

## DIMENSION INEQUALITY FOR A DEFINABLY COMPLETE UNIFORMLY LOCALLY O-MINIMAL STRUCTURE OF THE SECOND KIND

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**Abstract.** Consider a definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group. Let  $f : X \rightarrow R^n$  be a definable map, where  $X$  is a definable set and  $R$  is the universe of the structure. We demonstrate the inequality  $\dim(f(X)) \leq \dim(X)$  in this paper. As a corollary, we get that the set of the points at which  $f$  is discontinuous is of dimension smaller than  $\dim(X)$ . We also show that the structure is definably Baire in the course of the proof of the inequality.

**§1. Introduction.** A uniformly locally o-minimal structure of the second kind was first defined and investigated in the author's previous work [4]. It enjoys several tame properties such as local monotonicity. In addition, it admits local definable cell decomposition when it is definably complete.

In [4], the author defined dimension of a set definable in a locally o-minimal structure admitting local definable cell decomposition. Many assertions on dimension known in o-minimal structures [8] also hold true for locally o-minimal structures admitting local definable cell decomposition which are not necessarily definably complete [4, Section 5]. An exception is the inequality  $\dim(f(X)) \leq \dim(X)$ , where  $f : X \rightarrow R^n$  is a definable map. Here,  $R$  denotes the universe of the structure and  $X$  is a definable set. The author gave an example which does not satisfy the above dimension inequality in [4, Remark 5.5]. The structure in the example is not definably complete. A question is whether the dimension inequality holds true when the structure is definably complete. This paper gives an affirmative answer to this question. Our main theorem is as follows:

**THEOREM 1.1.** *Let  $\mathcal{R} = (R, <, +, 0, \dots)$  be a definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group. The inequality*

$$\dim(f(X)) \leq \dim(X)$$

*holds true for any definable map  $f : X \rightarrow R^n$ .*

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We get the following corollary:

**COROLLARY 1.2.** *Let  $\mathcal{R} = (R, <, +, 0, \dots)$  be as in Theorem 1.1. Let  $f : X \rightarrow R$  be a definable function. The set of the points at which  $f$  is discontinuous is of dimension smaller than  $\dim(X)$ .*

The author proved the dimension inequality in [3, Theorem 2.4] when the universe of the structure is the set of reals. This fact is not a direct corollary of the above theorem, that deals with expansions of abelian groups.

The paper is organized as follows. In Section 2, we first review the previous works relevant to this study. We prove several basic facts in Section 3. Satisfaction of the dimension inequality is relevant to definably Baire property introduced in [2]. Section 4 treats the definably Baire property. We show that a definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group is definably Baire in the section. We finally demonstrate Theorem 1.1 in Section 5.

We introduce the terms and notations used in this paper. The term ‘definable’ means ‘definable in the given structure with parameters.’ For any set  $X \subseteq R^{m+n}$  definable in a structure  $\mathcal{R} = (R, \dots)$  and for any  $x \in R^m$ , the notation  $X_x$  denotes the fiber defined as  $\{y \in R^n \mid (x, y) \in X\}$ . For a linearly ordered structure  $\mathcal{R} = (R, <, \dots)$ , an open interval is a definable set of the form  $\{x \in R \mid a < x < b\}$  for some  $a, b \in R$ . It is denoted by  $(a, b)$ . We define a closed interval in the same manner and it is denoted by  $[a, b]$ . An open box in  $R^n$  is the direct product of  $n$  open intervals. A closed box is defined similarly. A CBD set is a closed, bounded and definable set. Let  $A$  be a subset of a topological space. The notations  $\text{int}(A)$ ,  $\overline{A}$  and  $\partial A$  denote the interior, the closure and the frontier of the set  $A$ , respectively. The notation  $|S|$  denotes the cardinality of a set  $S$ .

**§2. Review of previous works.** We review the definitions and the assertions given in the previous works. A densely linearly ordered structure without endpoints  $\mathcal{R} = (R, <, \dots)$  is *definably complete* if every definable subset of  $R$  has both a supremum and an infimum in  $R \cup \{\pm\infty\}$ . The definition of a definably complete structure is found in [1, 6]. The structure  $\mathcal{R}$  is *locally o-minimal* if, for every point  $a \in R$  and every definable set  $X \subset R$ , there exists an open interval  $I$  containing the point  $a$  such that  $X \cap I$  is a finite union of points and intervals. A locally o-minimal structure is defined and investigated in [7]. We review the definition of uniformly locally o-minimal structures of the second kind [4].

**DEFINITION 2.1.** A locally o-minimal structure  $\mathcal{R} = (R, <, \dots)$  is a *uniformly locally o-minimal structure of the second kind* if, for any positive integer  $n$ , any definable set  $X \subseteq R^{n+1}$ ,  $a \in R^n$  and  $b \in R$ , there exist an open interval  $I$  containing the point  $b$  and an open box  $B$  containing  $a$  such that the definable sets  $X_x \cap I$  are finite unions of points and open intervals for all  $x \in B$ .

