Can. J. Math., Vol. XXX, No. 2, 1978, pp. 301–314

THE NUMBER OF CLOSED SUBSETS OF A TOPOLOGICAL SPACE

R. E. HODEL

Dedicated to L. Carlitz on his seventieth birthday

1. Introduction. Let X be an infinite topological space, let $\mathfrak{n}$ be an infinite cardinal number with $\mathfrak{n} \leqq |X|$. The basic problem in this paper is to find the number of closed sets in X of cardinality $\mathfrak{n}$. A complete answer to this question for the class of metrizable spaces has been given by A. H. Stone in [**31**], where he proves the following result. *Let X be an infinite metrizable space of weight $\mathfrak{m}$, let $\mathfrak{n} \leqq |X|$. Then the number of closed sets in X of cardinality $\mathfrak{n}$ is* (1) $|X|^{\mathfrak{n}}$ *if* $\mathfrak{n} \leqq \mathfrak{m}$, (2) $2^{\mathfrak{m}}$ *if* $\mathfrak{n} = |X|$, (3) 0 *or* $2^{\mathfrak{m}}$ *if* $\mathfrak{m} < \mathfrak{n} < |X|$. The main result of this paper (see § 3) is the extension of Stone's theorem to the class of regular σ-spaces satisfying property wD. (Although σ-spaces have many properties in common with metrizable spaces, they need not be paracompact nor first countable.) In §§ 4 and 5 we consider two special cases of the basic problem, namely the number of denumerable closed sets in X and the number of closed sets in X of cardinality $|X|$. In § 6 we consider the basic problem under the assumption of the generalized continuum hypothesis (hereafter abbreviated GCH). In § 7 we obtain upper bounds on $o(X)$, the number of open sets in X, in terms of other cardinal invariants of X.

2. Definitions, conventions, and known results. Unless otherwise stated, no separation axioms are assumed; however, regular, normal, and paracompact spaces are always Hausdorff. The set of positive integers is denoted by $\mathbf{N}$, and k and n denote elements of $\mathbf{N}$.

Let X be a topological space. A *net* for X is a collection $\mathscr{N}$ of subsets of X such that if V is open and $x \in V$, then there is some $N \in \mathscr{N}$ such that $x \in N \subseteq V$. A space with a σ-locally finite net is called a *σ-space* [**25**]. Nagata and Siwiec [**23**] have proved that a regular, σ-space has a σ-discrete net consisting of closed sets. Every regular σ-space is perfect (= every closed set is a G_δ). The space X is *θ-refinable* if for every open cover $\mathscr{U}$ of X there is a sequence $\{\mathscr{G}_n : n \in \mathbf{N}\}$ of open refinements of $\mathscr{U}$ such that, for each $p \in X$, there is some $n \in \mathbf{N}$ such that $\{G: G \in \mathscr{G}_n, p \in G\}$ is finite. The space X is *irreducible* if every open cover has a minimal open refinement; i.e., an open refinement $\mathscr{G}$ such that no proper subcollection of $\mathscr{G}$ covers X. The following implications hold: regular σ-space $\Rightarrow$ θ-refinable $\Rightarrow$ irreducible. Also, every subset of a perfect, θ-refinable space is θ-refinable. (See [**2**; **3**; **29**; **30**; **35**].)

Received September 14, 1976 and in revised form, June 27, 1977.

A subset A of a topological space X is *discrete* if each $x \in A$ has an open neighborhood V such that $V \cap A = \{x\}$. A T_1 space X has *property* wD if given any sequence $\langle x_n \rangle$ in X with no cluster points, there is a subsequence $\langle x_{nk} \rangle$ of $\langle x_n \rangle$ and a discrete open collection $\{V_k: k \in \mathbf{N}\}$ such that $x_{n_k} \in V_k$ for all $k \in \mathbf{N}$. (Note: The collection $\{V_k: k \in \mathbf{N}\}$ is closure-preserving and $\bar{V}_k \cap \bar{V}_n = \phi$ whenever $k \neq n$.) Property wD was introduced by Vaughan [**34**] as a generalization of property D of R. L. Moore [**22**]. (Also, see [**6**].) Every normal space and every regular, countably paracompact space has property wD, and it is easy to check that property wD is hereditary with respect to closed sets. See [**34**] for an example of a regular σ-space which does not have property wD and an example of a regular σ-space with property wD which does not have property D.

We adopt the following set-theoretic notation: $\mathfrak{m}$, $\mathfrak{n}$, and $\mathfrak{p}$ are cardinal numbers; α, β and γ are ordinal numbers; $|E|$ is the cardinality of the set E. The cofinality of $\mathfrak{m}$ is denoted by $cf(\mathfrak{m})$, and an infinite cardinal $\mathfrak{m}$ is said to be *sequential* if $cf(\mathfrak{m}) = \aleph_0$; i.e., if $\mathfrak{m}$ is the sum of a countable number of smaller cardinals. If E is a set and $\mathscr{V}$ is a collection of sets, then $\mathscr{P}_{\mathfrak{m}}(E) = \{A: A \subseteq E, |A| \leqq \mathfrak{m}\}$ and $(\mathscr{V})_{\mathfrak{m}} = \{G: G$ is the union of $\leqq \mathfrak{m}$ elements of $\mathscr{V}\}$.

We use w, L, d, c, and ψ to denote the following standard cardinal functions: weight, Lindelöf degree, density, cellularity, and pseudo-character. (For definitions, see Juhász [**17**].) Thus, if X is T_1, then $\psi(X) = \aleph_0$ (X has *countable pseudo-character*) if and only if every point in X is a G_δ. Also, if $c(X) = \aleph_0$, we say that X satisfies the *countable chain condition* (= CCC).

If ϕ is a cardinal function, then $h\phi$ denotes the hereditary version of ϕ; i.e., $h\phi(X) = \sup\{\phi(Y): Y \subseteq X\}$. (This notation is due to Engelking [**7**].) For example, a space X hereditarily satisfies the CCC if and only if $hc(X) = \aleph_0$. It is easy to check that $hc(X) = \aleph_0 \cdot \{\sup |A|: A$ is a discrete subset of $X\}$. (Thus, $hc(X) = s(X)$, where $s(X)$ is the *spread* of X.) Hajnal and Juhász [**11**] have proved that $|X| \leqq 2^{hc(X) \cdot \psi(X)}$ for X a T_1-space. This fundamental inequality is used on several occasions in this paper. Hajnal and Juhász [**12**] have also proved that if X is an infinite Hausdorff space and $|X|$ is a singular strong limit cardinal, then X has a discrete subset of cardinality $|X|$.

Let X be a topological space. The *tightness* of X, denoted $t(X)$, is the smallest infinite cardinal $\mathfrak{m}$ such that for each $x \in X$, if $x \in \bar{H}$, then there is some $K \subseteq H$ with $|K| \leqq \mathfrak{m}$ and $x \in \bar{K}$ (see [**1**]). If $t(X) = \aleph_0$, we say that X has *countable tightness*. It is clear that if $Y \subseteq X$, then $t(Y) \leqq t(X)$. The π-*character* of X, denoted $\pi\chi(X)$, is the smallest infinite cardinal $\mathfrak{m}$ such that for each $x \in X$, there is a collection $\mathscr{V}$ of open sets (not necessarily containing x) with $|\mathscr{V}| \leqq \mathfrak{m}$ such that if R is any open neighborhood of x, then $V \subseteq R$ for some $V \in \mathscr{V}$. Šapirovskiĭ [**28**] has proved that $w(X) \leqq \pi\chi(X)^{c(X)}$ whenever X is regular. The *net weight* of X, denoted $nw(X)$, is the smallest infinite cardinal $\mathfrak{m}$ such that X has a net of cardinality $\leqq \mathfrak{m}$. For a T_1 space X, let $\psi C(X) = \aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that every closed subset of X is the intersection of $\leqq \mathfrak{m}$ open sets. Note that $\psi C(X) = \aleph_0$ if and only if X is perfect, and

$\psi(X) \leqq \psi C(X)$ for every T_1 space X. The number of open sets in X is denoted by $o(X)$. Juhász [**18**] has proved that $o(X) \leqq |X|^{hd(X)}$, the inequality $o(X) \leqq w(X)^{hL(X)}$ appears in [**13**], and it is clear that $o(X) \leqq 2^{nw(X)}$.

