J. Aust. Math. Soc. **83** (2007), 87–104

CANONICAL VARIETIES OF COMPLETELY REGULAR SEMIGROUPS

MARIO PETRICH

(Received 15 April 2002; revised 1 August 2005)

Communicated by D. Easdown

Abstract

Completely regular semigroups $\mathcal{CR}$ are regarded here as algebras with multiplication and the unary operation of inversion. Their lattice of varieties is denoted by $\mathcal{L}(\mathcal{CR})$. Let $\mathcal{B}$ denote the variety of bands and $\mathcal{L}(\mathcal{B})$ the lattice of its subvarieties. The mapping $\mathcal{V} \to \mathcal{V} \cap \mathcal{B}$ is a complete homomorphism of $\mathcal{L}(\mathcal{CR})$ onto $\mathcal{L}(\mathcal{B})$. The congruence induced by it has classes that are intervals, say $\mathcal{V}B = [\mathcal{V}_B, \mathcal{V}^B]$ for $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$. Here $\mathcal{V}_B = \mathcal{V} \cap \mathcal{B}$. We characterize $\mathcal{V}^B$ in several ways, the principal one being an inductive way of constructing bases for $\vee$-irreducible band varieties. We term the latter canonical. We perform a similar analysis for the intersection of these varieties with the varieties $\mathcal{BG}$, $\mathcal{OBG}$ and $\mathcal{B}$.

2000 *Mathematics subject classification*: primary 20M07.
Keywords and phrases: completely regular semigroup, variety, band, congruence, interval, Polák's theorem, ladder.

1. Introduction and summary

Completely regular semigroups are those that are unions of their (pairwise disjoint maximal) subgroups. In the context of varieties, they are considered as algebras with the binary operation of multiplication and the unary operation of inversion. If S is such a semigroup and $a \in S$, then a^{-1} is the inverse of a in the maximal subgroup of S containing a. They form the variety $\mathcal{CR}$. We denote by $\mathcal{L}(\mathcal{CR})$ the lattice of all subvarieties of $\mathcal{CR}$.

The lattice $\mathcal{L}(\mathcal{CR})$ has attracted considerable attention. This is in view of the importance of the class of completely regular semigroups (with or without the unary operation) and the amenability of this lattice to various techniques. The principal of these consists of copious decompositions of $\mathcal{L}(\mathcal{CR})$, whose classes promise a

description usually in terms of varieties of groups or bands (idempotent semigroups). This approach to the structure of $\mathcal{L}(\mathcal{CR})$ has produced satisfactory results, but is by no means exhausted. For the study of these decompositions, and the construction of new ones, there is plenty of room for further research.

Denote by $\mathcal{B}$ the variety of bands. Our decomposition of $\mathcal{L}(\mathcal{CR})$ is the complete congruence induced by the mapping $\mathcal{V} \to \mathcal{V} \cap \mathcal{B}$. Its classes are intervals with the lower end $\mathcal{V} \cap \mathcal{B}$. We concentrate on $\vee$-irreducible band varieties since the others are obtained as finite intersections of these, and this property carries over to upper ends of these intervals. We characterize these in two ways and provide a single identity as a basis for their system of identities. This identity is defined inductively starting with two simple identities and their duals. This system of identities represents a variant of the systems of identities for $\vee$-irreducible varieties of bands constructed by Gerhard and Petrich [2]. A characterization of upper ends of these intervals in terms of Malcev product was devised by Reilly and Zhang [13]. An interesting extension of these ideas to a class of pseudovarieties was treated by Trotter and Weil [15]. We conclude the paper with a similar analysis for intersections of upper ends with the varieties $\mathcal{BG}$, $\mathcal{OBG}$ and $\mathcal{B}$, providing in each case a basis for their identities consisting of a single identity.

We term the upper ends of intervals belonging to $\vee$-irreducible band varieties, with a few exceptions at the bottom of the lattice, *canonical*. They play a seminal role in the context of varieties of completely regular semigroups.

2. Notation and terminology

We state here the needed symbolism and a few concepts. A complete list of these can be found in the book [7]. We shall refer to the book [7], as well as to its second volume [8], which is in preparation. In the latter case, we shall also state the original sources. Throughout we use set theoretic notation to describe and manipulate classes such as varieties.

For ρ an equivalence relation on a nonempty set X and $x \in X$, $x\rho$ denotes the ρ-class of x. In any lattice L, for $a, b \in L$ such that $a \leq b$, define the *interval* $[a, b]$ by $\{c \in L \mid a \leq c \leq b\}$.

Let S be a semigroup. Then $E(S)$ denotes its set of idempotents and $\mathcal{C}(S)$ the lattice of congruences on S. For $\rho \in \mathcal{C}(S)$, the *kernel* and the *trace* of ρ are given by $\ker \rho = \{a \in S \mid a\rho e \text{ for some } e \in E(S)\}$, $\operatorname{tr} \rho = \rho|_{E(S)}$, and the *left* and the *right traces* of ρ are $\operatorname{ltr} \rho = \operatorname{tr}(\rho \vee \mathcal{L})^0$ and $\operatorname{rtr} \rho = \operatorname{tr}(\rho \vee \mathcal{R})^0$, where the join is taken within equivalence relations on S. On S, the relations defined by

$$\lambda K \rho \iff \ker \lambda = \ker \rho \quad \text{and} \quad \lambda T_p \rho \iff p \operatorname{tr} \lambda = p \operatorname{tr} \rho,$$

https://doi.org/10.1017/S1446788700036405 Published online by Cambridge University Press

where $p \in \{l, r\}$, are the *kernel*, *left trace* and *right trace relations*, respectively. Their classes are intervals, and we write

$$\rho P = [\rho_P, \rho^P], \quad P \in \{K, T_l, T_r\}.$$

Finally let S be a completely regular semigroup. As S is a union of its (maximal) subgroups, on S we have a unary operation $a \to a^{-1}$, where a^{-1} is the inverse of a in the maximal subgroup of S containing a. Hence for the purpose of studying varieties of completely regular semigroups, they are considered with the binary operation of multiplication and the unary operation of inversion. We write $a^0 = aa^{-1}(= a^{-1}a)$ for any element a of S.

We denote by $\mathcal{CR}$ the variety of all completely regular semigroups and by $\mathcal{L}(\mathcal{CR})$ the lattice of its subvarieties. Via the usual antiisomorphism of the lattice of fully invariant congruences on a free completely regular semigroup of infinite rank and $\mathcal{L}(\mathcal{CR})$, the relations K, T_l and T_r defined above transfer to $\mathcal{L}(\mathcal{CR})$ in an obvious way. We use the same notation for these relations on $\mathcal{L}(\mathcal{CR})$ (and their intervals) as for the corresponding ones on semigroups. For these operators, we write for example, $\mathcal{V}_K$, $\mathcal{V}^{KT_r} = (\mathcal{V}^K)^{T_r}$, and so on. For $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, $\mathcal{L}(\mathcal{V})$ denotes the lattice of all subvarieties of $\mathcal{V}$.

If $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ has a basis $\{u_\alpha = v_\alpha\}_{\alpha \in A}$, we write $\mathcal{V} = [u_\alpha = v_\alpha]_{\alpha \in A}$, or simply $\mathcal{V} = [u = v]$ if A is a singleton. We shall mention a great number of subvarieties of $\mathcal{L}(\mathcal{CR})$ using for the notation their standard acronyms, for a full name and bases, we refer to the book [7].

Let X be a nonempty set. If $x_1, \ldots, x_n \in X$, then $w = x_1 \cdots x_n$ is a *word* over X, $h(w) = x_1$ is the *head* of w, $t(w) = x_n$ is the *tail* of w, and $c(w) = \{x_1, \ldots, x_n\}$ is the *content* of w.

We start by describing the specific construction needed throughout. Let Θ be the set of all (nonempty) words over the alphabet $\{T_l, T_r\}$ of the form $P_1 \cdots P_n$, where $P_i \in \{T_l, T_r\}$ and $P_i \neq P_{i+1}$ for $i = 1, \ldots, n-1$ with multiplication

$$(P_1 \cdots P_m)(Q_1 \cdots Q_n) = \begin{cases} P_1 \cdots P_m Q_1 \cdots Q_n & \text{if } P_m \neq Q_1, \\ P_1 \cdots P_m Q_2 \cdots Q_n & \text{otherwise.} \end{cases}$$

Clearly Θ is a semigroup. We adjoin the *empty word* $\emptyset$ to Θ to form Θ^1. We now coordinatize Θ as follows.