For a densely linearly ordered structure without endpoints  $\mathcal{R} = (R, <, \dots)$ , *cells* in  $R^n$  are definable subsets of  $R^n$  defined inductively as follows:

- A cell in  $R$  is a point or an open interval.
- A cell in  $R^n$  is the graph of a continuous definable function defined on a cell in  $R^{n-1}$  or a definable set of the form  $\{(x, y) \in C \times M \mid f(x) < y < g(x)\}$ , where

$C$  is a cell in  $R^{n-1}$  and  $f$  and  $g$  are definable continuous functions defined on  $C$  with  $f < g$ ,  $f = -\infty$  or  $g = +\infty$ .

We inductively define a *definable cell decomposition* of an open box  $B \subseteq R^n$ . For  $n = 1$ , a definable cell decomposition of  $B$  is a partition  $B = \bigcup_{i=1}^m C_i$  into finite cells. For  $n > 1$ , a definable cell decomposition of  $B$  is a partition  $B = \bigcup_{i=1}^m C_i$  into finite cells such that  $\pi(B) = \bigcup_{i=1}^m \pi(C_i)$  is a definable cell decomposition of  $\pi(B)$ , where  $\pi : R^n \rightarrow R^{n-1}$  is the projection forgetting the last coordinate. A *definable cell decomposition of  $B$  partitioning a finite family  $\{A_\lambda\}_{\lambda \in \Lambda}$*  of definable subsets of  $B$  is a definable cell decomposition of  $B$  such that the definable sets  $A_\lambda$  are unions of cells for all  $\lambda \in \Lambda$ .

The following local definable cell decomposition theorem is a main theorem of [4].

**THEOREM 2.2** (Local definable cell decomposition theorem). *Consider a definably complete uniformly locally o-minimal structure of the second kind  $\mathcal{R} = (R, <, \dots)$ . For any positive integer  $n$ , the following assertions hold true:*

- (D): *Let  $\{A_\lambda\}_{\lambda \in \Lambda}$  be a finite family of definable subsets of  $R^n$ . For any point  $a \in R^n$ , there exist an open box  $B$  containing the point  $a$  and a definable cell decomposition of  $B$  partitioning the finite family  $\{B \cap A_\lambda \mid \lambda \in \Lambda \text{ and } B \cap A_\lambda \neq \emptyset\}$ .*
- (C): *Let  $A \subseteq R^n$  be a definable subset and  $f : A \rightarrow R$  be a definable function. For any  $a \in R^n$ ,  $b \in R$  and any sufficiently small open interval  $J$  with  $b \in J$ , there exist an open box  $B$  containing the point  $a$  and a definable cell decomposition of  $B$  partitioning  $f^{-1}(J) \cap B$  such that the function  $f$  is continuous on any cell contained in  $f^{-1}(J) \cap B$ .*
- (U): *Let  $X$  be a definable subset of  $R^{n+1}$ . For any  $a \in R^n$  and  $b \in R$ , there exist an open interval  $I$  containing the point  $b$ , an open box  $B$  with  $a \in B$  and a positive integer  $N$  such that, for any  $x \in B$ , the definable set  $X_x \cap I$  contains an interval or  $|X_x \cap I| \leq N$  for any  $x \in B$ .*

**PROOF.** [4, Theorem 4.2] ◻

A locally o-minimal structure *admits local definable cell decomposition* if the assertions (D) hold true for all positive integers  $n$ . A definably complete locally o-minimal structure admits local definable cell decomposition if and only if it is a uniformly locally o-minimal structure of the second kind by [4, Corollary 4.1]. We can obtain the following corollary:

**COROLLARY 2.3.** *Consider a definably complete uniformly locally o-minimal structure of the second kind  $\mathcal{R} = (R, <, \dots)$ . Let  $X$  be a definable subset of  $R^{n+1}$  whose fibers  $X_x$  are closed for all  $x \in R^n$ . For any  $a \in R^n$  and  $b \in R$ , there exist a closed interval  $I$ , a closed box  $B$  in  $R^n$  and a positive integer  $N$  satisfying the following conditions:*

- *The point  $a$  is contained in the interior of  $B$ ;*
- *The point  $b$  is contained in the interior of  $I$ ;*
- *The definable set  $X_x \cap I$  is a disjoint union of at most  $N$  points and  $N$  closed intervals for any  $x \in B$ .*

PROOF. We consider the definable set

$$Y = \{(x, y) \in R^n \times R \mid X_x \text{ includes an open interval containing the point } y\}.$$

Take a sufficiently small open interval  $J$  with  $b \in J$  and a sufficiently small open box  $C$  with  $a \in C$ . For any  $x \in C$ , the definable set  $(X_x \setminus Y_x) \cap J$  is a union of at most  $N$  points for some positive integer  $N$  or contains an interval by Theorem 2.2(U). However, the latter situation never occurs by the definition of  $Y$ . Shrinking  $J$  and  $C$  if necessary, we may further assume that  $X_x$  is a finite union of points and open intervals because the structure  $\mathcal{R}$  is a uniformly locally o-minimal structure of the second kind. Take a closed interval  $I \subset J$  with  $b \in \text{int}(I)$  and a closed box  $B \subset C$  with  $a \in \text{int}(B)$ . Fix an arbitrary point  $x \in B$ . Since  $X_x \cap I$  is closed and a finite union of points and open intervals,  $X_x \cap I$  is a finite disjoint union of discrete points and closed intervals. The points in  $(X_x \setminus Y_x) \cap I$  are endpoints of the closed intervals or the discrete points. Therefore, the definable set  $X_x \cap I$  is a disjoint union of at most  $N$  points and  $N$  closed intervals.  $\dashv$

