




Cohomological invariants of root stacks and admissible double coverings

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Abstract. We give a formula for the cohomological invariants of a root stack, which we apply to compute the cohomological invariants and the Brauer group of the compactification of the stacks of hyperelliptic curves given by admissible double coverings.

1 Introduction

Cohomological invariants of algebraic groups are a classical topic, whose first examples date back to the beginning of the twentieth century [25]. The modern definition was developed by Serre in the late '90s, and the general theory can be found in Garibaldi, Merkurjev, and Serre's book [12]. From a more geometric point of view, they can be seen as invariants of the classifying stack BG rather than the group G . Following this idea, in [21], the second author defined an extension of the notion of cohomological invariants to general algebraic stacks.

Let \mathcal{X} be an algebraic stack over a field k and M a cycle module over k , as defined in [22]. A cohomological invariant for \mathcal{X} with coefficients in M is a functorial way to associate to any point $\text{Spec}(K) \rightarrow \mathcal{X}$ an element of $M^\bullet(K)$, satisfying a certain continuity condition. Originally, in [12], cohomological invariants were defined as taking values in the cycle module H_D , where D is an ℓ -torsion Galois module for some positive number ℓ not divisible by $\text{char}(k)$, and $H_D^\bullet(F) := \oplus_i H_{\text{Gal}}^i(F, D(i))$. The theory for general cycle modules was recently developed by Gille and Hirsch [13], and adapted to the setting of cohomological invariants of algebraic stacks by the authors in [9].

Our definition retrieves the classical theory of cohomological invariants of algebraic groups when $\mathcal{X} = BG$, and if $\mathcal{X} = X$ is a scheme, its cohomological invariants with coefficients in H_D are equal to its unramified cohomology $H_{\text{nr}}^\bullet(X, D)$. Moreover, in [9], the authors proved that the cohomological invariants compute the Brauer group, and used them to compute the Brauer groups of the moduli stacks \mathcal{H}_g of hyperelliptic curves of genus g .

In [8, Appendix A], the authors showed that the cohomological invariants (and consequently, by [9, Remark 6.11], the Brauer group) of the standard compactification $\overline{\mathcal{H}}_g$ of \mathcal{H}_g are trivial, while claiming this not to occur for the compactification $\widetilde{\mathcal{H}}_g$

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obtained via the theory of admissible double coverings. In this paper, we prove the aforementioned claim.

Theorem (5.7, 5.10) *Let M be an ℓ -torsion cycle module. Then:*

- *If ℓ is odd, we have $\text{Inv}^\bullet(\tilde{\mathcal{H}}_g, M) = M^\bullet(k)$.*
- *If $\ell = 2^m$, then we have an exact sequence*

$$0 \rightarrow \text{Inv}^\bullet(\text{BS}_{2g+2}, M) \rightarrow \text{Inv}^\bullet(\tilde{\mathcal{H}}_g, M) \rightarrow M^\bullet(k)_2,$$

where the last map lowers degree by $g + 2$. Moreover, if g is even, the last map is surjective and split.

Thanks to the result above, we are able compute the prime-to- c part of the Brauer group of $\tilde{\mathcal{H}}_g$, which coincides with the whole Brauer group of the stack when the base field has characteristic zero.

Corollary (5.11) *We have*

$${}^c\text{Br}(\tilde{\mathcal{H}}_g) = {}^c\text{Br}(k) \oplus H^1(k, \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

If one considers the analogy with unramified cohomology, this appears to be in stark contrast to what happens for proper smooth varieties, where unramified cohomology is a birational invariant. The stacks $\overline{\mathcal{H}}_g$ and $\tilde{\mathcal{H}}_g$ are smooth, proper (in fact, projective, according to Kresch’s definition of a projective Deligne–Mumford stack [19]), and birational to each other, but their cohomological invariants are very different. The reason why this happens is related to the so-called root stack construction.

1.1 Weak factorization, root stacks, and cohomological invariants

Given smooth projective varieties X, Y over a base field k of characteristic zero, the weak factorization theorem states that a birational map $X \dashrightarrow Y$, which is an isomorphism on an open subset U , always factors as a composition

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X_n = Y,$$

where for each i either f_i or f_i^{-1} is a blowup in a smooth center and each f_i is an isomorphism on U .

Recently, Harper [16] proved a similar result for Deligne–Mumford stacks over a field of characteristic zero. Given a proper birational map $\mathcal{X} \dashrightarrow \mathcal{Y}$, we always have a factorization

$$\mathcal{X} = \mathcal{X}_0 \xrightarrow{f_0} \mathcal{X}_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} \mathcal{X}_n = \mathcal{Y},$$

where for each i either f_i or f_i^{-1} is a *stacky blowup*, that is, either a regular blowup in a smooth center or a root stack along a smooth divisor. Given a Cartier divisor V , the order r root stack along V , denoted $\mathcal{X}_{(V,r)} \rightarrow \mathcal{X}$, is a canonical way to add a μ_r stabilizer along the points of V .

Unramified cohomology and cohomological invariants are left unchanged when taking a blowup, but can change drastically when taking a root stack. If one wants to be able to compare the cohomological invariants of birational proper Deligne–Mumford

stacks, it is critical to be able to understand what happens when taking a root stack. This is described in the following theorem.

Theorem (2.5) *Let \mathcal{X} be a smooth algebraic stack over a field k , and let $r \geq 2$ be an integer that is not divisible by $\text{char}(k)$. Let $V \subset \mathcal{X}$ be a smooth divisor.*

Then, $\text{Inv}^\bullet(\mathcal{X}_{(V,r)}, M)$ is equal to the subgroup of $\text{Inv}^\bullet(\mathcal{X} \setminus V, M)$ of invariants α such that the ramification of α at V is of r -torsion.

The stack $\widetilde{\mathcal{H}}_g$ is, up to codimension 2 closed substacks, a root stack over $\overline{\mathcal{H}}_g$. More precisely, after excising some loci, it can be obtained from an open substack of $\overline{\mathcal{H}}_g$ by adding a μ_2 stabilizer along the divisor of irreducible singular curves.

As cohomological invariants are unchanged when removing a closed subset of codimension ≥ 2 , we can (and we do) apply the theorem above to compute the cohomological invariants of $\widetilde{\mathcal{H}}_g$, leveraging our previous knowledge of the cohomological invariants of \mathcal{H}_g and $\overline{\mathcal{H}}_g$.

1.2 Cohomological invariants and Chow groups with coefficients

We briefly recall some properties of cohomological invariants which will be used in the paper. For a detailed introduction to cohomological invariants and Chow groups with coefficients, the reader can refer to [9, Section 1].

Given a field F , let $T^\bullet(F)$ be the tensor algebra $\bigoplus_{i \geq 0} (F^*)^{\otimes i}$. Milnor’s K-theory K_M is the functor from field extensions of k to graded-commutative rings which sends a field F to

$$K_M^\bullet(F) = T^\bullet(F) / \{a \otimes b \mid a, b \in F^*, a + b = 1\}.$$

Milnor’s K-theory is the basic, and most important, cycle module. A general cycle module M is a functor from fields over k to abelian groups which is functorially a K_M -module, and shares the main properties of Milnor’s K-theory. Given a torsion Galois module D over k , the functor

$$H_D^\bullet(F) = \bigoplus_{i \geq 0} H_{\text{Gal}}^i(F, D(i))$$

is a cycle module. In particular, if we take $D = \mathbb{Z}/\ell\mathbb{Z}$ by Voevodsky’s norm-residue isomorphism [24, Theorem 6.1], we have

$$H_{\mathbb{Z}/\ell\mathbb{Z}} = K_M / (\ell) \stackrel{\text{def}}{=} K_\ell.$$

If M is of ℓ -torsion, it is a K_ℓ -module.

Cohomological invariants with coefficients in a cycle module M form a graded abelian group. When M is given by K_M or K_ℓ , they form a graded-commutative ring. In general, $\text{Inv}^\bullet(\mathcal{X}, M)$ is an $\text{Inv}^\bullet(\mathcal{X}, K_M)$ -module, and if M is of ℓ -torsion, it is an $\text{Inv}^\bullet(\mathcal{X}, K_\ell)$ -module.

The functor $\text{Inv}^\bullet(-, M)$ forms a sheaf with respect to the *smooth-Nisnevich* topology, where the coverings are smooth representable maps $\mathcal{Y} \rightarrow \mathcal{X}$ such that any point $\text{Spec}(K) \rightarrow \mathcal{X}$ has a lifting $\text{Spec}(K) \rightarrow \mathcal{Y}$. Moreover, cohomological invariants are unchanged when passing to an affine bundle or an open subset whose complement has codimension ≥ 2 , and in general, restriction to a dense open subset is injective.

Our main tool for computations will be Rost’s Chow groups with coefficients [22]. These are an extension of ordinary Chow groups: instead of having \mathbb{Z} -linear sums of cycles, they are generated by sums in the form $\sum_i \alpha_i P_i$, where α_i belongs to $M^\bullet(k(P))$. The i -codimensional Chow group of X with coefficients in M is denoted by $A^i(X, M)$.

Chow groups with coefficients share the same properties of ordinary Chow groups, and moreover, given a closed subset V of X of pure codimension d , they have a long exact sequence

$$\dots \rightarrow A^i(X, D) \rightarrow A^i(X \setminus V, D) \rightarrow A^{i+1-d}(V, D) \rightarrow A^{i+1}(X, D) \rightarrow \dots$$

When X is a smooth scheme, we have an isomorphism $A^0(X, M) \simeq \text{Inv}^\bullet(X, M)$.

More generally, if X is a smooth scheme being acted upon by a smooth affine algebraic group G , we can use the Edidin–Graham–Totaro approximation process [11] to define equivariant Chow groups with coefficients $A_G^i(X, M)$. This was first done by Guillot [15]. Again, we have

$$\text{Inv}^\bullet([X/G], M) \simeq A_G^0(X, M).$$

1.3 Outline of the paper

In Section 2, we discuss the root stack construction in greater detail. In particular, we give a presentation of the root stack $X_{(V,r)}$ as a quotient (Proposition 2.2), which we use to deduce our formula for cohomological invariants (Theorem 2.4).