An important example should be mentioned to help keep the basic problem in perspective. The Stone-Čech compactification of **N** is a compact, Hausdorff, separable space such that every infinite closed set has cardinality $2^{2^{\aleph_0}}$. (See [**24**].) Thus $\beta\mathbf{N}$ has no denumerable closed sets and no closed sets of cardinality $2^{\aleph_0}$.

3. Closed sets in regular σ-spaces satisfying property wD. In this section we extend Stone's theorem [**31**] from metrizable spaces to the class of regular σ-spaces satisfying property wD. Although the generalization (Theorem 3.2) is stated in terms of the net weight rather than the weight, these two cardinal functions agree for metrizable spaces. Two key ideas due to Stone are used in the proof. One is easily described as follows: to prove that a space has $\mathfrak{m}^{\mathfrak{n}}$ closed sets of cardinality $\mathfrak{n}(\mathfrak{n} \leqq \mathfrak{m})$, obtain a discrete subset of cardinality $\mathfrak{m}$ which has at most one limit point; this immediately yields $\mathfrak{m}^{\mathfrak{n}}$ closed sets of cardinality $\mathfrak{n}$. Propositions 7 and 8 in this section are concerned with finding such sets; the difficult case is when $\mathfrak{m}$ is sequential. The other key idea of Stone's is a consideration of the two cases (a) and (b) which appear in the proof of 3.8.

The approach we take is to isolate four properties of a topological space X which are sufficient to determine, for each infinite cardinal $\mathfrak{n} \leqq |X|$, the number of closed sets in X of cardinality $\mathfrak{n}$. We then show that every regular σ-space with property wD satisfies these four properties.

I. $|X| \leqq nw(X)^{\aleph_0}$.

II. *If Y is an infinite subset of X, then Y has a discrete subset of cardinality $nw(Y)$.*

III. *If Y is a closed subset of X, and Y has a discrete subset of cardinality $\mathfrak{m} \geqq \aleph_0$, then Y has at least $2^{\mathfrak{m}}$ closed subsets of cardinality $|Y|$.*

IV. *If Y is an open subset of X, and Y has a discrete subset of cardinality $\mathfrak{m} \geqq \aleph_0$, then there is a discrete subset B of Y of cardinality $\mathfrak{m}$ which has at most one limit point in Y.*

THEOREM 3.1. *Let X be an infinite topological space satisfying I-IV, and let $nw(X) = \mathfrak{m}$. (1) The total number of closed sets in X is $2^{\mathfrak{m}}$, and there are $2^{\mathfrak{m}}$ closed sets of cardinality $|X|$. (2) If $\aleph_0 \leqq \mathfrak{n} \leqq \mathfrak{m}$, the number of closed sets of cardinality $\mathfrak{n}$ is $|X|^{\mathfrak{n}}(=\mathfrak{m}^{\mathfrak{n}})$. (3) If $\mathfrak{m} < \mathfrak{n} < |X|$, the number of closed sets of cardinality $\mathfrak{n}$ is 0 or $2^{\mathfrak{m}}$.*

Proof. Since $o(X) \leqq 2^{nw(X)}$, the total number of closed sets in X is $\leqq 2^{\mathfrak{m}}$. By II and III (with $Y = X$), X has $2^{\mathfrak{m}}$ closed sets of cardinality $|X|$. This completes the proof of (1). To prove (2), let $\mathfrak{n}$ be an infinite cardinal with $\mathfrak{n} \leqq \mathfrak{m}$. Clearly it suffices to construct $|X|^{\mathfrak{n}}$ closed sets of cardinality $\mathfrak{n}$. By I,

$|X| \leqq \mathfrak{m}^{\aleph_0}$, so $|X|^{\mathfrak{n}} = \mathfrak{m}^{\mathfrak{n}}$. By II and IV (with $Y = X$), there is a subset B of X of cardinality $\mathfrak{m}$ with at most one limit point. Hence there are $\mathfrak{m}^{\mathfrak{n}}$ $(= |X|^{\mathfrak{n}})$ closed sets in X of cardinality $\mathfrak{n}$.

To prove (3), assume that X has a closed subset L of cardinality $\mathfrak{n}$, where $\mathfrak{m} < \mathfrak{n} < |X|$. Note that $nw(L) = \mathfrak{m}$ or $nw(X - L) = \mathfrak{m}$. If $nw(L) = \mathfrak{m}$, then II and III applied to L immediately yields a collection of $2^{\mathfrak{m}}$ closed sets in X, each of cardinality $\mathfrak{n}$. Suppose that $nw(X - L) = \mathfrak{m}$. Then by II and IV applied to $(X - L)$, there is a discrete subset B of $(X - L)$ of cardinality $\mathfrak{m}$ which has at most one limit point in $(X - L)$. Hence $\{L \cup \bar{E}: E \subseteq B\}$ is a collection of $2^{\mathfrak{m}}$ closed sets in X, each of cardinality $\mathfrak{n}$.

We now show that every regular σ-space with property wD satisfies I-IV. Property I follows from 3.4, II from 3.5, III from 3.6, and IV from 3.7(2) and 3.8. This yields the following generalization of Stone's theorem.

THEOREM 3.2. *Let X be an infinite, regular σ-space satisfying property wD, let $nw(X) = \mathfrak{m}$. (1) The total number of closed sets in X is $2^{\mathfrak{m}}$, and there are $2^{\mathfrak{m}}$ closed sets of cardinality $|X|$. (2) If $\aleph_0 \leqq \mathfrak{n} \leqq \mathfrak{m}$, the number of closed sets of cardinality $\mathfrak{n}$ is $|X|^{\mathfrak{n}}$ $(= \mathfrak{m}^{\mathfrak{n}})$; in particular, the number of closed sets of cardinality $\mathfrak{m}$ is $2^{\mathfrak{m}}$. (3) If $\mathfrak{m} < \mathfrak{n} < |X|$, the number of closed sets of cardinality $\mathfrak{n}$ is 0 or $2^{\mathfrak{m}}$.*

Remark 3.3. Regarding part (3) of Theorem 3.2, if GCH holds, then there is no cardinal $\mathfrak{n}$ such that $\mathfrak{m} < \mathfrak{n} < |X|$. On the other hand, as proved by Stone in [**31**], if X is a complete metric space and $\mathfrak{n}$ is a cardinal such that $w(X) < \mathfrak{n} < |X|$, then X has no closed subset of cardinality $\mathfrak{n}$.

PROPOSITION 3.4. *If X is T_1, then $|X| \leqq nw(X)^{\psi(X)}$.*

Proof. We may assume that X is infinite. Let $\psi(X) = \mathfrak{n}$, let $\mathscr{N}$ be a net for X of cardinality $nw(X)$, and for each $p \in X$ let $\{V(\alpha, p): 0 \leqq \alpha < \mathfrak{n}\}$ be a collection of open neighborhoods of p such that $\bigcap V(\alpha, p) = \{p\}$. For each α and p, choose $N(\alpha, p) \in \mathscr{N}$ such that $p \in N(\alpha, p) \subseteq V(\alpha, p)$. Define $f: X \rightarrow \mathscr{P}_{\mathfrak{n}}(\mathscr{N})$ by $f(p) = \{N(\alpha, p): 0 \leqq \alpha < \mathfrak{n}\}$. Then f is one-one, so $|X| \leqq nw(X)^{\mathfrak{n}}$.

PROPOSITION 3.5. *If X is a regular σ-space, then $nw(X) = hc(X)$. Moreover, if X is infinite, then X has a discrete subset of cardinality $nw(X)$.*

Proof. First we show that $hc(X) = nw(X)$. Since $hc(X) \leqq nw(X)$ always, it suffices to show that $nw(X) \leqq hc(X)$. Let $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$ be a σ-discrete closed net for X. For each $n \in \mathbf{N}$, $|\mathscr{F}_n| \leqq hc(X)$, and so $|\mathscr{F}| \leqq hc(X)$. Since $\mathscr{F}$ is a net for X, $nw(X) \leqq hc(X)$.