The dimension of a set definable in a locally o-minimal structure admitting local definable cell decomposition is defined in [4, Section 5]. A definable set  $X \subseteq R^n$  is of  $\dim(X) \geq m$  if there exists an open box  $B \subseteq R^m$  and a definable continuous injective map  $f : B \rightarrow X$  which is homeomorphic onto its image. A definable set  $X \subseteq R^n$  is of  $\dim(X) = m$  if it is of  $\dim(X) \geq m$  and it is not of  $\dim(X) \geq m + 1$ . The empty set is defined to be of dimension  $-\infty$ . The dimension satisfies the following basic properties:

PROPOSITION 2.4. Consider a locally o-minimal structure  $\mathcal{R} = (R, <, \dots)$  admitting local definable cell decomposition. The following assertions hold true:

- (a): Let  $X \subseteq Y$  be definable sets. Then, the inequality  $\dim(X) \leq \dim(Y)$  holds true.
- (b): Let  $X$  and  $Y$  be definable subsets of  $R^n$ . We have

$$\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}.$$

- (c): The inequality  $\dim(\partial S) < \dim(S)$  is satisfied for any definable set  $S$ .

PROOF. (a) [4, Lemma 5.1]; (b) [4, Corollary 5.4(ii)]; (c) [4, Theorem 5.6].  $\dashv$

Recall that the purpose of this paper is to demonstrate another basic inequality  $\dim(f(X)) \leq \dim(X)$ , where  $f : X \rightarrow R^n$  is a definable map, when the structure  $\mathcal{R}$  is definably complete.

We get the following lemma on the dimension of the projection image. A lemma similar to it is found in [3], but we give a complete proof here.

LEMMA 2.5. Consider a locally o-minimal structure  $\mathcal{R} = (R, <, \dots)$  admitting local definable cell decomposition. Let  $X$  be a definable subset of  $R^{m+n}$  and  $\pi : R^{m+n} \rightarrow R^m$  be a coordinate projection. Assume that the fibers  $X_x$  are of dimension  $\leq 0$  for all  $x \in R^m$ . Then, we have  $\dim X \leq \dim \pi(X)$ .

PROOF. For any  $(a, b) \in R^m \times R^n$ , there exist open boxes  $B_a \subseteq R^m$  and  $B_b \subseteq R^n$  with  $(a, b) \in B_a \times B_b$  and  $\dim(X \cap (B_a \times B_b)) = \dim \pi(X \cap (B_a \times B_b))$  by [4, Lemma 5.4]. We have  $\dim \pi(X \cap (B_a \times B_b)) \leq \dim \pi(X)$  by Proposition 2.4(a). On the other

hand, we have  $\dim(X) = \sup_{(a,b) \in R^m \times R^n} \dim(X \cap (B_a \times B_b))$  by [4, Corollary 5.3]. We have finished the proof.  $\dashv$

We also review  $D_\Sigma$ -sets introduced in [1].

**DEFINITION 2.6** ( $D_\Sigma$ -sets). Consider an expansion of a linearly ordered structure  $\mathcal{R} = (R, <, 0, \dots)$ . A *parameterized family* of definable sets  $\{X\langle x \rangle\}_{x \in S}$  is the family of the fibers of a definable set; that is, there exists a definable set  $\mathcal{X}$  with  $X\langle x \rangle = \mathcal{X}_x$  for all  $x$  in a definable set  $S$ . A parameterized family  $\{X\langle r, s \rangle\}_{r>0, s>0}$  of CBD subsets  $X\langle r, s \rangle$  of  $R^n$  is called a  $D_\Sigma$ -family if  $X\langle r, s \rangle \subseteq X\langle r', s \rangle$  and  $X\langle r, s' \rangle \subseteq X\langle r, s \rangle$  whenever  $r < r'$  and  $s < s'$ . Note that  $X\langle r, s \rangle$  is not necessarily strictly contained in  $X\langle r', s \rangle$ . It is the same for the inclusion  $X\langle r, s' \rangle \subseteq X\langle r, s \rangle$ . A definable subset  $X$  of  $R^n$  is a  $D_\Sigma$ -set if  $X = \bigcup_{r>0, s>0} X\langle r, s \rangle$  for some  $D_\Sigma$ -family  $\{X\langle r, s \rangle\}_{r>0, s>0}$ .

A parameterized family of definable sets  $\{X\langle s \rangle\}_{s>0}$  is a *definable decreasing family of CBD sets* if we have  $X\langle s \rangle = X\langle r, s \rangle$  for some  $D_\Sigma$ -family  $\{X\langle r, s \rangle\}_{r>0, s>0}$  with  $X\langle r_1, s \rangle = X\langle r_2, s \rangle$  for all  $r_1, r_2$  and  $s$ .

We next review definably Baire property introduced in [2].

**DEFINITION 2.7.** Consider an expansion of a densely linearly ordered structure  $\mathcal{R} = (R, <, 0, \dots)$ . A parameterized family of definable sets  $\{X\langle r \rangle\}_{r>0}$  is called a *definable increasing family* if  $X\langle r \rangle \subseteq X\langle r' \rangle$  whenever  $0 < r < r'$ . A definably complete expansion of a densely linearly ordered structure is *definably Baire* if the union  $\bigcup_{r>0} X\langle r \rangle$  of any definable increasing family  $\{X\langle r \rangle\}_{r>0}$  with  $\text{int}(\overline{X\langle r \rangle}) = \emptyset$  has an empty interior.