For $p \in \{l, r\}$ and $n \geq 1$, let $\langle p, n \rangle$ be the product of n factors of alternating T_l and T_r starting with T_p. We also write $\bar{p} = q$, where $\{p, q\} = \{l, r\}$. For these symbols, we have the product

$$\langle p, m \rangle \langle q, n \rangle = \begin{cases} \langle p, m+n \rangle & \text{if } p = q, m \text{ is even or } p \neq q, m \text{ is odd,} \\ \langle p, m+n-1 \rangle & \text{otherwise,} \end{cases}$$

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and the dual

$$\overline{\langle p, n\rangle} = \begin{cases} \langle p, n\rangle & \text{if } n \text{ is odd,} \\ \langle \bar{p}, n\rangle & \text{if } n \text{ is even.} \end{cases}$$

We now return to $\mathcal{L}(\mathcal{CR})$. First let $\mathcal{K}_0 = \{\mathcal{V}_K \mid \mathcal{V} \in \mathcal{L}(\mathcal{CR})\}$. Then $\mathcal{K}_0$ is a complete lattice whose joins coincide with those in $\mathcal{L}(\mathcal{CR})$. We adjoin three elements to $\mathcal{K}_0$ by letting $\mathbb{N}_3^* = \{L^*, T^*, R^*\}$, $\mathcal{K} = \mathcal{K}_0 \cup \mathbb{N}_3^*$ with the ordering in $\mathcal{K}_0$, and for any $\mathcal{V} \in \mathcal{K}_0$,

$$\mathcal{V} > L^* > T^* < R^* < \mathcal{V}.$$

Also let $\mathbb{N}_3 = \{\mathcal{LNB}, \mathcal{S}, \mathcal{RNB}\}$ and $\mathbb{N}_6 = \{\mathcal{LZ}, \mathcal{T}, \mathcal{RZ}\} \cup \mathbb{N}_3$.

We may think of the elements L^*, T^* and R^* as 'markers'. In addition, we have the following 'markers' for the operators of K upper and lower. For any $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, we define

$$\mathcal{V}_{K^*} = \begin{cases} \mathcal{V}_K & \text{if } \mathcal{V} \notin \mathbb{N}_6, \\ L^* & \text{if } \mathcal{V} \in \{\mathcal{LZ}, \mathcal{LNB}\}, \\ T^* & \text{if } \mathcal{V} \in \{\mathcal{T}, \mathcal{S}\}, \\ R^* & \text{if } \mathcal{V} \in \{\mathcal{RZ}, \mathcal{RNB}\}, \end{cases}$$

and for any $\mathcal{V} \in \mathcal{K}$,

$$\mathcal{V}^{K^*} = \begin{cases} \mathcal{V}^K & \text{if } \mathcal{V} \in \mathcal{K}_0, \\ \mathcal{LNB} & \text{if } \mathcal{V} = L^*, \\ \mathcal{S} & \text{if } \mathcal{V} = T^*, \\ \mathcal{RNB} & \text{if } \mathcal{V} = R^*. \end{cases}$$

In particular, for any $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ and $\tau \in \Theta$, we write

$$\mathcal{V}_{\tau K^*} = (\mathcal{V}_\tau)_{K^*} \quad \text{and} \quad \mathcal{V}^{K^*\tau} = (\mathcal{V}^{K^*})^\tau.$$

The set $\mathcal{K}^{\Theta^1}$ of all functions $\Theta^1 \to \mathcal{K}$, ordered componentwise, is a complete lattice. Recall that both Θ^1 and $\mathcal{K}$ are partially ordered sets. We now identify a part of $\mathcal{K}^{\Theta^1}$.

Let Φ be the set of all functions $\varphi : \Theta^1 \to \mathcal{K}$ satisfying the following conditions:

(P1) $\emptyset\varphi \in \mathcal{K}_0$.

(P2) φ is order preserving.

(P3) If $\tau \in \Theta$ and $\tau\varphi = L^*$, then $t(\tau) = T_r$.

(P4) If $\tau \in \Theta$ and $\tau\varphi = R^*$, then $t(\tau) = T_l$.

(P5) If $\sigma \in \Theta^1$ and $\tau \in \Theta$ are such that $\sigma\varphi \in \mathcal{K}_0$ and either $\sigma = \emptyset$ or $t(\sigma) \neq h(\tau)$, then $(\sigma\varphi)_{\tau K^*} \subseteq (\sigma\tau)\varphi$.

We retain the set theoretical symbolism for $\mathcal{K}$.

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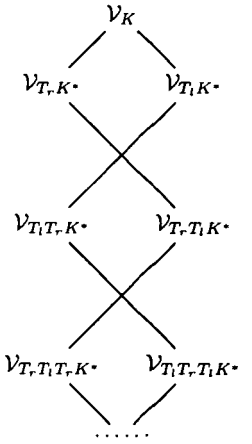
3. Preparation

We now state the results most frequently used throughout the paper; for infrequent results we refer to the books [7, 8] or the original sources. The central theorem is due to Polák [10] which we state in the following form.

THEOREM 3.1. *The mappings* $\chi : \mathcal{V} \to \chi_{\mathcal{V}}$, *where* $\chi_{\mathcal{V}} : \tau \to \mathcal{V}_{\tau K^*} (\tau \in \Theta)$ *and* $\emptyset \to \mathcal{V}_K$, *and* $\xi : \varphi \to \bigcap_{\tau \in \Theta^1} (\tau\varphi)^{K^* \bar{\tau}}$ *are mutually inverse isomorphisms between* $[\mathcal{S}, \mathcal{CR}]$ *and* Φ.

PROOF. See [8] or the original work [10]. □

It is convenient to think of $\chi_{\mathcal{V}}$ as a *ladder* in the form



We start with statements concerning K.

LEMMA 3.2.

(i) *The following intervals are* K*-classes*: $[\mathcal{T}, \mathcal{B}]$, $[\mathcal{G}, \mathcal{O}]$, $[\mathcal{CS}, \mathcal{LO}]$.

(ii) *For any* $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$,

$$\mathcal{V}_K = \begin{cases} \mathcal{V} \cap \mathcal{G} & \text{if } \mathcal{V} \subseteq \mathcal{O}, \\ \mathcal{V} \cap \mathcal{CS} & \text{if } \mathcal{V} \in \mathcal{L}(\mathcal{LO}) \backslash \mathcal{L}(\mathcal{O}). \end{cases}$$

(iii) *For any* $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$ *such that* $\mathcal{U} \subseteq \mathcal{V}$, *we have* $\mathcal{U}_K \subseteq \mathcal{V}_K$ *and* $\mathcal{U}^K \subseteq \mathcal{V}^K$.

PROOF. These statements follow from [8], for their origin, see [4, 6, 9, 11]. □

We now consider T_r. The statements concerning T_l are obtained by (left-right) duality.

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LEMMA 3.3. (i) *The following intervals are T_r-classes*:

$$\begin{aligned} &[\mathcal{T}, \mathcal{RG}], && \mathcal{RG} = [a = x^0 a], \\ &[\mathcal{LZ}, \mathcal{CS}], && \mathcal{CS} = [a = (ax)^0 a], \\ &[\mathcal{S}, \mathcal{RRO}], && \mathcal{RRO} = [xa = a^0 xa], \\ &[\mathcal{LNB}, \mathcal{LRRO}], && \mathcal{LRRO} = [yxa = (ya)^0 yxa]. \end{aligned}$$

(ii) *If $\mathcal{V} = [u_\alpha = v_\alpha]_{\alpha\in A} \in \mathcal{L}(\mathcal{CR})$, then $\mathcal{V}^{T_r} = [u_\alpha x = (v_\alpha x)^0 u_\alpha x]_{\alpha\in A}$, where $x \notin \bigcup_{\alpha\in A} c(u_\alpha v_\alpha)$.*

(iii) *If $\mathcal{V} = [u_\alpha = v_\alpha]_{\alpha\in A} \in [\mathcal{LRB}, \mathcal{CR}]$, then $\mathcal{V}_{T_r} = [s^k(u_\alpha) = s^k(v_\alpha)]_{\alpha\in A, k\geq 0}$.*

(iv) *For any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$ such that $\mathcal{U} \subseteq \mathcal{V}$, we have $\mathcal{U}_{T_r} \subseteq \mathcal{V}_{T_r}$ and $\mathcal{U}^{T_r} \subseteq \mathcal{V}^{T_r}$.*

PROOF. These statements follow from [8], for their origin, see Petrich and Reilly [7, Theorem IX.5.1], [4], [6] and [10]. □

Figure 1 depicts the lattice of proper band varieties as constructed in [2].

