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RESEARCH ARTICLE

Stochastic comparison of parallel systems with Pareto components

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Abstract
Pareto distribution is an important distribution in extreme value theory. In this paper, we consider parallel systems with Pareto components and study the effect of heterogeneity on skewness of such systems. It is shown that, when the lifetimes of components have different shape parameters, the parallel system with heterogeneous Pareto component lifetimes is more skewed than the system with independent and identically distributed Pareto components. However, for the case when the lifetimes of components have different scale parameters, the result gets reversed in the sense of star ordering. We also establish the relation between star ordering and dispersive ordering by extending the result of Deshpande and Kochar [(1983). Dispersive ordering is the same as tail ordering. *Advances in Applied Probability* 15(3): 686–687] from support $(0, \infty)$ to general supports (a, ∞), $a > 0$. As a consequence, we obtain some new results on dispersion of order statistics from heterogeneous Pareto samples with respect to dispersive ordering.

1. Introduction

Skewness is a measure of the degree of asymmetry in the distribution. Although symmetric distributions can be easily distinguished from the nonsymmetric ones, it is usually difficult to adjudge whether one nonsymmetric distribution is more skewed as compared with another. One approach to deal with this is to make use of various partial orders that have been introduced to compare the relative skewness of probability distributions. In this regard, van Zwet [21] introduced a convex transform order, and Oja [19] proposed a weaker ordering, known as a star order (see Section 2 for definitions). These orderings play an important role in reliability theory as they reflect whether one distribution ages faster than the other in some stochastic sense.

In the literature [6,8,10,11,12,13], the stochastic comparison of series and parallel systems has been effectively done through order statistics, which refers to an ordered arrangement of random variables (r.v.s) $X_1, X_2, \ldots, X_n$. It is denoted by $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$, where $X_{1:n}$ and $X_{n:n}$ are the smallest and the largest order statistics representing the lifetime of series and parallel systems, respectively. While studying the effect of heterogeneity on the skewness of order statistics, Kochar and Xu [14] established that the largest order statistics from heterogeneous exponential samples are more skewed than the one from homogeneous exponential samples in the sense of the convex transform order, and consequently, the Lorenz order. Da *et al.* [3] pointed out that an analogous result for general order statistics (i.e., kth order statistic) also holds in the sense of Lorenz ordering.

Recently, Fang and Zhang [7] extended these works by considering r.v.s with Weibull distributions as it includes exponential distributions. They proved that the smallest order statistics from heterogeneous

Weibull samples are more skewed than the one from homogeneous Weibull samples in the sense of the convex transform order without any restriction on the parameters. Kochar and Xu [15] further considered a general distribution framework and studied the skewness of order statistics from two samples. Specifically, they established star ordering for the proportional hazard rate model under some restrictions on parameters and presented results on Weibull and Pareto distributed samples. More recently, Ding *et al.* [5] considered a scale model framework and examined the effect of heterogeneity on the skewness of the largest order statistics from such samples in the sense of star ordering. They proved that, without any restriction on the scale parameters, the skewness of the largest order statistics from heterogeneous samples is more than that from homogeneous samples.

Thus, it can be observed from the above discussion that most of the works have compared order statistics from heterogeneous exponential and Weibull samples. However, very little work has been done, in this regard, on Pareto distribution which is an important distribution in actuarial science and extreme value theory. This motivated us to study the effect of heterogeneity on the skewness of order statistics from Pareto samples. An r.v. X is said to follow the Pareto distribution with shape parameter λ and scale parameter b (denoted by $X \sim \text{Pa}(\lambda, b)$) if its survival function is given by

$$\bar{F}(x) = \left(\frac{b}{x}\right)^{\lambda}, \quad \lambda > 0,\ x \geq b > 0. \tag{1}$$

In this paper, we compare the largest order statistics according to the convex transform order and the star order. More precisely, we consider two different models: In the first model, we consider $X_1, \ldots, X_n$ as independent Pareto r.v.s with different shape parameters λ_i, $i = 1, \ldots, n$, and the same scale parameter b, and $Y_1, \ldots, Y_n$ as a random sample from a Pareto distribution with shape parameter λ and scale parameter b. We prove that

$$\lambda \geq \bar{\lambda} \;\Rightarrow\; Y_{n:n} \leq_{\mathrm{c}} X_{n:n}, \tag{2}$$

where $\bar{\lambda} = (1/n) \sum_{i=1}^{n} \lambda_i$. In the second model, we consider $X_1, \ldots, X_n$ as independent Pareto r.v.s with the same shape parameter λ, and different scale parameters b_i, $i = 1, \ldots, n$, and consider $Y_1, \ldots, Y_n$ as a random sample from a Pareto distribution with shape parameter λ and scale parameter b. We prove that

$$X_{n:n} \leq_{*} Y_{n:n} \tag{3}$$

which is actually a correction of the result in [15] which establishes that $Y_{n:n} \leq_{*} X_{n:n}$. The relationship in (3) is actually interesting as it says that the largest order statistic from the homogeneous Pareto sample is more skewed than the one from the heterogeneous Pareto sample in the sense of star ordering. We further extend the second model to include different shape parameters λ_1 and λ_2, and show that

$$\lambda_1 \geq \lambda_2 \;\Rightarrow\; X_{n:n} \leq_{*} Y_{n:n}. \tag{4}$$

Furthermore, it is well-known that there exists a connection between star ordering and dispersive ordering, as claimed by Deshpande and Kochar [4]. However, their result focuses on distributions with support $(0, \infty)$ and remains silent on distributions with general support (a, ∞), where $a > 0$. Therefore, we fill this gap in the literature by establishing sufficient conditions under which star ordering implies dispersive ordering for distributions with support (a, ∞), $a > 0$. As a consequence, we obtain results that compare the largest order statistic from the heterogeneous Pareto sample with that from the homogeneous Pareto sample in the sense of dispersive ordering.