In Section 3, we recall some basic facts on the stack of stable hyperelliptic curves, and we provide a new description of an open substack of $\overline{\mathcal{H}}_g$, which will be useful for our purposes.

In Section 4, we describe the theory of admissible double coverings and its connection to hyperelliptic curves.

Finally, in Section 5, we compute the cohomological invariants of $\widetilde{\mathcal{H}}_g$ (Theorem 5.7 for the even genus case and Theorem 5.10 for the odd genus case), as well as its Brauer group (Theorem 5.11).

1.4 Notation and conventions

Throughout the paper, we will fix a base field k of characteristic $c \neq 2$. All stacks and schemes are assumed to be of finite type over k unless otherwise specified. By ℓ or r , we will always mean a positive integer that is not divisible by the characteristic of k . If A is an abelian group, we denote A_ℓ the ℓ -torsion subgroup. If moreover A is torsion, we denote ${}^c A$ the subgroup of elements whose order is not divisible by c .

2 Cohomological invariants of root stacks

Fix an integer $r \geq 1$, and let X be an integral scheme over some field k whose characteristic does not divide r . Choose an invertible sheaf \mathcal{L} over X and define $X_{(\mathcal{L},r)}$ as the root stack of order r of X along the invertible sheaf \mathcal{L} : this is the stack over the big étale site of X whose objects are triples $(S \rightarrow X, \mathcal{M}, \phi)$, where \mathcal{M} is an invertible sheaf over S together with an isomorphism $\phi : \mathcal{L}_S \simeq \mathcal{M}^{\otimes r}$. A morphism

$(S \rightarrow X, \mathcal{M}, \phi) \rightarrow (S \rightarrow X, \mathcal{M}', \phi')$ over S is given by an isomorphism of invertible sheaves $\psi : \mathcal{M} \simeq \mathcal{M}'$ such that $\phi' \circ \psi^{\otimes r} = \phi$.

The root stack $X_{(\mathcal{L},r)}$ has a simple presentation as quotient stack: let L be the line bundle associated to the invertible sheaf \mathcal{L} , i.e., $L := \underline{\text{Spec}}_{\mathcal{O}_X}(\text{Sym } \mathcal{L}^\vee)$, and denote L^* the complement of the zero section in L .

The natural \mathbb{G}_m -action on L turns $L^* \rightarrow X$ into a \mathbb{G}_m -torsor. For every $r \geq 1$, we can also twist the \mathbb{G}_m -action by the character $\chi_r : \mathbb{G}_m \xrightarrow{x^r} \mathbb{G}_m$. In what follows, for every scheme S endowed with a \mathbb{G}_m -action, we denote $[S/\mathbb{G}_{m,r}]$ the quotient stack where the \mathbb{G}_m -action is twisted by χ_r . The next result is [5, Proposition 2.3.5].

Proposition 2.1 *We have $X_{(\mathcal{L},r)} \simeq [L^*/\mathbb{G}_{m,r}]$.*

Consider a nonzero global section σ of \mathcal{L} , and let V be the divisor where the section σ vanishes. The stack $X_{(V,r)}$ is the root stack of order r of X along V : its objects are quadruples $(S \rightarrow X, \mathcal{M}, \tau, \phi)$ where \mathcal{M} is an invertible sheaf over S and τ is a global section of \mathcal{M} , together with an isomorphism $\phi : \mathcal{L}_S \simeq \mathcal{M}^{\otimes r}$ such that $\phi(\sigma) = \tau^{\otimes r}$.

The root stack $X_{(V,r)}$ also admits a simple description as a quotient stack: regard σ as a morphism from L to \mathbb{A}_X^1 , and let $(\cdot)^r : \mathbb{A}_X^1 \rightarrow \mathbb{A}_X^1$ be the r th-power morphism, so that we can form the Cartesian diagram:

$$\begin{array}{ccc} Y^* & \longrightarrow & \mathbb{A}_X^1 \\ \downarrow & & \downarrow (\cdot)^r \\ L^* & \xrightarrow{\sigma} & \mathbb{A}_X^1. \end{array}$$

Roughly, the scheme Y can also be thought as the closed subscheme of $L^* \times \mathbb{A}^1$ locally described by the equation $u^r = \sigma(x, t)$, where x and t are local coordinates for L^* and u is a coordinate for \mathbb{A}^1 .

We can define an action of \mathbb{G}_m on L^* and \mathbb{A}_X^1 , so that the morphisms σ and $(\cdot)^r$ become equivariant: namely, we let \mathbb{G}_m acts on both L^* and the \mathbb{A}_X^1 on the bottom-right corner via its r th power, and on the \mathbb{A}_X^1 in the top-right corner via the standard action. In this way, Y^* inherits a \mathbb{G}_m -action too, and we get a Cartesian diagram of quotient stacks as follows:

$$\begin{array}{ccc} [Y^*/\mathbb{G}_m] & \longrightarrow & [\mathbb{A}_X^1/\mathbb{G}_m] \\ \downarrow & & \downarrow \\ [L^*/\mathbb{G}_{m,r}] & \longrightarrow & [\mathbb{A}_X^1/\mathbb{G}_{m,r}]. \end{array}$$

Then, [5, Proposition 2.3.5] says the following.

Proposition 2.2 *We have $X_{(V,r)} \simeq [Y^*/\mathbb{G}_m]$.*

We can use Proposition 2.2 to investigate the cohomological invariants of $X_{(V,r)}$, for X smooth. We start with the case of a root stack along an invertible sheaf.

Proposition 2.3 *Let X be a smooth scheme over k . Consider an invertible sheaf \mathcal{L} over X , and let $X_{(\mathcal{L},r)}$ be the root stack of order r of X along \mathcal{L} . Then, there is an exact sequence of $M^\bullet(k)$ -modules:*

$$0 \rightarrow \text{Inv}^\bullet(X, M) \rightarrow \text{Inv}^\bullet(X_{(\mathcal{L},r)}, M) \rightarrow \ker(c_1(\mathcal{L}))_r \rightarrow 0,$$

where the term on the right stands for the subgroup of r -torsion elements in the kernel of $c_1(\mathcal{L}) : A^0(X, M) \rightarrow A^1(X, M)$.

Proof The open embedding of stacks $[L^*/\mathbb{G}_{m,r}] \hookrightarrow [L/\mathbb{G}_{m,r}]$ induces the following long exact sequence of equivariant Chow groups with coefficients:

$$0 \rightarrow A_{\mathbb{G}_{m,r}}^0(L, M) \rightarrow A_{\mathbb{G}_{m,r}}^0(L^*, M) \rightarrow A_{\mathbb{G}_{m,r}}^0(X, M) \xrightarrow{\sigma_{0*}} A_{\mathbb{G}_{m,r}}^1(L, M).$$

We have an isomorphism

$$A_{\mathbb{G}_{m,r}}^*(L, M) \simeq A_{\mathbb{G}_{m,r}}^*(X, M) \simeq A^*(X, M)[\xi],$$

where ξ is the generator of $A_{\mathbb{G}_m}^*(\text{Spec}(k))$ as $H^\bullet(k)$ -module. It has codimension 1 and cohomological degree 1. In particular,

$$A_{\mathbb{G}_{m,r}}^1(L, M) \simeq A^1(X, M) \oplus A^0(X, M) \cdot \xi.$$

With this identification, the morphism σ_{0*} is equal to

$$\alpha \mapsto c_1^{\mathbb{G}_{m,r}}(\mathcal{L})(\alpha) = c_1(\mathcal{L})(\alpha) + r\alpha \cdot \xi.$$

If α is an element in the kernel of σ_{0*} , then it must necessarily satisfy

$$c_1(L) \cap \alpha = 0, \quad r\alpha = 0.$$

We deduce that $\ker(\sigma_{0*})$ is equal to subgroup of r -torsion elements of the kernel of $c_1(\mathcal{L})$.

Combining what we have just found with the long exact sequence above, the proposition easily follows. ■

Next, we study the case of a root stack along a divisor.

Theorem 2.4 *Let X be a smooth scheme over k . Consider a smooth Cartier divisor $V \subset X$, and let $X_{(V,r)}$ be the root stack of order r of X along V . Then, $\text{Inv}^\bullet(X_{(V,r)}, M)$ is equal to the subset of $\text{Inv}^\bullet(X \setminus V, M)$ of invariants α such that $r \cdot \partial_V(\alpha) = 0$.*

Equivalently, there is an exact sequence of $M^\bullet(k)$ -modules:

$$0 \rightarrow \text{Inv}^\bullet(X, M) \rightarrow \text{Inv}^\bullet(X_{(V,r)}, M) \rightarrow \ker(i_*)_r \rightarrow 0,$$

where the term on the right stands for the subgroup of r -torsion elements in the kernel of $i_* : A^0(V, M) \rightarrow A^1(X, M)$.

Proof Recall from Proposition 2.2 that $X_{(V,r)} \simeq [Y^*/\mathbb{G}_m]$: here Y^* stands for the closed subscheme in $L^* \times \mathbb{A}^1$ of equation $\sigma(x, t) = u^r$, where $\sigma : L \rightarrow \mathbb{A}^1$ is determined by the section σ of \mathcal{L} defining V and u is a coordinate for \mathbb{A}^1 . The group \mathbb{G}_m acts in the standard way on \mathbb{A}^1 and as $\mathbb{G}_{m,r}$ on L .

Let Y be the closure of Y^* in $L \times \mathbb{A}^1$. Observe that we have $Y \setminus Y^* \simeq V$, as it is equal to the intersection of the $\sigma^{-1}(0)$ with the image of the zero section $t_0 : X \rightarrow L$

(the coordinate u must necessarily be equal to zero). Moreover, the scheme Y is \mathbb{G}_m -invariant.

In the following, we denote the root stack $V_{(\mathcal{O}_V(V),r)}$ by \tilde{V} . Observe that there is a closed immersion of stacks $\tilde{V} \hookrightarrow X_{(V,r)}$: the complement of \tilde{V} in $X_{(V,r)}$ is isomorphic to $X \setminus V$. Therefore, we have an exact sequence of Chow groups with coefficients as follows:

$$(2.1) \quad 0 \rightarrow A^0(X_{(V,r)}, M) \rightarrow A^0(X_{(V,r)} \setminus \tilde{V}, M) \simeq A^0(X \setminus V, M) \rightarrow A^0(\tilde{V}, M).$$

We need to understand the kernel of the last map in the sequence above.