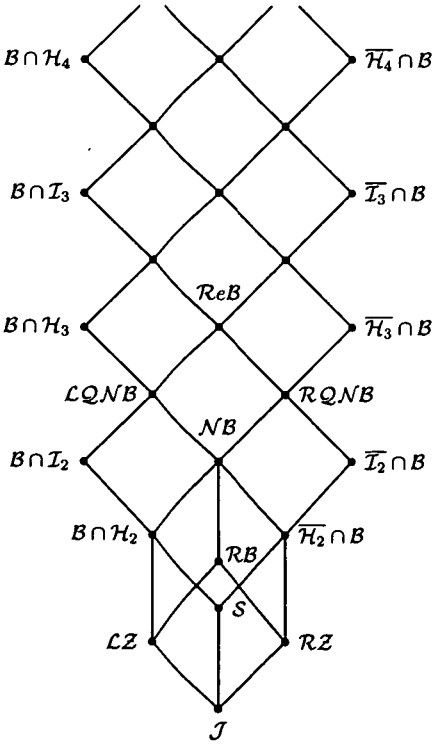

FIGURE 1. Proper band varieties

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4. Canonical varieties

We define the canonical varieties by means of their bases. We characterize some of their T_l- and T_r-classes and establish their relationship with the varieties in $\aleph_3$, as well as their ordering.

We start with two operations on the free unary semigroups U_X on X by stipulating

$$\begin{aligned} &\overline{x} = x, \quad \overline{uv} = \overline{u}\,\overline{v}, \quad \overline{u^{-1}} = \overline{u}^{-1} && (x \in X,\ u, v \in U_X),\\ &w^0 = ww^{-1} && (w \in U_X). \end{aligned}$$

It follows that $\overline{w^0} = \overline{w}^0$. The effect of the bar operation is to reverse the order of variables in w, while respecting the parentheses and exponents.

We introduce three systems of words G_n, I_n and H_n for $n \geq 2$ inductively as follows:

$$G_2 = x_2x_1, \quad I_2 = x_2x_1x_2^0, \quad H_2 = x_2,$$

and for $n > 2$,

$$G_n = x_n\overline{G_{n-1}}, \quad P_n = G_n\left(x_n\overline{P_{n-1}}\right)^0 \quad (P \in \{I, H\}).$$

We call the varieties

$$\begin{aligned} &\mathcal{I}_n = [G_n = I_n], && \overline{\mathcal{I}_n} = \left[\overline{G_n} = \overline{I_n}\right], && n \geq 2,\\ &\mathcal{H}_n = [G_n = H_n], && \overline{\mathcal{H}_n} = \left[\overline{G_n} = \overline{H_n}\right], && n \geq 3,\\ &\mathcal{H}_2 = \mathcal{LNO}, && \overline{\mathcal{H}_2} = \mathcal{RNO} \end{aligned}$$

canonical, for they will play an essential role in much of the discussion that follows.

When using canonical varieties, we may consider the alphabet $X = \{x_1, x_2, \dots\}$, or include further variables if it is convenient to change the notation.

We shall frequently have (left-right) duals of statements or formulae involving terms with bars. These are obtained by putting a bar on $\mathcal{P}_n$, $\overline{\mathcal{P}_n}$, G_n, $\overline{G_n}$, P_n or $\overline{P_n}$ for $P \in \{I, H\}$ where $\overline{\overline{W}} = W$.

In the next proposition we calculate T_l- and T_r-classes of canonical varieties. When restricted to the lattice of proper band varieties, this result coincides with Gerhard and Petrich [1, Corollary 7.10] and exhausts all T_l- and T_r-classes of proper band varieties within $\mathcal{L}(\mathcal{B})$.

PROPOSITION 4.1. *Let $\mathcal{P}_n = \mathcal{I}_n$ for $n \geq 2$ and $\mathcal{P}_n = \mathcal{H}_n$ for $n > 2$.*

(i) *The following intervals are T_l-classes:* $[\mathcal{S}, \mathcal{I}_2]$, $[\mathcal{RNB}, \mathcal{H}_3]$, $\left[\overline{\mathcal{P}_n}, \mathcal{P}_{n+1}\right]$.

(ii) *The following intervals are T_r-classes:* $\left[\mathcal{S}, \overline{\mathcal{I}_2}\right]$, $\left[\mathcal{LNB}, \overline{\mathcal{H}_3}\right]$, $\left[\mathcal{P}_n, \overline{\mathcal{P}_{n+1}}\right]$.

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PROOF. By duality, it suffices to prove part (ii). That $[\mathcal{S}, \overline{\mathcal{I}_2}]$ is a T_r-class is part of Lemma 3.3 (i). By Petrich and Reilly [7, Theorem IX.5.1 (iv)], we have $\mathcal{LRRO} = \overline{\mathcal{H}_3}$ and Lemma 3.3 (i) implies that $[\mathcal{LNB}, \overline{\mathcal{H}_3}]$ is a T_r-class. We verify first the lower end $\mathcal{P}_n$. To apply Lemma 3.3 (iii), we must check that $\mathcal{LRB} \subseteq \mathcal{P}_n$. By Petrich and Reilly [7, Lemma V.1.4] we have that $\mathcal{LRB}$ is generated by L_2^1, the 2-element left zero semigroup with an identity adjoined. For this, it suffices to show that L_2^1 satisfies the identity $G_n = P_n$. The form of this identity is $x_2x_1 = x_2x_1x_2^0$ for $\mathcal{P}_2 = \mathcal{I}_2$ and $G_n = G_n(x_n\overline{P}_{n-1})^0$ otherwise. Hence these identities are of the form $G_n = G_nW$ for a word W with $c(W) \subseteq c(G_n)$. It is now obvious that L_2^1 satisfies $G_n = P_n$. It follows by Lemma 3.3 (iii) that $(\mathcal{P}_n)_{T_r} = \mathcal{P}_n$. For the upper end, by Lemma 3.3 (ii) we get

$$\begin{aligned}\mathcal{P}_n^{T_r} &= [G_nx_{n+1} = (P_nx_{n+1})^0G_nx_{n+1}]\\ &= \left[\overline{x_{n+1}\overline{G_n}} = \overline{x_{n+1}\overline{G_n}(x_{n+1}\overline{P_n})^0}\right] = \left[\overline{G_{n+1}} = \overline{P_{n+1}}\right] = \overline{\mathcal{P}_{n+1}},\end{aligned}$$

which proves that $\left[\mathcal{P}_n, \overline{\mathcal{P}_{n+1}}\right]$ is a T_r-class. □

It is easy to check that

$$[\mathcal{S}, \mathcal{I}_2] \text{ is the } T_l\text{-class of } \mathcal{H}_2, \quad [\mathcal{RNB}, \mathcal{H}_3] \text{ is the } T_l\text{-class of } \overline{\mathcal{H}_2},$$
$$[\mathcal{S}, \overline{\mathcal{I}_2}] \text{ is the } T_r\text{-class of } \overline{\mathcal{H}_2}, \quad [\mathcal{LNB}, \overline{\mathcal{H}_3}] \text{ is the } T_r\text{-class of } \mathcal{H}_2.$$

Hence Proposition 4.1 takes care of all T_l- and T_r-classes of canonical varieties.

The next lemma characterizes $\mathcal{S}^r$, $\mathcal{LNB}^r$ and $\mathcal{RNB}^r$ in terms of canonical varieties. By turning things around, the lemma could be used to define canonical varieties in terms of $\mathcal{S}^r$, $\mathcal{LNB}^r$ and $\mathcal{RNB}^r$.