Now assume that X is infinite and $nw(X) = \mathfrak{m}$. First suppose that $\mathfrak{m}$ is not sequential. If $|\mathscr{F}_n| < \mathfrak{m}$ for all n, then $|\mathscr{F}| < \mathfrak{m}$, a contradiction of $nw(X) = \mathfrak{m}$. Thus for some $n \in \mathbf{N}$, $|\mathscr{F}_n| \geqq \mathfrak{m}$, from which it follows that X has a (closed) discrete subset of cardinality $\mathfrak{m}$. If $\mathfrak{m}$ is sequential, the existence of a discrete subset of cardinality $\mathfrak{m}$ follows from $hc(X) = \mathfrak{m}$. (See [**14**]).

PROPOSITION 3.6. *Let X be a regular, perfect space satisfying property* wD, *let A be a discrete subset of X of cardinality $\mathfrak{m} \geqq \aleph_0$. Then there are at least $2^{\mathfrak{m}}$ closed sets in X of cardinality $|X|$.*

Proof. Let $|X| = \mathfrak{p}$. Clearly we may assume that $\mathfrak{m} < \mathfrak{p}$. First suppose that $\mathfrak{p}$ is not sequential. Let $A = A_1 \cup A_2$, where $|A_1| = |A_2| = \mathfrak{m}$ and $A_1 \cap A_2 = \phi$. For each $x \in A_1$ let V_x be an open neighborhood of x such that $V_x \cap A = \{x\}$, and for each $B \subseteq A_1$ let $V(B) = \bigcup_{x \in B} V_x$. If for all $B \subseteq A_1$, $|V(B)| < \mathfrak{p}$, then $\{X - V(B) : B \subseteq A_1\}$ is a collection of $2^{\mathfrak{m}}$ closed sets in X, each of cardinality $\mathfrak{p}$. Suppose, then, that for some $B \subseteq A_1$, $|V(B)| = \mathfrak{p}$. Since X is perfect and $\mathfrak{p}$ is not sequential, there is a closed set $F \subset V(B)$ with $|F| = \mathfrak{p}$. Then $\{F \cup \bar{E} : E \subseteq A_2\}$ is a collection of $2^{\mathfrak{m}}$ closed sets in X, each of cardinality $\mathfrak{p}$.

Now suppose that $\mathfrak{p}$ is sequential, say $\mathfrak{p} = \sum_{k=1}^{\infty} \mathfrak{p}_k$, where $\mathfrak{m} < \mathfrak{p}_1 < \mathfrak{p}_2 < \ldots < \mathfrak{p}$. We consider two cases.

(a) *There is a cardinal $\mathfrak{n} < \mathfrak{p}$ such that for each $x \in A$, there is an open neighborhood of x of cardinality $\leqq \mathfrak{n}$.* For each $x \in A$ let V_x be an open neighborhood of x of cardinality $\leqq \mathfrak{n}$ such that $V_x \cap A = \{x\}$. For each $B \subseteq A$ let $V(B) = \bigcup_{x \in B} V_x$. Now $|V(B)| \leqq \mathfrak{m} \cdot \mathfrak{n} < \mathfrak{p}$, so $\{X - V(B) : B \subseteq A\}$ is a collection of $2^{\mathfrak{m}}$ closed sets in X, each of cardinality $\mathfrak{p}$.

(b) *There is a sequence $\langle x_n \rangle$ in A such that if V is any open neighborhood of x_n, then $|V| > \mathfrak{p}_n$.* Let $E = A - \{x_n : n \in \mathbf{N}\}$, and note that $|E| = \mathfrak{m}$. (It may be necessary to take a subsequence of $\langle x_n \rangle$ if $\mathfrak{m} = \aleph_0$.) First suppose that $\langle x_n \rangle$ has no cluster points. Since X has property wD, there is a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ and a discrete open collection $\{V_k : k \in \mathbf{N}\}$ such that $x_{n_k} \in V_k$ for all $k \in \mathbf{N}$. For each k let W_k be an open neighborhood of x_{n_k} such that $W_k \subseteq V_k$ and $\bar{W}_k \cap E = \phi$. Note that $|\bar{W}_k| > \mathfrak{p}_k$. For each $B \subseteq E$ let $L(B) = (\bigcup_{k=1}^{\infty} \bar{W}_k) \cup \bar{B}$. Then $\{L(B) : B \subseteq E\}$ is a collection of $2^{\mathfrak{m}}$ closed sets in X, each of cardinality $\mathfrak{p}$. Next suppose that $\langle x_n \rangle$ has a cluster point p. Let $\{V_k : k \in \mathbf{N}\}$ be a decreasing sequence of open neighborhoods of p such that $\bigcap_{k=1}^{\infty} \bar{V}_k = \{p\}$. Let $\langle x_{n_k} \rangle$ be a subsequence of $\langle x_n \rangle$ such that $x_{n_k} \in V_k$ for all $k \in \mathbf{N}$. Let W_k be an open neighborhood of x_{n_k} such that $W_k \subseteq V_k$ and $\bar{W}_k \cap E = \phi$. Recall that $|\bar{W}_k| > \mathfrak{p}_k$. For each $B \subseteq E$ let $L(B) = [(\bigcup_{k=1}^{\infty} \bar{W}_k) \cup \{p\}] \cup \bar{B}$. Then $\{L(B) : B \subseteq E\}$ is a collection of $2^{\mathfrak{m}}$ closed sets in X, each of cardinality $\mathfrak{p}$.

PROPOSITION 3.7. *Let X be a T_1, perfect space and let A be a discrete subset of X.* (1) *If $|A| > \mathfrak{m}$, then there is a subset B of A such that $|B| > \mathfrak{m}$ and B has no limit points.* (2) *If $|A| = \mathfrak{m}$, and $\mathfrak{m}$ is not sequential, then there is a subset B of A of cardinality $\mathfrak{m}$ with no limit points.*

Proof. We may assume that $\mathfrak{m}$ is infinite. For each $x \in A$ let V_x be an open neighborhood of x such that $V_x \cap A = \{x\}$. Now $V = \bigcup_{x \in A} V_x$ is open, so $V = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed. For $n \in \mathbf{N}$ let $H_n = F_n \cap A$. Then H_n has no limit points, and $A = \bigcup_{n=1}^{\infty} H_n$. If $|A| > \mathfrak{m}$, then $|H_n| > \mathfrak{m}$ for some $n \in \mathbf{N}$. If $|A| = \mathfrak{m}$, and $\mathfrak{m}$ is not sequential, then $|H_n| = \mathfrak{m}$ for some $n \in \mathbf{N}$.

PROPOSITION 3.8. *Let* $\mathfrak{m}$ *be a sequential cardinal, let* X *be a regular, perfect,* θ*-refinable space having property* wD, *let* Y *be an open subset of* X, *and let* A *be a discrete subset of* Y *of cardinality* $\mathfrak{m}$. *Then there is a discrete subset* B *of* Y *of cardinality* $\mathfrak{m}$ *such that* B *has at most one limit point in* Y.

Proof. Let $\mathfrak{m} = \sum_{k=1}^{\infty} \mathfrak{p}_k$, where $\mathfrak{p}_1 < \mathfrak{p}_2 < \ldots < \mathfrak{m}$. We consider two cases.

(a) *There is a cardinal* $\mathfrak{p} < \mathfrak{m}$ *such that for each* $x \in Y$, *there is an open neighborhood* V_x *of* x *such* that $|V_x \cap A| \leqq \mathfrak{p}$. We may assume that $V_x \subseteq Y$ for all $x \in Y$. Since every subspace of a perfect, θ-refinable space is θ-refinable, it follows that Y is irreducible. Let $\mathscr{G} = \{G_t \colon t \in T\}$ be a minimal open refinement of $\{V_x \colon x \in Y\}$. Since $A \subseteq \bigcup_{t \in T} (G_t \cap A)$, $|G_t \cap A| \leqq \mathfrak{p}$ for all $t \in T$, and $|A| = \mathfrak{m}$, it follows that $|T| \geqq \mathfrak{m}$. For each $t \in T$, let $x_t \in (G_t - \bigcup_{s \neq t} G_s)$, and let $B = \{x_t \colon t \in T\}$. Then $B \subseteq Y$, and B is a discrete subset of X of cardinality $\geqq \mathfrak{m}$ which has no limit points in Y.