The following proposition is a direct corollary of the local definable cell decomposition theorem.

**PROPOSITION 2.8.** *A definably complete uniformly locally o-minimal structure of the second kind is definably Baire if and only if the union  $\bigcup_{r>0} X\langle r \rangle$  of any definable increasing family  $\{X\langle r \rangle\}_{r>0}$  with  $\text{int}(X\langle r \rangle) = \emptyset$  has an empty interior.*

**PROOF.** Because  $\text{int}(\overline{X\langle r \rangle}) \neq \emptyset$  iff  $\text{int}(X\langle r \rangle) \neq \emptyset$  iff  $X\langle r \rangle$  contains an open cell in this case by Theorem 2.2(D).  $\dashv$

**§3. Preliminaries.** From now on, we consider a definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group  $\mathcal{R} = (R, <, +, 0, \dots)$ . We demonstrate several basic facts in this section.

The notation  $\text{dist}(x, S)$  denotes the distance of a point  $x = (x_1, \dots, x_n) \in R^n$  to a definable subset  $S$  of  $R^n$  given by

$$\text{dist}(x, S) = \inf \left\{ \max_{1 \leq i \leq n} \text{abs}(x_i - y_i) \mid y = (y_1, \dots, y_n) \in S \right\},$$

where  $\text{abs}(\cdot)$  denotes the absolute value. It is used in the proof of the following lemma. We need to employ the assumption that  $\mathcal{R}$  is a definably complete expansion of densely linearly ordered abelian group in order to define the distance.

**LEMMA 3.1.** *Let  $X$  be a bounded definable set. There exists a definable decreasing family of CBD sets  $\{X\langle s \rangle\}_{s>0}$  with  $X = \bigcup_{s>0} X\langle s \rangle$ .*

PROOF. We demonstrate the lemma by the induction on  $d = \dim(X)$ . When  $d = 0$ ,  $X$  is discrete and closed by [4, Corollary 5.3]. We have only to set  $X\langle s \rangle = X$  for all  $s > 0$  in this case.

We next consider the case in which  $d > 0$ . We have  $\dim \overline{\partial X} < d$  by Proposition 2.4(b),(c). We get  $\dim(X \cap \overline{\partial X}) < d$  by Proposition 2.4(a). There exists a definable decreasing family of CBD sets  $\{Y\langle s \rangle\}_{s>0}$  with  $X \cap \overline{\partial X} = \bigcup_{s>0} Y\langle s \rangle$  by the induction hypothesis. Set  $Z\langle s \rangle = \{x \in \overline{X} \mid \text{dist}(x, \overline{\partial X}) \geq s\}$  for all  $s > 0$ . They are CBD. It is obvious that  $\bigcup_{s>0} Z\langle s \rangle = \overline{X} \setminus \overline{\partial X} = X \setminus \overline{\partial X}$ . Set  $X\langle s \rangle = Y\langle s \rangle \cup Z\langle s \rangle$ . The family  $\{X\langle s \rangle\}_{s>0}$  is a definable decreasing family we are looking for.  $\dashv$

LEMMA 3.2. Any definable set  $X$  is a  $D_\Sigma$ -set. That is, there exists a  $D_\Sigma$ -family  $\{X\langle r, s \rangle\}_{r>0, s>0}$  with  $X = \bigcup_{r>0, s>0} X\langle r, s \rangle$ .

PROOF. Let  $X$  be a definable subset of  $R^n$ . Set  $X\langle r \rangle = X \cap [-r, r]^n$ . We can construct subsets  $X\langle r, s \rangle$  of  $X\langle r \rangle$  satisfying the condition in the same manner as the proof of Lemma 3.1. We omit the details.  $\dashv$

LEMMA 3.3. Let  $X$  be a bounded definable set and  $\{X\langle s \rangle\}_{s>0}$  be a definable decreasing family of CBD sets with  $X = \bigcup_{s>0} X\langle s \rangle$ . If  $X$  has a nonempty interior, then there exists a positive  $s \in R$  such that  $X\langle s \rangle$  has a nonempty interior.

PROOF. We prove the lemma following the same strategy as the proof of [1, 3.1]. Let  $X$  be a definable subset of  $R^n$ . We prove the lemma by induction on  $n$ . We first consider the case in which  $n = 1$ . Assume that  $\text{int}(X\langle s \rangle) = \emptyset$  for all  $s > 0$ . Fix an arbitrary point  $a \in R$ . Apply Theorem 2.2(U) to the point  $(0, a) \in R^2$ , then we can get a positive integer  $M$ , an interval  $I$  with  $a \in I$  and  $t > 0$  such that, for any  $0 < s < t$ ,  $I \cap X\langle s \rangle$  contains an open interval or consists of at most  $M$  points. The sets  $I \cap X\langle s \rangle$  consist of at most  $M$  points because  $\text{int}(X\langle s \rangle) = \emptyset$ . We get  $|X \cap I| = |\bigcup_{s>0} (I \cap X\langle s \rangle)| \leq M$ . We have demonstrated that, for any point  $a \in R$ , there exists an open interval  $I$  containing the point  $a$  such that  $X \cap I$  has an empty interior. It implies that  $X$  has an empty interior.