LEMMA 4.2. *Let $n \geq 1$.*

(i) $$\mathcal{S}^{\langle l,n\rangle} = \mathcal{LNB}^{\langle l,n\rangle} = \begin{cases}\mathcal{I}_{n+1}d & \text{if } n \text{ is odd,}\\ \overline{\mathcal{I}_{n+1}} & \text{if } n \text{ is even.}\end{cases}$$

(ii) $$\mathcal{S}^{\langle r,n\rangle} = \mathcal{RNB}^{\langle r,n\rangle} = \begin{cases}\overline{\mathcal{I}_{n+1}}d & \text{if } n \text{ is odd,}\\ \mathcal{I}_{n+1} & \text{if } n \text{ is even.}\end{cases}$$

(iii) $$\mathcal{RNB}^{\langle l,n\rangle} = \begin{cases}\mathcal{H}_{n+2} & \text{if } n \text{ is odd,}\\ \overline{\mathcal{H}_{n+2}} & \text{if } n \text{ is even.}\end{cases}$$

(iv) $$\mathcal{LNB}^{\langle r,n\rangle} = \begin{cases}\overline{\mathcal{H}_{n+2}} & \text{if } n \text{ is odd,}\\ \mathcal{H}_{n+2} & \text{if } n \text{ is even.}\end{cases}$$

PROOF. The proof is by induction on n.

(i) For $n = 1$, by the dual of Lemma 3.3 (i), we have $\mathcal{S}^{T_l} = \mathcal{I}_2$. Assume the statement is true for n. If n is odd, then by Proposition 4.1 (ii), we get

$$\mathcal{S}^{\langle l,n+1\rangle} = (\mathcal{S}^{\langle l,n\rangle})^{T_r} = \mathcal{I}_{n+1}^{T_r} = \overline{\mathcal{I}_{n+2}}$$

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and if n is even, then by Proposition 4.1 (i), we have

$$\mathcal{S}^{\langle l,n+1\rangle} = (\mathcal{S}^{\langle l,n\rangle})^{T_l} = \overline{\mathcal{I}_{n+1}}^{T_l} = \mathcal{I}_{n+2}.$$

This establishes the assertion for $\mathcal{S}^{\langle l,n\rangle}$. Since $\mathcal{LNB}^{T_l} = \mathcal{S}^{T_l}$, the remaining equality also holds.

(ii) This is the dual of part (i).

(iii) The argument here follows the same lines as in the proof of part (i) and is omitted.

(iv) This is the dual of part (iii). □

We determine next the ordering of canonical varieties.

LEMMA 4.3. *Let $n \geq 2$.*

(i) *For $\mathcal{P} \in \{\mathcal{H}, \mathcal{I}\}$, we have $\mathcal{P}_n \subseteq \overline{\mathcal{P}_{n+1}}$ and $\overline{\mathcal{P}_n} \subseteq \mathcal{P}_{n+1}$.*

(ii) *We have $\mathcal{I}_n \subseteq \mathcal{H}_{n+1}$, $\overline{\mathcal{I}_n} \subseteq \overline{\mathcal{H}_{n+1}}$, $\mathcal{H}_n \subseteq \mathcal{I}_n$ and $\overline{\mathcal{H}_n} \subseteq \overline{\mathcal{I}_n}$.*

PROOF. (i) This follows directly from Proposition 4.1.

(ii) The proof is by induction on n. Since $\mathcal{S} \subset \mathcal{RNB}$, by the dual of Lemma 3.3 (iv), we have $\mathcal{S}^{T_l} \subseteq \mathcal{RNB}^{T_l}$, which by Lemma 4.2 implies that $\mathcal{I}_2 \subseteq \mathcal{H}_3$. Dually, we get that $\overline{\mathcal{I}_2} \subseteq \overline{\mathcal{H}_3}$. This establishes the statements for $n = 2$. Suppose the validity of both inclusions for n. Then by Lemma 3.3 (iv) and its dual, we get $\mathcal{I}_n^{T_r} \subseteq \mathcal{H}_{n+1}^{T_r}$ and $\overline{\mathcal{I}_n}^{T_l} \subseteq \overline{\mathcal{H}_{n+1}}^{T_l}$, which by Lemma 4.2 yields $\overline{\mathcal{I}_{n+1}} \subseteq \overline{\mathcal{H}_{n+2}}$, as required.

The argument for the remaining formulae is similar by noting that $\mathcal{H}_2 \subset \mathcal{I}_2$ and starting with $\mathcal{RNB} \subset \mathcal{RRO}$ instead of $\mathcal{S} \subset \mathcal{RNB}$. □

The inclusions in Lemma 4.3 are in concordance with implications in Gerhard and Petrich [2, Lemma 6.2].

For some varieties $\mathcal{V}$ of completely regular semigroups, we compare here the varieties $\mathcal{V}_{\langle p,n\rangle}$ with certain varieties in $\mathbb{N}_6$. The first lemma concerns varieties of completely simple semigroups.

LEMMA 4.4. *Let $\mathcal{V} \in \mathcal{L}(\mathcal{CS})$.*

(i) $$\mathcal{V}_{\langle l,n\rangle} = \begin{cases} \mathcal{RZ} & \text{if } n = 1 \text{ and } \mathcal{V} \supseteq \mathcal{RZ}, \\ \mathcal{T} & \text{otherwise.} \end{cases}$$

(ii) $$\mathcal{V}_{\langle r,n\rangle} = \begin{cases} \mathcal{LZ} & \text{if } n = 1 \text{ and } \mathcal{V} \supseteq \mathcal{LZ}, \\ \mathcal{T} & \text{otherwise.} \end{cases}$$

PROOF. According to Lemma 3.3 (i) and its dual, we have

$$\mathcal{V}_{T_l} = \begin{cases} \mathcal{T} & \text{if } \mathcal{V} \subseteq \mathcal{LG}, \\ \mathcal{RZ} & \text{if } \mathcal{V} \supseteq \mathcal{RZ}, \end{cases} \qquad \mathcal{V}_{T_r} = \begin{cases} \mathcal{T} & \text{if } \mathcal{V} \subseteq \mathcal{RG}, \\ \mathcal{LZ} & \text{if } \mathcal{V} \supseteq \mathcal{LZ}. \end{cases}$$

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We consider two cases.

(1) $\mathcal{V} \subseteq \mathcal{LG}$. Then $\mathcal{V}_{T_l} = \mathcal{T}$ and

$$\mathcal{V}_{T_r} = \begin{cases} \mathcal{T} & \text{if } \mathcal{V} \subseteq \mathcal{G}, \\ \mathcal{LZ} & \text{if } \mathcal{V} \supseteq \mathcal{LZ}, \end{cases}$$

and thus $\mathcal{V}_{T_r T_l} = \mathcal{T}$.

(2) $\mathcal{V} \supseteq \mathcal{RZ}$. Then $\mathcal{V}_{T_l} = \mathcal{RZ}$, whence $\mathcal{V}_{T_l T_r} = \mathcal{T}$, and

$$\mathcal{V}_{T_r} = \begin{cases} \mathcal{T} & \text{if } \mathcal{V} \subseteq \mathcal{RG}, \\ \mathcal{LZ} & \text{if } \mathcal{V} \supseteq \mathcal{LZ}, \end{cases}$$

whence $\mathcal{V}_{T_l T_r} = \mathcal{T}$.

The assertion of the lemma follows. □

The next three lemmas concern arbitrary varieties of completely regular semigroups.

LEMMA 4.5. *Let* $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ *and* $n \geq 1$.

(i) $\mathcal{V}_{\langle l,n\rangle} = \mathcal{S}$ *if and only if* $\mathcal{S} \subseteq \mathcal{V} \subseteq \mathcal{I}_{n+1}$.

(ii) $\mathcal{V}_{\langle r,n\rangle} = \mathcal{S}$ *if and only if* $\mathcal{S} \subseteq \mathcal{V} \subseteq \overline{\mathcal{I}_{n+1}}$.