We claim that there is a factorization

$$(2.2) \quad A^0(X_{(V,r)} \setminus \tilde{V}, M) \simeq A^0(X \setminus V, M) \xrightarrow{r\partial_V} A^0(V, M) \hookrightarrow A^0(\tilde{V}, M),$$

where the first map is r times the boundary map, and the second map is the pullback along $\tilde{V} \rightarrow V$, whose injectivity follows from Proposition 2.3.

Let Z^* be the closed subscheme of Y^* where $u = 0$, which is isomorphic to $L^*|_V$, and set $U^* := Y^* \setminus Z^*$. Observe that we have $[Z^*/\mathbb{G}_{m,r}] \simeq \tilde{V}$ and $[U^*/\mathbb{G}_m] \simeq X \setminus V$.

Observe that $f : Y^* \rightarrow L^*$ is a cyclic cover of degree r branched along Z^* and ramified over $L^*|_V$. Moreover, $A^0_{\mathbb{G}_m}(U^*, M) \simeq A^0(X \setminus V, M)$. Let $g : L^*|_{X \setminus V} \rightarrow X \setminus V$ be the projection morphism. Then, it follows from what we just observed that every element of $A^0_{\mathbb{G}_m}(U^*, M)$ is of the form $f^* g^* \alpha$. By [22, Definition 1.1(R3a)], we have:

$$\partial_{Z^*}(f^* g^* \alpha) = r \cdot f_V^*(\partial_{L^*|_V}(g^* \alpha)),$$

where

$$\partial_{L^*|_V} : A^0_{\mathbb{G}_m}(L^*|_{X \setminus V}, M) \longrightarrow A^0_{\mathbb{G}_m}(L^*|_V, M)$$

is the boundary map induced by the closed embedding $L^*|_V \hookrightarrow L^*$, and $f_V : Z^* \rightarrow L^*|_V$ is the restriction of f to Z^* . Consider the Cartesian diagram:

$$\begin{array}{ccc} L^*|_V & \longrightarrow & L^* \\ \downarrow g_V & & \downarrow \\ V & \longrightarrow & X. \end{array}$$

Then, the compatibility formula implies

$$r \cdot \partial_{L^*|_V}(g^* \alpha) = r \cdot g_V^*(\partial_V \alpha),$$

where $\partial_V : A^0(X \setminus V, M) \rightarrow A^0(V, M)$ is the boundary map induced by the closed embedding $V \hookrightarrow X$. Putting all together, we have shown that (2.2) holds true.

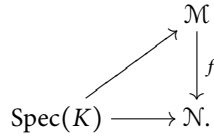
We deduce from (2.1) that

$$\text{Inv}^\bullet(X_{(V,r)}, M) = \ker(r \cdot \partial_V : A^0(X \setminus V, M) \longrightarrow A^0(V, M)).$$

In other words, the cohomological invariants of $X_{(V,r)}$ are the invariants of $X \setminus V$ whose ramification along V is of r -torsion.

It is easy to see that $A^0(X, M)$ injects into $\ker(r\partial_V)$. The quotient is then a subgroup of $\ker(A^0(V, M) \rightarrow A^1(X, M))$, and it is precisely the subgroup of r -torsion elements. This concludes the proof. ■

Let us recall the definition of a smooth-Nisnevich morphism [21, Definition 3.2]. A morphism of algebraic stacks $f : \mathcal{M} \rightarrow \mathcal{N}$ is said to be smooth-Nisnevich if the map f is representable, smooth, and for every field K and morphism $\text{Spec}(K) \rightarrow \mathcal{N}$, there exists a lifting $\text{Spec}(K) \rightarrow \mathcal{M}$ such that



Cohomological invariants of a stack form a sheaf in the smooth-Nisnevich topology [21, Theorem 3.8]. When the base field k is infinite, any finite type algebraic stack over k with affine geometric stabilizers admits a smooth-Nisnevich covering by a scheme of finite type over k [21, Theorem 3.6].

Theorem 2.5 *Let \mathcal{X} be a smooth algebraic stack over k , and let $r \geq 2$ be an integer that is not divisible by $\text{char}(k)$. Let $V \subset \mathcal{X}$ be a smooth divisor. Suppose moreover that \mathcal{X} admits a smooth-Nisnevich cover $X \xrightarrow{\pi} \mathcal{X}$ with X a scheme, and let V' be the inverse image of V .*

Then, $\text{Inv}^\bullet(\mathcal{X}_{(V,r)}, \mathcal{M})$ is equal to the subset of $\text{Inv}^\bullet(\mathcal{X} \setminus V, \mathcal{M})$ of invariants α such that $r \cdot \partial_{V'}(\pi^ \alpha) = 0$.*

If moreover $\mathcal{X} = [X/G]$ with X a smooth scheme and G a smooth affine group scheme, we have an equivariant ramification map

$$\partial_V : \text{Inv}^\bullet(\mathcal{X}, \mathcal{M}) = A_G^0(X, \mathcal{M}) \rightarrow A_G^0(V, \mathcal{M}).$$

Then, $\text{Inv}^\bullet(\mathcal{X}_{(V,r)}, \mathcal{M})$ is equal to the subset of $\text{Inv}^\bullet(\mathcal{X} \setminus V, \mathcal{M})$ of invariants α such that $r \cdot \partial(\alpha) = 0$.

Proof Let α be an invariant of $\mathcal{X}_{(V,r)}$, and let $\pi : X \rightarrow \mathcal{X}$ be any smooth-Nisnevich cover of \mathcal{X} . Call V' the preimage of V in X . The fiber product $X \times_{\mathcal{X}} \mathcal{X}_{(V,r)}$ is equal to $X_{(V',r)}$, the root stack of X along V' of order r , and $X_{(V',r)}$ is a smooth-Nisnevich cover of $\mathcal{X}_{(V,r)}$. Then, by Theorem 2.4, the ramification $\partial_{V'}(\pi^* \alpha)$ must be of r -torsion.

On the other hand, let α be an invariant of $\mathcal{X} \setminus V$ such that $r \cdot \partial_V(\alpha) = 0$. We claim that α extends to $\mathcal{X}_{(V,r)}$.

Again by Theorem 2.4, the pullback of α to $X \setminus V'$ extends to a cohomological invariant of $X_{(V',r)}$, so we only need to show that the gluing conditions are satisfied. This is trivially true, because the gluing conditions can be checked on the generic point of $X_{(V',r)}$ and we know that the invariant glues on the open subset isomorphic to $X \setminus V'$.

To prove the last point, it suffices to notice that $\pi^*(\partial_V(\alpha)) = \partial_{V'}(\pi^* \alpha)$. ■

Remark 2.6 By [21, Proposition 3.6], when k is infinite, the theorem above applies to all smooth algebraic stacks with affine stabilizers. In particular, it applies to the moduli stack $\overline{\mathcal{H}}_g$ of stable hyperelliptic curves.

When k is finite, note that by [20, p.276 13.2.23, p.187 8.4.13] the stack $\overline{\mathcal{M}}_g$ has a presentation as $[X/\text{GL}_{5g-5}]$, where X is a smooth scheme. Consequently, $\overline{\mathcal{H}}_g$ has a presentation as $[V/\text{GL}_{5g-5}]$, where $V \subset X$ is some closed, GL_{5g-5} -invariant

subscheme. The induced map $V \rightarrow [V/GL_{5g-5}]$ is smooth-Nisnevich, because GL_{5g-5} is special; hence, $\overline{\mathcal{H}}_g$ has a smooth-Nisnevich cover of finite type.

The precise description of the smooth-Nisnevich covers will not be relevant to our computations, as we will use the equivariant part of the theorem which only requires the existence of such a cover.

Example 2.7 Consider the scheme \mathbb{P}^1 with two marked points $0, \infty$. It is well known that \mathbb{P}^1 has no nontrivial cohomological invariants. If we take the root stack $\mathbb{P}^1_{(0,r)}$, Theorem 2.4 shows us that the cohomological invariants are still trivial, because the complement $\mathbb{P}^1 \setminus 0$ is isomorphic to the affine line \mathbb{A}^1 , which has trivial invariants.

On the other hand, if we take the divisor $V = 0 + \infty$, we have $\mathbb{P}^1 \setminus V \simeq \mathbb{G}_m$, and for every r ,

$$\text{Inv}^\bullet(\mathbb{P}^1_{(V,r)}, M) = M^\bullet(k) \oplus \alpha \cdot M^\bullet(k)_r.$$

Another way to see this is to note that $\mathbb{P}^1_{(V,r)} \simeq [\mathbb{P}^1/\mu_r]$, where μ_r acts by multiplication on either homogeneous coordinate.

3 The stack of stable hyperelliptic curves

A stable hyperelliptic curve (C, ι) over $\text{Spec}(k)$ is a stable curve C endowed with an involution $\iota : C \rightarrow C$ such that the quotient $f : C \rightarrow C/\langle \iota \rangle$ is a curve of genus 0.

Let $\overline{\mathcal{H}}_g$ be the stack of stable hyperelliptic curves, i.e., the closure of \mathcal{H}_g inside the stack $\overline{\mathcal{M}}_g$ of stable curves. The irreducible components of the boundary divisor $\partial\overline{\mathcal{H}}_g := \overline{\mathcal{H}}_g \setminus \mathcal{H}_g$ can be described as follows (see [6]):

- The irreducible divisor Δ_0 is the stack of stable hyperelliptic curves having a nonseparating node that is fixed by the hyperelliptic involution.
- The irreducible divisor Δ_i , for $i = 1, \dots, \lfloor \frac{g}{2} \rfloor$, is the stack of stable hyperelliptic curves with two components of arithmetic genus i and $g - i$ meeting at a separating node that is fixed by the hyperelliptic involution.
- The irreducible divisor Ξ_i , for $i = 1, \dots, \lfloor \frac{g}{2} \rfloor - 1$, is the stack of stable hyperelliptic curves with two components of genus i and $g - i - 1$ meeting at two points that are switched by the hyperelliptic involution.