The rest of the paper is organized as follows. In Section 2, we mention some relevant definitions on the skewness and dispersive orders. Section 3 contains results on the two models stated above with specific focus on the convex transform ordering and the star ordering. We then establish connection between the star ordering and the dispersive ordering in Section 4 and also present some new results on dispersive ordering among parallel systems from Pareto distributions. Finally, we present some applications in Section 5 and provide a further discussion on this topic in Section 6.

2. Preliminaries

In this section, we recall some notions of stochastic orders that will be used in the paper. Throughout this paper, the terms increasing and decreasing imply nondecreasing and nonincreasing, respectively. Let X and Y be two non-negative r.v.s having distribution functions F and G, survival functions $\overline{F}$ and $\overline{G}$, density functions f and g, and failure rates r_X and r_Y, respectively. Also, G^{-1} is the right continuous inverse of G.

Definition 2.1. *(i) X is said to be smaller than Y in the convex transform order (denoted by $X \leq_c Y$) if, and only if, $G^{-1}F(x)$ is convex in x on the support of X.*

(ii) X is said to be smaller than Y in the star order (denoted by $X \leq_* Y$) if $G^{-1}F(x)$ is starshaped in x, that is, $G^{-1}F(x)/x$ is increasing in x on the support of X.

(iii) X is said to be smaller than Y in the Lorenz order (denoted by $X \leq_{Lorenz} Y$) if

$$\frac{1}{E(Y)}\int_0^{G^{-1}(u)} x\,dG(x) \leq \frac{1}{E(X)}\int_0^{F^{-1}(u)} x\,dF(x), \quad \forall u \in (0,1].$$

The relation $X \leq_c Y$ means that X is less skewed than Y [16]. Moreover, the convex transform order is also called the more IFR order in reliability theory, and thus, $X \leq_c Y$ implies that X ages faster than Y in some sense. In reliability theory, the star order is also called the more $IFRA$ (increasing failure rate in average) order, and thus, $X \leq_* Y$ can be interpreted in terms of average failure rates, that is,

$$\frac{\overline{r}_X(F^{-1}(u))}{\overline{r}_Y(G^{-1}(u))}$$

is increasing in $u \in (0,1]$, where $\overline{r}_X(x) = -\ln \overline{F}(x)/x$ and $\overline{r}_Y(x) = -\ln \overline{G}(x)/x$ are the average failure rates of X and Y, respectively.

The above partial orders are scale invariant. From [16], it is known that

$$X \leq_c Y \;\Rightarrow\; X \leq_* Y \;\Rightarrow\; X \leq_{\text{Lorenz}} Y \Rightarrow\; cv(X) \leq cv(Y),$$

where $cv(X) = \sqrt{\text{Var}(X)}/E(X)$ is the coefficient of variation of X. Another important concept is of dispersive ordering which is useful for comparing spread among probability distributions.

Definition 2.2. *X is said to be smaller than Y in the dispersive order (denoted by $X \leq_{disp} Y$) if*

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha),$$

where $0 < \alpha \leq \beta < 1$, and F^{-1} and G^{-1} are the right continuous inverses of F and G, respectively.

This definition implies that the difference between any two quantiles of F is smaller than the difference between the corresponding quantiles of G. It can be easily seen that $X \leq_{\text{disp}} Y$ if, and only if,

$$g(G^{-1}F(x)) \leq f(x), \quad \forall x > 0.$$

For an extensive and comprehensive discussion on the theory of the above partial orders, one may refer to [2,16,20]. The following lemmas will be useful in proving the main result.

Lemma 2.3 [2 p. 120]**.** *Let $W(x)$ be a Lebesgue–Stieltjes measure, not necessarily positive, for which $\int_{-\infty}^{t} dW(x) \geq 0$, for all t, and let $h(x) \geq 0$ be decreasing. Then,*

$$\int_{-\infty}^{\infty} h(x)\,dW(x) \geq 0.$$

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Lemma 2.4 [17 p. 36]**.** *If* $\boldsymbol{a} = (a_1, \ldots, a_n)$ *and* $\boldsymbol{b} = (b_1, \ldots, b_n)$ *are two real sequences such that* $a_1 \le \cdots \le a_n$ *and* $b_1 \le \cdots \le b_n$*, or* $a_1 \ge \cdots \ge a_n$ *and* $b_1 \ge \cdots \ge b_n$*, then the following inequality is true:*

$$\sum_{j=1}^{n} a_j \sum_{j=1}^{n} b_j \le n \sum_{j=1}^{n} a_j b_j.$$

3. Main results

In this section, we establish the relation between the largest order statistics from heterogeneous Pareto samples and the one from homogeneous Pareto samples based on skewness orders (convex transform order and star order). Here, we focus on two different models: the first model considers the Pareto distribution with different shape parameters, while the second one covers results on the Pareto distribution with different scale parameters. To do so, we divide this section into two subsections that deal separately with the above-stated two models.

3.1. Model 1

Kochar and Xu [15] proved the following result on comparing two parallel systems in the sense of star ordering when the underlying r.v.s follow the Pareto distribution and their shape parameters are different.