PROOF. The argument is by induction on n for both statements simultaneously. For $n = 1$, by Lemma 3.3 (i) and its dual, we have

$$\mathcal{V}_{T_l} = \mathcal{S} \iff \mathcal{S} \subseteq \mathcal{V} \subseteq \mathcal{I}_2, \quad \mathcal{V}_{T_r} = \mathcal{S} \iff \mathcal{S} \subseteq \mathcal{V} \subseteq \overline{\mathcal{I}_2}.$$

Assume that the above statements are true for n. Then

$$\begin{aligned} \mathcal{V}_{\langle l,n+1\rangle} = \mathcal{S} &\iff (\mathcal{V}_{T_l})_{\langle r,n\rangle} = \mathcal{S} \\ &\iff \mathcal{S} \subseteq \mathcal{V}_{T_l} \subseteq \overline{\mathcal{I}_{n+1}} && \text{by the induction hypothesis} \\ &\implies \mathcal{S}^{T_l} \subseteq (\mathcal{V}_{T_l})^{T_l} \subseteq (\overline{\mathcal{I}_{n+1}})^{T_l} && \text{by Lemma 3.3 (iv)} \\ &\implies \mathcal{I}_2 \subseteq \mathcal{V}^{T_l} \subseteq \mathcal{I}_{n+2} && \text{by Proposition 4.1 (i)} \\ &\implies \mathcal{V} \subseteq \mathcal{V}^{T_l} \subseteq \mathcal{I}_{n+2} \end{aligned}$$

and also $\mathcal{S} = \mathcal{V}_{\langle l,n+1\rangle} \subseteq \mathcal{V}$, so that $\mathcal{S} \subseteq \mathcal{V} \subseteq \mathcal{I}_{n+2}$. Conversely,

$$\begin{aligned} \mathcal{S} \subseteq \mathcal{V} \subseteq \mathcal{I}_{n+2} &\implies \mathcal{S}_{T_l} \subseteq \mathcal{V}_{T_l} \subseteq (\mathcal{I}_{n+2})_{T_l} && \text{by Lemma 3.3 (iv)} \\ &\implies \mathcal{S} \subseteq \mathcal{V}_{T_l} \subseteq \overline{\mathcal{I}_{n+1}} && \text{by Proposition 4.1 (i)} \\ &\implies (\mathcal{V}_{T_l})_{\langle r,n\rangle} = \mathcal{S} && \text{by the induction hypothesis} \\ &\implies \mathcal{V}_{\langle l,n+1\rangle} = \mathcal{S}. \end{aligned}$$

This proves the statement for $\mathcal{V}_{\langle l,n\rangle}$; a dual argument is valid for $\mathcal{V}_{\langle r,n\rangle}$. □

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LEMMA 4.6. *Let $\mathcal{V} \in \mathcal{L}(\mathcal{CS})$ and $n \geq 1$.*

(i) $\mathcal{V}_{\langle l,n\rangle} \subseteq \mathcal{S}$ *if and only if* $\mathcal{V} \subseteq \mathcal{I}_{n+1}$.

(ii) $\mathcal{V}_{\langle r,n\rangle} \subseteq \mathcal{S}$ *implies* $\mathcal{V} \subseteq \overline{\mathcal{I}_{n+1}}$.

PROOF. (i) By Lemmas 4.4 and 4.5, we obtain

$$\begin{aligned}\mathcal{V}_{\langle l,n\rangle} \subseteq \mathcal{S} &\iff \mathcal{V}_{\langle l,n\rangle} = \mathcal{S} \text{ or } \mathcal{V}_{\langle l,n\rangle} = \mathcal{T} \\ &\iff \mathcal{S} \subseteq \mathcal{V} \subseteq \mathcal{I}_{n+1}, \text{ or } \mathcal{V} \subseteq \mathcal{LG} \text{ and } n = 1, \text{ or } \mathcal{V} \subseteq \mathcal{CS} \text{ and } n > 1 \\ &\iff \mathcal{V} \subseteq \mathcal{I}_{n+1},\end{aligned}$$

since $\mathcal{LG} \subseteq \mathcal{I}_2$ and $\mathcal{CS} \subseteq \mathcal{I}_{n+1}$ for $n > 1$.

(ii) This is the dual of part (i). □

LEMMA 4.7. *Let $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ and $\tau \in \Theta$. Then $\mathcal{V}_{\tau K^*} = T^*$ if and only if $\mathcal{V} \subseteq \mathcal{S}^{\overline{\tau}}$.*

PROOF. By Lemma 4.2 (i)–(ii) and Lemma 4.6, we get

$$\begin{aligned}\mathcal{V}_{\langle l,n\rangle K^*} = T^* &\iff \mathcal{V}_{\langle l,n\rangle} \subseteq \mathcal{S} \iff \mathcal{V} \subseteq \mathcal{I}_{n+1} \\ &\iff \begin{cases}\mathcal{V} \subseteq \mathcal{S}^{\langle l,n\rangle} & \text{if } n \text{ is odd} \\ \mathcal{V} \subseteq \mathcal{S}^{\langle r,n\rangle} & \text{if } n \text{ is even}\end{cases} \\ &\iff \mathcal{V} \subseteq \mathcal{S}^{\overline{\langle l,n\rangle}}\end{aligned}$$

and dually for $\mathcal{V}_{\langle r,n\rangle K^*}$. □

5. The variety $\mathcal{V}^B$

The mapping $\mathcal{V} \to \mathcal{V} \cap \mathcal{B}$ is a complete retraction of $\mathcal{L}(\mathcal{CR})$ onto $\mathcal{L}(\mathcal{B})$. The classes of the congruence induced on $\mathcal{L}(\mathcal{CR})$ by it are intervals of the form $[\mathcal{V} \cap \mathcal{B}, (\mathcal{V} \cap \mathcal{B})^B]$. The main result of this section is the determination of the variety $(\mathcal{V} \cap \mathcal{B})^B$.

We start with two lemmas and a figure depicting the ladder of a canonical variety. By Lemma 3.2 (i), we have $\mathcal{T}^K = \mathcal{B}$ and by Lemma 3.3 (iii) and its dual, $\mathcal{B}_{T_l} = \mathcal{B}_{T_r} = \mathcal{B}$. It follows that all the vertices of the ladder of $\mathcal{B}$ are labelled by $\mathcal{T}$. Let $\mathcal{V} \in [\mathcal{S}, \mathcal{CR}]$. If a vertex of the ladder of $\mathcal{V}$ is labelled by an element of $\mathcal{K}_0$, then the corresponding vertex of the ladder of $\mathcal{V} \cap \mathcal{B}$ is labelled by $\mathcal{T}$. The remaining vertices of the ladders of $\mathcal{V}$ and $\mathcal{V} \cap \mathcal{B}$ are the same (and are labelled by elements of $\mathbb{N}_3^*$).

LEMMA 5.1. *Let $n \geq 2$.*

(i) $(\mathcal{I}_n)_{\langle l,n-1\rangle K^*} = (\overline{\mathcal{I}_n})_{\langle r,n-1\rangle K^*} = T^*$.

(ii) $(\mathcal{I}_n)_{\langle r,n\rangle K^*} = (\overline{\mathcal{I}_n})_{\langle l,n\rangle K^*} = T^*$.

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PROOF. (i) The proof is by induction on n. For $n = 2$, we have by Lemma 3.3 (i) that $(\mathcal{I}_2)_{T_l K^\bullet} = \mathcal{S}_{K^\bullet} = T^*$ and dually for the second statement. Assume the statements are true for n. By Proposition 4.1 (i), we get $(\mathcal{I}_{n+1})_{\langle l,n\rangle K^\bullet} = (\overline{\mathcal{I}_n})_{T_l\langle r,n-1\rangle K^\bullet} = T^*$ by the induction hypothesis. The statement $(\overline{\mathcal{I}_{n+1}})_{\langle r,n\rangle K^\bullet} = T^*$ follows dually.

(ii) For $n = 2$, we have by Proposition 4.1 (ii) and Lemma 3.3 (i),

$$(\mathcal{I}_2)_{T_r T_l K^\bullet} = (\mathcal{I}_2)_{T_l K^\bullet} = \mathcal{S}_{K^\bullet} = T^*$$

and for $n > 2$, by part (i) and Proposition 4.1 (ii), we have that

$$(\mathcal{I}_n)_{\langle r,n\rangle K^\bullet} = ((\mathcal{I}_n)_{T_l})_{\langle l,n-1\rangle K^\bullet} = (\mathcal{I}_n)_{\langle l,n-1\rangle K^\bullet} = T^*$$

and dually for the second statement. □

The following lemma is crucial for the main result of the paper.