We next consider the case in which  $n > 1$ . Assume that  $X$  has a nonempty interior. We show that the definable set  $X\langle s \rangle$  has a nonempty interior for some  $s > 0$ . A closed box  $B = C \times I \subseteq R^{n-1} \times R$  is contained in  $X$ . We have  $B = \bigcup_{s>0} (B \cap X\langle s \rangle)$ . Hence, we may assume that  $X$  is a closed box  $B$  without loss of generality.

Shrinking  $B$  if necessary, we may assume that there exists  $N > 0$  such that, for every  $x \in C$ , the fiber  $(X\langle s \rangle)_x$  is a disjoint union of finite points and at most  $N$  closed intervals for any sufficiently small  $s > 0$  by Corollary 2.3. Set  $I = [c_1, c_2]$ . Take  $2N$  distinct points in the open interval  $(c_1, c_2)$ , say  $b_1, \dots, b_{2N}$ . We may assume that  $b_i < b_j$  whenever  $i < j$ . Set  $b_0 = c_1$  and  $b_{2N+1} = c_2$ . Put  $I_j = [b_{j-1}, b_j]$  for all  $1 \leq j \leq 2N + 1$ .

Consider the sets  $Y^k\langle s \rangle = \{x \in C \mid I_k \subseteq (X\langle s \rangle)_x\}$  for all  $s > 0$  and  $1 \leq k \leq 2N + 1$ . They are CBD. Therefore,  $\left\{ \bigcup_{k=1}^{2N+1} Y^k\langle s \rangle \right\}_{s>0}$  is a definable decreasing family of CBD sets. We demonstrate that  $C = \bigcup_{s>0} \bigcup_{k=1}^{2N+1} Y^k\langle s \rangle$ . Let  $x \in C$  be fixed. We have only to show that  $I_k \subseteq (X\langle s \rangle)_x$  for some  $k$  and  $s$ . For any  $k$ , there exists  $s_k > 0$  such that  $\text{int}(I_k \cap (X\langle s_k \rangle)_x) \neq \emptyset$  by the induction hypothesis because  $\{I_k \cap (X\langle s \rangle)_x\}_{s>0}$  is a decreasing family of CBD sets with  $I_k = \bigcup_s I_k \cap (X\langle s \rangle)_x$ . Take  $s_{\min} = \min\{s_k \mid 1 \leq k \leq 2N + 1\}$ . We have  $\text{int}(I_k \cap (X\langle s_{\min} \rangle)_x) \neq \emptyset$  for all  $1 \leq k \leq 2N + 1$ . Assume that  $I_k \not\subseteq (X\langle s_{\min} \rangle)_x$  for all  $k$ . Recall that  $(X\langle s_{\min} \rangle)_x$  is a disjoint union of finite points

and at most  $N$  closed intervals. The closed intervals should be contained in  $I_k$ ,  $I_k \cup I_{k+1}$  or  $I_{k-1} \cup I_k$  for some  $k$ . Therefore,  $\text{int}(I_{j_N} \cap (X \langle s_{\min} \rangle)_x)$  is empty for some  $1 \leq j_N \leq 2N + 1$ . Contradiction. We have proven that  $I_k \subseteq (X \langle s \rangle)_x$  for some  $k$  and  $s$ .

Apply the induction hypothesis to  $C = \bigcup_{s>0} \bigcup_{k=1}^{2N+1} Y^k \langle s \rangle$ . The set  $\bigcup_{k=1}^{2N+1} Y^k \langle s \rangle$  has a nonempty interior for some  $s > 0$ . The CBD set  $Y^k \langle s \rangle$  has a nonempty interior for some  $k$  by [4, Theorem 3.3]. The CBD set  $X \langle s \rangle$  has a nonempty interior because  $I_k \times Y^k \langle s \rangle$  is contained in  $X \langle s \rangle$ .  $\dashv$

**LEMMA 3.4.** *Assume that  $\mathcal{R}$  is definably Baire. Let  $X$  be a definable set and  $\{X \langle r, s \rangle\}_{r>0, s>0}$  be a  $D_\Sigma$ -family with  $X = \bigcup_{r>0, s>0} X \langle r, s \rangle$ . If  $X$  has a nonempty interior, the CBD set  $X \langle r, s \rangle$  has a nonempty interior for some  $r > 0$  and  $s > 0$ .*

**PROOF.** Let  $X$  be a definable subset of  $R^n$ . Set  $X' \langle r, s \rangle = X \langle r, s \rangle \cap [-r, r]^n$ . We have  $X = \bigcup_{r>0, s>0} X' \langle r, s \rangle$ . We may assume that  $X \langle r \rangle = \bigcup_{r>0, s>0} X \langle r, s \rangle$  is bounded considering  $X' \langle r, s \rangle$  instead of  $X \langle r, s \rangle$ . The lemma is now immediate from Proposition 2.8 and Lemma 3.3.  $\dashv$

**§4. On definably Baire property.** We demonstrate that the structure  $\mathcal{R}$  is definably Baire. We first show the following lemma.

