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REGENERATION OF BRANCHING PROCESSES WITH IMMIGRATION IN VARYING ENVIRONMENTS[‡]

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Abstract

We consider linear-fractional branching processes (one-type and two-type) with immigration in varying environments. For $n \geq 0$, let Z_n count the number of individuals of the nth generation, which excludes the immigrant who enters the system at time n. We call n a regeneration time if $Z_n = 0$. For both the one-type and two-type cases, we give criteria for the finiteness or infiniteness of the number of regeneration times. We then construct some concrete examples to exhibit the strange phenomena caused by the so-called varying environments. For example, it may happen that the process is extinct, but there are only finitely many regeneration times. We also study the asymptotics of the number of regeneration times of the model in the example.

Keywords: Branching processes,; immigration,; regeneration times,; continued fractions,; product of nonnegative matrices.

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1. Introduction

It is known that Galton–Watson processes are widely applied in nuclear physics, biology, ecology, epidemiology, and many other areas, and have been extensively studied; see [2, 10, 18] and references therein. The study of Galton–Watson processes can be extended directly in two directions. One popular extension is the branching process in a random environment (BPRE), which has attracted much attention. Many interesting results arise from the existence of the random environment; we refer the reader to [15] and references therein for details. Another interesting extension of the Galton–Watson process is the branching process in a varying environment (BPVE). Compared with BPREs, the the study of BPVEs has not been as successful. The main reason is that a BPVE is no longer a time-homogeneous Markov chain, but BPREs do have some homogeneous properties. Indeed, if the environments are assumed to be stationary

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and ergodic, then a BPRE is a time-homogeneous process under annealed probability. The emerging of the so-called varying environments also brings some strange phenomena to the branching processes. For example, the process may 'fall asleep' in some positive state [19], it may diverge at different exponential rates [21], and the tail probabilities of the surviving time may show some strange asymptotics [9, 29]. For other aspects of the study of BPVEs, we refer the reader to [3–5, 7, 11, 13, 14] and the references therein.

In this paper we study BPVEs with immigration. For simplicity, we assume that only one immigrant immigrates into the system in each generation. Roughly speaking, for $n \geq 0$, let Z_n be the number of individuals in the nth generation, which does not count the immigrants entering the system at time n. If $Z_n = 0$, we call n a regeneration time. Our aim is to provide the necessary and sufficient conditions to decide whether the process has finitely or infinitely many regeneration times. We should note that for Galton–Watson processes or BPREs with immigration, if the process has one regeneration time it must have infinitely many regeneration times. In other words, it will never happen that such a process owns finitely many regeneration times if there are any due to the time homogeneity.

Our motivation originates from two aspects, the regeneration structure of BPREs and the cutpoints of random walks in varying environments. On one hand, in [16], in order to study the stable limit law of random walks in random environments, a regeneration structure of a single-type BPRE was constructed, and the tail probabilities of the regeneration time and the number of total progeny before the first regeneration time were estimated. Related problems in the multitype case of this regeneration structure can be found in [17, 23, 25]. Along these lines, it is natural for us to consider the number of regeneration times for BPVEs. On the other hand, in [6, 12, 20, 26], a class of questions related to the cutpoints of random walks in varying environments was considered. We find that the regeneration structures for BPVEs and the excursions between successive cutpoints share some similarities, so we aim to study the regeneration of BPVEs in this paper.

We currently treat only the regeneration times of one-type and two-type BPVEs with linear fractional offspring distributions. Basically, the one-type case is much easier, and the two-type case is very complicated. But the ideas in studying the two models are similar, so we omit the proofs of the one-type case. The difficulty in studying the two-type case arises from the fact that the probability that n is a regeneration time is written in terms of the product of 2×2 nonnegative matrices, which are hard to estimate. To overcome this difficulty, we need some delicate analyses among the spectral radii, the tails of continued fractions, and the product of nonnegative matrices. These analyses lead to some interesting results in these fields, which may be of independent interest.

In Section 2, we precisely define the models and state the main results. In Section 3, we prove some properties of continued fractions that are useful for the proof of the main result. In Section 4, we focus on the proof of the main result. In Section 5, we construct some concrete examples that explicitly exhibit the new phenomena that arise from the existence of so-called varying environments.

2. Models and main results

Linear-fractional branching processes are of special interest as the iterations of their generating functions are again linear-fractional functions, allowing for explicit calculations of various entities of importance [2, pp. 7–8]. Such explicit results illuminate the known asymptotic results concerning more general branching processes, and, on the other hand, may bring insight into less-investigated aspects of the theory of branching processes. So, in order to discuss the properties of the generation time of branching processes in varying environments (one-type and two-type), we study linear-fractional branching processes first.

2.1. One-type case

For $k \geq 1$, suppose $0 < p_k \leq \frac{1}{2}$, $q_k > 0$ are numbers such that $p_k + q_k = 1$ and

$$f_k(s) = \frac{p_k}{1 - q_k s}, \quad s \in [0, 1].$$

Let $\{Z_n\}_{n\geq 0}$ be a Markov chain such that $Z_0 = 0$ and $E(s^{Z_n} \mid Z_0, \ldots, Z_{n-1}) = [f_n(s)]^{1+Z_{n-1}}$, $n \geq 1$. Clearly, $\{Z_n\}_{n\geq 0}$ forms a branching process in varying environments with exactly one immigrant in each generation. We now define the regeneration time with which we are concerned.

Definition 1. Let $C = \{n \geq 0 \colon Z_n = 0\}$, and for $k \geq 1$ let $C_k = \{n \colon n + i \in C, 0 \leq i \leq k - 1\}$. If $n \in C$, n is called the regeneration time of the process $\{Z_n\}$. If $n \in C_k$, we call n a k-strong regeneration time of the process $\{Z_n\}$.

Remark 1. Here, we slightly abuse the term 'regeneration time'. Notice that if $R \in C$ then $Z_R = 0$, i.e. the process temporarily dies out. But the process may get regenerated at time R, since there is an immigrant entering into the system at time R that may give birth to a number of individuals. In this point of view, we call R a regeneration time. We emphasize that the regeneration times here are different from the classical regeneration times of regenerative processes. In the literature (see, e.g., [24]), for a stochastic process $X = \{X(t)\}_{t\geq 0}$, if there is a random variable $R > 0$ such that $\{X(t + R)\}_{t\geq 0}$ is independent of $\{\{X(t)\}_{t<R}, R\}$, and $\{X(t + R)\}_{t\geq 0}$ equals $\{X(t)\}_{t\geq 0}$ in distribution, then X is call a regenerative process and R is called a regeneration time. In our setting, for a regeneration time R, due to the existence of the so-called varying environments the distribution of $\{Z_{R+n}\}_{n\geq 0}$ differs from that of $\{Z_n\}_{n\geq 0}$.

For $k \geq 1$, let $m_k = f'_k(1) = q_k/p_k$. For $n \geq k \geq 1$, set $D(k, n) := 1 + \sum_{j=k}^{n} m_j \cdots m_n$ and write, for simplicity, $D(n) \equiv D(1, n)$.

The following theorem provides a criterion for the finiteness of the number of regeneration times.

Theorem 1. *Suppose that, for some $\varepsilon > 0$, $\varepsilon < p_n \leq \frac{1}{2}$, $n \geq 1$. Let D(n), $n \geq 1$, be as defined above. If*

$$\sum_{n=2}^{\infty} \frac{1}{D(n) \log n} < \infty,$$

then $\{Z_n\}$ has at most finitely many regeneration times, almost surely. If there exists some $\delta > 0$ such that $D(n) \leq \delta n \log n$ for n large enough and

$$\sum_{n=2}^{\infty} \frac{1}{D(n) \log n} = \infty,$$

then $\{Z_n\}$ has infinitely many k-strong regeneration times, almost surely.

2.2. Two-type case

Suppose a_k, b_k, d_k, θ_k, $k \geq 1$, are positive real numbers and set

$$M_k := \begin{bmatrix} a_k & b_k \\ d_k & \theta_k \end{bmatrix}, \quad k \geq 1.$$

For $n \geq 1$ and $\mathbf{s} = (s_1, s_2)^\top \in [0, 1] \times [0, 1]$, let

$$\mathbf{f}_n(\mathbf{s}) = \left(f_n^{(1)}(\mathbf{s}), f_n^{(2)}(\mathbf{s})\right)^\top = \mathbf{1} - \frac{M_n(\mathbf{1} - \mathbf{s})}{1 + \mathbf{e}_1^\top M_n(\mathbf{1} - \mathbf{s})}, \tag{1}$$

which is known as the probability-generating function of a linear-fractional distribution. Here and in what follows, $a^\top$ denotes the transpose of the vector a, $\mathbf{e}_1 = (1, 0)^\top$, $\mathbf{e}_2 = (0, 1)^\top$, and $\mathbf{1} = \mathbf{e}_1 + \mathbf{e}_2 = (1, 1)^\top$.

Suppose $\mathbf{Z}_n = (Z_{n,1}, Z_{n,2})^\top$, $n \geq 0$, is a two-type branching process with immigration satisfying $E\left(s^{\mathbf{Z}_n} \mid \mathbf{Z}_0, \ldots, \mathbf{Z}_{n-1}\right) = \mathbf{f}_n(\mathbf{s})^{\mathbf{Z}_{n-1}+\mathbf{e}_1}$ for all $n \geq 1$. Here, $\mathbf{f}_n(\mathbf{s})^{\mathbf{Z}_{n-1}+\mathbf{e}_1} = \left[f_n^{(1)}(\mathbf{s})\right]^{Z_{n-1,1}+1}\left[f_n^{(2)}(\mathbf{s})\right]^{Z_{n-1,2}}$.

We now define the regeneration times and the k-strong regeneration times of the two-type process $\{\mathbf{Z}_n\}$ in a similar fashion to the one-type case.

