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# Blow-up phenomena in parabolic problems with time-dependent coefficients under Neumann boundary conditions

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This paper deals with the blow-up of solutions to a class of parabolic problems with time-dependent coefficients under homogeneous Neumann boundary conditions. For one set of problems in this class we show that no global solution can exist. For another we derive lower bounds for the time of blow-up when blow-up occurs.

## 1. Introduction

In recent years a strong interest in the phenomenon of blow-up of solutions to various classes of nonlinear problems has developed. Much of the earlier work on blow-up is referenced in the books of Straughan [7] and of Quittner and Souplet [6] as well as in the survey papers of Bandle and Brunner [1] and of Levine [2]. Problems analogous to those considered here but with constant coefficients were treated in [5], and problems with time-dependent coefficients under homogeneous Dirichlet boundary conditions were investigated by Payne and Philippin in [3] for a single equation, and in [4] for a related system. In the present paper it is shown that, with certain restrictions on the form of the nonlinear term in the governing equation, no non-zero global positive solution can exist in $L^1$. Under somewhat different conditions, a lower bound for the time of blow-up is obtained when blow-up occurs.

## 2. Non-existence

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. We consider the following problem, whose solution may blow up at some finite time $t^\star$:

$$\left.\begin{aligned} u_t = \Delta u + k(t)f(u), \quad \boldsymbol{x} = (x_1,\dots,x_N) \in \Omega, \quad t \in (0,t^\star), \\ \frac{\partial u(\boldsymbol{x},t)}{\partial \nu} = 0, \quad \boldsymbol{x} \in \partial\Omega, \quad t \in (0,t^\star), \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \geqslant 0, \quad (u_0(\boldsymbol{x}) \not\equiv 0), \quad \boldsymbol{x} \in \Omega, \end{aligned}\right\} \tag{2.1}$$

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Here $\Delta$ is the Laplace operator, $k(t)$ is a non-negative function of $t$ for $t > 0$ and $\partial u/\partial \nu := \nabla u \cdot \boldsymbol{\nu}$ is the normal derivative of $u$ on $\partial\Omega$. $\boldsymbol{\nu} := (\nu_1, \dots, \nu_N)$ is the unit exterior normal vector on the boundary $\partial\Omega$. Our results in this section will hold for functions $f$ that satisfy

$$f(s) \geqslant g(s) \geqslant 0, \quad s \geqslant 0, \tag{2.2}$$

where $g$ satisfies Jensen's inequality, i.e.

$$\frac{1}{|\Omega|}\int_\Omega g(u)\,\mathrm{d}\boldsymbol{x} \geqslant g\biggl(\frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}\boldsymbol{x}\biggr). \tag{2.3}$$

In (2.3), $|\Omega| := \int_\Omega \mathrm{d}\boldsymbol{x}$ is the volume of $\Omega$. We note that (2.3) is satisfied for convex functions $g$. In establishing blow-up, we introduce the auxiliary function

$$\Psi(t) := \frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}\boldsymbol{x}. \tag{2.4}$$

A differentiation of $\Psi$ gives

$$\Psi'(t) = \frac{1}{|\Omega|}k(t)\int_\Omega f(u)\,\mathrm{d}\boldsymbol{x} \geqslant k(t)g(\Psi(t)). \tag{2.5}$$

An integration over the time-interval of existence leads to

$$\int_{\Psi(0)}^{\Psi(t)} \frac{\mathrm{d}\eta}{g(\eta)} \geqslant \int_0^t k(\tau)\,\mathrm{d}\tau. \tag{2.6}$$

Clearly, if the condition

$$\int_{\Psi(0)}^{\infty} \frac{\mathrm{d}\eta}{g(\eta)} = M < \infty \tag{2.7}$$

is satisfied, and if the function $k(t)$ satisfies

$$\lim_{t\to\infty}\int_0^t k(\tau)\,\mathrm{d}\tau = \infty, \tag{2.8}$$

then no non-zero solution of (2.1) in $L^1$ can exist for all time, and an upper bound $T$ for $t^\star$ is given by

$$\int_0^T k(\tau)\,\mathrm{d}\tau = M. \tag{2.9}$$

We have established the following result.

Theorem 2.1. *If $f(u)$ and $g(u)$ satisfy (2.2), (2.3) and (2.7), and if $k(t)$ satisfies (2.8), then no non-zero $L^1$ solution of (2.1) can exist for all time, but the solution will blow up in $L^1$ at time $t^\star \leqslant T$, where $T$ is defined by (2.9).*

We note that the blow-up of $u(\boldsymbol{x}, t)$ in $L^1$ implies the blow-up in $L^q$ for any $q > 1$. This follows from Hölder's inequality.

