}|c_i|.$$

Therefore condition (i) of the previous theorem also holds (with $m_0=\frac{3}{2}$), and by the previous theorem we conclude that the basis $\{\varphi_i\}_{i=0}^{\infty}$ for $C[0,1]$ is monotonically boundedly complete.

REMARKS. 1. A semi-normalized basis $\{x_i\}_{i=1}^{\infty}$ for a Banach space X is said to be *of type P* [**4**, p. 308] if $\sup_n\left\|\sum_{i=1}^{n} x_i\right\|<+\infty$. It is known that a basis $\{x_i\}_{i=1}^{\infty}$ of type P has the property that if $\{a_i\}_{i=1}^{\infty}$ decreases monotonically to zero then $\sum_{i=1}^{\infty} a_i x_i$ converges in X [**4**, p. 308]. The fact that $\{\varphi_i\}_{i=0}^{\infty}$ is neither boundedly complete nor of type P in $C[0,1]$ gives significance to the preceding result.

2. One can show in a roughly analogous (yet simpler) way that the normalized Haar system in $L^{\infty}[0,1]$ is also monotonically boundedly complete (but not boundedly complete, nor of type P). A natural problem which arises is the investigation of other "classical" bases and basic sequences in regard to monotone bounded completeness. In particular, is the normalized Haar basis for $L^1[0,1]$ [**4**, p. 13] monotonically boundedly complete? What about the Franklin basis for $C[0,1]$ obtained by applying the Gram–Schmidt orthonormalization procedure to $\{\varphi_i\}_{i=0}^{\infty}$?

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