Let $\overline{\mathcal{H}}_g^o$ be the open substack of $\overline{\mathcal{H}}_g$ formed by those curves that either have at most one node that is fixed by the involution or exactly two nodes that are switched by the involution. Observe that the complement of $\overline{\mathcal{H}}_g^o$ in $\overline{\mathcal{H}}_g$ has codimension 2. We will call Δ_i^o and Ξ_i^o , respectively, the restriction of Δ_i and Ξ_i to $\overline{\mathcal{H}}_g^o$.

We give now an alternative description of the stack $\overline{\mathcal{H}}_g^o$ that resembles the one in [3, Corollary 4.7]. Let \mathcal{F}_g^o be the fibered category over the site of k -schemes

$$\mathcal{F}_g^o(S) := \{(P \rightarrow S, \mathcal{L}, s)\},$$

where:

- $P \rightarrow S$ is a family of twisted conics of rank ≥ 2 , i.e., each geometric fiber is a stacky curve whose coarse moduli space is a conic of rank ≥ 2 , and the only stacky structure

allowed is at most a $B\mu_2$ at the node (see [1, Definition 2.1.4] for the notion of twisted curve).

- \mathcal{L} is a line bundle on P of degree $-g - 1$ whose degree on each irreducible component is >1 . Moreover, if the fiber possesses a twisted node, the degree of \mathcal{L} restricted to an irreducible component must be an half-integer (more on this a few lines below).
- s is a global section of $\mathcal{L}^{\otimes(-2)}$. For every geometric point z of S , if the conic P_z is smooth, then s can have at most one double root, otherwise no double roots are allowed.

More explicitly, if $P \rightarrow \text{Spec}(\bar{k})$ is a rank 2 twisted conic, the restriction of \mathcal{L} to an irreducible component can have degree $\frac{r}{2}$ with $r = 3, 4, \dots, 2\lfloor \frac{g}{2} \rfloor + 1$. Indeed, consider an auxiliary rank 2 conic P' , so that P is the rigidification of $[P'/\mu_2]$ away from the node (here μ_2 acts by sending $(x : y) \mapsto (x : -y)$). A line bundle of bidegree $(\frac{r}{2}, \frac{l}{2})$ can then be thought of as a line bundle on P whose lifting to P' has bidegree (r, l) : by this we mean that, regarding P' as two copies of \mathbb{P}^1 glued in one point, the restriction of the line bundle has degree r on one component and degree l on the other.

The category \mathcal{F}_g^o is actually a Deligne–Mumford stack. Observe that \mathcal{F}_g^o contains an open substack isomorphic to \mathcal{H}_g , namely the substack of objects $(P \rightarrow S, \mathcal{L}, s)$ where $P \rightarrow S$ is smooth and the vanishing locus of s is étale over the base.

Proposition 3.1 *There is an isomorphism $\overline{\mathcal{H}}_g^o \simeq \mathcal{F}_g^o$ that extends the one of [3, Corollary 4.7].*

Proof Let $(C \rightarrow S, \iota)$ be an object of $\overline{\mathcal{H}}_g^o$. We can form the stacky quotient $[C/\iota]$, which is by construction a stacky conic of rank ≥ 2 . As the stacky structure is supported on a divisor, we can apply the rigidification process of [1]*5.1 to rigidify $[C/\iota]$ away from the singular locus: the resulting stack $P \rightarrow S$ is a family of twisted conics of rank ≥ 2 .

Observe that the morphisms $C \rightarrow [C/\iota]$ and $[C/\iota] \rightarrow P$ are both flat, hence so must be their composition $f : C \rightarrow P$. In particular, this easily implies that $f_*\mathcal{O}_C$ is a locally free sheaf of rank 2. Moreover, the inclusion $\mathcal{O}_P \hookrightarrow f_*\mathcal{O}_C$ of ι -invariant elements has a splitting given by the Reynolds morphism, hence $f_*\mathcal{O}_C \simeq \mathcal{O}_P \oplus \mathcal{L}$ for some line bundle \mathcal{L} on P .

The structure of $f_*\mathcal{O}_C$ as $\mathbb{Z}/2\mathbb{Z}$ -graded algebra determines a morphism $\mathcal{L}^{\otimes 2} \rightarrow \mathcal{O}_P$ or, equivalently, a global section $s : \mathcal{O}_P \rightarrow \mathcal{L}^{\otimes(-2)}$. The vanishing locus of s coincides with the branch locus of $C \rightarrow P$, which has degree $2g + 2$ by the Hurwitz formula: this implies that \mathcal{L} has total degree $-g - 1$. As C is stable, the degree of \mathcal{L} restricted to every irreducible component must be ≥ 2 . Eventually, stability of C also implies that the vanishing locus of s satisfies the required assumptions.

In this way, we have constructed a morphism of stacks $\varphi : \overline{\mathcal{H}}_g^o \rightarrow \mathcal{F}_g^o$. To construct its quasi-inverse, we can do the following: given an object $(P \rightarrow S, \mathcal{L}, s)$ of \mathcal{F}_g^o , consider the \mathcal{O}_P -module $\mathcal{O}_P \oplus \mathcal{L}$. We can give to $\mathcal{O}_P \oplus \mathcal{L}$ the structure of an \mathcal{O}_P -algebra as follows. First observe that the multiplication by the global section s of $\mathcal{L}^{\otimes(-2)}$ defines a morphism $\mathcal{O}_P \rightarrow \mathcal{L}^{\otimes(-2)}$. We can dualize this morphism of \mathcal{O}_P -modules, obtaining in this way $s^\vee : \mathcal{L}^{\otimes 2} \rightarrow \mathcal{O}_P$.

Given $U \rightarrow P$ and elements $a \oplus b$ and $a' \oplus b'$ of $\mathcal{O}_P(U) \oplus \mathcal{L}(U)$, we set

$$(a \oplus b) \cdot (a' \oplus b') = (aa' + s^\vee(b \otimes b'), ab' + a'b).$$

This gives a multiplicative structure to $\mathcal{O}_P \oplus \mathcal{L}$, turning it into an \mathcal{O}_P -algebra. Define $C := \underline{\text{Spec}}_{\mathcal{O}_P}(\mathcal{O}_P \oplus \mathcal{L})$: the involution on $\mathcal{O}_P \oplus \mathcal{L}$ defined as

$$a \oplus b \longmapsto a \oplus -b$$

is a morphism of algebras and determines an involution $\iota : C \rightarrow C$, whose quotient C/ι is the relative spectrum of the ι -invariant part of $(\mathcal{O}_P \oplus \mathcal{L})$, that is, \mathcal{O}_P . In other terms, $[C/\iota]$ has the same coarse moduli space of P , and the rigidification of $[C/\iota]$ at the vanishing locus of s coincides with P .

This shows that $C \rightarrow P$ is a ramified cover of degree 2. We claim that $(C \rightarrow S, \iota)$ is an object of $\overline{\mathcal{H}}_g^o$. For this, it is enough to check that in the case $S = \text{Spec}(k)$, the curve C is a stable hyperelliptic curve belonging to $\overline{\mathcal{H}}_g^o$.

First assume that p is a point in the smooth locus of P . Then, we have $\mathcal{O}_{P,p} \simeq k[x]_{(x)}$, and there exists a trivialization $\mathcal{L}_p \simeq \mathcal{O}_{P,p}$. Then, we have that locally over p , the curve C is the localization at (x) of $\text{Spec}(k[x, y]/(y^2 = f))$, where f is the image of 1 along the map $\mathcal{O}_{P,p} \xrightarrow{s} \mathcal{L}_p^{\otimes -2} \simeq \mathcal{O}_{P,p}$. This shows that if P is smooth, then C is a double cover ramified over the vanishing locus of s , which is a divisor of degree $2g + 2$ with at most one double point. This implies that C is an irreducible stable hyperelliptic curve of genus g with at most one node.

Now, suppose that p is an untwisted node of P : as the section s does not vanish on p , we have that $C \rightarrow P$ is étale locally around p . In particular, this together with the hypotheses on the vanishing locus of s forces C to have two irreducible components of genus i and $g - i - 1$ meeting in the two nodes over p . The two irreducible components are double covers of the two components of p ; hence, C is again a stable hyperelliptic curve in $\overline{\mathcal{H}}_g^o$.

Finally, suppose that p is a twisted node of P : then again $C \rightarrow P$ is étale of degree two around p , but the presence of the twist implies that the map from C to the coarse moduli space of P ramifies at the node (see the proof of [10, Theorem 2.5]) and that C is the union of two curves of genus i and $g - i$ meeting in a point. Moreover, the two components are double covers of the two components of P ; hence, C is a stable hyperelliptic curve belonging to $\overline{\mathcal{H}}_g^o$.

In this way, we have constructed a morphism of stacks $\mathcal{F}_g^o \rightarrow \overline{\mathcal{H}}_g^o$, and is straightforward to check that this gives a quasi-inverse to φ . ■

Remark 3.2 The necessity of dealing with twisted conics comes from the fact that the morphism $C \rightarrow P$, whenever it ramifies over the node, cannot be flat unless the node is twisted. This is explained in greater detail in the introduction of [10] (see also [10, Section 2]).

On the other hand, the line bundle $\mathcal{L}^{\otimes (-2)}$ comes from the coarse moduli space of P , so the global section s does as well. This shows that we still have a morphism from $\overline{\mathcal{H}}_g^o$ to the stack of conics of rank ≥ 2 with a line bundle \mathcal{M} of degree $2g + 2$ and a global section s of \mathcal{M} , but this is not an isomorphism.

The divisor Δ_0^o is sent by the isomorphism of Proposition 3.1 to the closed substack in \mathcal{F}_g^o whose objects are triples $(P \rightarrow S, \mathcal{L}, s)$ with $P \rightarrow S$ smooth. We denote this divisor by \mathcal{D}_0^o .

The divisors Δ_i^o for $i > 0$ are isomorphic to the divisors in \mathcal{F}_g^o whose objects are $(P \rightarrow S, \mathcal{L}, s)$ are such that P is a family of conics of rank 2 and \mathcal{L} has bidegree $(-\frac{2i+1}{2}, -\frac{2g-2i+1}{2})$ on each geometric fiber. In particular, every fiber of $P \rightarrow S$ must have a twist in the node. We denote these divisors by \mathcal{D}_i^o .