## 3. Lower bound for $t^\star$

In this section we assume that $\Omega$ is a bounded convex region in $\mathbb{R}^3$, and we make the following assumptions on the data:

$$0 \leqslant f(s) \leqslant s^p, \quad p > 1, \tag{3.1}$$

$$k(t) > 0, \qquad \frac{k'(t)}{k(t)} \leqslant \beta, \tag{3.2}$$

for some constant $\beta \geqslant 0$. The case where $k(t) = 0$ for some value of $t$ will be considered at the end of this section. In deriving a lower bound for $t^\star$, we introduce the auxiliary function

$$\Phi(t) := (k(t))^{2n} \int_\Omega u^{2n(p-1)} \, \mathrm{d}\boldsymbol{x}, \tag{3.3}$$

where $n$ is a constant which satisfies

$$n \geqslant \max\left\{\frac{1}{p-1}, 1\right\}. \tag{3.4}$$

A differentiation of (3.3) gives

$$\Phi'(t) = 2n\frac{k'}{k}\Phi + 2n(p-1)k^{2n} \int_\Omega u^{2n(p-1)-1}[\Delta u + ku^p] \, \mathrm{d}\boldsymbol{x}. \tag{3.5}$$

For convenience we set

$$v(\boldsymbol{x}, t) := u^{n(p-1)}. \tag{3.6}$$

Making use of the divergence theorem and of (3.2), we obtain

$$\Phi'(t) \leqslant 2n\beta\Phi - \frac{2[2n(p-1)-1]}{n(p-1)}k^{2n} \int_\Omega |\nabla v|^2 \, \mathrm{d}\boldsymbol{x} + 2n(p-1)k^{2n+1} \int_\Omega v^{2+1/n} \, \mathrm{d}\boldsymbol{x}. \tag{3.7}$$

The last term in (3.7) may be bounded as follows:

$$\begin{aligned} k^{2n+1} \int_\Omega v^{2+1/n} \, \mathrm{d}\boldsymbol{x} &\leqslant \left(k^{2n} \int_\Omega v^2 \, \mathrm{d}\boldsymbol{x}\right)^{1-1/n} \left(k^{3n} \int_\Omega v^3 \, \mathrm{d}\boldsymbol{x}\right)^{1/n} \\ &\leqslant \frac{n-1}{\gamma n} k^{2n} \int_\Omega v^2 \, \mathrm{d}\boldsymbol{x} + \frac{\gamma^{n-1}}{n} k^{3n} \int_\Omega v^3 \, \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{3.8}$$

where $\gamma$ is a positive constant to be chosen later. The first inequality in (3.8) follows from Hölder's inequality and the second inequality follows from the inequality

$$a^r b^{1-r} \leqslant ra + (1-r)b, \tag{3.9}$$

valid for $a > 0$, $b > 0$, $r \in (0, 1)$. To obtain a bound for the last term in (3.8), we make use of the following Sobolev-type inequality:

$$k^{3n} \int_\Omega v^3 \, \mathrm{d}\boldsymbol{x} \leqslant \left\{\lambda k^{2n} \int_\Omega v^2 \, \mathrm{d}\boldsymbol{x} + \mu k^{2n} \left(\int_\Omega v^2 \, \mathrm{d}\boldsymbol{x} \int_\Omega |\nabla v|^2 \, \mathrm{d}\boldsymbol{x}\right)^{1/2}\right\}^{3/2}, \tag{3.10}$$

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valid for $\Omega$ bounded convex in $\mathbb{R}^3$, where $\lambda$ and $\mu$ are defined as

$$\lambda := \frac{\sqrt{3}}{2\rho}, \qquad \mu := \frac{1}{\sqrt{3}}\left(1 + \frac{d}{\rho}\right), \tag{3.11}$$

with

$$\rho := \min_{\partial\Omega}\left(\sum_{i=1}^{3} x_i \nu_i\right) > 0, \qquad d^2 := \max_{\Omega}\left(\sum_{i=1}^{3} x_i^2\right). \tag{3.12}$$

The inequality (3.10) was derived by Payne and Schaefer in [5]. It is a particular case of their inequality (2.16). From (3.10), we obtain

$$\begin{aligned} k^{3n}\int_\Omega v^3\,\mathrm{d}\boldsymbol{x} &\leqslant \left\{\lambda\Phi + \mu\left(\Phi k^{2n}\int_\Omega |\nabla v|^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}\right\}^{3/2} \\ &\leqslant \sqrt{2}\left\{\lambda^{3/2}\Phi^{3/2} + \mu^{3/2}\Phi^{3/4}\left(k^{2n}\int_\Omega |\nabla v|^2\,\mathrm{d}\boldsymbol{x}\right)^{3/4}\right\} \\ &\leqslant \sqrt{2}\left\{\lambda^{3/2}\Phi^{3/2} + \mu^{3/2}\left[\frac{1}{4\sigma^3}\Phi^3 + \frac{3\sigma}{4}k^{2n}\int_\Omega |\nabla v|^2\,\mathrm{d}\boldsymbol{x}\right]\right\}, \end{aligned} \tag{3.13}$$