(b) *There is a sequence* $\langle x_n \rangle$ *in* Y *such that if* V *is any open neighborhood of* x_n, *then* $|V \cap A| > \mathfrak{p}_n$. First suppose that $\langle x_n \rangle$ has no cluster points. Since X has property wD, there is a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ and a discrete open collection $\{V_k \colon k \in \mathbf{N}\}$ such that $x_{n_k} \in V_k$ for all $k \in \mathbf{N}$. Now $|V_k \cap A| > \mathfrak{p}_{n_k} \geqq \mathfrak{p}_k$, so by 3.7(1) there is a subset B_k of $V_k \cap A$ such that $|B_k| > \mathfrak{p}_k$ and B_k has no limit points. Let $B = \bigcup_{k=1}^{\infty} B_k$. Then $B \subseteq Y$ and B is a discrete subset of X of cardinality $\mathfrak{m}$ with no limit points. Next suppose that $\langle x_n \rangle$ has a cluster point p. Let $\{V_k \colon k \in \mathbf{N}\}$ be a decreasing collection of open neighborhoods of p such that $\bigcap_{k=1}^{\infty} \bar{V}_k = \{p\}$. Let $\langle x_{n_k} \rangle$ be a subsequence of $\langle x_n \rangle$ such that $x_{n_k} \in V_k$ for all $k \in \mathbf{N}$. Now $|V_k \cap A| > \mathfrak{p}_{n_k} \geqq \mathfrak{p}_k$, so by 3.7(1) there is a subset B_k of $V_k \cap A$ such that $|B_k| > \mathfrak{p}_k$ and B_k has no limit points. Let $B = \bigcup_{k=1}^{\infty} B_k$. Then $B \subseteq Y$, and B is a discrete subset of X of cardinality $\mathfrak{m}$ with at most one limit point, namely p.

4. Denumerable closed sets. In this section we want to find the number of denumerable closed sets in an infinite Hausdorff space X. The number of such sets may be 0 (take $X = \beta\mathbf{N}$), and the maximum number is $|X|^{\aleph_0}$. In Theorem 4.1 below we show that this maximum number is achieved under fairly weak conditions on X. We then give several examples; of special interest is 4.5, an example of a compact, Hausdorff space X in which the number of denumerable closed sets is neither 0 nor $|X|^{\aleph_0}$.

THEOREM 4.1. *Let* X *be an infinite, regular space with countable pseudo-character. Then the number of denumerable closed sets in* X *is* $|X|^{\aleph_0}$.

Proof. Let $\mathscr{E}$ be a collection of denumerable subsets of X such that $|\mathscr{E}| = |X|^{\aleph_0}$ and the intersection of any two distinct elements of $\mathscr{E}$ is finite. (See [**33**].) For each $E \in \mathscr{E}$, construct a denumerable closed set E^* as follows. (1) If E has no limit points, let $E^* = E$. (2) If E has a limit point p, let $\{V_n \colon n \in \mathbf{N}\}$ be a decreasing collection of open neighborhoods of p such that

$\cap \bar{V}_n = \{p\}$, let $\{x_n: n \in \mathbf{N}\}$ be a denumerable subset cf E such that $x_n \in V_n$ for all n, and let $E^* = \{x_n: n \in \mathbf{N}\} \cup \{p\}$.

Now let D and E be distinct elements of $\mathscr{E}$. Since $D \cap E$ is finite, it easily follows that $D^* \cap E^*$ is also finite, and hence $D^* \neq E^*$. Thus $\{E^*: E \in \mathscr{E}\}$ is a collection of $|X|^{\aleph_0}$ denumerable closed sets in X.

Remark 4.2. The above result cannot be extended to higher cardinals; $\psi(\beta\mathbf{N}) \leqq 2^{\aleph_0}$, but $\beta\mathbf{N}$ has no closed sets of cardinality $2^{\aleph_0}$. However, the following question is suggested. If X is a regular space with countable pseudo-character, and $|X| > 2^{\aleph_0}$, does X have a closed subset of cardinality $2^{\aleph_0}$?

Example 4.3. *Assuming the axiom of constructibility, there is a compact, Hausdorff, hereditarily separable, hereditarily normal space such that every infinite closed set has cardinality* $2^{2^{\aleph_0}}$. Such a space has been constructed by Fedorcŭk [**8**].

Example 4.4. *There is a compact, Hausdorff space X with $|X| = \mathfrak{m}$, $\mathfrak{m}$ is sequential, X has a discrete subset of cardinality $\mathfrak{m}$, and every infinite closed subset of X has cardinality* $\geqq 2^{2^{\aleph_0}}$. The basic idea in constructing X is to replace each positive integer k in $\beta\mathbf{N}$ with a suitable compact space. We recall some facts about Stone-Čech compactifications. (1) $\beta\mathbf{N}$ is the set of all ultrafilters on $\mathbf{N}$, and a base for the topology on $\beta\mathbf{N}$ is $\{U': U \subseteq \mathbf{N}\}$, where $U' = \{q \in \beta\mathbf{N}: U \in q\}$. (2) If D is a discrete space of cardinality $\mathfrak{p} \geqq \aleph_0$, then $|\beta D| = 2^{2^{\mathfrak{p}}}$, βD has a discrete subset of cardinality $\mathfrak{p}$, and every infinite closed subset of βD has cardinality $\geqq 2^{2^{\aleph_0}}$. (See [**9**; **26**].) Notation: for a cardinal $\mathfrak{p}$, $\exp_1(\mathfrak{p}) = 2^{\mathfrak{p}}$ and $\exp_{k+1}(\mathfrak{p}) = 2^{\exp_k(\mathfrak{p})}$.

For each positive integer k let D_k be a discrete space of cardinality $\exp_k(\aleph_0)$. Then $|\beta D_k| = \exp_{k+2}(\aleph_0)$, βD_k has a discrete subset of cardinality $\exp_k(\aleph_0)$, and every infinite closed subset of βD_k has cardinality $\geqq 2^{2^{\aleph_0}}$. Let $\mathfrak{m} = \sum_{k=1}^{\infty} \exp_k(\aleph_0)$, and note that $\mathfrak{m}$ is sequential. Let $\mathbf{N}^* = \{q: q$ is a free ultrafilter on $\mathbf{N}\}$, and let $X = \mathbf{N}^* \cup (\bigcup_{k=1}^{\infty} \beta D_k)$. (We assume that $\mathbf{N}^* \cap \beta D_k = \phi$ for all k and $\beta D_k \cap \beta D_n = \phi$ for $n \neq k$.) For $U \subseteq \mathbf{N}$, let $U'' = \{q \in \mathbf{N}^*: U \in q\} \cup (\bigcup_{k \in U} \beta D_k)$, and let $\mathscr{B} = \{U: U$ is open in βD_k for some $k\} \cup \{U'': U$ an infinite subset of $\mathbf{N}\}$. Then $\mathscr{B}$ is a base for a compact, Hausdorff topology on X. (The proof is similar to the corresponding result for $\beta\mathbf{N}$; see [**26**].) Note that $|X| = \mathfrak{m}$ and X has a discrete subset of cardinality $\mathfrak{m}$.