On the other hand, the divisors Ξ_i^o are isomorphic to the stacks whose objects are triples $(P \rightarrow S, \mathcal{L}, s)$ such that $P \rightarrow S$ is a family of rank 2 conics and \mathcal{L} has bidegree $(-i, -g - i - 1)$ on each fiber. These divisors will be denoted \mathcal{X}_i^o .

4 The stack of admissible double coverings

In the following, a *nodal curve* is a connected, reduced, and proper scheme C over $\text{Spec}(k)$ of dimension 1 whose singularities are all nodal, i.e., the formal completion of the local ring of C at a singular point p is isomorphic to $k[[x, y]]/(xy)$, for every singular point of C .

A *family of nodal curves* is a proper and flat morphism of schemes $C \rightarrow S$ such that every geometric fiber is a nodal curve.

Definition 4.1 [17, Section 4] An *admissible double covering* of genus g is a finite morphism $C \rightarrow P$ between a genus g nodal curve C and a genus 0 nodal curve P such that:

- On the complement of the singular locus of P , the morphism $C \rightarrow P$ has degree 2.
- The image of every node q of C is a node p of P , and we have that

$$\widehat{\mathcal{O}_{V,p}} \simeq k[[x, y]]/(xy) \longrightarrow \widehat{\mathcal{O}_{C,q}} \simeq k[[u, v]]/(u, v)$$

sends $x \mapsto u^e$ and $y \mapsto v^e$ with $1 \leq e \leq 2$.

- Only a finite number of automorphisms of P fixes both the ramification divisor of $C \rightarrow P$ and the singular locus.

Given a scheme S , a *family of admissible double coverings* of genus g over S is an S -morphism $C \rightarrow P$ between families of nodal curves over S such that over every geometric point \bar{s} of S , the morphism $C_{\bar{s}} \rightarrow P_{\bar{s}}$ is an admissible covering of degree 2 and genus g .

Remark 4.1 The definition above differs from the one of [17], because we do not ask for $2g + 2$ sections $\sigma_i : \text{Spec}(k) \rightarrow P$ such that the ramification divisor, away from the nodes, is equal to $\sum \sigma_i$.

We can form the fibered category $\widetilde{\mathcal{H}}_g$ of admissible double coverings of genus g , that is,

$$\widetilde{\mathcal{H}}_g(S) := \{(C \rightarrow P \rightarrow S)\},$$

where a morphism $(C \rightarrow P \rightarrow S) \rightarrow (C' \rightarrow P' \rightarrow S')$ consists of a morphism $S \rightarrow S'$ and two isomorphisms $C \simeq C'_S$ and $P \simeq P'_S$ which commute with the covering maps.

The stack $\widetilde{\mathcal{H}}_g$ is a smooth Deligne–Mumford stack. One way to see this is to consider the stack $\mathcal{B}_{0,2g+2}(S_2)$ of twisted admissible coverings (see [1, Section 2.2]), which

is isomorphic to the stack of admissible coverings of degree 2 of Harris–Mumford by [1, Proposition 4.2.2]. The stack $\mathcal{B}_{0,2g+2}(S_2)$ is a smooth Deligne–Mumford stack, and the automorphism group of every twisted S_2 -admissible covering contains the symmetric group S_{2g+2} , which acts by permutation on the markings. By [1, Section 5.1], we can take the rigidification $[\mathcal{B}_{0,2g+2}(S_2)/\mathcal{B}S_{2g+2}]$, and it is easy to see that this last stack is isomorphic to $\widetilde{\mathcal{H}}_g$.

We can recast Definition 4.1 using the notion of admissible involution.

Definition 4.2 [4, Section 3] An *admissible involution* of a nodal curve C over $\text{Spec}(k)$ is an involution $\iota : C \rightarrow C$ such that:

- No irreducible component is fixed by ι .
- The quotient $C/\langle \iota \rangle$ is a nodal curve of genus 0.
- If a node is fixed by ι , then ι acts on the local ring $\mathcal{O}_{C,p}$, hence on its formal completion $\widehat{\mathcal{O}}_{C,p}$, and it does not switch the two branches of the node.

The definition above extends in an obvious way to families of nodal curves.

Proposition 4.2 [23, Proposition 3.3.7] Let $C \rightarrow P \rightarrow S$ be a family of admissible double coverings over S . Then, there exists a unique admissible involution $\iota : C \rightarrow C$ over S such that $P = C/\langle \iota \rangle$.

Conversely, given an admissible involution $\iota : C \rightarrow C$ of a family of nodal curves, the morphism $C \rightarrow C/\langle \iota \rangle$ is an admissible double covering.

The following results give a somewhat more concrete description of how an admissible double covering looks like.

Proposition 4.3 [23, Lemma 4.1.2 and Proposition 4.1.3] Let $C \rightarrow P$ be an admissible double covering, and let $\iota : C \rightarrow C$ be the admissible involution of Proposition 4.2. Then:

- (1) The nodal curve $C \rightarrow S$ is semistable.
- (2) Every irreducible component of C is smooth. Moreover, if $\iota(C_i) = C_i$, then $C_i \rightarrow C_i/\langle \iota \rangle$ is a hyperelliptic curve, otherwise C_i and $\iota(C_i)$ are two disconnected projective lines.
- (3) Every nonstable component is a projective line intersecting the rest of the curve in two points, which get switched by ι .

An immediate consequence of the proposition above is that the admissible involution ι of an admissible double covering $C \rightarrow P$ descends to a hyperelliptic involution $\bar{\iota}$ of the stable model \bar{C} of C : in other terms, the stable curve \bar{C} is a stable hyperelliptic curve. This argument works for families too.

Corollary 4.4 Let $C \rightarrow P \rightarrow S$ be a family of admissible double covering, and let $\bar{C} \rightarrow S$ be its stable model. Then, the admissible involution $\iota : C \rightarrow C$ descends to a hyperelliptic involution $\bar{\iota} : \bar{C} \rightarrow \bar{C}$, and the pair $(\bar{C} \rightarrow S, \bar{\iota})$ is a family of stable hyperelliptic curves.

The stabilization morphism commutes with base change; hence, the corollary above determines a morphism of stacks

$$\varphi : \widetilde{\mathcal{H}}_g \longrightarrow \overline{\mathcal{H}}_g, \quad (C \rightarrow P \rightarrow S) \longmapsto (\bar{C} \rightarrow S, \bar{\iota})$$

called *the collapsing morphism* [23, Definition 4.1.13].

Observe that the collapsing morphism induces a bijection at the level of geometric points, and it is obviously an isomorphism over the open substack of smooth curves. This in turn implies that the irreducible components of the boundary $\mathcal{H}_g \setminus \mathcal{H}_g$ are $\tilde{\Delta}_i := \varphi^{-1}(\Delta_i)$ for $i = 0, 1, \dots, \lfloor \frac{g}{2} \rfloor$ and $\tilde{\Xi}_j := \varphi^{-1}(\Xi_j)$ for $j = 1, 2, \dots, \lfloor \frac{g}{2} \rfloor$.

Recall that in Section 3, we defined the open substack \mathcal{H}_g^o of $\overline{\mathcal{H}}_g$. Set then $\tilde{\mathcal{H}}_g^o := \varphi^{-1}(\mathcal{H}_g^o)$. As for the case of stable hyperelliptic curves, the complement of $\tilde{\mathcal{H}}_g^o$ in $\tilde{\mathcal{H}}_g$ has codimension 2, and the irreducible components of $\tilde{\mathcal{H}}_g^o \setminus \mathcal{H}_g$ are $\tilde{\Delta}_i^o := \varphi^{-1}(\Delta_i^o)$ and $\tilde{\Xi}_j^o := \varphi^{-1}(\Xi_j^o)$.

Observe that the collapsing map induces an isomorphism of stacks for almost all the irreducible components of the boundary of $\tilde{\mathcal{H}}_g^o$ and $\overline{\mathcal{H}}_g^o$, with the only exception of $\tilde{\Delta}_0^o \rightarrow \Delta_0^o$: indeed, compared to its stable model, an admissible covering in $\tilde{\Delta}_0^o$ has an additional automorphism, namely the involution of the projective line that switches the two intersection points.

Proposition 4.5 *The stack $\tilde{\mathcal{H}}_g^o$ is a root stack of $\overline{\mathcal{H}}_g^o$ of order 2 along the divisor Δ_0^o .*

Proof Denote $\widehat{\mathcal{H}}_g^o$ the root stack of $\overline{\mathcal{H}}_g^o$ of order 2 along Δ_0^o . We want to prove that $\tilde{\mathcal{H}}_g^o \simeq \widehat{\mathcal{H}}_g^o$. First, we are going to lift the collapsing morphism $\tilde{\mathcal{H}}_g^o \rightarrow \overline{\mathcal{H}}_g^o$ to a morphism to $\widehat{\mathcal{H}}_g^o$. Second, we will show that the lifted morphism is actually an isomorphism.

For the first point, we need to show that if $(C \rightarrow S, \iota)$ is a family of admissible coverings, and $(\overline{C} \rightarrow S, \bar{\iota})$ is its stabilization, then the divisor $D \subset S$ parametrizing those fibers in \overline{C} with a nonseparating node is of the form $2E$ for some other divisor $E \subset S$. Here, we are implicitly assuming that D is a divisor and does not coincide with the whole S nor is empty.

Let $Y \subset \overline{C}$ be the singular locus of $\overline{C} \rightarrow S$ supported over D . Up to passing to a Zariski cover of S , we can assume that $D = \{f = 0\}$ is principal, and that étale-locally around Y the family $\overline{C} \rightarrow S$ has equation $x'y' + f = 0$. After a change of coordinates, this can be rewritten as

$$y^2 - x^2 + f = 0.$$

A key point here is that for every geometric point $s \in D$, the two nodes of C_s are switched by the admissible involution. This implies that the two branches of \overline{C}_s at the node are switched by the hyperelliptic involution, hence at the level of coordinates $x' \mapsto y'$ and $y' \mapsto x'$ or equivalently $x \mapsto x$ and $y \mapsto -y$.