LEMMA 5.2. *Let $\mathcal{U}$ be a canonical variety and $\mathcal{V} \in [\mathcal{S}, \mathcal{CR}]$ be such that $\mathcal{V} \cap \mathcal{B} \subseteq \mathcal{U}$. Then $\mathcal{V} \subseteq \mathcal{U}$.*

PROOF. For the four choices for $\mathcal{U}$, namely $\mathcal{I}_n$, $\overline{\mathcal{I}_n}$, $\mathcal{H}_n$ and $\overline{\mathcal{H}_n}$, we consider only $\mathcal{I}_n$ when n is even; the other cases are treated by the same method, but with different details. We let $\varphi = \chi_{\mathcal{I}_n}$ with n even and $\psi = \chi_{\mathcal{V}}$. The hypothesis implies that for any $\tau \in \Theta$, $\tau\varphi \in \mathbb{N}_3^*$ implies that $\tau\psi = \tau\varphi$. In view of Theorem 3.1, it suffices to prove that $\tau\psi \subseteq \tau\varphi$ whenever $\tau\varphi \in \mathcal{K}_0$.

The proof utilizes condition (P5) to obtain bounds on the values of ψ, which turns out to be just the corresponding values of φ. Recall that (P5) is of the form

(i) $\tau \in \Theta$ implies $(\emptyset\psi)_{\tau K^\bullet} \subseteq \tau\psi$,

(ii) $\sigma\tau \in \Theta, \sigma\psi \in \mathcal{K}_0, t(\sigma) \neq h(\tau)$ implies $(\sigma\psi)_{\tau K^\bullet} \subseteq (\sigma\tau)\psi$.

According to the above, we need only consider the case $\sigma\psi \in \mathcal{K}_0$. By Lemma 5.1, we have $\langle l, n-1\rangle\varphi = T^*$ and thus $\langle l, n-1\rangle\psi = T^*$, which by condition (i) yields

$$(\emptyset\psi)_{\langle l,n-1\rangle K^\bullet} \subseteq \langle l, n-1\rangle\psi = T^*.$$

We shall use condition (ii) in the cases: $\sigma\tau = \langle l, n-1\rangle$ and $\sigma\tau = \langle r, n\rangle$. According to Lemma 5.1, we have $(\sigma\tau)\varphi = T^*$ and thus also $(\sigma\tau)\psi = T^*$ in both cases. Hence $(\sigma\psi)_{\tau K^\bullet} \subseteq (\sigma\tau)\psi = T^*$, which by Lemma 4.7 gives $\sigma\psi \subseteq \mathcal{S}^\tau$. Now considering all relevant factorizations of $\langle l, n-1\rangle$ and $\langle r, n\rangle$, we obtain the following results which we present in the form of a table.

i	σ	τ	ψ	$\mathcal{I}$	φ
$2, 4, \ldots, n-2$	$\langle l, i\rangle$	$\langle l, n-1-i\rangle$	$\langle l, i\rangle\psi \subseteq \mathcal{S}^{\langle l,n-1-i\rangle}$	$\mathcal{I}_{n-i}$	$\langle l, i\rangle\varphi$
$1, 3, \ldots, n-3$	$\langle l, i\rangle$	$\langle r, n-1-i\rangle$	$\langle l, i\rangle\psi \subseteq \mathcal{S}^{\langle r,n-1-i\rangle}$	$\overline{\mathcal{I}_{n-i}}$	$\langle l, i\rangle\varphi$
$2, 4, \ldots, n-2$	$\langle r, i\rangle$	$\langle r, n-i\rangle$	$\langle r, i\rangle\psi \subseteq \mathcal{S}^{\langle l,n-i\rangle}$	$\mathcal{I}_{n+1-i}$	$\langle r, i\rangle\varphi$
$1, 3, \ldots, n-1$	$\langle r, i\rangle$	$\langle l, n-i\rangle$	$\langle r, i\rangle\psi \subseteq \mathcal{S}^{\langle r,n-i\rangle}$	$\overline{\mathcal{I}_{n+1-i}}$	$\langle r, i\rangle\varphi$

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The first column of the table gives the range for i, the second the choice of σ, the third the choice for τ, the fourth uses condition (ii) above and Lemma 4.7, the fifth equals the bound in the fourth column by Lemma 4.2 (i)–(ii), the sixth the corresponding values of φ. That the fifth and the sixth columns are equal follows by an easy inductive process, see Figure 2.

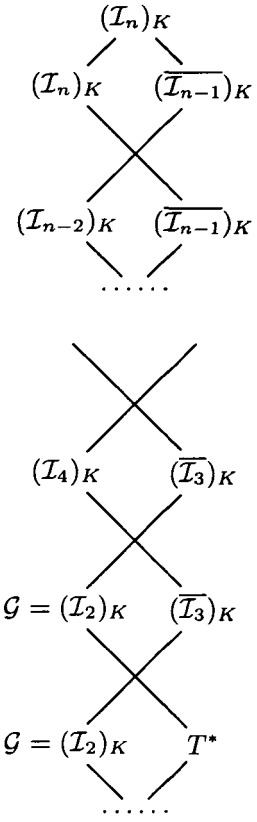

FIGURE 2. The ladder of $\mathcal{I}_n$ for n even

We have thus proved that $\sigma\psi \subseteq \sigma\varphi$ for all $\sigma \in \Theta^1$ for which $\sigma\psi \in \mathcal{K}_0$; for the remaining σ we have $\sigma\psi = \sigma\varphi$. Therefore $\psi \subseteq \varphi$ which by Theorem 3.1 implies that $\mathcal{V} \subseteq \mathcal{U}$. □

We first define a relation B by

$$\mathcal{U}\ B\ \mathcal{V} \Longleftrightarrow \mathcal{U} \cap \mathcal{B} = \mathcal{V} \cap \mathcal{B}\ (\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})).$$

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We also define the correspondence $\mathcal{V} \to \mathcal{V}^B = (\mathcal{V} \cap \mathcal{B})^B$ ($\mathcal{V} \in \mathcal{L}(\mathcal{CR})$) as follows:

$$(\mathcal{V} \cap \mathcal{B})^B = \mathcal{V} \quad \text{if} \quad \mathcal{V} \in \{\mathcal{G}, \mathcal{LG}, \mathcal{RG}, \mathcal{CS}, \mathcal{P}, \mathcal{X} \cap \mathcal{Y}, \mathcal{CR}\},$$

where $\mathcal{P}$ is canonical, $\mathcal{P} \neq \mathcal{H}_2, \overline{\mathcal{H}_2}$, and

$$(\mathcal{X}, \mathcal{Y}) \in \left\{(\mathcal{I}_n, \overline{\mathcal{H}_{n+1}}), (\mathcal{I}_n, \overline{\mathcal{I}_n}), (\mathcal{H}_{n+1}, \overline{\mathcal{I}_n}) \;\middle|\; n \geq 2\right\}$$
$$\cup \left\{(\mathcal{H}_n, \overline{\mathcal{I}_n}), (\mathcal{H}_n, \overline{\mathcal{H}_n}), (\mathcal{I}_n, \overline{\mathcal{H}_n}) \;\middle|\; n > 2\right\}.$$

According to Figure 1, $\mathcal{V}^B$ is defined for all $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ and is thus a function on $\mathcal{L}(\mathcal{CR})$ into itself.

The part of the ladder L labelled by elements of $\mathbb{N}_3^*$ including its position in L is the *socle* of L to be denoted by $\operatorname{soc}(L)$. We are now ready for the principal result of the paper.