valid for arbitrary positive $\sigma$. In (3.13), the first inequality is equivalent to (3.10); the second inequality follows from Hölder's inequality:

$$(a+b)^{3/2} \leqslant \sqrt{2}(a^{3/2} + b^{3/2}) \tag{3.14}$$

with $a > 0$, $b > 0$; the third inequality follows from (3.9). Combining (3.13), (3.8) and (3.7), we obtain

$$\begin{aligned} \Phi'(t) &\leqslant c_1\Phi + c_2\Phi^{3/2} + c_3\Phi^3 \\ &\quad + \left\{\frac{3}{\sqrt{2}}(p-1)\sigma\gamma^{n-1}\mu^{3/2} - \frac{2[2n(p-1)-1]}{n(p-1)}\right\}k^{2n}\int_\Omega |\nabla v|^2\,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{3.15}$$

with

$$\left.\begin{aligned} c_1 &:= 2n\beta + 2(p-1)(n-1)\gamma^{-1}, \\ c_2 &:= (2\lambda)^{3/2}(p-1)\gamma^{n-1}, \\ c_3 &:= \frac{(p-1)\gamma^{n-1}\mu^{3/2}}{\sqrt{2}\sigma^3}. \end{aligned}\right\} \tag{3.16}$$

We now choose

$$\sigma := \frac{2^{3/2}[2n(p-1)-1]}{3n(p-1)^2\gamma^{n-1}\mu^{3/2}}, \tag{3.17}$$

leading to the first-order differential inequality

$$\Phi'(t) \leqslant c_1\Phi + c_2\Phi^{3/2} + c_3\Phi^3. \tag{3.18}$$

Note that the constant $\gamma(>0)$ is still at our disposal for $n \neq 1$. If the solution blows up at time $t^\star$, then there is a time $t_1$ (which might be 0) beyond which $\Phi(t) > \Phi(0)$.

Then, integrating (3.18) from $t_1$ to $t^\star$, we obtain

$$\int_{\Phi(0)}^{\infty} \frac{\mathrm{d}\eta}{c_1\eta + c_2\eta^{3/2} + c_3\eta^3} \leqslant t^\star - t_1 \leqslant t^\star. \tag{3.19}$$

This gives the desired lower bound for $t^\star$ and leads to the following result.

THEOREM 3.1. *Let $u(\boldsymbol{x}, t)$ be the solution of (2.1) in a bounded convex region of $\mathbb{R}^3$. Assume that the data $f$ and $k$ satisfy the conditions (3.1), (3.2). We conclude that if $u(\boldsymbol{x}, t)$ blows up in $\Phi$ norm at $t^\star$, then a lower bound for $t^\star$ is given by (3.19), where the $c_i$ are defined in (3.16).*

Note that for $\Phi(t) \geqslant \Phi(0)$ we have

$$(\Phi(t))^{3/2} \leqslant (\Phi(t))^3(\Phi(0))^{-3/2}. \tag{3.20}$$

Substituting (3.20) into (3.18), we obtain the weaker differential inequality

$$\Phi'(t) \leqslant c_1\Phi + \tilde{c}_3\Phi^3, \tag{3.21}$$

with

$$\tilde{c}_3 := c_3 + c_2(\Phi(0))^{-3/2}. \tag{3.22}$$

Integrating (3.21), we obtain

$$(\Phi(t))^{-2} \geqslant \left((\Phi(0))^{-2} + \frac{\tilde{c}_3}{c_1}\right)\mathrm{e}^{-2c_1 t} - \frac{\tilde{c}_3}{c_1}, \tag{3.23}$$

from which we obtain the cruder lower bound

$$t^\star \geqslant T := \frac{1}{2c_1}\log\left[1 + \frac{c_1}{\tilde{c}_3}(\Phi(0))^{-2}\right]. \tag{3.24}$$

For $p \geqslant 2$, an appropriate choice of $n$ is $n = 1$. In this case the values of the constants $c_i$ and $\sigma$ are greatly simplified. For $p \in (1, 2)$, we may choose $n = (p-1)^{-1}$. For $k = 1$ and $n = 1$, we retrieve the lower bound for $t^\star$ derived in [5].