This means that, locally around Y , the equation $y^2 = x^2 - f$ describes $\overline{C} \rightarrow S$ as a double cover of $\overline{P} \rightarrow S$ with branching divisor $W \subset \overline{P}$ of equation $x^2 - f = 0$. Let $W' \subset P$ be the branching divisor of the admissible double covering $C \rightarrow P$: then around Y , the induced map $W' \rightarrow W$ is an isomorphism away from the locus supported over $D \subset S$, because by construction $W'_D \rightarrow W_D$ is of degree 2. In other terms, locally around Y , the map $W' \rightarrow W$ is a normalization.

This implies that W must be singular along $D = \{f = 0\}$: a straightforward computation shows that then the vanishing locus of f must be everywhere singular,

hence $f = g^k$ for some $k > 1$. Moreover, the geometry of the normalization morphism $W' \rightarrow W$ forces $k = 2n$ for some $n \geq 1$. This shows that $D = 2E$, where E is the divisor of equation $g^n = 0$.

Observe that the Cartier divisor E descends along the Zariski cover that we considered at the beginning of the proof, so we actually defined a morphism $\widetilde{\mathcal{H}}_g^o(S) \rightarrow \widehat{\mathcal{H}}_g^o(S)$ for every S such that $D \subset S$ is not empty nor equal to S . If $D = \emptyset$, there is nothing to prove. If $D = S$, we can consider a deformation S' of S : as we just proved that $S = 2E$, we deduce that $\mathcal{O}(E)|_S$ is a square root of the normal bundle $\mathcal{O}(S)|_S$, implying that as above we have a lifting $\widetilde{\mathcal{H}}_g^o(S) \rightarrow \widehat{\mathcal{H}}_g^o(S)$.

We claim that $\widetilde{\mathcal{H}}_g^o \rightarrow \widehat{\mathcal{H}}_g^o$ is representable, finite, and birational. Concerning representability, it is enough to show that the induced homomorphism between the automorphism groups of the objects is always injective. Let $C_s \rightarrow P_s \rightarrow \text{Spec}(k)$ be an admissible double covering, and let $\overline{C}_s \rightarrow \text{Spec}(k)$ be its stabilization: the kernel of the induced homomorphism

$$\underline{\text{Aut}}_k((C_s, \iota_s)) \longrightarrow \underline{\text{Aut}}_k((\overline{C}_s, \bar{\iota}_s))$$

is trivial unless \overline{C}_s has a nonseparating node, in which case the kernel is generated by the involution φ_s that on the genus zero component of C_s fixes the two nodes and on the other component is the identity.

Consider a generic one-parameter deformation $\overline{C}' \rightarrow S'$ of \overline{C}_s , where S' is a DVR with parameter t . Let $S \rightarrow S'$ be the degree 2 ramified cover $S \rightarrow S'$ given by $t \mapsto t^2$, and let $\overline{C} \rightarrow S$ be the pullback of $\overline{C}' \rightarrow S'$. In this case, the divisor D is supported on the closed point, and it is defined by the equation $\{t^2 = 0\}$. In particular, $D = 2E$, where E is the divisor of equation $\{t = 0\}$.

Call $p \in \overline{C}_s$ the nonseparating node: for what we have seen, locally around p , the double cover $\overline{C} \rightarrow \overline{P} \rightarrow S$ is described by the equation $y^2 = (x^2 - t^2)$. The blowup of \overline{C} at p is a family of admissible double coverings $C \rightarrow S$, whose central fiber is isomorphic to C_s .

The involution $S \rightarrow S$ defined by $t \mapsto -t$ induces an involution φ of $C \rightarrow S$ which on the central fiber coincides with φ_s . By construction, the morphism $\widetilde{\mathcal{H}}_g^o(S) \rightarrow \widehat{\mathcal{H}}_g^o(S)$ sends

$$(C \rightarrow S, \iota) \longmapsto (\overline{C} \rightarrow S, \bar{\iota}, \mathcal{O} \cdot t^{-1}, t),$$

hence the automorphism φ_s is sent to the automorphism of $((\overline{C}_s, \bar{\iota}), k \cdot t^{-1}, t)$ that acts as the identity on \overline{C}_s and as -1 on $k \cdot t^{-1}$. We deduce that the homomorphism between the automorphism groups is injective and that the map of stacks is representable.

The same argument can be used to show that $\widetilde{\mathcal{H}}_g^o \rightarrow \widehat{\mathcal{H}}_g^o$ is proper and quasi-finite, hence finite. As this map is representable, birational, and finite, and the target is normal, we deduce that it is actually an isomorphism, thus concluding the proof. ■

Remark 4.6 Proposition 4.5 can also be deduced from [23, Proposition 4.4.5].

5 Computation of cohomological invariants

We start by recalling some facts on the cohomological invariants of the stack of smooth hyperelliptic curves. A more in depth discussion of this subject can be found in the introduction to [9].

Let ét_n be the stack of étale algebras of degree n . It is isomorphic to the classifying stack BS_n . Given a family of smooth hyperelliptic curves $C \rightarrow S$ with involution ι , the ramification divisor of $C \rightarrow C/\langle \iota \rangle$ is the spectrum of an étale \mathcal{O}_S -algebra of degree $2g + 2$. This defines a map

$$\mathcal{H}_g \longrightarrow \text{ét}_{2g+2} \simeq \text{BS}_{2g+2},$$

which is smooth-Nisnevich, so that the pullback $\text{Inv}^\bullet(\text{BS}_{2g+2}, M) \rightarrow \text{Inv}^\bullet(\mathcal{H}_g, M)$ is injective. The cohomological invariants of BS_{2g+2} are isomorphic to

$$M^\bullet(k) \oplus \alpha_1 \cdot M^\bullet(k)_2 \oplus \dots \oplus \alpha_{g+1} \cdot M^\bullet(k)_2,$$

where the classes α_i are the *arithmetic Stiefel–Whitney classes* coming from the invariants of BO_{2g+2} (see [12, p.53 23.3] for the case $M = H_D$, and [9, Proposition 5.10] for the general case).

Theorem 5.1 [9, Theorem 6.9 and Remark 6.10] *Let M be an ℓ -torsion cycle module, and let I_g be the submodule of $\text{Inv}^\bullet(\text{BS}_{2g+2}, M)$ given by*

$$I_g = M^\bullet(k) \oplus \bigoplus_{i=2}^{g+1} \alpha_i \cdot M^\bullet(k)_2.$$

Let n_g be equal to $4g + 2$ if g is even and $8g + 4$ if g is odd. If g is even, we have

$$\text{Inv}^\bullet(\mathcal{H}_g, M) = \alpha'_1 \cdot M^\bullet(k)_{n_g} \oplus I_g \oplus \beta_{2g+2} \cdot M^\bullet(k)_2,$$

where α'_1 and β_{2g+2} are elements of degrees, respectively, 1 and $2g + 2$.

If g is odd, there is an exact sequence

$$0 \rightarrow \alpha'_1 \cdot M^\bullet(k)_{n_g} \oplus I_g \oplus w_2 \cdot M^\bullet(k)_2 \rightarrow \text{Inv}^\bullet(\mathcal{H}_g, M) \rightarrow M^\bullet(k)_2,$$

where α'_1 and w_2 are elements of degrees, respectively, 1 and 2, and the last map lowers degree by $g + 2$. When $k = \bar{k}$, the last map is surjective and the sequence splits.

As the pullback of an open immersion is injective, the invariants of $\widetilde{\mathcal{H}}_g$ inject into those of \mathcal{H}_g . We want to understand which cohomological invariants extend to the stack $\widetilde{\mathcal{H}}_g^o$ introduced in Section 4, which in turn has the same invariants as $\widetilde{\mathcal{H}}_g$ as they differ by a subset of codimension 2.

Recall that for $g \geq 3$ odd, there is a morphism $\mathcal{H}_g \rightarrow \text{BPGL}_2$ that sends an object $(C \rightarrow S, \iota)$ to the Severi–Brauer variety $P = C/\iota \rightarrow S$ (for g even, this map still exists, but it is not very interesting, as in this case C/ι is always the projectivization of a rank 2 vector bundle on S).

Proposition 5.2 *We have:*

- (1) *For every $g \geq 2$ and for every ℓ -torsion cycle module M , the cohomological invariants of \mathcal{H}_g with coefficients in M that is pulled back from BS_{2g+2} extend to cohomological invariants of $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$.*

(2) For $g \geq 3$ odd, the invariant w_2 pulled back along $\mathcal{H}_g \rightarrow \text{BPGL}_2$ does not extend to an invariant of $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$.

Proof To prove (1), we show that the map $\mathcal{H}_g \rightarrow \text{BS}_{2g+2}$ extends to the whole of $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$. Because of the isomorphism $\overline{\mathcal{H}}_g^o \simeq \mathcal{F}_g^o$ of Proposition 3.1, we can equivalently prove that there is a morphism $\mathcal{F}_g^o \setminus \mathcal{D}_0^o \rightarrow \text{BS}_{2g+2}$ extending the morphism on \mathcal{H}_g .

This last task is readily accomplished: to every object $(P \rightarrow S, L, s)$ of $\mathcal{F}_g^o \setminus \mathcal{D}_0^o$, we associate the vanishing locus of s , which by construction is étale over S of degree $2g + 2$, determining in this way the desired map $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o \rightarrow \text{BS}_{2g+2}$.

Point (2) has already been shown in the proof of [8, Theorem A.1]. ■

Proposition 5.3 *Let M be a p^n -primary torsion cycle module, with $2, \text{char}(k) \nmid p$. Then,*

$$\text{Inv}^\bullet(\overline{\mathcal{H}}_g^o \setminus \Delta_0^o, M) \simeq M^\bullet(k).$$

Proof We know from [9, Proposition 7.5] that

$$(5.1) \quad \text{Inv}^\bullet(\mathcal{H}_g, M) \simeq M^\bullet(k) \oplus M^\bullet(k)_{p^m}[1]$$

as $M^\bullet(k)$ -modules, where $p^m = \text{gcd}(p^n, 2g + 1)$.