Definition 2. Let $C = \{n \geq 0 \colon \mathbf{Z}_n = (0, 0)^\top\}$ and, for $k \geq 1$, let $C_k = \{n \colon n + i \in C, 0 \leq i \leq k - 1\}$. If $n \in C$, we call n a regeneration time of the process $\{\mathbf{Z}_n\}$. If $n \in C_k$, we call n a k-strong regeneration time of the process $\{\mathbf{Z}_n\}$.

To study the regeneration times of the two-type branching process, we need the following condition on the sequences $a_k, b_k, d_k, \theta_k, k \geq 1$.

Assumption 1. *Suppose that*

$$\sum_{k=2}^{\infty} |a_k - a_{k-1}| + |b_k - b_{k-1}| + |d_k - d_{k-1}| + |\theta_k - \theta_{k-1}| < \infty, \tag{2}$$

and $a_k \to a$, $b_k \to b$, $d_k \to d$, $\theta_k \to \theta$ as $k \to \infty$, where $b, d > 0$ and $a, \theta \geq 0$ are certain numbers such that $a + \theta > 0$ and $bd - a\theta \neq 0$.

In what follows, for a matrix M we denote by $\varrho(M)$ its spectral radius (the largest eigenvalue). For $n \geq m \geq 1$, we set

$$L(m, n) = 1 + \sum_{j=m}^{n} \prod_{i=j}^{n} \varrho(M_i), \tag{3}$$

and write $L(1,n)$ as $L(n)$ for simplicity. In the two-type case, we have the following criteria.

Theorem 2. *Assume Assumption 1 holds, and that $\varrho(M_k) \geq 1$ for all $k \geq 1$.*

(i) If

$$\sum_{n=2}^{\infty} \frac{1}{L(n) \log n} < \infty$$

then $\{\mathbf{Z}_n\}$ has at most finitely many regeneration times, almost surely.

(ii) If there exists some $\delta > 0$ such that $L(n) \leq \delta n \log n$ for n large enough and

$$\sum_{n=2}^{\infty} \frac{1}{L(n) \log n} = \infty,$$

then $\{\mathbf{Z}_n\}$ has infinitely many k-strong regeneration times, almost surely.

Remark 2. For branching processes with general offspring distributions, the probability-generating function of the population size cannot be computed explicitly in general, but, as seen in [1, 15], it can be controlled from below and above by that of a branching process with linear-fractional offspring distributions. Based on this observation, our results may be generalized to branching processes with general offspring distributions.

Remark 3. We give some explanation on Assumption 1. The assumption in (2) allows us to show that

$$\zeta \le \frac{\varrho(M_m \cdots M_n)}{\varrho(M_m)\cdots\varrho(M_n)} \le \gamma$$

for some universal constant $0 < \zeta < \gamma < \infty$, i.e. the spectral radius of the product of matrices can be uniformly bounded from below and above by the product of the spectral radii of these matrices; see Lemma 5. Such a result plays an important role in proving Theorem 2.

Remark 4. In the following, we discuss only the two-type case and give the proof of Theorem 2. The proof of Theorem 1 is omitted since it is similar to that of Theorem 2.

3. Products of 2 × 2 matrices and continued fractions

The probability-generating function of $\mathbf{Z}_n$ can be written in terms of the products of the mean offspring matrices, which are hard to compute directly since they are inhomogeneous. But it is known that the products of 2×2 matrices can be written in terms of the products of the tails of certain continued fractions.

In this section, we focus on how to estimate the products of the mean offspring matrices by means of continued fractions. To begin with, we introduce some new matrices related to M_k, $k \ge 1$. For $k \ge 1$, set

$$A_k := \begin{pmatrix} \tilde{a}_k & \tilde{b}_k \\ \tilde{d}_k & 0 \end{pmatrix}, \quad \tilde{a}_k = a_k + \frac{b_k\theta_{k+1}}{b_{k+1}},\ \tilde{b}_k = b_k,\ \tilde{d}_k = d_k - \frac{a_k\theta_k}{b_k},$$

and write

$$\Lambda_k := \begin{pmatrix} 1 & 0 \\ \theta_k/b_k & 1 \end{pmatrix}.$$

Then, for $n \ge k \ge 1$, we have

$$A_k = \Lambda_k^{-1} M_k \Lambda_{k+1}, \qquad \mathbf{e}_1^\top \prod_{i=k}^{n} M_i \mathbf{1} = \mathbf{e}_1^\top \prod_{i=k}^{n} A_i (1,\, 1 - \theta_{n+1}/b_{n+1})^\top, \tag{4}$$

and $A_k \cdots A_n = \Lambda_k^{-1} M_k \cdots M_n \Lambda_{n+1}$, $n \ge k \ge 1$. Since a_k, b_k, d_k, and θ_k are all positive numbers, we have

$$\begin{aligned} \mathbf{e}_1^\top A_k \cdots A_n \mathbf{e}_1 &= \mathbf{e}_1^\top M_k \cdots M_n (1,\, \theta_{n+1}/b_{n+1})^\top > 0, & n > k \ge 1, \\ \mathbf{e}_1^\top A_k \cdots A_n \mathbf{e}_2 &= \mathbf{e}_1^\top M_k \cdots M_n \mathbf{e}_2 > 0, & n > k \ge 1, \\ \mathbf{e}_2^\top A_k \cdots A_n \mathbf{e}_1 &= (-\theta_k/b_k,\, 1) M_k \cdots M_n (1,\, \theta_{n+1}/b_{n+1})^\top, & n \ge k \ge 1. \end{aligned} \tag{5}$$

Under Assumption 1, we have

$$\lim_{k\to\infty} M_k = M := \begin{pmatrix} a & b \\ d & \theta \end{pmatrix}, \qquad \lim_{k\to\infty} A_k = A := \begin{pmatrix} a+\theta & b \\ d-a\theta/b & 0 \end{pmatrix},$$

whose spectral radii are

$$\varrho := \varrho(M) = \varrho(A) = \frac{a+\theta+\sqrt{(a+\theta)^2+4(bd-a\theta)}}{2},$$
$$\varrho_1 := \varrho_1(M) = \varrho_1(A) = \frac{a+\theta-\sqrt{(a+\theta)^2+4(bd-a\theta)}}{2}.$$

Next, we introduce some basics on continued fractions. Let β_k, α_k, $k\ge 1$ be certain real numbers. For $1\le k\le n$, we denote by

$$\xi_{k,n} \equiv \frac{\beta_k}{\alpha_k} + \frac{\beta_{k+1}}{\alpha_{k+1}} + \cdots + \frac{\beta_n}{\alpha_n} := \cfrac{\beta_k}{\alpha_k + \cfrac{\beta_{k+1}}{\alpha_{k+1} + \ddots + \cfrac{\beta_n}{\alpha_n}}} \tag{6}$$

the $(n-k+1)$th approximant of a continued fraction, and

$$\xi_k := \frac{\beta_k}{\alpha_k} + \frac{\beta_{k+1}}{\alpha_{k+1}} + \frac{\beta_{k+2}}{\alpha_{k+2}} + \cdots. \tag{7}$$

If $\lim_{n\to\infty}\xi_{k,n}$ exists, we say that the continued fraction ξ_k is convergent and its value is defined as $\lim_{n\to\infty}\xi_{k,n}$. We call ξ_k, $k\ge 1$, in (7) the tails, and

$$h_k := \frac{\beta_k}{\alpha_{k-1}} + \frac{\beta_{k-1}}{\alpha_{k-2}} + \cdots + \frac{\beta_2}{\alpha_1},$$

$k\ge 2$, the critical tails of the continued fraction

$$\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \cdots,$$

respectively.

The following lemma gives the convergence of the tails and the critical tails of the continued fractions.

Lemma 1. *If* $\lim_{n\to\infty}\alpha_n = \alpha \ne 0$, $\lim_{n\to\infty}\beta_n = \beta$, *and* $\alpha^2+4\beta\ge 0$, *then, for any* $k\ge 1$, $\lim_{n\to\infty}\xi_{k,n}$ *exists and, furthermore,*

$$\lim_{k\to\infty} h_k = \lim_{k\to\infty}\xi_k = \frac{\alpha}{2}\left(\sqrt{1+4\beta/\alpha^2}-1\right).$$

The proof of Lemma 1 can be found in many references. We refer the reader to [20] (see the discussion between (4.1) and (4.2) on p. 81 therein).

For $n\ge k\ge 1$, let

$$y_{k,n} = \mathbf{e}_1^\top \prod_{i=k}^{n} A_i \mathbf{e}_1, \qquad \xi_{k,n} = \frac{y_{k+1,n}}{y_{k,n}}, \tag{8}$$

where we stipulate that $y_{n+1,n}=1$. Then we have

$$\mathbf{e}_1^\top \prod_{i=k}^{n} A_i \mathbf{e}_1 = \xi_{k,n}^{-1}\cdots\xi_{n,n}^{-1}, \quad n\ge k\ge 1. \tag{9}$$

Lemma 2. *For $1\le k\le n$, $\xi_{k,n}$ defined in (8) coincides with the one in (6) with $\beta_k=\tilde{b}_k^{-1}\tilde{d}_{k+1}^{-1}$ and $\alpha_k=\tilde{a}_k\tilde{b}_k^{-1}\tilde{d}_{k+1}^{-1}$.*

Proof. Clearly,

$$\xi_{n,n}=\frac{1}{y_{n,n}}=\frac{1}{\tilde{a}_n}=\frac{\tilde{b}_n^{-1}\tilde{d}_{n+1}^{-1}}{\tilde{a}_n\tilde{b}_n^{-1}\tilde{d}_{n+1}^{-1}}=\frac{\beta_n}{\alpha_n}.$$