**LEMMA 4.1.** *Let  $X$  be a bounded definable subset of  $R^{n+1}$ . Set*

$$S = \{x \in R^n \mid X_x \text{ contains an open interval}\}.$$

*The set  $S$  has an empty interior if  $X$  has an empty interior.*

**PROOF.** Assume that  $S$  has a nonempty interior. There exists a definable decreasing family of CBD sets  $\{X \langle s \rangle\}_{s>0}$  with  $X = \bigcup_{s>0} X \langle s \rangle$  by Lemma 3.1. Set  $S \langle s \rangle = \{x \in R^n \mid \exists t \in R, [t - s, t + s] \subseteq (X \langle s \rangle)_x\}$  for all  $s > 0$ . They are CBD by [6, Lemma 1.7] because they are the projection images of the CBD sets  $\{(x, t) \in R^n \times R \mid [t - s, t + s] \subseteq (X \langle s \rangle)_x\}$ . We have  $S = \bigcup_{s>0} S \langle s \rangle$ . In fact, it is obvious that  $\bigcup_{s>0} S \langle s \rangle \subseteq S$  by the definition. Take a point  $x \in S$ , then we have  $\text{int}(X_x) \neq \emptyset$ . We have  $\text{int}(X \langle s_1 \rangle)_x \neq \emptyset$  for some  $s_1 > 0$  by Lemma 3.3. We get  $[t - s_2, t + s_2] \subseteq (X \langle s_1 \rangle)_x$  for some  $s_2 > 0$  and  $t$ . Set  $s = \min\{s_1, s_2\}$ , then we have  $x \in S \langle s \rangle$ . We have demonstrated that  $S = \bigcup_{s>0} S \langle s \rangle$ .

Again by Lemma 3.3, we have  $\text{int}(S \langle s \rangle) \neq \emptyset$  for some  $s > 0$ . We obtain  $\text{int}(X \langle s \rangle) \neq \emptyset$  by [1, 2.8(2)]. We get  $\text{int}(X) \neq \emptyset$ .  $\dashv$

We reduce to the one-dimensional case.

**LEMMA 4.2.** *The structure  $\mathcal{R}$  is definably Baire if the union  $\bigcup_{r>0} S \langle r \rangle$  of any definable increasing family  $\{S \langle r \rangle\}_{r>0}$  of subsets of  $R$  has an empty interior whenever  $S \langle r \rangle$  have empty interiors for all  $r > 0$ .*

**PROOF.** Let  $\{X \langle r \rangle\}_{r>0}$  be a definable increasing family of subsets of  $R^n$ . Set  $X = \bigcup_{r>0} X \langle r \rangle$ . We have only to show that the definable set  $X \langle r \rangle$  has a nonempty interior for some  $r > 0$  under the assumption that  $X$  has a nonempty interior by Proposition 2.8. Under this assumption, the definable set  $X$  contains a bounded open box  $B$ . We may assume that  $X$  is a bounded open box  $B$  without loss of generality by considering  $B$  and  $\{X \langle r \rangle \cap B\}_{r>0}$  in place of  $X$  and  $\{X \langle r \rangle\}_{r>0}$ , respectively.

We prove the lemma by the induction on  $n$ . The lemma is obvious by the assumption of the lemma when  $n = 1$ . We next consider the case in which  $n > 1$ . We lead to a contradiction assuming that  $X\langle r \rangle$  have empty interiors for all  $r > 0$ . Let  $\pi : R^n \rightarrow R^{n-1}$  be the projection forgetting the last coordinate. We have  $B = B_1 \times I$  for some open box  $B_1$  in  $R^{n-1}$  and some open interval  $I$ . Consider the set  $Y\langle r \rangle = \{x \in B_1 \mid \text{the fiber } (X\langle r \rangle)_x \text{ contains an open interval}\}$  for all  $r > 0$ . They have empty interiors by Lemma 4.1. The union  $\bigcup_{r>0} Y\langle r \rangle$  has an empty interior by the induction hypothesis. In particular, we have  $B_1 \neq \bigcup_{r>0} Y\langle r \rangle$  and we can take a point  $x \in B_1 \setminus (\bigcup_{r>0} Y\langle r \rangle)$ . Since  $x \notin \bigcup_{r>0} Y\langle r \rangle$ , the fiber  $(X\langle r \rangle)_x$  does not contain an open interval for any  $r > 0$ . Therefore, the union  $\bigcup_{r>0} (X\langle r \rangle)_x$  has an empty interior by the assumption. On the other hand, we have  $I = \bigcup_{r>0} (X\langle r \rangle)_x$  because  $B = \bigcup_{r>0} X\langle r \rangle$ . It is a contradiction.  $\dashv$

We prove that  $\mathcal{R}$  is definably Baire now.

**THEOREM 4.3.** *A definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group is definably Baire.*

**PROOF.** Let  $\mathcal{R} = (R, <, +, 0, \dots)$  be the considered structure. Let  $\{X\langle r \rangle\}_{r>0}$  be a definable increasing family of subsets of  $R$ . Set  $X = \bigcup_{r>0} X\langle r \rangle$ . We have only to show that the definable set  $X$  has an empty interior under the assumption that  $X\langle r \rangle$  have empty interiors for all  $r > 0$  by Lemma 4.2. Under this assumption,  $X\langle r \rangle$  are discrete and closed because the structure is locally o-minimal.