Let $[\mathcal{L}]$ be the generator of the p^n -torsion subgroup of

$$\text{Pic}(\mathcal{H}_g) \simeq \mathbb{Z}/2^{1+r_g}(2g + 1)\mathbb{Z},$$

where $r_g \in \{0, 1\}$ is such that $r_g \equiv g \pmod{2}$. Recall [9, Lemma 1.13] that $\text{Inv}^1(\mathcal{H}_g, K_\ell) = H_{\text{ét}}^1(\mathcal{H}_g, \mu_\ell)$. Then, in (5.1) above, we have

$$M^\bullet(k)_{p^m}[1] \simeq M^\bullet(k)_{p^m} \cdot [\mathcal{L}],$$

with the product induced by the K_ℓ -module structure of M .

First, suppose that $M = K_{p^n}$. In order to prove that $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$ has only trivial invariants, it is enough to check that its Picard group has trivial p^n -torsion: this easily follows from the computation of $\text{Pic}(\overline{\mathcal{H}}_g)$ (see [6, Theorem 2] in characteristic 0 and [8, Theorem B.1] in positive characteristic). Moreover, this also shows that the ramification of $[\mathcal{L}]$ along the boundary divisor $\cup \Delta_i^o \cup \Xi_j^o$, regarded as an element of $\mathbb{Z}/p^m\mathbb{Z}$, is invertible.

For a general p^n -primary torsion cycle module M and $\gamma \in M_{p^m}^\bullet(k)$, let $[\mathcal{L}] \cdot \gamma$ be an invariant of \mathcal{H}_g that extends to an invariant of $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$: its ramification is equal to γ times the ramification of $[\mathcal{L}]$, which we know to be invertible as an element of $\mathbb{Z}/p^m\mathbb{Z}$. This implies $\gamma = 0$, and we are done. ■

Corollary 5.4 *Let M be a p^n -primary torsion Galois module, with $\text{char}(k), 2 \nmid p$. Then,*

$$\text{Inv}^\bullet(\widetilde{\mathcal{H}}_g, M) \simeq M^\bullet(k).$$

Proof The stack $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$ is an open substack of $\widetilde{\mathcal{H}}_g$, hence the invariants of the latter inject into the ring of invariants of the former, which is trivial by Proposition 5.3. ■

5.1 The g even case

Proposition 5.5 *Let $g \geq 2$ be even, and let M be a 2^n -primary torsion Galois module. Then, the exceptional invariant β_{g+2} of \mathcal{H}_g extends to $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$.*

Proof We give the proof for $M = K_2$, the general case follows easily from this one.

Let us recall how β_{g+2} is constructed. Let $\mathcal{C} \rightarrow \mathcal{H}_g$ be the universal smooth conic, and let \mathcal{L} and \mathbf{s} be, respectively, the universal line bundle on \mathcal{C} and the universal global section of $\mathcal{L}^{\otimes(-2)}$. The existence of such objects over \mathcal{H}_g is a consequence of [3, Proposition 2.2].

Let \mathcal{U} be the complement in \mathcal{C} of the vanishing locus \mathcal{V} of \mathbf{s} : in particular, we have that $\mathcal{L}^{\otimes 2}$ is trivial on \mathcal{U} or equivalently $\mathcal{L}_{\mathcal{U}}$ belongs to $H^1(\mathcal{U}, \mu_2)$.

Then, we can define a degree 1 cohomological invariant t of \mathcal{U} as follows: given a morphism $p : \text{Spec}(K) \rightarrow \mathcal{U}$ we set $t(p) = [\mathcal{L}_K]$, which belongs to $H^1(\text{Spec}(K), \mu_2)$.

Another way to see the invariant t is the following. Given a morphism $p : \text{Spec}(K) \rightarrow \mathcal{U}$, consider any lifting $p' : \text{Spec}(K) \rightarrow \mathcal{C}$. The element $p'^*(\mathbf{s}) \in K^*$ is well defined up to squares, and it is easy to see that this glues to the cohomological invariant t . In particular, this shows that the boundary of t in the complement of \mathcal{U} is equal to 1.

This invariant does not extend to a global invariant of \mathcal{C} , but $\alpha_{g+1} \cdot t$ does, where α_{g+1} is the degree $g + 1$ generator of $\text{Inv}^*(\text{BS}_{2g+2}, M)$. The reason is that

$$\partial_{\mathcal{V}}(t \cdot \alpha_{g+1}) = \alpha_{g+1},$$

and the restriction of α_g to \mathcal{V} is zero. This is explained in detail in [8, Section 2].

Using the well-known property that \mathcal{C} is actually the projectivization of a rank 2 vector bundle, we deduce that $\text{Inv}^*(\mathcal{C}) \simeq \text{Inv}^*(\mathcal{H}_g)$, hence the construction above gives the exceptional invariant β_{g+2} of \mathcal{H}_g .

The construction of the exceptional invariant for $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$ proceeds along the same lines. Let $\mathcal{P}_{sm} \rightarrow \overline{\mathcal{H}}_g^o \setminus \Delta_0^o$ be the smooth locus of the universal conic, and let \mathcal{L} and \mathbf{s} be, respectively, the universal line bundle restricted to \mathcal{P}_{sm} and the universal global section of $\mathcal{L}^{\otimes(-2)}$. Again, the existence of such objects is assured by the isomorphism $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o \simeq \mathcal{F}_g^o$ of Proposition 3.1.

Observe that $\mathcal{P}_{sm} \rightarrow \overline{\mathcal{H}}_g^o \setminus \Delta_0^o$ is a smooth-Nisnevich covering of $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$, hence the cohomological invariants of the latter inject into $\text{Inv}^*(\mathcal{P}_{sm}, M)$ via pullback.

Moreover, if an invariant of \mathcal{C} , the universal smooth conic, can be extended to \mathcal{P}_{sm} , then it necessarily comes from $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$. Indeed, let ξ be such an invariant: due to $\mathcal{P}_{sm} \rightarrow \overline{\mathcal{H}}_g^o \setminus \Delta_0^o$ being a smooth-Nisnevich cover, we have that ξ comes from $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$ if and only if it glues, i.e., $\text{pr}_1^* \xi - \text{pr}_2^* \xi = 0$ in $\text{Inv}^*(\mathcal{P}_{sm} \times_{\overline{\mathcal{H}}_g^o \setminus \Delta_0^o} \mathcal{P}_{sm}, M)$.

On the other hand, the restriction of $\text{pr}_1^* \xi - \text{pr}_2^* \xi$ to $\mathcal{C} \times_{\mathcal{H}_g} \mathcal{C}$ is zero, as we already know that the invariant glues over \mathcal{H}_g . As $\mathcal{C} \times_{\mathcal{H}_g} \mathcal{C}$ is an open substack of $\mathcal{P}_{sm} \times_{\overline{\mathcal{H}}_g^o \setminus \Delta_0^o} \mathcal{P}_{sm}$, the invariants of the latter injects into the invariants of the former. Therefore, $\text{pr}_1^* \xi - \text{pr}_2^* \xi$ is zero as well, hence ξ glues to an invariant of $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$, as claimed.

The same argument used to produce the invariant β_{g+2} of \mathcal{C} applies in this situation, giving us an invariant on \mathcal{P}_{sm} which obviously extends β_{g+2} . More precisely, one

considers the complement of the vanishing locus of \mathbf{s} in \mathcal{P}_{sm} : here, we have the invariant t of cohomological degree 1 given by

$$t = [\mathcal{L}] \in H^1(\mathcal{P}_{sm} \setminus \{\mathbf{s} = 0\}, \mu_2).$$

The product $\alpha_{g+1} \cdot t$, due to the very same reason for which $\alpha_{g+1} \cdot t$ extended in the smooth case, extends to a global invariant of \mathcal{P}_{sm} . With a little abuse of notation, this invariant is also denoted β_{g+2} . Then, for what we observed a few lines above, the invariant β_{g+2} must come from $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$. This concludes the proof. ■

Putting together Propositions 5.2 and 5.5, we derive the following result.

Corollary 5.6 *Let $g \geq 2$ be an even integer and M a 2-primary torsion cycle module. Then, we have*

$$\text{Inv}^\bullet(\mathcal{H}_g, M) \simeq \text{Inv}^\bullet(\overline{\mathcal{H}}_g^o \setminus \Delta_0^o, M).$$

We can now apply the formula for the invariants of a root stack to compute the invariants of $\widetilde{\mathcal{H}}_g$.

Theorem 5.7 *Let $g \geq 2$ be even and M a cycle module of p^n -primary torsion, with $p \neq 2$ prime. Then, if $p = 2$, we have*

$$\text{Inv}^\bullet(\widetilde{\mathcal{H}}_g, M) \simeq \text{Inv}^\bullet(\mathcal{H}_g, M),$$

otherwise the invariants of $\widetilde{\mathcal{H}}_g$ are trivial.

Proof The complement in $\widetilde{\mathcal{H}}_g$ of the open substack $\widetilde{\mathcal{H}}_g^o$ has codimension 2, hence these two stacks have the same cohomological invariants.

We know from Proposition 4.5 that $\widetilde{\mathcal{H}}_g^o$ is a root stack of order 2 of $\overline{\mathcal{H}}_g^o$ along the divisor Δ_0^o . Therefore, by Theorem 2.4, the cohomological invariants of $\widetilde{\mathcal{H}}_g^o$ are equal to the cohomological invariants of $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$ whose ramification along Δ_0^o is of 2-torsion.

It follows from Corollary 5.6 that the ramification of all the cohomological invariants of $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$ along Δ_0^o is of 2-torsion. This concludes the proof. ■

Remark 5.8 Theorem 5.7 should be compared with [8, Theorem A.1], where it is shown that the cohomological invariants of \mathcal{H}_g are all trivial. This does not come as a surprise: the compactification of \mathcal{H}_g by admissible double coverings preserves most of the structure needed to construct the cohomological invariants of smooth hyperelliptic curves. On the other hand, this is not the case for the compactification by stable hyperelliptic curves.

5.2 The g odd case

When g is odd and M is a 2^n -primary torsion cycle module with $n \geq 2$, the stack \mathcal{H}_g has basically two more invariants than in the even genus case (see Theorem 5.1). One of the two is the invariant w_2 of cohomological degree 2 pulled back from B PGL_2 , and the other comes from the 4-primary torsion of $\text{Pic}(\mathcal{H}_g)$, which for g odd is equal to $\mathbb{Z}/4(2g + 1)\mathbb{Z}$. This second invariant, denoted α'_1 , is a square root of α_1 , the degree 1 invariant coming from BS_{2g+2} .