Theorem 3.1. *Let* $X_1, \ldots, X_n$ *be independent Pareto r.v.s with shape parameter* λ_i*,* $i = 1, \ldots, n$*, and the same scale parameter* $b > 0$*. Let* $Y_1, \ldots, Y_n$ *be a random sample from a Pareto distribution with shape parameter* $\lambda \ge \tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$ *and scale parameter* $b > 0$*. Then,*

$$Y_{n:n} \le_* X_{n:n}.$$

In the next theorem, we extend the above result to the case when the two parallel systems are compared in the sense of stronger skewness ordering, that is, convex transform ordering.

Theorem 3.2. *Let* $X_1, \ldots, X_n$ *be independent Pareto r.v.s with the different shape parameters* λ_i*,* $i = 1, \ldots, n$*, and the same scale parameter* $b > 0$*. Let* $Y_1, \ldots, Y_n$ *be a random sample from a Pareto distribution with shape parameter* $\lambda \ge \bar{\lambda} = (1/n) \sum_{i=1}^{n} \lambda_i$ *and scale parameter* $b > 0$*. Then,*

$$Y_{n:n} \le_{\rm c} X_{n:n}.$$

Proof. For $x \ge b$, the distribution function of $X_{n:n}$ is

$$F(x) = P(X_{n:n} \le x) = \prod_{i=1}^{n} \left[1 - \left(\frac{b}{x} \right)^{\lambda_i} \right],$$

with the density function as

$$f(x) = \frac{1}{x} \sum_{i=1}^{n} \frac{\lambda_i (b/x)^{\lambda_i}}{1 - (b/x)^{\lambda_i}} \prod_{i=1}^{n} \left[1 - \left(\frac{b}{x} \right)^{\lambda_i} \right].$$

Similarly, the distribution function of $Y_{n:n}$, for $x \ge b$, is

$$G(x) = P(Y_{n:n} \le x) = \left[1 - \left(\frac{b}{x} \right)^{\lambda} \right]^n,$$

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with the density function as

$$g(x) = n\frac{\lambda}{x}\left(\frac{b}{x}\right)^{\lambda}\left[1-\left(\frac{b}{x}\right)^{\lambda}\right]^{n-1}.$$

To prove $Y_{n:n} \leq_{\rm c} X_{n:n}$, it suffices to show that $G^{-1}(F(x))$ is concave on x. Differentiating

$$G^{-1}F(x) = b(1-F^{1/n}(x))^{-1/\lambda} = b\left(1-\prod_{i=1}^{n}\left[1-\left(\frac{b}{x}\right)^{\lambda_i}\right]^{1/n}\right)^{-1/\lambda}$$

with respect to x, we obtain

$$\frac{f(x)}{g(G^{-1}F(x))} = \frac{\frac{1}{x}\sum_{i=1}^{n}\frac{\lambda_i(b/x)^{\lambda_i}}{1-(b/x)^{\lambda_i}}}{\frac{n\lambda}{b}(1-\prod_{i=1}^{n}[1-(\frac{b}{x})^{\lambda_i}]^{1/n})^{1/\lambda}(\prod_{i=1}^{n}[1-(\frac{b}{x})^{\lambda_i}]^{-1/n}-1)}. \tag{5}$$

To show that (5) is further decreasing in x, it is equivalent to proving that

$$\phi_1'(x)\phi_2(x) \leq \phi_1(x)\phi_2'(x), \tag{6}$$

where

$$\phi_1(x) = \frac{1}{x}\sum_{i=1}^{n}\frac{\lambda_i(b/x)^{\lambda_i}}{1-(b/x)^{\lambda_i}}; \quad \phi_1'(x) = -\frac{1}{x^2}\sum_{i=1}^{n}\frac{\lambda_i(b/x)^{\lambda_i}}{1-(b/x)^{\lambda_i}} - \frac{1}{x^2}\sum_{i=1}^{n}\frac{\lambda_i^2(b/x)^{\lambda_i}}{(1-(b/x)^{\lambda_i})^2},$$

and

$$\phi_2(x) = \left(1-\prod_{i=1}^{n}\left[1-\left(\frac{b}{x}\right)^{\lambda_i}\right]^{1/n}\right)^{1/\lambda}\left(\prod_{i=1}^{n}\left[1-\left(\frac{b}{x}\right)^{\lambda_i}\right]^{-1/n}-1\right);$$

$$\phi_2'(x) = -\frac{1}{nx}\left(1-\prod_{i=1}^{n}\left[1-\left(\frac{b}{x}\right)^{\lambda_i}\right]^{1/n}\right)^{1/\lambda}\sum_{i=1}^{n}\frac{\lambda_i(b/x)^{\lambda_i}}{1-(b/x)^{\lambda_i}}\left[\frac{1}{\lambda}+\prod_{i=1}^{n}\left[1-\left(\frac{b}{x}\right)^{\lambda_i}\right]^{-1/n}\right].$$