Assume that  $X$  has a nonempty interior. The definable set  $X$  contains an open interval. Take a point  $a$  contained in the open interval. Consider the function  $f : \{r \in R \mid r > 0\} \rightarrow \{x \in R \mid x > a\}$  defined by  $f(r) = \inf\{x > a \mid x \in X\langle r \rangle\}$ . It is definable because the set  $\{(r, x) \in R^2 \mid a < x, x \in X\langle r \rangle\}$  is definable and taking the infimum of a parameterized definable set yields a definable map. It is obvious that  $f$  is a decreasing function because  $\{X\langle r \rangle\}_{r>0}$  is a definable increasing family. We demonstrate that  $\lim_{r \rightarrow \infty} f(r) = a$ . Let  $b$  be an arbitrary point sufficiently close to  $a$  with  $b > a$ . Since  $X = \bigcup_{r>0} X\langle r \rangle$  contains a neighborhood of  $a$ , there exists a positive element  $r \in R$  with  $b \in X\langle r \rangle$ . We have  $a < f(r) \leq b$  by the definition of  $f$ . We have shown that  $\lim_{r \rightarrow \infty} f(r) = a$ .

Consider the image  $\text{Im}(f)$  of the function  $f$ . Take a sufficiently small open interval  $I \subseteq X$  containing the point  $a$ . The intersection  $I \cap \text{Im}(f)$  is a finite union of points and open intervals because it is definable in the locally o-minimal structure  $\mathcal{R}$ . Take an arbitrary point  $b \in \text{Im}(f)$  and a point  $r > 0$  with  $b = f(r)$ . Since  $X\langle r \rangle$  is closed, we have  $b \in X\langle r \rangle$ . Any point  $b' \in \text{Im}(f)$  with  $b' > b$  is also contained in  $X\langle r \rangle$ . In fact, take a point  $r' > 0$  with  $b' = f(r')$ . If  $r' > r$ , the set  $X\langle r' \rangle$  contains the point  $b$  because  $X\langle r \rangle \subseteq X\langle r' \rangle$ . Then we have  $b' = f(r') \leq b$  by the definition of the function  $f$ , and this is a contradiction. If  $r' < r$ , we have  $b' \in X\langle r' \rangle \subseteq X\langle r \rangle$ .

Set  $b_1 = \inf\{b' \in \text{Im}(f) \mid b' > b\}$ . We have  $b_1 \in X\langle r \rangle$  and  $b_1 > b$  because  $\{b' \in \text{Im}(f) \mid b' > b\} \subseteq X\langle r \rangle$  and  $X\langle r \rangle$  is closed and discrete. The open interval  $(b, b_1)$  has an empty intersection with  $\text{Im}(f)$ . We have shown that  $I \cap \text{Im}(f)$  does not contain an open interval. The set  $I \cap \text{Im}(f)$  consists of finite points. This contradicts the fact that  $\lim_{r \rightarrow \infty} f(r) = a$ .  $\dashv$

**REMARK 4.4.** It is already known that a definably complete expansion of an ordered field is definably Baire [5]. Our research target is a uniformly locally



o-minimal structure of the second kind. A uniformly locally o-minimal expansion of the second kind of an ordered field is o-minimal by [4, Proposition 2.1]. In this case, it is trivially definably Baire by the definable cell decomposition theorem [8, Chapter 3, (2,11)]. We have more interest in the case in which the structure is not an expansion of an ordered field.

**§5. The proof of the main theorem.** We demonstrate Theorem 1.1 in this section. Recall that we consider a definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group  $\mathcal{R} = (R, <, +, 0, \dots)$ . We first show that a definable map is continuous on an open subset of the domain of definition.

LEMMA 5.1. *A definable map  $f : U \rightarrow R^n$  defined on an open set  $U$  is continuous on a nonempty definable open subset of  $U$ .*

PROOF. The structure  $\mathcal{R}$  is definably Baire by Theorem 4.3. So we can use Lemma 3.4.

Let  $U$  be a definable open subset of  $R^m$ . Consider the projection  $\pi : R^{m+n} \rightarrow R^m$  onto the first  $m$  coordinates. Let  $\Gamma(f)$  denote the graph of  $f$ . There exists a  $D_\Sigma$ -family  $\{X\langle r, s \rangle\}_{r>0, s>0}$  with  $\Gamma(f) = \bigcup_{r>0, s>0} X\langle r, s \rangle$  by Lemma 3.2. Note that  $\pi(X\langle r, s \rangle)$  is CBD by [6, Lemma 1.7]. The family  $\{\pi(X\langle r, s \rangle)\}_{r>0, s>0}$  is a  $D_\Sigma$ -family and we have  $U = \bigcup_{r>0, s>0} \pi(X\langle r, s \rangle)$ . The CBD set  $\pi(X\langle r, s \rangle)$  has a nonempty interior for some  $r$  and  $s$  by Lemma 3.4. The fiber  $\pi^{-1}(x) \cap \Gamma(f)$  is a singleton for any  $x \in U$ . The CBD set  $X\langle r, s \rangle$  is the graph of the restriction of  $f$  to  $\pi(X\langle r, s \rangle)$ . Since the graph of the restriction of  $f$  to  $\pi(X\langle r, s \rangle)$  is closed,  $f$  is continuous on  $\text{int}(\pi(X\langle r, s \rangle))$ .  $\dashv$

We finally prove Theorem 1.1.