We have already seen in Proposition 5.2 that w_2 does not extend to an invariant of $\overline{\mathcal{H}}_g^o \setminus \Delta_0^o$, so in order to have a complete picture in degree $\leq g + 1$ of $\text{Inv}^\bullet(\overline{\mathcal{H}}_g^o \setminus \Delta_0^o, M)$, we only need to understand if α'_1 extends or not. A bit surprisingly, the value of the genus plays a role here.

Proposition 5.9 *Let $g \geq 3$ be an odd integer, and let M be a 2-primary torsion cycle module. There are submodules N_g^\bullet of $M^\bullet(k)_2$ such that:*

(1) *If $g \geq 5$, we have an exact sequence*

$$0 \rightarrow \text{Inv}^\bullet(\text{BS}_{2g+2}, M) \rightarrow \text{Inv}^\bullet(\overline{\mathcal{H}}_g^o \setminus \Delta_0^o, M) \rightarrow N_g^\bullet \rightarrow 0$$

such that the inverse image of a nonzero element in N_g^\bullet has degree at least $g + 2$.

(2) *If $g = 3$, define*

$$I = \alpha_2 \cdot M^\bullet(k)_2 \oplus \alpha_3 \cdot M^\bullet(k)_2 \oplus \alpha_4 \cdot M^\bullet(k)_2 \subset \text{Inv}^\bullet(\text{BS}_4, M).$$

Then, there is an exact sequence

$$0 \rightarrow M^\bullet(k) \oplus M^\bullet(k)_4[1] \oplus I \rightarrow \text{Inv}^\bullet(\overline{\mathcal{H}}_3^o \setminus \Delta_0^o, M) \rightarrow N_3^\bullet \rightarrow 0$$

such that the inverse image of a nonzero element in N_3^\bullet has degree at least 5. The $M^\bullet(k)_4[1]$ is equal to $\alpha'_1 \cdot M^\bullet(k)_4$, where α'_1 is a square root of α_1 .

Proof First suppose $M = K_{2^n}$, $n \geq 2$.

When g is odd, the Picard group of \mathcal{H}_g is generated by the Hodge line bundle λ [14]. Consequently, $\text{Pic}(\overline{\mathcal{H}}_g)$ is generated by λ , which extends to the compactification, and the boundary divisor classes $[\Xi_j], [\Delta_i]$.

In [7], Cornalba and Harris showed that, over a field of characteristic zero, the group $\text{Pic}(\overline{\mathcal{H}}_g) \otimes \mathbb{Q}$ is defined by a single relation. The proof of \mathbb{Q} -linear independence of the boundary divisors in $\text{Pic}(\overline{\mathcal{H}}_g) \otimes \mathbb{Q}$ contained in [2, Chapter 13, Section 8], where is formulated over fields of characteristic zero, can be easily verified to hold true also over fields of characteristic $c > 2$.

Concerning the Cornalba–Harris relation, this was proved in positive characteristic $c > 2$ by Kausz [18, Theorem 3.1] and by Yamaki [26, Theorem 1.7] in characteristic 2. These results imply that $\text{Pic}(\overline{\mathcal{H}}_g) \otimes \mathbb{Q}$ has rank g also over fields of positive characteristic $c > 2$.

From [6, Theorem 2] in characteristic zero and [8, Theorem B.1] in positive characteristic $c > 2$, we moreover know that the Picard group of $\overline{\mathcal{H}}_g$ is torsion-free, and thus free of rank g .

Consider the restriction map $\text{Pic}(\overline{\mathcal{H}}_g^o)_4 \rightarrow \text{Pic}(\mathcal{H}_g)_4 = \mathbb{Z}/4\mathbb{Z}$. Using the equality $H^1(\mathcal{X}, \mu_\ell) = \text{Inv}^1(\mathcal{X}, K_\ell)$ [9, Lemma 1.13] and the Kummer exact sequence, we get a commutative diagram:

$$\begin{array}{ccccccc} \mathcal{O}^*(\overline{\mathcal{H}}_g^o \setminus \Delta_0^o) / \mathcal{O}^*(\overline{\mathcal{H}}_g^o \setminus \Delta_0^o)^4 & \rightarrow & \text{Inv}^1(\overline{\mathcal{H}}_g^o \setminus \Delta_0^o, K_4) & \rightarrow & \text{Pic}(\overline{\mathcal{H}}_g^o \setminus \Delta_0^o)_4 & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{O}^*(\mathcal{H}_g) / \mathcal{O}^*(\mathcal{H}_g)^4 & \longrightarrow & \text{Inv}^1(\mathcal{H}_g, K_4) & \longrightarrow & \text{Pic}(\mathcal{H}_g)_4 & \longrightarrow & 0. \end{array}$$

The rows are exact sequences, the first vertical arrow is an isomorphism (both are equal to $k^*/(k^*)^4$), and the second vertical arrow is injective. We conclude that a primitive 4-torsion element in $\text{Pic}(\overline{\mathcal{H}}_g^o \setminus \Delta_0^o)$, if any, would have to map to a generator of the 4-torsion of $\text{Pic}(\mathcal{H}_g)$, which is generated by $(2g + 1)\lambda$. In particular, if there is a primitive 4-torsion element in $\text{Pic}(\overline{\mathcal{H}}_g^o \setminus \Delta_0^o)$, we may get one in the form $(2g + 1)\lambda + \beta$, where β is a linear combination of the boundary divisors.

If $g = 3$, it is immediate to verify that $(2g + 1)\lambda + 2[\Xi_1] + 2[\Delta_1]$ is a primitive 4-torsion element. Then, the description of $\text{Inv}^\bullet(\mathcal{H}_3, D)$ of Theorem 5.1, combined with Proposition 5.2 and the observations above, yields the desired result.

If $g > 3$, let $(2g + 1)\lambda + \beta$ be a 4-torsion element. Then,

$$(8g + 4)\lambda - (8g + 4)\lambda - 4\beta = 2 \sum_{j=1}^{\lfloor \frac{g-1}{2} \rfloor} (j + 1)(g - j)[\Xi_j] + 4 \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} i(g - i)[\Delta_i].$$

Note that the coefficient of $[\Xi_2]$ on the RHS is equal to $6(g - 2)$, which is not divisible by 4. This gives a nontrivial relation between the $[\Xi_i]$ and $[\Delta_j]$ classes, which is impossible as we know that $\text{Pic}(\overline{\mathcal{H}}_g^o) \otimes \mathbb{Q}$ has rank $g - 1$.

What we have just shown implies that for $g > 3$, the ramification of α'_1 along the boundary divisor is of 2-torsion, because we know that $\alpha_1 = 2\alpha'_1$ can be extended: this implies that an invariant $\gamma \cdot \alpha'_1$ of $M^\bullet(k)_4 \cdot \alpha'_1$ is unramified if and only if γ is a multiple of 2, hence the subgroup of $M^\bullet(k)_4 \cdot \alpha'_1$ of unramified elements coincides with $M^\bullet(k)_2 \cdot \alpha_1$.

As in the genus 3 case, combining this with Proposition 5.2 and Theorem 5.1, we get the claimed description. ■

This allows us to give an almost complete description of the cohomological invariants of $\widetilde{\mathcal{H}}_g$ for g odd.

Theorem 5.10 *Let $g \geq 3$ be an odd integer, and let M be a p^n -primary torsion cycle module, with $p \neq c$ prime. If $p = 2$, there is a (possibly trivial) submodule N_g^\bullet of $M^\bullet(k)_2$ and an exact sequence*

$$0 \rightarrow \text{Inv}^\bullet(\text{BS}_{2g+2}, M) \rightarrow \text{Inv}^\bullet(\widetilde{\mathcal{H}}_g, M) \rightarrow N_g^\bullet \rightarrow 0$$

such that the inverse image of a nonzero element in N_g^\bullet has degree at least $g + 2$. If $p > 2$, then the cohomological invariants of $\widetilde{\mathcal{H}}_g$ are trivial.

Proof For $g \geq 5$, we can apply exactly the same argument that we used to prove Theorem 5.7.

In the case $g = 3$, the ramification of the invariant α'_1 along Δ_0 cannot be of 2-torsion, otherwise we would be able to extend the invariant $\alpha_1 = 2\alpha'_1$ to \mathcal{H}_g . Therefore, this case can be worked out as in the proof of Theorem 5.7 as well. ■

5.3 Brauer groups

Theorems 5.7 and 5.10 give us all the information needed to determine the prime-to-the-characteristic part of the Brauer group of $\widetilde{\mathcal{H}}_g$.

Theorem 5.11 *Let $g \geq 2$ and set $c = \text{char}(k)$. Let ${}^c\text{Br}(-)$ denote the prime-to- c part of the Brauer group. Then:*

$${}^c\text{Br}(\tilde{\mathcal{H}}_g) \simeq {}^c\text{Br}(k) \oplus H^1(k, \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

In particular, when $c = 0$, the formula above describes the whole Brauer group of $\tilde{\mathcal{H}}_g$.

Proof The proof works exactly as in [9, Theorem 8.1]. The starting point is the isomorphism

$$(5.2) \quad \text{Inv}^2(\tilde{\mathcal{H}}_g, \mathbb{Z}/\ell\mathbb{Z}(-1)) \simeq \text{Br}'(\tilde{\mathcal{H}}_g)_\ell,$$

where, on the right, we have the ℓ -torsion part of the cohomological Brauer group.

Theorems 5.7 and 5.10 can then be used to compute the left-hand side of (5.2): one has only to observe that all the elements in the cohomological Brauer group that do not come from the ground field are of 2-primary torsion.

To conclude, by results of Edidin–Hassett–Kresch–Vistoli and Kresch–Vistoli [9, Theorem 1.1], we have that the prime-to- $\text{char}(k)$ part of the cohomological Brauer group of a smooth, quasi-projective, tame Deligne Mumford stack belongs to the Brauer group, so in particular ${}^c\text{Br}(\tilde{\mathcal{H}}_g) = {}^c\text{Br}'(\tilde{\mathcal{H}}_g)$. ■

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