Substituting these values in (6), we obtain

$$\begin{aligned}
&\left[\sum_{i=1}^{n}\frac{\lambda_i(b/x)^{\lambda_i}}{1-(b/x)^{\lambda_i}}+\sum_{i=1}^{n}\frac{\lambda_i^2(b/x)^{\lambda_i}}{(1-(b/x)^{\lambda_i})^2}\right]\left[1-\prod_{i=1}^{n}\left[1-\left(\frac{b}{x}\right)^{\lambda_i}\right]^{1/n}\right] \\
&\quad \geq \frac{1}{n\lambda}\left(\sum_{i=1}^{n}\frac{\lambda_i(b/x)^{\lambda_i}}{1-(b/x)^{\lambda_i}}\right)^2\left[\lambda+\prod_{i=1}^{n}\left[1-\left(\frac{b}{x}\right)^{\lambda_i}\right]^{1/n}\right] \\
\Longleftrightarrow &\left[\sum_{i=1}^{n}\frac{\lambda_i z^{\lambda_i}}{(1-z^{\lambda_i})^2}(1-z^{\lambda_i}+\lambda_i)\right]\left[1-\prod_{i=1}^{n}(1-z^{\lambda_i})^{1/n}\right] \\
&\quad \geq \frac{1}{n}\left(\sum_{i=1}^{n}\frac{\lambda_i z^{\lambda_i}}{1-z^{\lambda_i}}\right)^2\left[1+\frac{1}{\lambda}\prod_{i=1}^{n}(1-z^{\lambda_i})^{1/n}\right],
\end{aligned} \tag{7}$$

where $z = b/x$. From Lemma 2.4 and Cauchy–Schwarz inequality, it follows that,

$$\sum_{i=1}^{n} \frac{\lambda_i z^{\lambda_i}}{(1-z^{\lambda_i})^2}(1 - z^{\lambda_i} + \lambda_i) \geq \frac{1}{n}\sum_{i=1}^{n} \frac{\lambda_i z^{\lambda_i}}{(1-z^{\lambda_i})^2}\sum_{i=1}^{n}(1 - z^{\lambda_i} + \lambda_i)$$

and

$$\sum_{i=1}^{n} \frac{\lambda_i z^{\lambda_i}}{(1-z^{\lambda_i})^2}\sum_{i=1}^{n}\lambda_i z^{\lambda_i} \geq \left(\sum_{i=1}^{n}\frac{\lambda_i z^{\lambda_i}}{1-z^{\lambda_i}}\right)^2.$$

On applying these two inequalities on left- and right-hand side of (7), we obtain

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} \frac{\lambda_i z^{\lambda_i}}{(1-z^{\lambda_i})^2}\sum_{i=1}^{n}(1 - z^{\lambda_i} + \lambda_i)\left[1 - \prod_{i=1}^{n}(1-z^{\lambda_i})^{1/n}\right] \\
&\quad \geq \frac{1}{n}\sum_{i=1}^{n} \frac{\lambda_i z^{\lambda_i}}{(1-z^{\lambda_i})^2}\sum_{i=1}^{n}\lambda_i z^{\lambda_i}\left[1 + \frac{1}{\lambda}\prod_{i=1}^{n}(1-z^{\lambda_i})^{1/n}\right] \\
&\Longleftrightarrow \sum_{i=1}^{n}(1 - z^{\lambda_i} + \lambda_i)\left[1 - \prod_{i=1}^{n}(1-z^{\lambda_i})^{1/n}\right] \geq \sum_{i=1}^{n}\lambda_i z^{\lambda_i}\left[1 + \frac{1}{\lambda}\prod_{i=1}^{n}(1-z^{\lambda_i})^{1/n}\right] \\
&\Longleftrightarrow \sum_{i=1}^{n}(1 - z^{\lambda_i})(1 + \lambda_i) \geq \prod_{i=1}^{n}(1-z^{\lambda_i})^{1/n}\left[\sum_{i=1}^{n}\left(\frac{\lambda_i}{\lambda} - 1\right)z^{\lambda_i} + \sum_{i=1}^{n}(1+\lambda_i)\right] \\
&\Longleftrightarrow \sum_{i=1}^{n}(1 + \lambda_i) \geq \sum_{i=1}^{n}\left(\frac{\lambda_i}{\lambda} - 1\right)z^{\lambda_i} + \sum_{i=1}^{n}(1+\lambda_i),
\end{aligned} \tag{8}$$

where the last inequality follows from Lemma 2.4 and arithmetic–geometric mean inequality, that is,

$$\sum_{i=1}^{n}(1 - z^{\lambda_i})(1 + \lambda_i) \geq \frac{1}{n}\sum_{i=1}^{n}(1 - z^{\lambda_i})\sum_{i=1}^{n}(1+\lambda_i) \geq \prod_{i=1}^{n}(1-z^{\lambda_i})^{1/n}\sum_{i=1}^{n}(1+\lambda_i).$$

Also, note that (8) holds if

$$\sum_{i=1}^{n}\left(1 - \frac{\lambda_i}{\lambda}\right)z^{\lambda_i} \geq 0. \tag{9}$$

Here, for $z \in (0, 1)$, $h(x) = z^x$ is a decreasing function in x, and for $j \in \{1, \ldots, n\}$, $\sum_{i=1}^{j}(1-\lambda_i/\lambda) \geq 0$ since $\lambda \geq (1/n)\sum_{i=1}^{n}\lambda_i$. Thus, it follows from Lemma 2.3 that (9) holds true, and the desired result follows immediately. □

The following result, which establishes star ordering between the largest order statistics in a general scenario, is an extension of Theorem 3.1. The proof is trivial and is so omitted.