For $1\le k<n$, note that

$$\begin{aligned}\xi_{k,n}=\frac{y_{k+1,n}}{y_{k,n}}=\frac{\mathbf{e}_1^\top A_{k+1}\cdots A_n\mathbf{e}_1}{\mathbf{e}_1^\top A_k\cdots A_n\mathbf{e}_1}&=\frac{\mathbf{e}_1^\top A_{k+1}\cdots A_n\mathbf{e}_1}{(\tilde{a}_k\mathbf{e}_1^\top+\tilde{b}_k\mathbf{e}_2^\top)A_{k+1}\cdots A_n\mathbf{e}_1}\\ &=\frac{1}{\tilde{a}_k+\tilde{b}_k\dfrac{\mathbf{e}_2^\top A_{k+1}\cdots A_n\mathbf{e}_1}{\mathbf{e}_1^\top A_{k+1}\cdots A_n\mathbf{e}_1}}\\ &=\frac{1}{\tilde{a}_k+\tilde{b}_k\tilde{d}_{k+1}\dfrac{\mathbf{e}_1^\top A_{k+2}\cdots A_n\mathbf{e}_1}{\mathbf{e}_1^\top A_{k+1}\cdots A_n\mathbf{e}_1}}\\ &=\frac{\tilde{b}_k^{-1}\tilde{d}_{k+1}^{-1}}{\tilde{a}_k\tilde{b}_k^{-1}\tilde{d}_{k+1}^{-1}+\xi_{k+1,n}}=\frac{\beta_k}{\alpha_k+\xi_{k+1,n}}.\end{aligned}$$

We come to the conclusion that the lemma is true by iterating this equation. □

In the remainder of this section, we always assume that Assumption 1 holds, and ξ_k, $\xi_{k,n}$, $n\ge k\ge 1$, are as defined in (6) and (7) with $\beta_k=\tilde{b}_k^{-1}\tilde{d}_{k+1}^{-1}$ and $\alpha_k=\tilde{a}_k\tilde{b}_k^{-1}\tilde{d}_{k+1}^{-1}$. Since

$$\lim_{k\to\infty}\beta_k=:\beta=(bd-a\theta)^{-1}\neq 0,\quad \lim_{k\to\infty}\alpha_k=:\alpha=\frac{a+\theta}{bd-a\theta}\neq 0,\quad \alpha^2+4\beta=\frac{(a-\theta)^2+4bd}{(bd-a\theta)^2}>0,$$

it follows from Lemma 1 that

$$\lim_{n\to\infty}\xi_{k,n}=\xi_k,\qquad \lim_{k\to\infty}\xi_k=:\xi=\frac{\alpha}{2}\left(\sqrt{1+4\beta/\alpha^2}-1\right)=\varrho^{-1}>0. \tag{10}$$

Moreover, consulting (5), (8), and the relationship $\xi_k=\beta_k/(\alpha_k+\xi_{k+1})$, we have

$$\xi_k>0,\ \ \xi_{k,n}>0\quad \text{for all } n\ge k\ge 1. \tag{11}$$

The relationship between the entries $\xi_{k+1}^{-1}\cdots\xi_{k+n}^{-1}$ and $A_{k+1}\cdots A_{k+n}$, which plays an important role in the proof of our main result, was established in [28, Theorem 2]. For convenience, we state it here.

Proposition 1. ([28, Theorem 2].) *Let ξ_k be as in (7) for all $k\ge 1$. Suppose $M_k\to M$, $a+\theta\neq 0$, $b\neq 0$, and $bd\neq a\theta$. Then there exists a $k_0>0$ such that, for $k\ge k_0$ and $i,j=1,2$, we have*

$$\lim_{n\to\infty}\frac{e_i^\top A_{k+1}\cdots A_{k+n}e_j}{\xi_{k+1}^{-1}\cdots\xi_{k+n}^{-1}}=\varphi(i,j,k), \tag{12}$$

where the convergence is uniform in k, and

$$\varphi(1) := \varphi(1, 1, k) = \frac{\varrho}{\varrho - \varrho_1}, \qquad \varphi(2) := \varphi(1, 2, k) = \frac{b}{\varrho - \varrho_1},$$
$$\varphi(2, 1, k) = \frac{\rho}{\varrho - \varrho_1} \frac{\tilde{d}_{k+1}}{\xi_{k+1}^{-1}}, \qquad \varphi(2, 2, k) = \frac{b}{\varrho - \varrho_1} \frac{\tilde{d}_{k+1}}{\xi_{k+1}^{-1}}.$$

Furthermore, if $\rho \geq 1$ then, for $k \geq k_0$, with the above $i, j = 1, 2$,

$$\lim_{n\to\infty} \frac{\sum_{s=1}^{n+1} e_i^\top A_{k+s} \cdots A_{k+n} e_j}{\sum_{s=1}^{n+1} \xi_{k+s}^{-1} \cdots \xi_{k+n}^{-1}} = \varphi(i, j, k), \tag{13}$$

where the convergence is uniform in k.

4. Proof of the main result

Keep in mind that in what follows, unless otherwise specfied, c (with or without an index) is a positive constant whose value may be different from line to line.

For $n \geq k \geq 1$, write $\mathbf{f}_{k,n}(\mathbf{s}) := \mathbf{f}_k(\mathbf{f}_{k+1} \cdots (\mathbf{f}_n(\mathbf{s})) \cdots)$. By iterating (1), we see that

$$\mathbf{f}_{k,n}(\mathbf{s}) = \mathbf{1} - \frac{M_k \cdots M_n(\mathbf{1} - \mathbf{s})}{1 + \sum_{j=k}^n \mathbf{e}_1^\top M_j \cdots M_n(\mathbf{1} - \mathbf{s})}.$$

As a consequence,

$$E\big(\mathbf{s}^{\mathbf{Z}_n} \mid \mathbf{Z}_0 = 0\big) = \prod_{k=1}^n \mathbf{e}_1^\top f_{k,n}(\mathbf{s}) = \frac{1}{1 + \sum_{j=1}^n \mathbf{e}_1^\top M_j \cdots M_n(\mathbf{1} - \mathbf{s})},$$

which implies that

$$P(\mathbf{Z}_n = 0 \mid \mathbf{Z}_0 = 0) = \frac{1}{1 + \sum_{j=1}^n \mathbf{e}_1^\top M_j \cdots M_n \mathbf{1}}. \tag{14}$$

Let $G(k, n) := 1 + \sum_{j=k}^n \mathbf{e}_1^\top M_j \cdots M_n \mathbf{1}$, $n \geq k \geq 1$.

In order to study the regeneration times of the process $\{\mathbf{Z}_n\}$, we should estimate $G(k,n)$. With Proposition 1 in hand, and using some other estimates, we can control $G(k,n)$ by the entries $\sum_{j=k}^n \varrho(M_j) \cdots \varrho(M_n)$. We state the methods of the estimates in the following lemmas.

Lemma 3. *Assume Assumption 1 holds, and $\varrho(M_k) \geq 1$. Then, for $\varepsilon > 0$, there exist constants k_0 and N such that, for all $k > k_0$ and $n - k > N$,*

$$\varphi(1) + \left(1 - \frac{\theta}{b}\right)\varphi(2) - \varepsilon \leq \frac{G(k, n)}{1 + \sum_{j=k}^n \xi_j^{-1} \cdots \xi_n^{-1}} \leq \varphi(1) + \left(1 - \frac{\theta}{b}\right)\varphi(2) + \varepsilon, \tag{15}$$

where $\varphi(1)$ and $\varphi(2)$ are as defined in Proposition 1. Furthermore, for $n > N$,

$$\varphi(1) + \left(1 - \frac{\theta}{b}\right)\varphi(2) - \varepsilon \leq \frac{G(1, n)}{1 + \sum_{j=1}^n \xi_j^{-1} \cdots \xi_n^{-1}} \leq \varphi(1) + \left(1 - \frac{\theta}{b}\right)\varphi(2) + \varepsilon. \tag{16}$$

Proof. In view of (4), we have

$$G(k, n) = 1 + \sum_{j=k}^n \left(\mathbf{e}_1^\top \prod_{i=j}^n A_i \mathbf{e}_1\right) + \left(1 - \frac{\theta_{n+1}}{b_{n+1}}\right) \sum_{j=k}^n \left(\mathbf{e}_1^\top \prod_{i=j}^n A_i \mathbf{e}_2\right) \tag{17}$$

for all $n \geq k \geq 1$. Assumption 1 means that all the conditions of Proposition 1 are fulfilled. Then, in view of (13), we can see that for each $\varepsilon > 0$ there exists a constant $N' > 0$ such that, for all $n > N'$ and $k \geq k_0$,

$$\varphi(l) - \varepsilon < \frac{\sum_{s=1}^{n+1} \mathbf{e}_1^\top A_{k+s} \cdots A_{k+n} \mathbf{e}_l}{\sum_{s=1}^{n+1} \xi_{k+s}^{-1} \cdots \xi_{k+n}^{-1}} < \varphi(l) + \varepsilon, \quad l = 1, 2.$$

Thus, for $k > k_0$ and $n - k \geq N'$,

$$\left| \frac{\sum_{j=k}^{n+1} \mathbf{e}_1^\top \prod_{i=j}^{n} A_i \mathbf{e}_l}{\sum_{j=k}^{n+1} \xi_j^{-1} \cdots \xi_n^{-1}} - \varphi(l) \right| = \left| \frac{\sum_{s=1}^{n-k+2} \mathbf{e}_1^\top \prod_{i=(k-1)+s}^{(k-1)+n-(k-1)} A_i \mathbf{e}_l}{\sum_{s=1}^{n-k+2} \xi_{(k-1)+s}^{-1} \cdots \xi_n^{-1}} - \varphi(l) \right| < \varepsilon \tag{18}$$

for $l = 1, 2$. Noticing that $1 - (\theta_{n+1}/b_{n+1}) \to 1 - (\theta/b)$ as $n \to \infty$, we conclude that (15) is true.