PROOF OF THEOREM 1.1. Recall that we have to show:

(\*) The inequality  $\dim(f(X)) \leq \dim(X)$  holds true for any definable map  $f : X \rightarrow R^n$ .

Lemma 3.4 is available as in the proof of Lemma 5.1 for the same reason.

Set  $d = \dim(f(X))$ . We demonstrate that  $\dim(X) \geq d$ . We can reduce to the case in which the image  $f(X)$  is an open box  $B$  of dimension  $d$ . In fact, there exist an open box  $B$  in  $R^d$  and a definable map  $g : B \rightarrow f(X)$  such that the map  $g$  is a definable homeomorphism onto its image by the definition of dimension [4, Definition 5.1]. Set  $Y = f^{-1}(g(B))$  and  $h = g^{-1} \circ f|_Y : Y \rightarrow B$ . When  $\dim(Y) \geq d$ , we get  $\dim(X) \geq d$  by Proposition 2.4(a) because  $Y$  is a subset of  $X$ . We may assume that  $f(X) = B$  by considering  $Y$  and  $h$  instead of  $X$  and  $f$ , respectively.

We next reduce to the case in which the map  $f$  is the restriction of a coordinate projection. Consider the graph  $G = \Gamma(f) \subseteq R^{m+d}$  of the definable map  $f$ . Let  $\pi : R^{m+d} \rightarrow R^d$  be the projection onto the last  $d$  coordinates. Assume that  $\dim(G) \geq d$ , then we have  $\dim(X) \geq \dim(G) \geq d = \dim(f(X))$  by Lemma 2.5 because  $X$  is the image of  $G$  under the projection  $\pi'$  onto the first  $m$  coordinates and  $G \cap (\pi')^{-1}(x)$  is a singleton and of dimension  $\leq 0$  for any  $x \in X$ . The assertion (\*) holds true in this case. We have succeeded in reducing to the case in which  $X = G$  and  $f$  is the restriction of  $\pi$  to  $G$ .

We have a  $D_\Sigma$ -family  $\{X\langle r, s \rangle\}_{r>0, s>0}$  with  $X = \bigcup_{r>0, s>0} X\langle r, s \rangle$  by Lemma 3.2. The family  $\{f(X\langle r, s \rangle)\}_{r>0, s>0}$  is also a  $D_\Sigma$ -family by [6, Lemma 1.7] because  $f$  is the restriction of a projection. We have  $B = \bigcup_{r>0, s>0} f(X\langle r, s \rangle)$ . The CBD set  $f(X\langle r, s \rangle)$  has a nonempty interior for some  $r > 0$  and  $s > 0$  by Lemma 3.4. We fix such  $r > 0$  and  $s > 0$ . Take an open box  $U$  contained in  $f(X\langle r, s \rangle)$ . Note that the inverse image  $\{y \in X\langle r, s \rangle \mid f(y) = x\}$  of  $x \in U$  is CBD because the restriction of the projection  $f|_{X\langle r, s \rangle}$  is continuous. Consider a definable function  $\varphi : U \rightarrow X\langle r, s \rangle$  given by  $\varphi(x) = \mathbf{lexmin}\{y \in X\langle r, s \rangle \mid f(y) = x\}$ , where the notation  $\mathbf{lexmin}$  denotes the lexicographic minimum defined in [6]. We can get an open box  $V$  contained in  $U$  such that the restriction  $\varphi|_V$  of  $\varphi$  to  $V$  is continuous by Lemma 5.1. The definable set  $X\langle r, s \rangle$  is of dimension  $\geq d$  by the definition of dimension because it contains the graph of the definable continuous map  $\varphi|_V$  defined on the open box  $V$  in  $R^d$ . We have  $\dim X \geq \dim(X\langle r, s \rangle) \geq d$  by Proposition 2.4(a). We have proven Theorem 1.1.  $\dashv$

The proof of Corollary 1.2 is the same as that of [3, Corollary 2.6]. As it is short, we give it here.

PROOF OF COROLLARY 1.2. Let  $\mathcal{D}$  be the set of points at which the definable function  $f$  is discontinuous. Assume that the domain of definition  $X$  is a definable subset of  $R^m$ . Let  $G$  be the graph of  $f$ . We have  $\dim(G) = \dim(X)$  by Lemma 2.5 and Theorem 1.1. Set  $\mathcal{E} = \{(x, y) \in X \times R \mid y = f(x) \text{ and } f \text{ is discontinuous at } x\}$ . We get  $\dim(\mathcal{E}) < \dim(G)$  by Theorem 2.2(C) and [4, Corollary 5.3]. Let  $\pi : R^{m+1} \rightarrow R^m$  be the projection forgetting the last coordinate. We have  $\mathcal{D} = \pi(\mathcal{E})$  by the definitions of  $\mathcal{D}$  and  $\mathcal{E}$ . We finally obtain  $\dim(\mathcal{D}) = \dim(\pi(\mathcal{E})) \leq \dim(\mathcal{E}) < \dim(G) = \dim(X)$  by Theorem 1.1.  $\dashv$

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