THEOREM 5.3. *The mapping $\eta : \mathcal{V} \to \mathcal{V} \cap \mathcal{B}$ ($\mathcal{V} \in \mathcal{L}(\mathcal{CR})$) is a complete retraction of $\mathcal{L}(\mathcal{CR})$ onto $\mathcal{L}(\mathcal{B})$, which induces B. Moreover, for any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, we have*

(i) *$\mathcal{U}\ B\ \mathcal{V}$ if and only if $\operatorname{soc}(\chi_\mathcal{U}) = \operatorname{soc}(\chi_\mathcal{V})$,*
(ii) *$\mathcal{V}B = [\mathcal{V} \cap \mathcal{B}, (\mathcal{V} \cap \mathcal{B})^B]$.*

PROOF. By [8], (and of origin [12, 14]), $\mathcal{B}$ is a neutral element of $\mathcal{L}(\mathcal{CR})$. This implies that the mapping η is a homomorphism of $\mathcal{L}(\mathcal{CR})$ into $\mathcal{L}(\mathcal{B})$. Evidently, η leaves the members of $\mathcal{L}(\mathcal{B})$ fixed. We have seen in Figure 1 that the lattice $\mathcal{L}(\mathcal{B})$ is of finite width. Hence the join of an arbitrary family $\mathcal{F}$ of band varieties equals the join of a finite subfamily of $\mathcal{F}$. It follows that η is a complete $\vee$-homomorphism and is trivially a complete $\cap$-homomorphism. Clearly η induces the relation B.

(i) From the discussion at the beginning of this section, we immediately see that any two varieties have the same intersection with $\mathcal{B}$ if and only if their ladders have the same socle.

(ii) Given $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, it is obvious that $\mathcal{V} \cap \mathcal{B}$ is the least variety having the same intersection with $\mathcal{B}$ as $\mathcal{V}$. For the upper end of the interval $\mathcal{V}B$, we proceed as follows.

We show first that for all $\mathcal{W} \in \mathcal{L}(\mathcal{B})$, we have $\mathcal{W}^B \cap \mathcal{B} = \mathcal{W}$. This obviously holds for $\mathcal{W} \in \mathcal{L}(\mathcal{RB})$. For a canonical variety $\mathcal{P}$ and $\mathcal{W} = \mathcal{P} \cap \mathcal{B}$, we get

$$\mathcal{W}^B \cap \mathcal{B} = (\mathcal{P} \cap \mathcal{B})^B \cap \mathcal{B} = \mathcal{P} \cap \mathcal{B} = \mathcal{W},$$

and for the remaining varieties $\mathcal{W} = \mathcal{X} \cap \mathcal{Y} \cap \mathcal{B}$,

$$\mathcal{W}^B \cap \mathcal{B} = (\mathcal{X} \cap \mathcal{Y} \cap \mathcal{B})^B \cap \mathcal{B} = \mathcal{X} \cap \mathcal{Y} \cap \mathcal{B} = \mathcal{W}.$$

We still have to show the required maximality.

Let $\mathcal{W} \in \mathcal{L}(\mathcal{B})$ and $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ be such that $\mathcal{V} \cap \mathcal{B} = \mathcal{W}$. We show that $\mathcal{V} \subseteq \mathcal{W}^B$ by starting from the bottom of Figure 1. The required statement holds if $\mathcal{W} \in \mathcal{L}(\mathcal{RB})$.

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Lemma 5.2 takes care of the case $\mathcal{W} = \mathcal{P} \cap \mathcal{B}$ where $\mathcal{P}$ is a canonical variety. For the remaining varieties, let $\mathcal{X}$ and $\mathcal{Y}$ be canonical varieties and consider the band variety $\mathcal{W} = \mathcal{X} \cap \mathcal{Y} \cap \mathcal{B}$. We have seen above that $\mathcal{B}$ is a neutral element of $\mathcal{L}(\mathcal{CR})$. Hence

$$[\mathcal{V} \vee (\mathcal{X} \cap \mathcal{B})] \cap \mathcal{B} = (\mathcal{V} \cap \mathcal{B}) \vee (\mathcal{X} \cap \mathcal{B}) = \mathcal{X} \cap \mathcal{B}.$$

Since $\mathcal{X}$ is a canonical variety, by the above we obtain that $\mathcal{V} \vee (\mathcal{X} \cap \mathcal{B}) \subseteq \mathcal{X}$, and thus $\mathcal{V} \subseteq \mathcal{X}$. Analogously, we get $\mathcal{V} \subseteq \mathcal{Y}$ whence $\mathcal{V} \subseteq \mathcal{X} \cap \mathcal{Y}$, as required.

Therefore $\mathcal{W}^B$ has the requisite properties, which implies that for any $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, we have $\mathcal{V}B = [\mathcal{V} \cap \mathcal{B}, (\mathcal{V} \cap \mathcal{B})^B]$, as asserted. □

If, for $\mathcal{V} \in \mathcal{L}(\mathcal{B})$, we let $\mathcal{V}_B = \mathcal{V}$, we obtain the familiar form $\mathcal{V}B = [\mathcal{V}_B, \mathcal{V}^B]$. We now relate the ladders with the variety $\mathcal{V}^B$.

PROPOSITION 5.4. *Let $\mathcal{V} \in [\mathcal{S}, \mathcal{B}]$ and $\mathcal{U}$ be the intersection of all canonical varieties containing $\mathcal{V}$. Then $\mathcal{V}^B = \mathcal{U} = \bigcap \{(\tau\chi_{\mathcal{V}})^{K^*\overline{\tau}} \mid \tau\chi_{\mathcal{V}} \in \mathbb{N}_3^*\}$.*

PROOF. Let $\{\mathcal{U}_\alpha \mid \alpha \in A\}$ be the set of all canonical varieties containing $\mathcal{V}$. According to Figure 1, $\mathcal{V} = \bigcap_{\alpha \in A} (\mathcal{U}_\alpha \cap \mathcal{B})$ and thus, by the dual of Pastijn and Petrich [3, Lemma 4.10], we obtain

$$\mathcal{V}^B = \left(\bigcap_{\alpha \in A} (\mathcal{U}_\alpha \cap \mathcal{B})\right)^B = \bigcap_{\alpha \in A} (\mathcal{U}_\alpha \cap \mathcal{B})^B = \bigcap_{\alpha \in A} \mathcal{U}_\alpha = \mathcal{U}.$$

This proves the first equality in the statement of the proposition. For the second, we proceed as follows.

If $\tau\chi_{\mathcal{V}} \in \mathbb{N}_3^*$, then $(\tau\chi_{\mathcal{V}})^{K^*} \in \mathbb{N}_3$ and thus, by Lemma 4.2, we have that $(\tau\chi_{\mathcal{V}})^{K^*\overline{\tau}}$ is canonical. Hence

$$\mathcal{U} \subseteq \bigcap \{(\tau\chi_{\mathcal{V}})^{K^*\overline{\tau}} \mid \tau\chi_{\mathcal{V}} \in \mathbb{N}_3^*\}. \tag{5.1}$$

For the opposite inclusion, let $\mathcal{W}$ be a canonical variety such that $\mathcal{V} \subseteq \mathcal{W}$. By the cited reference, $\mathcal{W} = \mathcal{X}^\tau$ for some $\mathcal{X} \in \mathbb{N}_3$ and $\tau \in \Theta$. Lemma 3.3 (iv) implies that $\mathcal{V}_{\overline{\tau}} \subseteq (\mathcal{X}^\tau)_{\overline{\tau}}$ which, by [8], (of origin [10]) yields that $\mathcal{V}_{\overline{\tau}}$ is contained either in $\mathcal{X}_{T_l}$ or in $\mathcal{X}_{T_r}$. In either case, we get $\mathcal{V}_{\overline{\tau}} \in \mathbb{N}_3$ whence $\mathcal{V}_{\overline{\tau}K^*} \in \mathbb{N}_3^*$. Further, using the cited references, we get $(\overline{\tau}\chi_{\mathcal{V}})^{K^*\tau} = (\mathcal{V}_{\overline{\tau}K^*})^{K^*\tau} \subseteq ((\mathcal{X}^\tau)_{\overline{\tau}K^*})^{K^*\tau} \subseteq \mathcal{X}^\tau = \mathcal{W}$. The opposite inclusion in (5.1) follows. This proves the second equality in the statement of the proposition. □

If we include the varieties $\mathcal{G}$, $\mathcal{LG}$, $\mathcal{RG}$, $\mathcal{CS}$ and $\mathcal{CR}$ in the set of canonical varieties, then by Proposition 5.4, for any $\mathcal{V} \in \mathcal{L}(\mathcal{B})$, $\mathcal{V}^B$ is the intersection of all canonical varieties containing $\mathcal{V}$.

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COROLLARY 5.5. *The mapping* $\mathcal{V} \to \mathcal{V}^B$ $(\mathcal{V} \in \mathcal{L}(\mathcal{CR}))$ *is a complete* $\cap$*-endomorphism but is not a* $\vee$*-homomorphism of* $\mathcal{L}(\mathcal{CR})$.