Now we turn to (16). For the above ε, it follows from (12) that there exist constants k_0 and N" such that, for all $k > k_0$ and $n - k \geq N''$,

$$\left| \frac{e_1^\top A_k \cdots A_n e_l}{\xi_k^{-1} \cdots \xi_n^{-1}} - \varphi(l) \right| < \varepsilon, \quad l = 1, 2. \tag{19}$$

Taking (9) into account, we rewrite

$$\begin{aligned}
\sum_{j=1}^{n} \left(\mathbf{e}_1^\top \prod_{i=j}^{n} A_i \mathbf{e}_1 \right) &= \sum_{j=1}^{k_0} \xi_{j,n}^{-1} \cdots \xi_{n,n}^{-1} + \sum_{j=k_0+1}^{n} \left(\mathbf{e}_1^\top \prod_{i=j}^{n} A_i \mathbf{e}_1 \right) \\
&= \xi_{k_0+1,n}^{-1} \cdots \xi_{n,n}^{-1} \sum_{j=1}^{k_0} \xi_{j,n}^{-1} \cdots \xi_{k_0,n}^{-1} + \sum_{j=k_0+1}^{n} \left(\mathbf{e}_1^\top \prod_{i=j}^{n} A_i \mathbf{e}_1 \right) \\
&= e_1^\top A_{k_0+1} \cdots A_n e_1 \sum_{j=1}^{k_0} \xi_{j,n}^{-1} \cdots \xi_{k_0,n}^{-1} + \sum_{j=k_0}^{n} \left(\mathbf{e}_1^\top \prod_{i=j}^{n} A_i \mathbf{e}_1 \right).
\end{aligned} \tag{20}$$

It follows from (10) that

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{k_0} \xi_{j,n}^{-1} \cdots \xi_{k_0,n}^{-1}}{\sum_{j=1}^{k_0+1} \xi_j^{-1} \cdots \xi_{k_0}^{-1}} = 1. \tag{21}$$

Then, using (18), (19), and (21) to estimate (20), we get, for $n > \max\{N' + k_0, N'' + k_0\}$,

$$\left| \frac{\sum_{j=1}^{n} \left(\mathbf{e}_1^\top \prod_{i=j}^{n} A_i \mathbf{e}_1 \right)}{\sum_{j=1}^{n} \xi_j^{-1} \cdots \xi_n^{-1}} - \varphi(1) \right| < c\varepsilon. \tag{22}$$

We can rewrite the second summand in (17) as

$$\frac{\sum_{j=1}^{n} \left(\mathbf{e}_1^\top \prod_{i=j}^{n} A_i \mathbf{e}_2 \right)}{\sum_{j=1}^{n} \xi_j^{-1} \cdots \xi_n^{-1}} = \frac{\sum_{j=1}^{n} \left(\mathbf{e}_1^\top A_j \cdots A_{n-1} \mathbf{e}_1 \right)}{\sum_{j=1}^{n} \xi_j^{-1} \cdots \xi_{n-1}^{-1}} \frac{\tilde{b}_n}{\xi_n^{-1}}.$$

Then, taking the fact $\varphi(1)\cdot\lim_{n\to\infty}\tilde{b}_n/\xi_n^{-1}=\varphi(2)$ and (22) into consideration, we get that there exists a constant K such that, for $n>K$,

$$\left|\frac{\sum_{j=1}^{n}\left(\mathbf{e}_1^\top\prod_{i=j}^{n}A_i\mathbf{e}_2\right)}{\sum_{j=1}^{n}\xi_j^{-1}\cdots\xi_n^{-1}}-\varphi(2)\right|<c\varepsilon. \tag{23}$$

Therefore, taking (22), (23), and (17) together, we see that (16) is true. □

Lemma 4. *Suppose that Assumption 1 holds. Then there are constants $0<c_1<c_2<\infty$ and numbers N_1, N_2, which may depend on c_1 and c_2, such that, for all $n-m>N_1$, $m>N_2$,*

$$c_1<\frac{\mathbf{e}_1A_m\cdots A_n\mathbf{e}_1^\top}{\varrho(M_m)\cdots\varrho(M_n)}<c_2.$$

Proof. The lemma is a direct consequence of Lemmas 5, 6, and 7. □

The following is [27, Lemma 4]. For convenience, we state it here.

Lemma 5. ([27, Lemma 4].) *Suppose that Assumption 1 holds. Then, for $n\geq m\geq 1$,*

$$\zeta\leq\frac{\varrho(M_m\cdots M_n)}{\varrho(M_m)\cdots\varrho(M_n)}\leq\gamma$$

for some constants $0<\zeta<\gamma<\infty$ independent of m and n.

Lemma 6. *Suppose that Assumption 1 holds. Then there exist constants $c_3>0$, $c_4>0$ and numbers $N_1, N_3>0$, which may depend on c_3 and c_4, such that, for all $n-m\geq N_1$ and $m>N_3$,*

$$c_3<\frac{\varrho(A_m\cdots A_n)}{\mathbf{e}_1A_m\cdots A_n\mathbf{e}_1^\top}<c_4.$$

The proof of this lemma is similar to that of [27, Lemma 5]; we just point out the differences.

Proof of Lemma 6. We write

$$A_{m,n}:=A_m\cdots A_n=\begin{pmatrix}A_{m,n}(11) & A_{m,n}(12)\\ A_{m,n}(21) & A_{m,n}(22)\end{pmatrix},\quad n\geq m\geq 1.$$

From the proof of [27, Lemma 5], we get that

$$\varrho(A_{m,n})=\frac{A_{m,n}(11)+A_{m,n}(22)}{2}+\frac{\sqrt{(A_{m,n}(11)+A_{m,n}(22))^2+4P_{m,n}}}{2}, \tag{24}$$

where $P_{m,n}=A_{m,n}(12)A_{m,n}(21)-A_{m,n}(11)A_{m,n}(22)$.

Applying Proposition 1, we have that for $\varepsilon>0$ there exist $k_0>0$ and $N_1>0$ such that, for all $m>k_0$, $n-m\geq N_1$,

$$\left|\frac{A_{m,n}(ij)}{A_{m,n}(11)}-\frac{\varphi(i,j,m-1)}{\varphi(1,1,m-1)}\right|<\varepsilon,\quad i,j=1,2. \tag{25}$$

It follows from Lemma 1 that $\lim_{m\to\infty}\xi_m=-\varrho_1/(bd-a\theta)$. As a result,

$$\lim_{m\to\infty}\left[1+\frac{\varphi(2,2,m-1)}{\varphi(1,1,m-1)}\right]=\lim_{m\to\infty}\left[1+\left(\frac{\theta_m}{b_m}-\xi_m\frac{a_m\theta_m-d_mb_m}{b_m}\right)\frac{b}{\varrho-\theta}\right]=\frac{\varrho-\varrho_1}{\varrho-\theta}>0.$$

On the other hand, note that

$$R_m := \varphi(1, 2, m-1)\varphi(2, 1, m-1) - \varphi(1, 1, m-1)\varphi(2, 2, m-1) = 0. \tag{26}$$

We thus see that

$$\lim_{m\to\infty} V_m := \lim_{m\to\infty}\left[\frac{\varphi(1, 1, m-1)+\varphi(2, 2, m-1)}{2\varphi(1, 1, m-1)} + \frac{\sqrt{(\varphi(1, 1, m-1)+\varphi(2, 2, m-1))^2+4R_m}}{2\varphi(1, 1, m-1)}\right]$$
$$= \frac{\varrho-\varrho_1}{\varrho-\theta} > 0.$$

Consequently, there exist constants $c_3' > 0$, $c_4' > 0$, and $k_1 > 0$ such that, for $m > k_1$,

$$c_3' < V_m < c_4'. \tag{27}$$

Taking (25) and (24) into consideration, we have that, for all $m > k_0$ and $n - m \geq N_1$, there exists a constant c' such that

$$-c'\varepsilon < \frac{\varrho(A_m\cdots A_n)}{\mathbf{e}_1 A_m\cdots A_n\mathbf{e}_1^\top} - V_m < c'\varepsilon.$$

Therefore, in view of (27), we conclude that the lemma is true. □

Lemma 7. *Suppose Assumption 1 is fulfilled. Then there exist constants* $c_5 < c_6 < \infty$ *and numbers* N_1, N_4 *such that, for all* $n - m > N_1$, $m > N_4$,

$$c_5\varrho(M_m\cdots M_n) < \varrho(A_m\cdots A_n) < c_6\varrho(M_m\cdots M_n).$$

Proof. Let $A_{m,n}$ and $P_{m,n}$ be as defined in Lemma 6. Define

$$Q_{m,n} := A_{m,n}(11) + A_{m,n}(22) + A_{m,n}(12)\left(\frac{\theta_m}{b_m} - \frac{\theta_{n+1}}{b_{n+1}}\right).$$

By some easy calculation, we have $\varrho(M_m\cdots M_n) = \frac{1}{2}(Q_{m,n} + \sqrt{Q_{m,n}^2 + 4P_{m,n}})$.

Note that

$$\lim_{n\to\infty}\lim_{m\to\infty}\left[1 + \frac{\varphi(2, 2, k)}{\varphi(1, 1, k)} + \frac{\varphi(1, 2, k)}{\varphi(1, 1, k)}\left(\frac{\theta_m}{b_m} - \frac{\theta_{n+1}}{b_{n+1}}\right)\right] = \frac{\varrho-\varrho_1}{\varrho-\theta} > 0.$$

Thus, in view of (25) we get that there exist constants $0 < c_5' < c_6' < \infty$ and $N_1 > 0$, $N_3' > k_0$ such that. for all $m > N_3'$, $n - m > N_1$,

$$c_5' < \frac{Q_{k,m}}{A_{m,n}(11)} < c_6'. \tag{28}$$

Consulting (25) and (26), we have that there exists a constant $c' > 0$ such that, for all $n - m > N_1$ and $m > k_0$,

$$-c'\varepsilon < \frac{P_{m,n}}{A_{m,n}(11)} < c'\varepsilon. \tag{29}$$

Taking (28) and (29) into account, we get that there exist constants $0 < c_5'' < c_6'' < \infty$ such that, for all $m > N_3'$ and $n - m > N_1$,

$$c_5'' < \frac{\varrho(M_n \cdots M_n)}{A_{m,n}(11)} < c_6''.$$

Thus, in view of Lemma 6, we conclude that Lemma 7 is true. □

For $1 \le m < n$, let $L(m,n)$ be as in (3) and write

$$H(m, n) = 1 + \sum_{j=m}^{n} \xi_j^{-1} \cdots \xi_n^{-1}. \tag{30}$$

Also, we write $H(1,n)$ as $H(n)$ for simplicity.