Now let F be an infinite closed set in X, and let us show that $|F| \geqq 2^{2^{\aleph_0}}$. We may assume that $F \cap \beta D_k$ is finite for all $k \in \mathbf{N}$. Let $W = \{k: k \in \mathbf{N}, F \cap \beta D_k \neq \phi\}$. First suppose that W is infinite. Then W is an infinite subset of $\beta\mathbf{N}$, and so there are $2^{2^{\aleph_0}}$ points in N^* which are limit points of W for the space $\beta\mathbf{N}$. One can easily check that each of these points is also a limit point of F for the space X. Next suppose that W is finite. Since $F \cap \beta D_k$ is finite for all k, $F \cap \mathbf{N}^*$ is infinite. Again, there exist $2^{2^{\aleph_0}}$ points in $\mathbf{N}^*$, each of which is a limit point of $F \cap \mathbf{N}^*$ for the space $\beta\mathbf{N}$. Each of these points is also a limit point of F for the space X.

Example 4.5. *There is a compact, Hausdorff space X such that $|X| = \mathfrak{m} > 2^{2^{\aleph_0}}$, $\mathfrak{m}$ is sequential, X has a discrete subset of cardinality $\mathfrak{m}$, and the number of denumerable closed sets in X is neither 0 nor $|X|^{\aleph_0}$.* Let Y be constructed as in 4.4, let $X = Y \oplus [0, 1]$. Then $|X| = \mathfrak{m} < \mathfrak{m}^{\aleph_0}$, and the number of denumerable closed sets in X is $\mathfrak{m}$. (The number of closed sets in X of cardinality $2^{\aleph_0}$ is also $\mathfrak{m}$.)

Example 4.6. *For every infinite cardinal $\mathfrak{m}$ such that $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$, there is a countably compact regular space X such that $|X| = \mathfrak{m}$, X has a discrete subset of cardinality $\mathfrak{m}$, and X has no denumerable closed sets.* (I am grateful to K. Kunen for the suggestion that a regular space of cardinality $2^{\aleph_0}$ with no denumerable closed subsets can be constructed in $\beta\mathbf{N}$). Let D be a discrete space of cardinality $\mathfrak{m}$; recall that every infinite closed set in βD has cardinality $\geqq 2^{2^{\aleph_0}}$. Construct a sequence $\{H_\alpha: 0 \leqq \alpha < \aleph_1\}$ of subsets of βD such that (1) $|H_0| = \mathfrak{m}$ and H_0 is a discrete subset of βD; (2) $|H_\alpha| \leqq \mathfrak{m}$, $0 \leqq \alpha < \aleph_1$; (3) for $0 < \alpha < \aleph_1$, if E is a denumerable subset of $\bigcup_{0 \leqq \beta < \alpha} H_\beta$, then $|\bar{E} \cap H_\alpha| \geqq \aleph_1$. Then $X = \bigcup_{0 \leqq \alpha < \aleph_1} H_\alpha$ is the desired space.

Example 4.7. *Assume* GCH. *Then for every cardinal $\mathfrak{m} > \aleph_0$ there is a countably compact regular space X such that $|X| = \mathfrak{m}$, X has a discrete subset of cardinality $\mathfrak{m}$, and X has no denumerable closed sets.* First suppose that $\mathfrak{m}$ is not sequential. By GCH, $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$, so the existence of such a space follows from 4.6. Now assume that $\mathfrak{m}$ is sequential. Under GCH the construction in 4.4 can be carried out for $\mathfrak{m}$.

5. Closed sets in X of cardinality $|X|$. Let X be an infinite Hausdorff space. In this section we want to find the number of closed sets in X of cardinality $|X|$. We show that the number of such sets is at least $|X|$, and under GCH the number of such sets is $o(X)$. (Thus, under GCH, there are at least as many closed sets in X of cardinality $|X|$ as there are closed sets of any other cardinality.)

THEOREM 5.1. *Let X be an infinite Hausdorff space. Then there are at least $|X|$ closed sets in X of cardinality $|X|$.*

Proof. Let $|X| = \mathfrak{m}$. It suffices to show that if $\mathfrak{n}$ is a regular cardinal and $\mathfrak{n} \leqq \mathfrak{m}$, then X has at least $\mathfrak{n}$ closed sets of cardinality $\mathfrak{m}$. In proving this, we make the following assumption: if F is a closed subset of X, and W is an open subset of F with $|W| \geqq \mathfrak{n}$, then $|F - W| < \mathfrak{m}$. (Indeed, if $|F - W| = \mathfrak{m}$, then $\{(F - W) \cup \{p\}: p \in W\}$ is a collection of at least $\mathfrak{n}$ closed sets in X, each of cardinality $\mathfrak{m}$.) Under this assumption, we construct a sequence $\{F_\alpha: 0 \leqq \alpha < \mathfrak{n}\}$ of closed sets in X such that (1) for $0 \leqq \beta < \alpha < \mathfrak{n}$, $F_\alpha \subseteq F_\beta$ and $F_\alpha \neq F_\beta$; (2) $|X - F_\alpha| < \mathfrak{n}$, $0 \leqq \alpha < \mathfrak{n}$. Let $F_0 = X$. Now let α be fixed, $0 < \alpha < \mathfrak{n}$, and assume that $\{F_\beta: 0 \leqq \beta < \alpha\}$ have already been constructed so that (1) and (2) hold. We consider two cases.

(a) *α is a limit ordinal.* Let $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$. Clearly F_α is closed and (1) holds. Since $X - F_\alpha = \bigcup_{0\leqq\beta<\alpha} (X - F_\beta)$, and $\mathfrak{n}$ is regular, $|X - F_\alpha| < \mathfrak{n}$. Hence (2) holds.

(b) *α is a successor ordinal, say* $\alpha = \gamma + 1$. In this case F_γ is a closed set and $|X - F_\gamma| < \mathfrak{n}$, $|F_\gamma| = \mathfrak{m}$. Let U and V be disjoint, non-empty open sets in F_γ. If $|U| < \mathfrak{n}$, then $F_\alpha = F_\gamma - U$ is the desired closed set. Suppose that $|U| \geqq \mathfrak{n}$. Then we may assume that $|F_\gamma - U| < \mathfrak{m}$, and so $|U| = \mathfrak{m}$. It follows that $|V| < \mathfrak{n}$, and so $F_\alpha = F_\gamma - V$ is the desired closed set. (If $|V| \geqq \mathfrak{n}$, then $|F_\gamma - V| < \mathfrak{m}$, contradicting $U \subseteq F_\gamma - V$ and $|U| = \mathfrak{m}$.)

THEOREM 5.2. *Assume* GCH, *let X be an infinite Hausdorff space. Then X has $o(X)$ closed sets of cardinality $|X|$.*

Proof. Let $|X| = \mathfrak{m}$. By GCH, $o(X) = \mathfrak{m}$ or $o(X) = 2^\mathfrak{m}$. We assume that $o(X) = 2^\mathfrak{m}$. (If $o(X) = \mathfrak{m}$, we are finished by 5.1.) First suppose that $\mathfrak{m}$ is regular. Then for $\mathfrak{n} < \mathfrak{m}$, there are at most $\mathfrak{m}^\mathfrak{n} = \mathfrak{m}$ closed sets in X of cardinality $\mathfrak{n}$; thus there must be $2^\mathfrak{m}$ closed sets in X of cardinality $\mathfrak{m}$. Next suppose that $\mathfrak{m}$ is singular. Then there is a discrete subset of X of cardinality $\mathfrak{m}$ (see [**12**]), and hence there are $2^\mathfrak{m}$ closed sets in X, each of cardinality $\mathfrak{m}$.

6. The basic problem assuming GCH. If X is a Hausdorff space and $\mathfrak{n}$ is an infinite cardinal with $\mathfrak{n} < |X|$, then the maximum number of closed sets in X of cardinality $\mathfrak{n}$ is $|X|^\mathfrak{n}$. In this section we assume GCH and show that, under fairly general topological conditions on X, this maximum number is achieved for all $\mathfrak{n} < |X|$. In proving the main result (Theorem 6.7), we consider two cases, namely $\mathfrak{n}$ not sequential (Theorem 6.4) and $\mathfrak{n}$ sequential (Theorem 6.6).