Theorem 3.3. *Let $X_1, \ldots, X_n$ be independent Pareto r.v.s with different shape parameters λ_i, $i = 1, \ldots, n$, and the same scale parameter b_1. Let $Y_1, \ldots, Y_n$ be a random sample from a Pareto distribution with shape parameter $\lambda \geq \tilde{\lambda}$ and scale parameter b_2. Then, $Y_{n:n} \leq_* X_{n:n}$.*

Proof. Let $Z_1, \ldots, Z_n$ be independent Pareto r.v.s with different shape parameters λ_i, $i = 1, \ldots, n$, and the same scale parameter b_2. Since $\lambda \geq \tilde{\lambda}$, we know from Theorem 3.1 that $Y_{n:n} \leq_* Z_{n:n}$. Now, to prove

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the assertion that $Y_{n:n} \leq_* X_{n:n}$, it suffices to show that $Z_{n:n} \leq_* X_{n:n}$. Assume that the r.v.s $X_{n:n}$ and $Z_{n:n}$ have distribution functions F and H, and right continuous inverses F^{-1} and H^{-1}, respectively. Note that

$$P\left(\frac{b_1}{b_2}Z_i > t\right) = P\left(Z_i > \frac{b_2}{b_1}t\right) = \left(\frac{b_1}{t}\right)^{\lambda_i} = P(X_i > t).$$

This implies that $(b_1/b_2)Z_i \stackrel{d}{=} X_i$, where $\stackrel{d}{=}$ denotes equality in distribution. It is easy to verify that $H^{-1}F(x)/x$ is a constant, which leads to the conclusion that $Z_{n:n} \leq_* X_{n:n}$. Consequently, $Y_{n:n} \leq_* X_{n:n}$. □

This result also justifies the fact that the star ordering is scale invariant. In a similar vein, we have the following result which follows from Theorem 3.2 and can be proved in the same way as above.

Theorem 3.4. *Let $X_1, \ldots, X_n$ be independent Pareto r.v.s with different shape parameters λ_i, $i = 1, \ldots, n$, and the same scale parameter b_1. Let $Y_1, \ldots, Y_n$ be a random sample from a Pareto distribution with shape parameter $\lambda \geq \bar{\lambda}$ and scale parameter b_2. Then, $Y_{n:n} \leq_{\mathrm{c}} X_{n:n}$.*

3.2. Model 2

Kochar and Xu [15] claimed that if X_i's are independent Pareto r.v.s with the same shape parameter λ, and different scale parameters b_i, $i = 1, \ldots, n$, and Y_i's, $i = 1, \ldots, n$, is a random sample from a Pareto distribution with shape parameter λ and scale parameter b, then $Y_{n:n} \leq_* X_{n:n}$. However, we found that the correct result should be as follows:

Theorem 3.5. *Let $X_1, \ldots, X_n$ be independent Pareto r.v.s with the same shape parameter λ, and different scale parameters b_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample from a Pareto distribution with shape parameter λ and scale parameter b. Then,*

$$X_{n:n} \leq_* Y_{n:n}.$$

Proof. For $x \geq \max\{b_1, \ldots, b_n\}$, the distribution function of $X_{n:n}$ is,

$$F(x) = P(X_{n:n} \leq x) = \prod_{i=1}^{n}\left[1 - \left(\frac{b_i}{x}\right)^{\lambda}\right],$$

and, for $x \geq b$, the distribution function of $Y_{n:n}$ is

$$G(x) = P(Y_{n:n} \leq x) = \left[1 - \left(\frac{b}{x}\right)^{\lambda}\right]^{n}.$$

To prove $X_{n:n} \leq_* Y_{n:n}$, it suffices to show that

$$\frac{G^{-1}F(x)}{x} = \frac{b}{x}\left(1 - \prod_{i=1}^{n}\left[1 - \left(\frac{b_i}{x}\right)^{\lambda}\right]^{1/n}\right)^{-\frac{1}{\lambda}}$$

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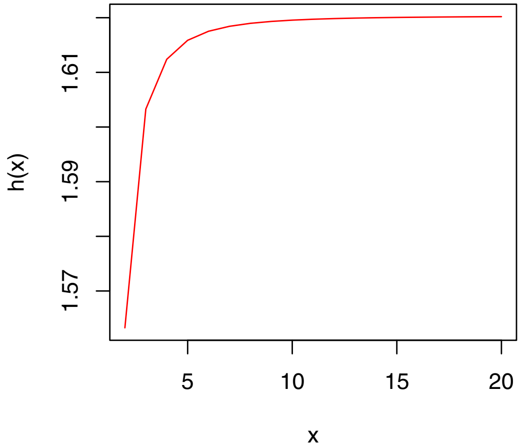

Figure 1. *Plot of $h(x)$ for different values of $x \in (2, 20)$.*

is increasing in $x \geq \max\{b_1, \ldots, b_n\}$, which is equivalent to showing that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{(b_i/x)^{\lambda}}{1-(b_i/x)^{\lambda}} \geq \prod_{i=1}^{n}\left[1-\left(\frac{b_i}{x}\right)^{\lambda}\right]^{-1/n} - 1$$

$$\Longleftrightarrow \frac{1}{n}\sum_{i=1}^{n}\left[1-\left(\frac{b_i}{x}\right)^{\lambda}\right]^{-1} \geq \prod_{i=1}^{n}\left[1-\left(\frac{b_i}{x}\right)^{\lambda}\right]^{-1/n}.$$

This holds true from arithmetic–geometric mean inequality, and hence, the result follows. □

The above result is of interest as it claims that the largest order statistic from the homogeneous Pareto sample is more skewed than the one from the heterogeneous Pareto sample in the sense of star ordering. In fact, a reverse relation holds true in case of exponential and Weibull samples [14,15]. The possible explanation for this difference is due to the fact that the range of Pareto distribution depends upon the scale parameter, whereas the other distributions considered in the literature, such as exponential and Weibull, have ranges independent of the parameter. It is for the same reason Pareto distribution should be studied separately, and not under the general setup of scale or proportional hazard rate models. The following example illustrates the application of Theorem 3.5.