If $k'/k$ becomes infinite for some value of $t$ we choose a different measure defined as

$$\chi(t) := \int_\Omega u^{2n(p-1)}\,\mathrm{d}\boldsymbol{x}. \tag{3.25}$$

Proceeding as before, we obtain

$$\chi'(t) \leqslant -\frac{2[2n(p-1)-1]}{n(p-1)}\int_\Omega |\nabla v|^2\,\mathrm{d}\boldsymbol{x} + 2n(p-1)k(t)\int_\Omega v^{2+1/n}\,\mathrm{d}\boldsymbol{x}, \tag{3.26}$$

where $v(\boldsymbol{x}, t)$ is defined in (3.6). For the next step, we assume that $n$ satisfies the conditions

$$n \geqslant (p-1)^{-1} \quad \text{and} \quad n > 1, \tag{3.27}$$

and make use of the inequality

$$\begin{aligned} k\int_\Omega v^{2+1/n}\,\mathrm{d}\boldsymbol{x} &\leqslant \left(k^{n/(n-1)}\int_\Omega v^2\,\mathrm{d}\boldsymbol{x}\right)^{1-1/n}\left(\int_\Omega v^3\,\mathrm{d}\boldsymbol{x}\right)^{1/n} \\ &\leqslant \frac{n-1}{n\gamma}k^{n/(n-1)}\int_\Omega v^2\,\mathrm{d}\boldsymbol{x} + \frac{\gamma^{n-1}}{n}\int_\Omega v^3\,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{3.28}$$

which is analogous to (3.8). Setting $k = 1$ in (3.13), we obtain

$$\int_{\Omega} v^3 \,\mathrm{d}\boldsymbol{x} \leqslant \sqrt{2}\Big\{\lambda^{3/2}\chi^{3/2} + \mu^{3/2}\Big[\frac{1}{4\sigma^3}\chi^3 + \frac{3\sigma}{4}\int_{\Omega} |\nabla v|^2 \,\mathrm{d}\boldsymbol{x}\Big]\Big\}. \tag{3.29}$$

Inserting (3.28), (3.29) into (3.26), and choosing $\sigma$ according to (3.17), we obtain the differential inequality

$$\chi'(t) \leqslant \tilde{c}_1 (k(t))^{n/(n-1)}\chi + c_2\chi^{3/2} + c_3\chi^3, \tag{3.30}$$

with

$$\tilde{c}_1 := 2(p-1)(n-1)\gamma^{-1}, \tag{3.31}$$

where $c_2$ and $c_3$ are defined in (3.16). We note again that the positive constant $\gamma$ is at our disposal for $n \neq 1$. It is unlikely that (3.30) can be solved explicitly, but a crude bound may be obtained using again the fact that if $\chi(t)$ blows up, there is a time $t_1$ beyond which $\chi(t) > \chi(0)$, so that we have

$$\chi(t) \leqslant (\chi(t))^3(\chi(0))^{-2}, \quad (\chi(t))^{3/2} \leqslant (\chi(t))^3(\chi(0))^{-3/2}, \quad t \in (t_1, t^\star). \tag{3.32}$$

Inserting the inequalities (3.32) into (3.30) and integrating, we obtain

$$\tfrac{1}{2}(\chi(0))^{-2} \leqslant \int_{t_1}^{t^\star} (\tilde{c}_1(\chi(0))^{-2}(k(t))^{n/(n-1)} + c_2(\chi(0))^{-3/2} + c_3)\,\mathrm{d}t, \tag{3.33}$$

i.e.

$$\frac{1}{2} \leqslant \int_0^{t^\star} (\tilde{c}_1(k(t))^{n/(n-1)} + c_2(\chi(0))^{1/2} + c_3(\chi(0))^2)\,\mathrm{d}t. \tag{3.34}$$

We may take $n = 2$ for $p \geqslant 2$ and $n = (p-1)^{-1}$ for $p \in (1, 2)$. We have established the following result.

THEOREM 3.2. *Let $u(\boldsymbol{x}, t)$ be the solution of (2.1) in a bounded convex region of $\mathbb{R}^3$. Assume that the data $f$ satisfy the condition (3.1). We then conclude that if $u(\boldsymbol{x}, t)$ blows up at time $t^\star$, then a lower bound for $t^\star$ is implicitly given by (3.34).*

## 4. Concluding remarks

Suppose that, instead of equation (2.1), for positive $k_i(t)$, $u(\boldsymbol{x}, t)$ satisfies

$$\frac{1}{k_1(t)}u_t = k_2(t)\Delta u + k_3(t)f(u), \tag{4.1}$$

with the same boundary and initial conditions as in (2.1). Then, as in [3], we may introduce a new variable

$$z(t) := \int_0^t k_1(\tau)k_2(\tau)\,\mathrm{d}\tau. \tag{4.2}$$

Setting

$$K(z) := \frac{k_3(t(z))}{k_2(t(z))}, \tag{4.3}$$

the solution $u(\boldsymbol{x}, z)$ now satisfies

$$u_z = \Delta u + K(z)f(u). \tag{4.4}$$

Under appropriate assumptions we may then read off a lower bound for $t^\star$ directly from the results of § 3.

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