We establish the relationship between $L(n)$ and $H(n)$ as follows.

Lemma 8. *Suppose that Assumption 1 holds. Then there exist constants* $0 < c_7 < c_8 < \infty$, $0 < c_9 < c_{10} < \infty$ *and positive integers* N_5, N_6, N_7 *such that, for* $n - m \ge N_5$, $m \ge N_6$,

$$c_7 < \frac{\xi_m \cdots \xi_n}{\varrho^{-1}(M_m) \cdots \varrho^{-1}(M_n)} < c_8, \tag{31}$$

and for $n > N_7$,

$$c_9 L(n) \le H(n) \le c_{10} L(n). \tag{32}$$

Proof. Clearly, taking (12) and Lemma 4 together, we get (31).

For (32), first, by (31) we have that, for $m > N_6$ and $n > N_5 + N_6$,

$$c_7 \sum_{j=m}^{n-N_5} \varrho(M_j) \cdots \varrho(M_n) < \sum_{j=m}^{n-N_5} \xi_j^{-1} \cdots \xi_n^{-1} < c_8 \sum_{j=m}^{n-N_5} \varrho(M_j) \cdots \varrho(M_n). \tag{33}$$

For $n > N_5 + N_6$, we rewrite

$$H(n) = 1 + \sum_{j=1}^{N_6} \xi_j^{-1} \cdots \xi_n^{-1} + \sum_{j=N_6+1}^{n-N_5} \xi_j^{-1} \cdots \xi_n^{-1} + \sum_{j=n-N_5+1}^{n} \xi_j^{-1} \cdots \xi_n^{-1}. \tag{34}$$

Since

$$\sum_{j=n-N_5+1}^{n} \xi_j^{-1} \cdots \xi_n^{-1} \to \left(\frac{a\theta - bd}{\rho_1}\right)^{N_5} + \cdots + \frac{a\theta - bd}{\rho_1} > 0,$$

$\sum_{j=n-N_5+1}^{n} \varrho(M_j) \cdots \varrho(M_n) \to \varrho^{N_5} + \cdots + \varrho > 0$ as $n \to \infty$. Then there exist constants $0 < c_{11} < c_{12} < \infty$ and $N_7' > 0$ such that, for all $n > N_7'$,

$$c_{11} \sum_{j=n-N_5+1}^{n} \varrho(M_j) \cdots \varrho(M_n) < \sum_{j=n-N_5+1}^{n} \xi_j^{-1} \cdots \xi_n^{-1} < c_{12} \sum_{j=n-N_5+1}^{n} \varrho(M_j) \cdots \varrho(M_n). \tag{35}$$

We rewrite

$$\sum_{j=1}^{N_6} \xi_j^{-1} \cdots \xi_n^{-1} = \xi_{N_6}^{-1} \cdots \xi_n^{-1} \frac{\sum_{j=1}^{N_6} \xi_j^{-1} \cdots \xi_{N_6-1}^{-1}}{\sum_{j=1}^{N_6} \varrho(M_j) \cdots \varrho(M_{N_6-1})} \sum_{j=1}^{N_6} \varrho(M_j) \cdots \varrho(M_{N_6-1})$$
$$=: \xi_{N_6}^{-1} \cdots \xi_n^{-1} \sum_{j=1}^{N_6} \varrho(M_j) \cdots \varrho(M_{N_6-1})\Delta.$$

Recalling that, by (11), $\xi_j > 0$ for all $j > 0$, in view of the fact that $\varrho(M_j) > 0$ for all $j > 0$ we get that Δ is a positive constant. Therefore, it follows from (31) that there exist constants $0 < c_{13} < c_{14} < \infty$ such that

$$c_{13} \sum_{j=1}^{N_6} \varrho(M_j) \cdots \varrho(M_n) \leq \sum_{j=1}^{N_6} \xi_j^{-1} \cdots \xi_n^{-1} \leq c_{14} \sum_{j=1}^{N_2} \varrho(M_j) \cdots \varrho(M_n). \tag{36}$$

Taking (34), (33), (35), and (36) into consideration, we conclude that, for all $n > N_7 := \max\{N_5 + N_6, N_7'\}$, (32) is true. □

Lemma 9. *For every n and k,*

$$P(n \in C_k) = \frac{1}{G(1, n)} \prod_{i=n+1}^{n+k-1} \frac{1}{1 + a_i + b_i}.$$

Also, for every $l > n + k$,

$$P(n \in C_k, l \in C_k) = \frac{1}{G(1, n)} \frac{1}{G(n + k, l)} \prod_{i=n+1}^{n+k-1} \frac{1}{1 + a_i + b_i} \prod_{i=l+1}^{l+k-1} \frac{1}{1 + a_i + b_i}.$$

Proof. Using the Markov property and (14), we have

$$\begin{aligned} P(n \in C_k) &= P(\mathbf{Z}_n = 0, \ldots, \mathbf{Z}_{n+k-1} = 0) \\ &= P(\mathbf{Z}_n = 0) \prod_{j=0}^{k-2} P(\mathbf{Z}_{n+j+1} = 0 \mid \mathbf{Z}_{n+j} = 0) \\ &= \frac{1}{G(1, n)} \prod_{j=n+1}^{n+k-1} \frac{1}{1 + a_j + b_j}, \\ P(n \in C_k, l \in C_k) &= P(n \in C_k) P(l \in C_k \mid n \in C_k) \\ &= P(n \in C_k) P(l \in C_k \mid \mathbf{Z}_{n+k-1} = 0) \\ &= \frac{1}{G(1, n)} \prod_{i=n+1}^{n+k-1} \frac{1}{1 + a_i + b_i} \times \frac{1}{G(n + k, l)} \prod_{i=l+1}^{l+k-1} \frac{1}{1 + a_i + b_i}. \end{aligned}$$

We thus complete the proof of the lemma. □

Recalling the definitions in (30) and (3), by some easy computations we see that

$$L(n + 1) = 1 + \rho(M_{n+1}) L(n) \tag{37}$$

and

$$H(n+1)=1+\xi_{n+1}^{-1}H(n), \qquad \frac{H(k,n)}{H(n)}=1-\prod_{j=k-1}^{n}\left(1-\frac{1}{H(j)}\right). \tag{38}$$

Now we are ready to prove the main results.

Proof of Theorem 2. For $j<i$, set $C_{j,i}=\{x\colon x\in(2^j, 2^i], x\in C\}$ and let $A_{j,i}=|C_{j,i}|$ be the cardinality of the set $C_{j,i}$. On the event $\{A_{m,m+1}>0\}$, let $l_m=\max\{k\colon k\in C_{m,m+1}\}$ be the largest regeneration time in $C_{m,m+1}$. Then, for $m\geq 1$ we have

$$\begin{aligned}
\sum_{j=2^{m-1}+1}^{2^{m+1}} P(j\in C) &= E(A_{m-1,m+1})\\
&\geq \sum_{n=2^m+1}^{2^{m+1}} E(A_{m-1,m+1}, A_{m,m+1}>0, l_m=n)\\
&= \sum_{n=2^m+1}^{2^{m+1}} P(A_{m,m+1}>0, l_m=n)E(A_{m-1,m+1}\mid A_{m,m+1}>0, l_m=n)\\
&= \sum_{n=2^m+1}^{2^{m+1}} P(A_{m,m+1}>0, l_m=n)\sum_{i=2^{m-1}+1}^{n} P(i\in C\mid A_{m,m+1}>0, l_m=n)\\
&\geq P(A_{m,m+1}>0)\min_{2^m<n\leq 2^{m+1}}\sum_{i=2^{m-1}+1}^{n} P(i\in C\mid A_{m,m+1}>0, l_m=n) =: a_m b_m.
\end{aligned} \tag{39}$$

Fix $2^m+1\leq n\leq 2^{m+1}$ and $2^{m-1}+1\leq i\leq n$. Using Lemma 9 and the Markov property, we get that

$$\begin{aligned}
&P(i\in C\mid A_{m,m+1}>0, l_m=n)\\
&\quad=\frac{P(Z_i=0, Z_n=0, Z_t\neq 0, n+1\leq t\leq 2^{m+1})}{P(Z_n=0, Z_t\neq 0, n+1\leq t\leq 2^{m+1})}\\
&\quad=\frac{P(Z_i=0, Z_n=0)}{P(Z_n=0)}\frac{P(Z_t\neq 0, n+1\leq t\leq 2^{m+1}\mid Z_i=0, Z_n=0)}{P(Z_t\neq 0, n+1\leq t\leq 2^{m+1}\mid Z_n=0)}\\
&\quad=\frac{P(Z_i=0, Z_n=0)}{P(Z_n=0)}=\frac{G(n)}{G(i)G(i+1,n)}.
\end{aligned} \tag{40}$$