Example 3.6. *Assume that X_1, X_2, X_3, X_4 are independent Pareto r.v.s $Pa(2.5, 0.4)$, $Pa(2.5, 1.2)$, $Pa(2.5, 1.5)$, and $Pa(2.5, 0.9)$, respectively, and Y_1, Y_2, Y_3, Y_4 is a random sample from $Pa(2.5, 1.8)$. It can be seen from Figure 1 that $h(x) = G^{-1}F(x)/x$ is increasing in $x \geq \max\{0.4, 1.2, 1.5, 0.9\}$, and thus verifies $X_{4:4} \leq_* Y_{4:4}$.*

Now, we prove the next result that extends Theorem 3.5 to a more general situation.

Theorem 3.7. *Let $X_1, \ldots, X_n$ be independent Pareto r.v.s with the same shape parameter λ_1, and different scale parameters b_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample from a Pareto distribution with shape parameter λ_2 and scale parameter b. If $\lambda_1 \geq \lambda_2$, then $X_{n:n} \leq_* Y_{n:n}$.*

Proof. For $x \geq \max\{b_1, \ldots, b_n\}$, the distribution function of $X_{n:n}$ is,

$$F(x) = P(X_{n:n} \leq x) = \prod_{i=1}^{n}\left[1-\left(\frac{b_i}{x}\right)^{\lambda_1}\right].$$

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Similarly, the distribution function of $Y_{n:n}$, for $x \geq b$, is

$$G(x) = P(Y_{n:n} \leq x) = \left[1 - \left(\frac{b}{x}\right)^{\lambda_2}\right]^n.$$

To prove $X_{n:n} \leq_* Y_{n:n}$, it suffices to show that

$$\frac{G^{-1}F(x)}{x} = \frac{b}{x}\left(1 - \prod_{i=1}^{n}\left[1 - \left(\frac{b_i}{x}\right)^{\lambda_1}\right]^{1/n}\right)^{-1/\lambda_2}$$

is increasing in $x \geq \max\{b_1, \ldots, b_n\}$, which is equivalent to proving that

$$\frac{\lambda_1}{\lambda_2}\frac{1}{n}\sum_{i=1}^{n}\frac{(b_i/x)^{\lambda_1}}{1-(b_i/x)^{\lambda_1}} \geq \prod_{i=1}^{n}\left[1 - \left(\frac{b_i}{x}\right)^{\lambda_1}\right]^{-1/n} - 1. \tag{10}$$

Since $\lambda_1 \geq \lambda_2$, it follows that $\lambda_1/\lambda_2 \geq 1$, and therefore,

$$\frac{\lambda_1}{\lambda_2}\frac{1}{n}\sum_{i=1}^{n}\frac{(b_i/x)^{\lambda_1}}{1-(b_i/x)^{\lambda_1}} \geq \frac{1}{n}\sum_{i=1}^{n}\frac{(b_i/x)^{\lambda_1}}{1-(b_i/x)^{\lambda_1}}.$$

Using this in (10), we get

$$\frac{1}{n}\sum_{i=1}^{n}\left[1 + \frac{(b_i/x)^{\lambda}}{1-(b_i/x)^{\lambda}}\right] \geq \prod_{i=1}^{n}\left[1 - \left(\frac{b_i}{x}\right)^{\lambda}\right]^{-1/n}$$

$$\Longleftrightarrow \frac{1}{n}\sum_{i=1}^{n}\left[1 - \left(\frac{b_i}{x}\right)^{\lambda}\right]^{-1} \geq \prod_{i=1}^{n}\left[1 - \left(\frac{b_i}{x}\right)^{\lambda}\right]^{-1/n}$$

which is guaranteed by arithmetic–geometric mean inequality. □

Till now, we focussed only on the two particular models wherein either shape parameters or scale parameters were different. In the next result, we focus on a much more general setting where both shape and scale parameters are different, or is a combination of the previous two models.

Theorem 3.8. *Let $X_1, \ldots, X_n$ be independent Pareto r.v.s with different shape parameters λ_i, $i = 1, \ldots, n$, and the same scale parameter b. Let $Y_1, \ldots, Y_n$ be independent Pareto r.v.s with the same shape parameter $\lambda \geq \tilde{\lambda}$ and different scale parameters b_i, $i = 1, \ldots, n$. Then,*

$$Y_{n:n} \leq_* X_{n:n}.$$

Proof. Let $Z_1, \ldots, Z_n$ be independent Pareto r.v.s with shape parameter λ and scale parameter b. Then, we know from Theorem 3.5 that $Y_{n:n} \leq_* Z_{n:n}$. Furthermore, as $\lambda \geq \tilde{\lambda}$, it follows from Theorem 3.1 that $Z_{n:n} \leq_* X_{n:n}$. These observations complete the proof. □

4. Dispersion

Deshpande and Kochar [4] provided the following connection between the star ordering and the dispersive ordering.

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Theorem 4.1. *Let X and Y be two r.v.s having distribution functions F and G such that* $F(0) = G(0) = 0$ *and density functions f and g such that* $f(0) \geq g(0) > 0$. *Then,*

$$X \leq_* Y \implies X \leq_{disp} Y.$$

It can be seen that the above result holds only for distributions having support $(0, \infty)$, and neglects the distributions like Pareto by not focussing on general supports (a, ∞), $a > 0$. In this regard, we prove the following result which establishes the relation between the star ordering and the dispersive ordering in a general situation.