PROPOSITION 6.1. *Assume* GCH, *let X be a Hausdorff space, and let $\mathfrak{n}$ be an infinite cardinal such that $\mathfrak{n} < |X|$.*

(1) *If $|X|$ is regular, then the number of closed sets in X of cardinality $\mathfrak{n}$ is 0 or $|X|^\mathfrak{n}$ $(= |X|)$.*

(2) *If $|X|$ is singular, and $\mathfrak{n} < cf(|X|)$ or $2^{cf(|X|)} < \mathfrak{n} < |X|$, then the number of closed sets in X of cardinality $\mathfrak{n}$ is 0 or $|X|^\mathfrak{n}$.*

Proof. The following observation is easy to prove using GCH: for $\mathfrak{n} < |X|$, the number of closed sets in X of cardinality $\mathfrak{n}$ is 0, $|X|$, or $|X|^\mathfrak{n}$. Note that (1) and (2) for the case $\mathfrak{n} < cf(|X|)$ follow from this observation and the fact that, under GCH, $|X|^\mathfrak{n} = |X|$ whenever $\mathfrak{n} < cf(|X|)$. It remains to consider the case in which $|X|$ is singular and $2^{cf(|X|)} < \mathfrak{n} < |X|$. Suppose that X has a closed set L of cardinality $\mathfrak{n}$, and let us construct $|X|^\mathfrak{n}$ $(= 2^{|X|})$ closed sets, each of cardinality $\mathfrak{n}$. Let A be a discrete subset of X of cardinality $|X|$ (see [**12**]), and assume that $A \cap L = \phi$. If $B \subseteq A$, and $|B| = cf(|X|)$, then $|\bar{B}| \leqq 2^{2^{cf(|X|)}}$. (See [**10**].) Hence $\{L \cup \bar{B}: B \subset A, |B| = cf(|X|)\}$ is a collection of $2^{|X|}$ closed sets in X, each of cardinality $\mathfrak{n}$.

LEMMA 6.2. *Let X be a regular space, let $\psi(X) = \mathfrak{n}$, and let A be an infinite subset of X such that $X = \bigcup\{\bar{B}: B \subseteq A, |B| \leqq \mathfrak{n}\}$. Then $|X| \leqq |A|^{\mathfrak{n}}$.*

Proof. For each $p \in X$, let B_p be a subset of A of cardinality $\leqq \mathfrak{n}$ such that $p \in \bar{B}_p$, and let $\{V(\alpha, p): 0 \leqq \alpha < \mathfrak{n}\}$ be a collection of open neighborhoods of p such that $\bigcap_{\alpha<\mathfrak{n}} \overline{V(\alpha, p)} = \{p\}$. Let $f: X \rightarrow \mathscr{P}_{\mathfrak{n}}(\mathscr{P}_{\mathfrak{n}}(A))$ be defined by $f(p) = \{V(\alpha, p) \cap B_p: 0 \leqq \alpha < \mathfrak{n}\}$. Then f is one-one, so $|X| \leqq |A|^{\mathfrak{n}}$.

PROPOSITION 6.3. *Let X be a regular space. Then $|X| \leqq d(X)^{\psi(X)\cdot t(X)}$ and $|X| \leqq d(X)^{\psi(X)\pi\chi(X)}$.*

Proof. We may assume that X is infinite. Let $\psi(X) \cdot t(X) = \mathfrak{n}$, let A be a dense subset of X of cardinality $d(X)$. Since $t(X) \leqq \mathfrak{n}$, $X = \bigcup \{\bar{B}: B \subseteq A, |B| \leqq \mathfrak{n}\}$. Hence $|X| \leqq d(X)^{\mathfrak{n}}$ by 6.2. The proof for $\psi(X) \cdot \pi\chi(X) = \mathfrak{n}$ is similar.

THEOREM 6.4. *Assume* GCH. *Let X be a regular space with countable pseudo-character and countable tightness, and let $\mathfrak{n}$ be an infinite cardinal with $\mathfrak{n} < |X|$. If $\mathfrak{n} = \aleph_0$, or $\mathfrak{n}$ is not sequential, then the number of closed sets in X of cardinality $\mathfrak{n}$ is $|X|^{\mathfrak{n}}$.*

Proof. For $\mathfrak{n} = \aleph_0$, the result follows from 4.1. Assume, then, that $\mathfrak{n} \geqq 2^{\aleph_0}$ and $\mathfrak{n}$ is not sequential. First we show that X has one closed set of cardinality $\mathfrak{n}$. Let $H \subseteq X$, $|H| = \mathfrak{n}$. By 6.3, $|\bar{H}| \leqq \mathfrak{n}^{\aleph_0}$, and since $\mathfrak{n}$ is not sequential and GCH is assumed, $\mathfrak{n}^{\aleph_0} = \mathfrak{n}$. Thus $\bar{H}$ is a closed set of cardinality $\mathfrak{n}$. Let $|X| = \mathfrak{m}$. By 6.1, we need only consider the case in which $\mathfrak{m}$ is singular and $\mathfrak{n} = cf(\mathfrak{m})$ or $\mathfrak{n} = 2^{cf(\mathfrak{m})}$. By GCH, $\mathfrak{m}^{\mathfrak{n}} = 2^{\mathfrak{m}}$, and so we must construct $2^{\mathfrak{m}}$ closed sets in X, each of cardinality $\mathfrak{n}$. Let A be a discrete subset of X of cardinality $\mathfrak{m}$ (see [**12**]), let L be a closed subset of X of cardinality $\mathfrak{n}$, and assume that $A \cap L = \phi$. If $B \subseteq A$ and $|B| = cf(\mathfrak{m})$, then $|\bar{B}| \leqq cf(\mathfrak{m})^{\aleph_0}$ by 6.3. Now $cf(\mathfrak{m})$ is a regular cardinal, so $cf(\mathfrak{m})^{\aleph_0} = cf(\mathfrak{m})$ or $cf(\mathfrak{m})^{\aleph_0} = 2^{\aleph_0}$, according as $cf(\mathfrak{m}) > \aleph_0$ or $cf(\mathfrak{m}) = \aleph_0$. In either case, since $\mathfrak{n} \geqq 2^{\aleph_0}$, it follows that $cf(\mathfrak{m})^{\aleph_0} \leqq \mathfrak{n}$. Hence $\{\bar{B} \cup L: B \subseteq A, |B| = cf(\mathfrak{m})\}$ is a collection of $2^{\mathfrak{m}}$ closed sets in X, each of cardinality $\mathfrak{n}$.

Example 6.5. *Assuming the axiom of constructibility, there is a regular, first countable, countably compact space X such that $|X| = hc(X) = 2^{\aleph_\omega}$ and X has no closed subsets of cardinality $\aleph_\omega$.* (Thus the hypothesis in 6.4 that $cf(\mathfrak{n}) > \aleph_0$ is not superfluous.) The example is due to Eric van Douwen (personal communication), and is based on results of Juhász, Nagy, and Weiss. Call a space *good* if it is regular, countably compact, and locally countable (i.e., every point has a countable open neighborhood). The following two results about good spaces are proved in [**19**]. (1) If X is an uncountable good space, then $cf(|X|) > \aleph_0$. (2) If the axiom of constructibility holds, and $cf(\mathfrak{m}) > \aleph_0$, then there is a good space of cardinality $\mathfrak{m}$. Now let X be a good space with $|X| = 2^{\aleph_\omega}$. By (1), X does not have a closed subset of cardinality $\aleph_\omega$. The fact that $hc(X) = |X|$ follows from the lemma on p. 40 of [**17**].

THEOREM 6.6. *Assume* GCH. *Let X be a regular, irreducible space with countable pseudo-character and property* wD, *and let $\mathfrak{n}$ be a sequential cardinal with $\mathfrak{n} < |X|$. Then the number of closed sets in X of cardinality $\mathfrak{n}$ is $|X|^{\mathfrak{n}}$.*

Proof. By 4.1, we may assume that $\aleph_0 < \mathfrak{n}$. By 6.1, it suffices to show that X has a closed set of cardinality $\mathfrak{n}$. Let $\mathfrak{n} = \sum_{k=1}^{\infty} \mathfrak{p}_k$, where $\mathfrak{p}_1 < \mathfrak{p}_2 < \ldots < \mathfrak{n}$. Note that $2^{2^{\mathfrak{p}_k}} < \mathfrak{n}$ for all k. We consider two cases.