It follows from Lemma 3 that for fixed $\varepsilon>0$ there exists a constant $K_1>0$ such that, for all $i>K_1$ and $n-i>K_1$,

$$\frac{G(n)}{G(i)G(i+1,n)}>\frac{H(n)}{H(i)H(i+1,n)}\cdot\frac{\varphi(1)+(1-\theta/b)\varphi(2)-\varepsilon}{[\varphi(1)+(1-\theta/b)\varphi(2)+\varepsilon]^2}, \tag{41}$$

$$G(i)>H(i)(\varphi(1)+(1-\theta/b)\varphi(2)-\varepsilon). \tag{42}$$

Recall that $\xi_k > 0$, $k \geq 1$ by (11). Then, by some easy computation, we have

$$\frac{H(n)}{H(i)H(i+1,n)} = \frac{\sum_{j=1}^{n+1} \xi_j^{-1} \cdots \xi_n^{-1}}{\left(\sum_{j=1}^{i+1} \xi_j^{-1} \cdots \xi_i^{-1}\right)\left(\sum_{j=i+1}^{n+1} \xi_j^{-1} \cdots \xi_n^{-1}\right)}$$
$$= \frac{\sum_{j=1}^{n+1} \xi_1 \cdots \xi_{j-1}}{\left(\sum_{j=1}^{i+1} \xi_1 \cdots \xi_{j-1}\right)\left(\sum_{j=i+1}^{n+1} \xi_{i+1} \cdots \xi_{j-1}\right)} \geq \frac{1}{\sum_{j=i+1}^{n+1} \xi_{i+1} \cdots \xi_{j-1}}. \quad (43)$$

It follows from (31) that, for all $i > N_6$ and $j - i \geq N_5 + 2$,

$$\xi_{i+1} \cdots \xi_{j-1} \leq c_8 \varrho(M_{i+1})^{-1} \cdots \varrho(M_{j-1})^{-1}.$$

Thus, for all $i > N_6$,

$$\sum_{j=i+N_5+2}^{n+1} \xi_{i+1} \cdots \xi_{j-1} < c_8 \sum_{j=i+N_5+2}^{n+1} \varrho(M_{i+1})^{-1} \cdots \varrho(M_{j-1})^{-1}. \quad (44)$$

Then, under the assumption $\varrho(M_{i+1}) \geq 1$, we have

$$\sum_{j=i+N_5+2}^{n+1} \xi_{i+1} \cdots \xi_{j-1} \leq c_8(n - i - N_5). \quad (45)$$

On the other hand, since $\xi_n \to \varrho_1/(a\theta - bd) =: \xi > 0$ as $n \to \infty$, there exists a constant K_2 such that, for all $n \geq K_2$, $\xi_n < \xi + 1$. As a result, for all $i > K_2$,

$$\sum_{j=i+1}^{i+N_5+1} \xi_{i+1} \cdots \xi_{j-1} \leq N_5(\xi + 1)^{N_5}.$$

Consulting (43), (44), and (45), we have, for all $i > K_3 := \max\{K_2, N_6\}$,

$$\frac{H(n)}{H(i)H(i+1,n)} \geq \frac{c}{n-i+1}.$$

In view of (41), we have thus shown that, for all $i \geq K := \max\{K_3, K_1\}$ and $n - i > K_1$,

$$\frac{G(n)}{G(i)G(i+1,n)} \geq \frac{c}{n-i+1}.$$

Taking this and (40) together, we get, for $m > \log K$ (which implies $2^{m-1} + 1 > K$),

$$b_m = \min_{2^m < n \leq 2^{m+1}} \sum_{i=2^{m-1}+1}^{n} P(i \in C \mid A_{m,m+1} > 0, l_m = n)$$
$$\geq c \min_{2^m < n \leq 2^{m+1}} \sum_{i=2^{m-1}+1}^{n-K_1-1} \frac{1}{n-i+1}$$
$$= c \min_{2^m < n \leq 2^{m+1}} \sum_{j=K_1+2}^{n-2^{m-1}} \frac{1}{j} = c \sum_{j=K_1+2}^{2^{m-1}} \frac{1}{j} \geq c \int_{K_1+2}^{2^{m-1}+1} \frac{1}{x}\, \mathrm{d}x \geq cm \log 2.$$

Substituting this into (39), using Lemma 9 and (42), we see that

$$\begin{aligned}\sum_{m=K+1}^{\infty} P(A_{m,m+1}>0) &\le \sum_{m=K+1}^{\infty}\frac{1}{b_m}\sum_{j=2^{m-1}+1}^{2^{m+1}} P(j\in C)\\ &= \sum_{m=K+1}^{\infty}\frac{1}{b_m}\sum_{j=2^{m-1}+1}^{2^{m+1}}\frac{1}{G(j)}\\ &\le c\sum_{m=K+1}^{\infty}\frac{1}{m}\sum_{j=2^{m-1}+1}^{2^{m+1}}\frac{1}{H(j)}\\ &\le c\sum_{m=K+1}^{\infty}\sum_{j=2^{m-1}+1}^{2^{m+1}}\frac{1}{H(j)\log j}\le c\sum_{n=2^K+1}^{\infty}\frac{1}{H(n)\log n}. \end{aligned} \tag{46}$$

Note that under the condition in case (i), $\sum_{n=2}^{\infty} 1/(L(n)\log n)<\infty$. Thus, it follows from (32) that the right-hand side side of (46) is finite, so is $\sum_{m=K+1}^{\infty} P(A_{m,m+1}>0)$. Applying the Borel–Cantelli lemma, we conclude that with probability 1, at most finitely many of the events $\{A_{m,m+1}>0\}$, $m\ge 1$, occur, which completes the first part of Theorem 2.

Next, we turn to the second part. Suppose there exists some $\delta>0$ such that $L(n)\le \delta n\log n$ for n large enough and $\sum_{n=2}^{\infty} 1/(L(n)\log n)=\infty$.

We also use Borel–Cantelli Lemma to prove the result in this case, but here we need to estimate not only the sum of $P(A_j)$, but also the sum of $P(A_jA_l)$.

First, let's study the probability $P(A_j)$. For $j\ge 1$, let $n_j=[j\log j]$ be the integer part of $j\log j$ and set $A_j=\{n_j\in C_k\}$. For fixed $\varepsilon>0$, in view of Lemma 3 there exist constants $L_1>k_0$ and L_2 such that, for all $n-k\ge L_1$, $k\ge k_0$,

$$\varphi(1)+\left(1-\frac{\theta}{b}\right)\varphi(2)-\varepsilon\le\frac{G(k,n)}{H(k,n)}\le\varphi(1)+\left(1-\frac{\theta}{b}\right)\varphi(2)+\varepsilon, \tag{47}$$

and, for all $m>L_2$,

$$\varphi(1)+\left(1-\frac{\theta}{b}\right)\varphi(2)-\varepsilon\le\frac{G(1,m)}{H(1,m)}\le\varphi(1)+\left(1-\frac{\theta}{b}\right)\varphi(2)+\varepsilon. \tag{48}$$

Notice that the sequence a_n+b_n, $n\ge 0$, is bounded away from 0 and positive. Then, taking (48) and Lemma 9 into account, we obtain

$$\sum_{j=L_2}^{\infty}P(A_j)=\sum_{j=L_2}^{\infty}\frac{1}{G(1,n_j)}\prod_{i=n_j+1}^{n_j+k-1}\frac{1}{1+a_i+b_i}\ge c\sum_{j=L_2}^{\infty}\frac{1}{H([j\log j])}. \tag{49}$$

Under the assumption $\varrho(M_j)\ge 1$ for all $j\ge 1$, we can see that $L(n)$ is increasing from (37). Applying [6, Lemma 2.2], we conclude that $\sum_{j=2}^{\infty} 1/(L([j\log j]))$ and $\sum_{j=2}^{\infty} 1/(L(j)\log j)$ converge or diverge simultaneously. As a result, it follows from (49) and Lemma 8 that

$$\sum_{j=\max\{L_2,N_7\}}^{\infty}P(A_j)\ge c\sum_{j=\max\{L_2,N_7\}}^{\infty}\frac{1}{L([j\log j])}=\infty. \tag{50}$$

Second, we study the probability $P(A_jA_l)$. Define $\mathcal{C}_k = \{(j, l)\colon 2 \le j < l, l\log l > j\log j + k\}$. Note that when $j > \mathrm{e}^{\max\{L_1, L_2\}+k}$ and $l \ge j+1$, we have $n_l - n_j - k > L_1$ and $n_l > L_2$. It thus follows from Lemma 9 and (47) that for the ε above, when $(j, l) \in \mathcal{C}_k$ satisfying $j > \mathrm{e}^{\max\{L_1, L_2\}+k}$,

$$\begin{aligned} P(A_jA_l) &= P(n_j \in C_k, n_l \in C_k) \\ &= \frac{1}{G(n_j)} \frac{1}{G(n_j+k, n_l)} \prod_{i=n_j+1}^{n_j+k-1} \frac{1}{1+a_i+b_i} \prod_{i=n_l+1}^{n_l+k-1} \frac{1}{1+a_i+b_i} \\ &= P(A_j)P(A_l) \frac{G(n_l)}{G(n_j+k, n_l)} \\ &\le P(A_j)P(A_l) \frac{H(n_l)}{H(n_j+k, n_l)} \frac{\varphi(1)+(1-\theta/b)\varphi(2)+\varepsilon}{\varphi(1)+(1-\theta/b)\varphi(2)-\varepsilon} \\ &= P(A_j)P(A_l) \frac{H(n_l)}{H(n_j+k, n_l)} (1+c\varepsilon). \end{aligned}$$

Then, taking (38) and the fact $\log(1-x) \le -x$ for all $0 < x < 1$ into account, we have

$$P(A_jA_l) \le P(A_j)P(A_l) \left(1 - \exp\left\{-\sum_{i=n_j+k-1}^{n_l} \frac{1}{H(i)}\right\}\right)^{-1} (1+c\varepsilon). \tag{51}$$

For the above $\varepsilon > 0$, let

$$\ell = \min\left\{l \ge j+1\colon \sum_{i=n_j+k-1}^{n_l} \frac{1}{H(i)} \ge \log\frac{1+\varepsilon}{\varepsilon}\right\}.$$

Obviously, for $l \ge \ell$, $\left(1 - \exp\left\{-\sum_{i=n_j+k-1}^{n_l} 1/H(i)\right\}\right)^{-1} \le 1+\varepsilon$. Thus, it follows from (51) that

$$P(A_jA_l) \le (1+c\varepsilon)(1+\varepsilon)P(A_j)P(A_l) \le (1+c\varepsilon)P(A_j)P(A_l) \text{ for all } l \ge \ell,\ (j, l) \in \mathcal{C}_k. \tag{52}$$