Theorem 4.2. *Let X and Y be two non-negative r.v.s having supports* (a, ∞) *and* (b, ∞), *respectively, such that* $b \geq a > 0$. *Then,*

$$X \leq_* Y \implies X \leq_{disp} Y.$$

Proof. Assume that $X \leq_* Y$. We know, by Definition 2.2, that

$$X \leq_* Y \iff \frac{G^{-1}F(x)}{x} \text{ is increasing in } x \text{ on the support of } X$$

$$\iff \frac{f(x)}{g(G^{-1}F(x))} \geq \frac{G^{-1}F(x)}{x},$$

where the last step follows on differentiation. Note that

$$\lim_{x \to a} \frac{G^{-1}F(x)}{x} = \frac{G^{-1}(0)}{a} \geq 1$$

since $G^{-1}(0) = b \geq a$. Thus, using this and the fact that $G^{-1}F(x)/x$ is increasing in x on the support of X, it follows that

$$f(x) \geq g(G^{-1}F(x)), \quad x \in (a, \infty)$$
$$\iff X \leq_{\text{disp}} Y.$$

These observations complete the proof. □

As mentioned in the previous section, we first consider the results of Model 1. It is known that if an r.v. X follows the Pareto distribution with shape parameter λ and scale parameter b, then its support depends upon the scale parameter. Thus, the first result here involves two different scale parameters b_1 and b_2, and consequently, the supports of $X_{n:n}$ and $Y_{n:n}$ are (b_1, ∞) and (b_2, ∞), respectively. The proof is omitted as it directly follows on employing Theorems 3.3 and 4.2.

Corollary 4.3. *Let* $X_1, \ldots, X_n$ *be independent Pareto r.v.s with different shape parameter* λ_i, $i = 1, \ldots, n$, *and the same scale parameter* b_1. *Let* $Y_1, \ldots, Y_n$ *be a random sample from a Pareto distribution with shape parameter* $\lambda \geq \tilde{\lambda}$ *and scale parameter* b_2. *Then,*

$$b_1 \geq b_2 \implies Y_{n:n} \leq_{disp} X_{n:n}.$$

It can be easily seen that the above result also holds true when $b_1 = b_2 = b$, that is, with the same scale parameter b. Now, we come to our second model and obtain the following result as a direct consequence of Theorem 3.7, and by employing Theorem 4.2.

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Corollary 4.4. *Let $X_1, \ldots, X_n$ be independent Pareto r.v.s with the same shape parameter λ_1, and different scale parameters b_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample from a Pareto distribution with shape parameter λ_2 $(\leq \lambda_1)$ and scale parameter b. Then,*

$$b \geq \max\{b_1, \ldots b_n\} \implies X_{n:n} \leq_{disp} Y_{n:n}.$$

Similarly, it is straightforward to see that this result holds when X_i's and Y_i's have the same shape parameter, that is, $\lambda_1 = \lambda_2 = \lambda$. The following corollary, which is based on a general model, can be seen to easily follow from Theorems 3.8 and 4.2.

Corollary 4.5. *Let $X_1, \ldots, X_n$ be independent Pareto r.v.s with different shape parameters λ_i, $i = 1, \ldots, n$, and the same scale parameter b. Let $Y_1, \ldots, Y_n$ be a random sample from a Pareto distribution with shape parameter $\lambda \geq \tilde{\lambda}$ and scale parameter b_i, $i = 1, \ldots, n$. Then,*

$$b \geq \max\{b_1, \ldots b_n\} \implies Y_{n:n} \leq_{disp} X_{n:n}.$$

5. Applications

In this section, we first present some applications of the results derived in the paper. Recall that the Pareto distribution is often used to describe the allocation of wealth among individuals as in any society, a large portion of the wealth is in hands of a small percentage of people in that society. It is also useful in analyzing stock price fluctuations, studying the claim amounts in insurance companies and in comparing different systems [1,18]. Below, we discuss two such applications:

(i) Consider an insurance company which is interested to know the maximum claim amount in 10 cities. For this purpose, the insurer divides the operators based on their gender and wishes to know whether the maximum claim for male operators is larger than that of female operators. Let $X_1, \ldots, X_{10}$ and $Y_1, \ldots, Y_{10}$, respectively, denote the claims for female and male operators in those 10 cities. It is reasonable to assume that $X_1, \ldots, X_{10}$ (and/or, $Y_1, \ldots, Y_{10}$) are independent r.v.s as they represent different cities. The results in our study would help the insurer in determining how the maximum claim amount $X_{10:10}$ and $Y_{10:10}$ are ordered.

(ii) For a design engineer, finding out the best scheme for assembling a parallel system is of great interest. Suppose there are two sets of components A_i, $i = 1, 2, 3$, following $\text{Pa}(\lambda_i, b)$, $b > 0$, and B_i, $i = 1, 2, 3$, following $\text{Pa}(\lambda, b)$, $b > 0$. Thus, if $\lambda \geq \overline{\lambda}$, the engineer can claim that the assembly of a parallel system using components of set A is better than that of set B by applying Theorem 3.2. Let the failure times of A_1, A_2, and A_3 be $\text{Pa}(2.5, 1)$, $\text{Pa}(3.2, 1)$, and $\text{Pa}(1.7, 1)$, respectively. Furthermore, let the failure times of B_1, B_2, and B_3 follow $\text{Pa}(3, 1)$. Clearly, $\lambda \geq \overline{\lambda}$. Thus, it follows from Theorem 3.2 that system A is preferable as System B ages faster than System A.

6. Discussion

Now, we turn our attention to star ordering results established between parallel systems for Model 2 (see Section 3). A natural question that arises is if we can strengthen these results by establishing the convex transform order between parallel systems. The answer to this is negative and will be evident from the following counter-example.