(a) *There is a cardinal $\mathfrak{p} < \mathfrak{n}$ such that for all $x \in X$, there is an open neighborhood V_x of x such that $|V_x| \leqq \mathfrak{p}$.* As in (a) of 3.8 (with $Y = X$), there is a subset of X of cardinality $\geqq \mathfrak{n}$ with no limit points.

(b) *There is a sequence $\langle x_n \rangle$ in X such that if V is any open neighborhood of x_n, then $|V| > \mathfrak{p}_n$.* The proof is similar to (b) in 3.8, with these modifications. Let $\langle x_{n_k} \rangle$ and $\{V_k: k \in \mathbf{N}\}$ be as in (b). For each $k \in \mathbf{N}$ let L_k be a subset of V_k of cardinality $\mathfrak{p}_k$, and note that $|\bar{L}_k| \leqq 2^{2^{\mathfrak{p}_k}}$. (See [**10**].) Let $H = \bigcup_{k=1}^{\infty} \bar{L}_k$. Then H (or $H \cup \{p\}$) is a closed set of cardinality $\mathfrak{n}$.

THEOREM 6.7. *Assume* GCH. *Let X be a regular, irreducible space with countable pseudo-character, countable tightness, and property* wD, *and let $\mathfrak{n}$ be an infinite cardinal with $\mathfrak{n} < |X|$. Then the number of closed sets in X of cardinality $\mathfrak{n}$ is $|X|_{\mathfrak{n}}$.*

COROLLARY 6.8. *Assume* GCH, *let X be a paracompact, first countable space, and let $\mathfrak{n}$ be an infinite cardinal with $\mathfrak{n} < |X|$. Then the number of closed sets in X of cardinality $\mathfrak{n}$ is $|X|^{\mathfrak{n}}$.*

Remark 6.9. In 6.6 and 6.7, "irreducible" can be replaced by "meta-Lindelöf." (Recall that a space is *meta-Lindelöf* if every open cover has a point-countable open refinement.) The necessary modifications in the proof of 6.6 are as follows. Assume that $\mathfrak{n} > \aleph_0$, and consider case (a). Let $\mathscr{G}$ be a point-countable open refinement of $\{V_x: x \in X\}$, and let A be a subset of X which is maximal with respect to the property that if x and y are distinct elements of A, then $y \notin \mathrm{st}\,(x, \mathscr{G})$. Note that A has no limit points. Moreover, by maximality of A, the collection $\mathscr{G}_1 = \{G \in \mathscr{G}: G \cap A \neq \phi\}$ covers X. Suppose $|A| < \mathfrak{n}$. Then $|\mathscr{G}_1| \leqq |A| \cdot \aleph_0 < \mathfrak{n}$, and since $|G| \leqq \mathfrak{p}$ for all $G \in \mathscr{G}_1$, it follows that $|X| < \mathfrak{n}$, a contradiction. Hence $|A| \geqq \mathfrak{n}$, and so X has a closed subset of cardinality $\mathfrak{n}$.

7. Upper bounds on $o(X)$. In this section we obtain upper bounds on $o(X)$ in terms of other cardinal invariants of X. Recall that an infinite, Hausdorff space has at least $2^{\aleph_0}$ open sets.

THEOREM 7.1. *If X is T_1, then $o(X) \leqq 2^{hd(X)\cdot\psi(X)}$. In particular, if X is an infinite, Hausdorff, hereditarily separable sbare with countable pseudo-character, then $o(X) = 2^{\aleph_0}$.*

Proof. This result follows immediately from the two inequalities $|X| \leqq 2^{hc(X)\cdot\psi(X)}$ and $o(X) \leqq |X|^{hd(X)}$.

LEMMA 7.2 (Šapirovskiĭ [**27**]). *Let $\mathscr{B}$ be an open cover of a topological space X, let $hc(X) \leqq \mathfrak{m}$. Then there is some G in $(\mathscr{B})_{\mathfrak{m}}$ and some A in $\mathscr{P}_{\mathfrak{m}}(X)$ such that $X = G \cup \bar{A}$.*

THEOREM 7.3. *If X is regular, then $o(X) \leqq \pi\chi(X)^{hc(X)\cdot\psi C(X)}$. In particular, if X is an infinite, regular, hereditarily* CCC, *perfect space with π-character $\leqq 2^{\aleph_0}$, then $o(X) = 2^{\aleph_0}$.*

Proof. Let $hc(X) \cdot \psi C(X) = \mathfrak{m}$, let $\pi\chi(X) = \mathfrak{n}$. Then $|X| \leqq 2^{hc(X)\cdot\psi(X)} \leqq 2^{\mathfrak{m}}$ and $w(X) \leqq \pi\chi(X)^{c(X)} \leqq \mathfrak{n}^{\mathfrak{m}}$ (see [**28**]). Let $\mathscr{B}$ be a base for X of cardinality $\leqq \mathfrak{n}^{\mathfrak{m}}$, and let $\mathscr{V} = \{G \cup \bar{A}: G \in (\mathscr{B})_{\mathfrak{m}}, A \in \mathscr{P}_{\mathfrak{m}}(X)\}$. Now $|(\mathscr{V})_{\mathfrak{m}}| \leqq \mathfrak{n}^{\mathfrak{m}}$, and so it suffices to show that each open set in X belongs to $(\mathscr{V})^{\mathfrak{m}}$. If W is open, then $W = \bigcup_{\alpha<\mathfrak{m}} F_\alpha$, where each F_α is closed. Fix $\alpha < \mathfrak{m}$. For each $p \in F_\alpha$, choose $B_p \in \mathscr{B}$ such that $B_p \subseteq W$. Apply 7.2 to $\{B_p: p \in F_\alpha\}$ and F_α to obtain A_α and E_α, each a subset of F_α of cardinality $\leqq \mathfrak{m}$, such that $F_\alpha \subseteq [(\bigcup_{p\in E_\alpha} B_p) \cup \bar{A}_\alpha] = V_\alpha$. Note that $V_\alpha \in \mathscr{V}$ and $V_\alpha \subseteq W$. Since $W = \bigcup_{\alpha<\mathfrak{m}} V_\alpha$ and $(\bigcup_{\alpha<\mathfrak{m}} V_\alpha) \in (\mathscr{V})_{\mathfrak{m}}$, the proof is complete.

THEOREM 7.4. *If X is normal, then $o(X) \leqq 2^{d(X)\cdot\psi C(X)}$. In particular, if X is an infinite, separable, perfectly normal space, then $o(X) = 2^{\aleph_0}$.*

Proof. Let $d(X) \cdot \psi C(X) = \mathfrak{m}$, let A be a dense subset of X of cardinality $\leqq \mathfrak{m}$. Let $\mathscr{H} = \{\bar{H}: H \subseteq A\}$, and let $\mathscr{L} = \{L: L$ is the intersection of $\leqq \mathfrak{m}$ elements of $\mathscr{H}\}$. Then $|\mathscr{L}| \leqq 2^{\mathfrak{m}}$, and so it suffices to show that every closed set in X belongs to $\mathscr{L}$. If F is closed, then $F = \bigcap_{\alpha<\mathfrak{m}} V_\alpha$, where each V_α is an open set. For each $\alpha < \mathfrak{m}$ let W_α be an open set such that $F \subseteq W_\alpha \subseteq \bar{W}_\alpha \subseteq V_\alpha$, and let $H_\alpha = A \cap W_\alpha$. Then $F = \bigcap_{\alpha<\mathfrak{m}} \bar{H}_\alpha$, so $F \in \mathscr{L}$.