Next, suppose $l < \ell$. Note that for $0 < u < \log((1+\varepsilon)/\varepsilon)$ we have $1 - \mathrm{e}^{-u} \ge cu$ for some $c := c(\varepsilon) > 0$ small enough. Then, in view of (51) and (32), we have

$$\begin{aligned} P(A_jA_l) &\le c(1+c\varepsilon)P(A_j)P(A_l) \left(\sum_{i=n_j+k-1}^{n_l} \frac{1}{H(i)}\right)^{-1} \\ &\le c(1+c\varepsilon)P(A_j)P(A_l) \left(\sum_{i=n_j+k-1}^{n_l} \frac{1}{L(i)}\right)^{-1} \quad \text{for } j > N_7. \end{aligned} \tag{53}$$

Since $L(n)$ is increasing, we have

$$\left(\sum_{i=n_j+k-1}^{n_l} \frac{1}{L(i)}\right)^{-1} \le \frac{L(n_l)}{n_l - n_j - k + 2}$$

for all $j \ge N_7$. Again by (32), we have

$$\left(\sum_{i=n_j+k-1}^{n_l} \frac{1}{L(i)}\right)^{-1} \le \frac{1}{c_9}\frac{H(n_l)}{n_l - n_j - k + 2}.$$

Therefore, taking (53) and (48) into account, we get that for $(j, l) \in \mathcal{C}_k$ satisfying $\ell \ge l \ge j+1$ and $j \ge \max\{N_7, \mathrm{e}^{\max\{L_1, L_2\}+k}\} =: M$,

$$\begin{aligned} P(A_jA_l) &\le c\frac{G(n_l)}{n_l - n_j - k + 2} P(A_j)P(A_l) \\ &= c\left(\prod_{i=1}^{k-1} \frac{1}{1 + a_{n_l+i} + b_{n_l+i}}\right)\frac{P(A_j)}{n_l - n_j - k + 2} \le \frac{cP(A_j)}{l\log l - j\log j}. \end{aligned}$$

Consequently, for $j \ge M$,

$$\begin{aligned} \sum_{j+1\le l<\ell, (j,l)\in\mathcal{C}_k} P(A_jA_l) &\le \sum_{j+1\le l<\ell, (j,l)\in\mathcal{C}_k} \frac{cP(A_j)}{l\log l - j\log j} \\ &\le cP(A_j)\sum_{l=j+1}^{\ell-1} \frac{1}{l\log l - j\log j} \\ &\le cP(A_j)\frac{1}{\log j}\sum_{l=j+1}^{\ell-1}\frac{1}{l-j} \le cP(A_j)\frac{\log \ell}{\log j}. \end{aligned} \tag{54}$$

Recall that

$$\sum_{i=n_j+k-1}^{n_l} \frac{1}{H(i)} < \log\frac{1+\varepsilon}{\varepsilon}, \quad j+1 \le l < \ell, \tag{55}$$

and $L(n) \le \delta n \log n$ for some $\delta > 0$ and n large enough. We claim that if j is large enough, then

$$\ell < j^\gamma \quad \text{if } \gamma > \left(\frac{1+\varepsilon}{\varepsilon}\right)^{c_{10}\delta} + \varepsilon. \tag{56}$$

Suppose on the contrary that $\ell \ge j^\gamma$. Then, for $j > N_7$,

$$\begin{aligned} \sum_{i=n_j+k-1}^{n_\ell} \frac{1}{H(i)} &\ge \sum_{i=n_j+k-1}^{n_\ell} \frac{1}{c_{10}L(i)} \\ &\ge \frac{1}{c_{10}\delta}\sum_{i=n_j+k-1}^{n_\ell}\frac{1}{i\log i} \\ &\ge \frac{1}{c_{10}\delta}(\log\log n_\ell - \log\log(n_j + k - 1)) \\ &\ge \frac{1}{c_{10}\delta}(\log\log n_\ell - \log\log n_{j+k}) \\ &\ge \frac{1}{c_{10}\delta}\log\frac{\gamma\log j + \log\gamma + \log\log j}{\log(j+k) + \log\log(j+k)} \ge \frac{1}{c_{10}\delta}\log(\gamma - \varepsilon) \end{aligned}$$

for j large enough. Since $\gamma > ((1+\varepsilon)/\varepsilon)^{c_{10}\delta} + \varepsilon$, we have

$$\sum_{i=n_j+k-1}^{n_\ell} \frac{1}{H(i)} \geq \log \frac{1+\varepsilon}{\varepsilon},$$

which contradicts (55). This means (56) is right.

Applying (56) and (54), we obtain, for j large enough,

$$\sum_{j+1\leq l<\ell,(j,l)\in\mathcal{C}_k} P(A_jA_l) \leq cP(A_j).$$

Taking this together with (52), we conclude that, for some $j_0 > 0$,

$$\sum_{j=j_0}^{n} \sum_{j<l\leq n,(j,l)\in\mathcal{C}_k} P(A_jA_l) \leq \sum_{j=j_0}^{n} \sum_{j<l\leq n,(j,l)\in\mathcal{C}_k} (1+c\varepsilon)P(A_j)P(A_l) + c\sum_{j=j_0}^{n} P(A_j).$$

Therefore, taking (50) into account, we have

$$\begin{aligned}\alpha &:= \varliminf_{n\to\infty} \frac{\sum_{j=j_0}^{n}\sum_{j<l\leq n,(j,l)\in\mathcal{C}_k} P(A_jA_l) - \sum_{j=j_0}^{n}\sum_{j<l\leq n,(j,l)\in\mathcal{C}_k} (1+c\varepsilon)P(A_j)P(A_l)}{\big(\sum_{j=j_0}^{n} P(A_j)\big)^2} \\ &\leq \varliminf_{n\to\infty} \frac{c}{\sum_{j=j_0}^{n} P(A_j)} = 0.\end{aligned}$$

An application of the Borel–Cantelli lemma [22, p. 235] yields

$$\begin{aligned}P(A_j,\ j\geq 1 \text{ occur infinitely often}) &\geq P(A_j,\ j\geq j_0 \text{ occur infinitely often}) \\ &\geq \frac{1}{1+\varepsilon+2\alpha} \geq \frac{1}{1+\varepsilon}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can conclude that $P(A_j,\ j\geq 1 \text{ occur infinitely often}) = 1$. So, the second part of the theorem is proved. □

5. Examples

For $n\geq 1$, let $q_n, p_n > 0$ be numbers such that $q_n + p_n = 1$. Suppose that $\mathbf{Z}_n$, $n\geq 0$, is a two-type branching process with immigration satisfying $\mathbf{Z}_0 = 0$, and there is a fixed immigration $\mathbf{e}_1$ in each generation, with offspring distributions

$$\begin{aligned}&\mathbb{P}(\mathbf{Z}_n = (0,j) \mid \mathbf{Z}_{n-1} = \mathbf{e}_1) = q_n^j p_n, \\ &\mathbb{P}(\mathbf{Z}_n = (1,j) \mid \mathbf{Z}_{n-1} = \mathbf{e}_2) = q_n^j p_n, \quad j\geq 0,\ n\geq 1.\end{aligned}$$

Some computation yields the mean matrix

$$M_n = \begin{pmatrix} 0 & b_n \\ 1 & b_n \end{pmatrix}$$

with $b_n = q_n/p_n$, $n\geq 1$. Then $\varrho(M_k) = (b_k + \sqrt{b_k^2 + 4b_k})/2$.

Fix $B \geq 0$. Set $i_0 := \min\left\{i\colon B/3i < \frac{2}{3}\right\}$ and let

$$p_i := \begin{cases} \frac{2}{3} - B/3i, & i \geq i_0, \\ \frac{2}{3}, & i < i_0. \end{cases}$$

Theorem 3. *Fix $B \geq 0$. If $B \geq 1$, then $\{\mathbf{Z}_n\}$ has finitely many regeneration times, almost surely. If $B < 1$, then $\{\mathbf{Z}_n\}$ has infinitely many k-strong regeneration points, almost surely.*

Proof. For $B \geq 0$, let $r_n = B/3n$. It is easy to see that $\lim_{n\to\infty} n^2(r_n - r_{n+1}) = B/3$. Thus, by some computation, we obtain

$$|b_{k+1} - b_k| \sim |r_{n+1} - r_n| \sim \frac{1}{n^2}, \quad n \to \infty,$$

which implies that $\sum_{k=1}^{\infty} |b_{k+1} - b_k| < \infty$. Since $p_k = \frac{2}{3} - r_k$, $k \geq 1$, we see that $b_k \geq \frac{1}{2}$. As a result, $\varrho(M_k) \geq 1$. Thus, the conditions in Theorem 2 are fulfilled.

By Taylor expansion of $\varrho(M_k)$ at 0, we get $\varrho(M_k) = 1 + 3r_k + O(r_k^2)$ as $k \to \infty$. And then, by Euler's asymptotic formula for the harmonic series, we get that

$$\varrho(M_1) \cdots \varrho(M_n) \sim cn^B \quad \text{as } n \to \infty. \tag{57}$$

When $B > 1$, $c\int_{k_0}^{n} 1/x^B \,\mathrm{d}x$ is convergence, so is $\sum_{k=k_0}^{n} \varrho(M_1)^{-1} \cdots \varrho(M_k)^{-1}$ by (57). Then it follows from (57) that

$$L(n) = \sum_{k=1}^{n} \varrho(M_k) \cdots \varrho(M_n) = \varrho(M_1) \cdots \varrho(M_n) \sum_{k=k_0}^{n} \varrho(M_1)^{-1} \cdots \varrho(M_{k-1})^{-1} \sim cn^B$$

as $n \to \infty$, which implies that $\sum_{n=2}^{\infty} 1/(L(n)\log n) < \infty$. So the conditions in Theorem 2(i) are satisfied. Then, by Theorem 2(i), the branching process has finitely many regeneration times almost surely.