PROOF. The first assertion follows easily from the first equality in Proposition 5.4 and the above remark. For the second,

$$\mathcal{LZ}^B \vee \mathcal{RZ}^B = \mathcal{LG} \vee \mathcal{RG} = \mathcal{ReG}, \quad (\mathcal{LZ} \vee \mathcal{RZ})^B = \mathcal{RB}^B = \mathcal{CS}. \qquad \square$$

6. Intersections with $\mathcal{BG}$, $\mathcal{OBG}$ and $\mathcal{B}$

For any canonical variety $\mathcal{P}_n$ we provide here a single identity that serves as a basis of identities for $\mathcal{P}_n \cap \mathcal{BG}$, $\mathcal{P}_n \cap \mathcal{OBG}$ and $\mathcal{P}_n \cap \mathcal{B}$, respectively. These identities are variants of $G_n = P_n$ with suitable interspersing of x_i^0 instead of x_i and G_n^0 and P_n^0, instead of G_n and P_n, respectively. We also characterize the first two intersections.

We consider only $\mathcal{P}_n$; the statements for $\overline{\mathcal{P}_n}$ are dual. There is nothing to say about $\mathcal{H}_2$, while $\mathcal{I}_2$ requires a somewhat different treatment. Hence we start with $\mathcal{I}_2$ and then treat the remaining cases in a unified way.

PROPOSITION 6.1.

(i) $\mathcal{I}_2 \cap \mathcal{BG} = \mathcal{I}_2 \cap \mathcal{OBG} = [(x_2x_1)^0 = x_2^0x_1^0x_2^0] = \mathcal{LRBG} = \mathcal{LRB} \vee \mathcal{G}$.

(ii) $\mathcal{I}_2 \cap \mathcal{B} = [x_2x_1 = x_2x_1x_2]$.

PROOF. (i) The first equality follows from the well-known fact that $\mathcal{I}_2 \subseteq \mathcal{O}$. The identities for $\mathcal{I}_2$ and $\mathcal{OBG}$ yield $(x_2x_1)^0 = (x_2x_1x_2^0)^0 = x_2^0x_1^0x_2^0$, proving that the second variety is contained in the third. The third, fourth and fifth equalities form part of Petrich and Reilly [7, Lemma V.5.2]. The last variety is obviously contained in the first.

(ii) This is well-known. $\square$

For the remaining cases, it is convenient to use the following notation:

(i) for a word w, possibly involving products, inversion and parentheses, we write $\hat{w}$ for the word obtained from w by deleting all powers of words, positive, negative or zero, as well as all parentheses;

(ii) $G_0 = P_0 = \emptyset$, the empty word, for $P \in \{H, I\}$;

(iii) $G_n(x_i^0)$ stands for the word G_n, in which every variable x_i is replaced by x_i^0.

THEOREM 6.2. *For* $\mathcal{P} \in \{\mathcal{H}, \mathcal{I}\}$ *and* $n > 2$, *the following equalities hold. Set* $G_1 = x_1$.

(i) $\mathcal{P}_n \cap \mathcal{BG} = \left[(x_n^0 G_{n-2} x_{n-1}^0)^0 = P_n^0\right] = \left[G_n^0 = (P_n(x_1, \ldots, x_{n-2}, x_{n-1}^0, x_n^0))^0\right]$
$= (\mathcal{P}_n \cap \mathcal{B})$*-band of groups.*

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(ii) $\mathcal{P}_n \cap \mathcal{OBG} = [G_n(x_i^0) = P_n^0] = (\mathcal{P}_n \cap \mathcal{B}) \vee \mathcal{G}$.
(iii) $\mathcal{P}_n \cap \mathcal{B} = [G_n = \hat{P}_n]$.

PROOF. (i) Denote these four varieties by $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{X}$, respectively.

$\mathcal{U} \subseteq \mathcal{V}$. First

$$G_n = x_n\overline{G_{n-1}} = x_n\overline{x_{n-1}\overline{G_{n-2}}} = x_nG_{n-2}x_{n-1}, \tag{6.1}$$

which together with the identity of $\mathcal{BG}$ yields

$$G_n^0 = (x_nG_{n-2}x_{n-1})^0 = (x_n^0G_{n-2}x_{n-1}^0)^0.$$

It remains to observe that $G_n = P_n$ implies $G_n^0 = P_n^0$.

$\mathcal{V} \subseteq \mathcal{W}$. We consider first the case $n = 3$. The substitution $x_1 \to x_2^0$ in the identity of $\mathcal{V}$ yields the identity of $\mathcal{BG}$. The two together immediately imply the identity of $\mathcal{W}$. Hence assume that $n > 3$. Then

$$\begin{aligned} P_n &= G_n(x_n\overline{P_{n-1}})^0 = G_n(x_n\overline{(G_{n-1}(x_{n-1}\overline{P_{n-2}})^0)^0} \\ &= G_n(x_n(P_{n-2}x_{n-1})^0\overline{G_{n-1}})^0 \\ &= x_nG_{n-2}x_{n-1}(x_n(P_{n-2}x_{n-1})^0G_{n-2}x_{n-1})^0. \end{aligned} \tag{6.2}$$

The substitution $x_i \to x_{n-1}$ for $i = 1, 2, \ldots, n-2$ in the identity given by (6.2) yields

$$(x_n^0x_{n-1}^0)^0 = (x_nx_{n-1}(x_nx_{n-1}^0x_{n-1})^0)^0 = (x_nx_{n-1})^0,$$

the identity of $\mathcal{BG}$. Now using (6.1) and (6.2), we obtain

$$\begin{aligned} G_n^0 &= (x_nG_{n-2}x_{n-1})^0 = (x_n^0G_{n-2}x_{n-1}^0)^0 = P_n^0 \\ &= (P_n(x_1, \ldots, x_{n-2}, x_{n-1}^0, x_n^0))^0. \end{aligned}$$

$\mathcal{W} \subseteq \mathcal{X}$. The same substitution as above in the identity given by (6.1) and (6.2) yields

$$(x_nx_{n-1})^0 = \left(x_n^0x_{n-1}^0(x_n^0x_{n-1}^0x_{n-1}^0)^0\right)^0 = (x_n^0x_{n-1}^0)^0, \tag{6.3}$$

the identity of $\mathcal{BG}$. Hence if $S \in \mathcal{W}$, then $S \in \mathcal{BG}$ and $S/\mathcal{H} \in \mathcal{P}_n \cap \mathcal{B}$ and thus $S \in \mathcal{X}$.

$\mathcal{X} \subseteq \mathcal{U}$. This case follows from Lemma 5.2.

(ii) Denote these three varieties by $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$, respectively. In the case $n = 3$, the identity for $\mathcal{OBG}$ implies that $\mathcal{U} = \mathcal{V} = \mathcal{W}$ using similar arguments as those above. So assume that $n > 3$.

https://doi.org/10.1017/S1446788700036405 Published online by Cambridge University Press

$\mathcal{U} \subseteq \mathcal{V}$. This is a direct consequence of $\mathcal{OBG} = [a^0x^0 = (ax)^0]$.

$\mathcal{V} \subseteq \mathcal{W}$. The same substitution as above now yields $x_n^0 x_{n-1}^0 = (x_n x_{n-1})^0$, see (6.3), giving the identity of $\mathcal{OBG}$.

By Petrich [5], we obtain $\mathcal{V} = (\mathcal{V} \cap \mathcal{B}) \vee (\mathcal{V} \cap \mathcal{G}) = (\mathcal{P}_n \cap \mathcal{B}) \vee \mathcal{G} = \mathcal{W}$.

$\mathcal{W} \subseteq \mathcal{U}$. This is obvious.

(iii) Denote these two varieties by $\mathcal{U}$ and $\mathcal{V}$, respectively.

$\mathcal{U} \subseteq \mathcal{V}$ This is trivial.

$\mathcal{V} \subseteq \mathcal{U}$. The substitution $x_i \to x_n^0$ for $i = 1, 2, \ldots, n-1$ by (6.1) and (6.2) yields $x_n = x_n^2$. The identity now gives $G_n = P_n$ also. □

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21420 Bol, Brač
Croatia