Example 6.1. *Let $X_1, \ldots, X_5$ and $Y_1, \ldots, Y_5$ are independent r.v.s with $X_i \sim Pa(\lambda, b_i)$ and $Y_i \sim Pa(\lambda, b)$, $i = 1, \ldots, 5$, respectively. Suppose $b_1 = 0.5$, $b_2 = 0.2$, $b_3 = 0.3$, $b_4 = 0.4$, $b_5 = 0.9$, $b = 1$, and $\lambda = 6$. Then, $h(100) = 1.44169$, $h(120) = 1.44886$, and $h(200) = 1.41019$, where $h(x) = (G^{-1}F(x))'$. It is clear that the values of the function $h(x)$ are not monotonically decreasing, and consequently, one cannot establish the convex transform ordering between $X_{n:n}$ and $Y_{n:n}$ when $X_i \sim Pa(\lambda, b_i)$ and $Y_i \sim Pa(\lambda, b)$, $i = 1, \ldots, n$.*

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Moreover, we have not been able to claim whether such an ordering holds for the other case when $X_i \sim \text{Pa}(\lambda_1, b_i)$ and $Y_i \sim \text{Pa}(\lambda_2, b)$, $i = 1, \ldots, n$, and it remains an open problem.

Another interesting problem is if we assume $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ as independent r.v.s with $X_i \sim \text{Pa}(\lambda_i, b_i)$ and $Y_i \sim \text{Pa}(\lambda, b_i)$, $i = 1, \ldots, n$, then under what restrictions on the parameters, the largest order statistics will be ordered with respect to star ordering and/or convex transform ordering, that is, $Y_{n:n} \leq_{*,c} X_{n:n}$. To prove this, one can consider $Z_i \sim \text{Pa}(\lambda_i, b)$, $i = 1, \ldots, n$. It can be seen from Theorem 3.8 that $Y_{n:n} \leq_* Z_{n:n}$. Thus, what remains to prove is that $Z_{n:n} \leq_* X_{n:n}$. Although it is possible that the result may hold true, it remains as an open problem.

In order to further justify that probability distributions with supports involving parameters should be treated separately, we now discuss results on series systems with component lifetimes having a Power law distribution. An r.v. X is said to follow a Power Law distribution with shape parameter α and scale parameter β (denoted by $X \sim Pow\,(\alpha, \beta)$) if its survival function is given by

$$\bar{F}(x) = \left(\frac{x}{\beta}\right)^{\alpha}, \quad \alpha > 0,\ 0 < x < \beta. \tag{11}$$

It is well-known that the Power law distribution is an inverse of Pareto distribution. So, to prove the following result, we adopt the methodology given in [9] see Lem. 3.2 and [22] see Lem. 4.12 for the proportional reversed hazard rate model and apply it to our inverse model.

Theorem 6.2. *Let $X_1, \ldots, X_n$ be independent Power r.v.s with the same shape parameter $\alpha > 0$ and different scale parameters β_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample from a Power Law distribution with shape parameter $\alpha > 0$ and scale parameter β. Then,*

$$Y_{1:n} \leq_* X_{1:n}.$$

Proof. Define $X'_i = 1/X_i$ and $Y'_i = 1/Y_i$, $i = 1, \ldots, n$. Then, $X_{1:n} = 1/X'_{n:n}$ and $Y_{1:n} = 1/Y'_{n:n}$. Note

$$F_{X_{1:n}}(x) = P(X_{1:n} \leq x) = P\left(\frac{1}{X'_{n:n}} \leq x\right) = P\left(X'_{n:n} \geq \frac{1}{x}\right) = \overline{F}_{X'_{n:n}}\left(\frac{1}{x}\right)$$

Similarly, $F_{X_{1:n}}(x) = \overline{F}_{Y'_{n:n}}(1/x)$. Furthermore, it is easy to check that

$$\frac{F^{-1}_{X_{1:n}} F_{Y_{1:n}}(x)}{x} = \frac{1}{\overline{F}^{-1}_{X'_{n:n}} \overline{F}_{Y'_{n:n}}(\frac{1}{x})x} = \frac{1}{F^{-1}_{X'_{n:n}} F_{Y'_{n:n}}(\frac{1}{x})x}.$$

Then,

$$\begin{aligned} \frac{F^{-1}_{X_{1:n}} F_{Y_{1:n}}(x)}{x} \text{ increases in } x &\iff F^{-1}_{X'_{n:n}} F_{Y'_{n:n}}\left(\frac{1}{x}\right) x \text{ decreases in } x \\ &\iff \frac{F^{-1}_{X'_{n:n}} F_{Y'_{n:n}}(x)}{x} \text{ increases in } x \end{aligned}$$

which is true from Theorem 3.5. □

On similar lines, we have the next result which follows on employing Theorem 3.1.

Theorem 6.3. *Let $X_1, \ldots, X_n$ be independent Power r.v.s with shape parameter α_i, $i = 1, \ldots, n$, and the same scale parameter $\beta > 0$. Let $Y_1, \ldots, Y_n$ be a random sample from a Power Law distribution with shape parameter $\alpha > 0$ and scale parameter $\beta = \tilde{\beta} = (\prod_{i=1}^{n} \beta_i)^{1/n}$. Then,*

$$X_{1:n} \leq_* Y_{1:n}.$$

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As it can be seen, the results on star ordering follow the same pattern as that of parallel systems with Pareto components. Although we have provided the results on star ordering only, we believe that results on other dispersion orderings would follow a similar pattern and are left as open problems.

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