When $B \leq 1$, $\int_{i_0}^{n} 1/x^B \,\mathrm{d}x$ is divergent, so is $\sum_{k=k_0}^{n} \varrho(M_1)^{-1} \cdots \varrho(M_{k-1})^{-1}$, and we have $\sum_{k=k_0}^{n} \varrho(M_1)^{-1} \cdots \varrho(M_{k-1})^{-1} \sim \int_{i_0}^{n} 1/x^B \,\mathrm{d}x$ by (57), while

$$\lim_{n\to\infty} \int_{i_0}^{n} \frac{1}{x^B} \,\mathrm{d}x = \begin{cases} \dfrac{1}{1-B} n^{1-B} - \dfrac{1}{(1-B)} i_0^{1-B}, & B < 1, \\ \log n - \log i_0, & B = 1. \end{cases} \tag{58}$$

In view of (57), we thus have

$$L(n) \sim \begin{cases} cn, & B < 1, \\ cn\log n, & B = 1 \end{cases} \quad \text{as } n \to \infty. \tag{59}$$

So if $B = 1$, then $\sum_{n=2}^{\infty} 1/(L(n)\log n) < \infty$. Applying Theorem 2(i), we can see that the branching process has finitely many regeneration times almost surely.

Assuming $B < 1$, it follows from (59) that $\sum_{n=2}^{\infty} 1/(L(n)\log n) = \infty$ and $L(n)\log n \leq cn\log n$ for n large enough. Then, by Theorem 2(ii), we conclude that the process has infinitely many k-strong regeneration times almost surely. □

Remark 5. Let $\{\mathbf{Y}_n\}_{n\geq 0}$ be a branching process in varying environments with $\mathbf{Y}_0 = \mathbf{e}_1$, which shares the same branching mechanism as $\{\mathbf{Z}_n\}_{n\geq 0}$ in this section. By the results in [27, Theorem 1], we can see that the tail probability of surviving time ν of the process $\{\mathbf{Y}_n\}_{n\geq 0}$ satisfies

$$P(\nu > n) \sim \frac{c}{1 + \sum_{k=1}^{n} \varrho(M_1)^{-1} \cdots \varrho(M_k)^{-1}}.$$

When $B = 1$, it follows from (58) that $P(\nu > n) \sim c/(n \log n) \to 0$. So $\{\mathbf{Y}_n\}_{n\geq 0}$ is extinct in this situition. Then we conclude that $\{\mathbf{Z}_n\}_{n\geq 0}$ should have a regeneration time. But, by Theorem 3, in this case, the process $\{\mathbf{Z}_n\}_{n\geq 0}$ has finitely many regeneration times. Such a phenomenon never happens for the time-homogenous branching process, since by the time-homogenous property, if the process owns one regeneration time, it must have infinitely many regeneration times.

When there are infinitely many regeneration times, we have established the asymptotic property of the number of regeneration times in $[0,n]$ as follows.

Theorem 4. *Fix* $0 \leq B < 1$. *Then*

$$\lim_{n\to\infty} \frac{E\#\{k\colon k \in C \cap [0, n]\}}{\log n} = c > 0 \tag{60}$$

and, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\#\{k\colon k \in C \cap [0, n]\}}{(\log n)^{1+\varepsilon}} = 0 \tag{61}$$

almost surely.

Remark 6. Notice that Theorem 4 contains the case $B = 0$. In this case, $p_i \equiv \frac{2}{3}$ and $\varrho(M_i) \equiv 1$ for all $i \geq 1$, so that $\{\mathbf{Z}_n\}$ is indeed a critical Galton–Watson process with immigration. As shown by Theorem 4, it seems that, up to multiplication by a positive constant, the value of $0 \leq B < 1$ does not affect the order of the number of regeneration times in $[0,n]$.

Proof. Let $S_n = \#\{k\colon k \in C \cap [0, n]\}$. By (14), we can see that

$$E(S_n) = \sum_{i=1}^{n} P(Z_i = 0) = \sum_{i=1}^{n} \frac{1}{G(i)}.$$

Consulting (16) and (32), there exist positive constants C_1 and C_2 such that

$$C_1 \sum_{i=1}^{n} \frac{1}{L(k)} \leq \sum_{i=1}^{n} \frac{1}{G(i)} \leq C_2 \sum_{i=1}^{n} \frac{1}{L(k)}.$$

It follows from (59) that when $B < 1$, $L(n) \sim cn$. As a result,

$$E(S_n) \leq \sum_{1\leq i\leq n} \frac{c}{i}. \tag{62}$$

We thus get (60).

Now we turn to the second part. Noticing that S_n is positive and nondecreasing, by (62) we have $E(\max_{1\le k\le n} S_k) = E(S_n) < \sum_{1\le i\le n} c/i$. We know that, for each $\varepsilon > 0$, $\sum_{i=2}^{\infty} 1/(i(\log i)^{1+\varepsilon}) < \infty$. Therefore, by [8, Theorem 2.1], we have

$$\frac{S_n}{(\log n)^{1+\varepsilon}} \to 0$$

almost surely as $n \to \infty$, which completes the proof of (61). □

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References

[1] Agresti, A. (1975). On the extinction times of varying and random environment branching processes. *J. Appl. Prob.* **12**, 39–46.

[2] Athreya, K. B. and Ney, P. E. (1972). *Branching Processes*. Springer, New York.

[3] Bhattacharya, N. and Perlman, M. (2017). Time inhomogeneous branching processes conditioned on non-extinction. Preprint, arXiv:1703.00337 [math.PR].

[4] Biggins, J. D., Cohn, H. and Nerman, O. (1999). Multi-type branching in varying environment. *Stoch. Process. Appl.* **83**, 357–400.

[5] Cohn, H. and Wang, Q. (2003). Multitype branching limit behavior. *Ann. Appl. Prob.* **13**, 490–500.

[6] Csáki, E., Földes, A. and Révész, P. (2010). On the number of cutpoints of the transient nearest neighbor random walk on the line. *J. Theoret. Prob.* **23**, 624–638.

[7] Dolgopyat, D., Hebbar, P., Koralov, L. and Perlman, M. (2018). Multi-type branching processes with time-dependent branching rates. *J. Appl. Prob.* **55**, 701–727.

[8] Fazekas, I. and Klesov, O. (2001). A general approach to the strong law of large numbers. *Theory Prob. Appl.* **45**, 436–449.

[9] Fujimagari, T. (1980). On the extinction time distribution of a branching process in varying environments. *Adv. Appl. Prob.* **12**, 350–366.

[10] Haccou, P., Jagers, P. and Vatutin, V. A. (2005). *Branching Processes: Variation, Growth, and Extinction of Populations*. Cambridge University Press.

[11] Jagers, P. (1974). Galton–Watson processes in varying environment. *J. Appl. Prob.* **11**, 174–178.

[12] James, N., Lyons, R. and Peres, Y. (2008). A transient Markov chain with finitely many cutpoints. In *Probability and Statistics: Essays in Honor of David A. Freedman* (IMS Collections 2), eds D. Nolan and T. Speed. Institute of Mathematical Statistics, pp. 24–29.

[13] Jones, O. D. (1997). On the convergence of multitype branching processes with varying environments. *Ann. Appl. Prob.* **7**, 772–801.

[14] Kersting, G. (2020). A unifying approach to branching processes in a varying environment. *J. Appl. Prob.* **57**, 196–220.

[15] Kersting, G. and Vatutin, V. (2017). *Discrete Time Branching Processes in Random Environment*. John Wiley, New York.

[16] Kesten, H., Kozlov, M. V. and Spitzer, F. (1975). A limit law for random walk in a random environment. *Compositio Math.* **30**, 145–168.

[17] KEY, E. S. (1987). Limiting distributions and regeneration times for multitype branching processes with immigration in a random environment. *Ann. Prob.* **15**, 344–353.
[18] KIMMEL, M. AND AXELROD, D. E. (2015). *Branching Processes in Biology*. Springer, New York.
[19] LINDVALL, T. (1974). Almost sure convergence of branching processes in varying and random environments. *Ann. Prob.* **2**, 344–346.
[20] LORENTZEN, L. (1995). Computation of limit periodic continued fractions. A survey. *Numer. Algorithms* **10**, 69–111.
[21] MACPHEE, I. M. AND SCHUH, H. J. (1983). A Galton–Watson branching process in varying environments with essentially constant means and two rates of growth. *Austrl. N. Z. J. Statist.* **25**, 329–338.
[22] PETROV, V. V. (2004). A generalization of the Borel–Cantelli lemma. *Statist. Prob. Lett.* **67**, 233–239.
[23] ROITERSHTEIN, A. (2007). A note on multitype branching processes with immigration in a random environment. *Ann. Prob.* **35**, 1573–1592.
[24] SIGMAN, K. AND WOLFF, R. W. (1993). A review of regenerative process. *SIAM Rev.* **35**, 269–288.
[25] WANG, H. M. (2013). A note on multitype branching process with bounded immigration in random environment. *Acta Math. Sin. (Engl. Ser.)* **29**, 1095–1110.
[26] WANG, H. M. (2019). On the number of points skipped by a transient (1,2) random walk on the lattice of the positive half line. *Markov Process. Relat. Fields* **25**, 125–148.
[27] WANG, H. M. (2021). On extinction time distribution of a 2-type linear-fractional branching process in a varying environment with asymptotically constant mean matrices. Preprint, arXiv:2106.01203 [math.PR].
[28] WANG, H. M. (2021). Asymptotics of entries of products of nonnegative 2-by-2 matrices. Preprint, arXiv:2111.10232 [math.PR].
[29] WANG, H. M. AND YAO, H. (2022). Two-type linear fractional branching processes in varying environments with asymptotically constant mean matrices. *J. Appl. Prob.* **59**, 224–255.