THEOREM 7.5. *If X is normal, then $o(X) \leqq \pi\chi(X)^{c(X)\cdot\psi C(X)}$. In particular, if X is an infinite, perfectly normal,* CCC *space with π-character $\leqq 2^{\aleph_0}$, then $o(X) = 2^{\aleph_0}$.*

Proof. Let $c(X) \cdot \psi C(X) = \mathfrak{m}$, let $\pi\chi(X) = \mathfrak{n}$. Now $w(X) \leqq \pi\chi(X)^{c(X)} \leqq \mathfrak{n}^{\mathfrak{m}}$, so X has a base $\mathscr{B}$ with $|\mathscr{B}| \leqq \mathfrak{n}^{\mathfrak{m}}$. Let $\mathscr{G} = \{\bar{G}: G$ is the union of $\leqq \mathfrak{m}$ elements of $\mathscr{B}\}$, and let $\mathscr{H} = \{H: H$ is the intersection of $\leqq \mathfrak{m}$ elements of $\mathscr{G}\}$. Now $|\mathscr{H}| \leqq \mathfrak{n}^{\mathfrak{m}}$, and so it suffices to show that every closed set in X belongs to $\mathscr{H}$. Let F be a closed set, and let $F = \bigcap_{\alpha<\mathfrak{m}} V_\alpha$, where each V_α is an open set. Fix $\alpha < \mathfrak{m}$. Let W_α be an open set such that $F \subseteq W_\alpha \subseteq \bar{W}_\alpha \subseteq V_\alpha$, and let $\mathscr{B}_\alpha$ be a maximal, disjoint subcollection of $\{B: B \in \mathscr{B}, B \subseteq W_\alpha\}$. Since $c(X) \leqq \mathfrak{m}$, $|\mathscr{B}_\alpha| \leqq \mathfrak{m}$. Let $G_\alpha = \bigcup \mathscr{B}_\alpha$, and note that $F \subseteq \bar{G}_\alpha \subseteq \bar{W}_\alpha$ and $\bar{G}_\alpha \in \mathscr{G}$. Since $F = \bigcap_{\alpha<\mathfrak{m}} \bar{G}_\alpha$, $F \in \mathscr{H}$ and the proof is complete.

Example 7.6. *Assuming $V = L$, there is a compact, Hausdorff, hereditarily separable, completely normal space in which the number of open sets is $2^{2^{\aleph_0}}$.* Such a space has been constructed by Fedorcŭk [**8**].

Example 7.7. *It is consistent that there exists a regular, hereditarily Lindelof space in which the number of open sets is $2^{2^{\aleph_0}}$.* Such a space has been constructed by Hajnal and Juhász [**16**].

Example 7.8. *There is a separable, perfect, first countable, regular space in which the number of open sets is* $2^{2^{\aleph_0}}$. The tangent disc space (see p. 36 of [**36**]) has the required properties.

Example 7.9. *There is a compact, Hausdorff, separable, first countable space in which the number of open sets is* $2^{2^{\aleph_0}}$. Let I^* be the top and bottom line of the lexicographically ordered square with the order topology. Then the space $X = I^* \times I^*$ has the required properties. (See [**4**; **17**].)

Question 7.10. It follows from the inequalities $|X| \leqq 2^{hL(X)}$ (X Hausdorff) and $o(X) \leqq w(X)^{hL(X)}$ that a hereditarily Lindelöf, first countable, Hausdorff space has $\leqq 2^{\aleph_0}$ closed sets. As a consequence of 7.1, a hereditarily separable, first countable Hausdorff space has $\leqq 2^{\aleph_0}$ closed sets. Does a regular, hereditarily CCC, first countable space have $\leqq 2^{\aleph_0}$ closed sets?

References

1. A. V. Arhangel'skiĭ, *On bicompacta hereditarily satisfying Suslin's condition. Tightness and free sequences*, Soviet Math. Dokl. *12* (1971), 1253–1256.

2. J. Boone, *On irreducible spaces*, Bull. Austral. Math. Soc. *12* (1975), 143–148.

3. D. K. Burke, *On p-spaces and wΔ-spaces*, Pacific J. Math. *35* (1970), 285–296.

4. D. K. Burke and R. E. Hodel, *The number of compact subsets of a topological space*, Proc. Amer. Math. Soc. *58* (1976), 363–368.

5. W. W. Comfort and A. W. Hager, *Estimates for the number of real-valued continuous functions*, Trans. Amer. Math. Soc. *150* (1970), 619–631.

6. E. K. van Douwen, *Functions from the integers to the integers and topology*, to appear.

7. R. Engelking, *General topology* (Warszawa, 1977).

8. V. V. Fedorcŭk, *On the cardinality of hereditarily separable compact Hausdorff spaces*, Soviet Math. Dokl. *16* (1975), 651–655.

9. L. Gillman and M. Jerison, *Rings of continuous functions* (Van Nostrand, Princeton, 1960).

10. J. de Groot, *Discrete subspaces of Hausdorff spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. *13* (1965), 537–544.

11. A. Hajnal and I. Juhász, *Discrete subspaces of topological spaces*, Indag. Math. *29* (1967), 343–356.

12. ——— *Discrete subspaces of topological spaces, II*, Indag. Math. *31* (1969), 18–30.

13. ——— *On hereditarily α-Lindelöf and hereditarily α-separable spaces*, Ann. Univ. Sci. Budapest Eötvös Sect. Math. *11* (1968), 115–124.

14. ——— *Some remarks on a property of topological cardinal functions*, Acta Math. Acad. Sci. Hung. *20* (1969), 25–37.

15. ——— *On the number of open sets*, Ann. Univ. Sci. Budapest, Sect. Math. *16* (1973), 99–102.

16. ——— *A consistency result concerning hereditarily α-Lindelöf spaces*, Acta Math. Acad. Sci. Hung. *24* (1973), 307–312.

17. I. Juhász, *Cardinal functions in topology*, Math. Centre Tract *34*, Amsterdam, 1971.

18. ——— *Remarks on cardinal functions*, Colloquia Mathematica Societatis János Bolyai *8*, Topics in Topology, Keszthely (Hungary), 1972, 449–450.

19. I. Juhász, Zs. Nagy, and W. Weiss, *On countably compact, locally countable spaces*, Preprint of the Mathematical Institute of the Hungarian Academy of Sciences, No. 7, 1976.

20. A. I. Krivoručko, *On the cardinality of the set of continuous functions*, Soviet Math. Dokl. *13* (1972), 1364–1367.

21. ——— *The cardinality and density of spaces of mappings*, Soviet Math. Dokl. *16* (1975), 281–285.

22. R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publ. *13*, rev. ed. 1962.

23. J. Nagata and F. Siwiec, *A note on nets and metrization*, Proc. Japan Acad. *44* (1968), 623–627.

24. J. Novák, *On the Cartesian product of two compact spaces*, Fund. Math. *40* (1953), 106–112.

25. A. Okuyama, *Some generalizations of metric spaces, their metrization theorems and product theorems*, Sci. Rep. Tokyo Kyoiku Daigaku, Ser. A *9* (1967), 236–254.

26. W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. *23* (1955), 409–419.

27. B. Šapirovskiĭ, *On discrete subspaces of topological spaces; weight, tightness and Suslin number*, Soviet Math. Dokl. *13* (1972), 215–219.

28. ——— *Canonical sets and character. Density and weight in compact spaces*, Soviet Math. Dokl. *15* (1974), 1282–1287.

29. J. C. Smith, *A remark on irreducible spaces*, Proc. Amer. Math. Soc. *57* (1976), 133-138.

30. ——— *Characterizations of generalized paracompact spaces*, Colloq. Math. *35* (1976), 29–39.

31. A. H. Stone, *Cardinals of closed sets*, Mathematika *6* (1959), 99–107.

32. ——— *Non-separable Borel sets*, Rozprawy Matematyczne *28* (1962), 1–40.

33. A. Tarski, *Sur la decomposition des ensembles en sous-ensembles presque disjoints*, Fund. Math. *12* (1928), 188–205.

34. J. Vaughan, *Discrete sequences of points*, to appear.

35. H. H. Wicke and J. M. Worrell Jr., *Characterizations of developable spaces*, Can. J. Math. *17* (1965), 820–830.

36. S. Willard, *General topology* (Addison-Wesley, Reading, 1968).

Duke University,
Durham, North Carolina 27706