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ON ALMOST STABLE CMC HYPERSURFACES IN MANIFOLDS OF BOUNDED SECTIONAL CURVATURE

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Abstract

We show that almost stable constant mean curvature hypersurfaces contained in a sufficiently small ball of a manifold of bounded sectional curvature are close to geodesic spheres.

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1. Introduction and statement of the result

Let $\phi : (M^n, g) \to (N^{n+1}, \overline{g})$ be an isometric immersion of an oriented closed n-dimensional Riemannian manifold M in an $(n+1)$-dimensional oriented manifold $(N, \overline{g})$. We assume that M is oriented by the global unit normal field ν so that ν is compatible with the orientations of M and N. We will denote by B the second fundamental form of ϕ and its mean curvature by H. Let $F : (-\varepsilon, \varepsilon) \times M \to N$ be a variation of ϕ so that $F(0, \cdot) = \phi$. The balance volume associated with the variation F is the function $V : (-\varepsilon, \varepsilon) \to \mathbb{R}$ defined by

$$\int_{[0,t]\times M} F^\star \, dv_{\overline{g}},$$

where $dv_{\overline{g}}$ is the volume element associated to the metric $\overline{g}$ on N. We will denote simply by dv the volume element of g. It is a classical fact that

$$V'(0) = \int_M f \, dv,$$

where $f(x) = \langle \partial F/\partial t(0, x), \nu \rangle$. Moreover, the area function $A(t) = \int_M dv_{F_t^\star \overline{g}}$ satisfies

$$A'(0) = -n \int_M H f \, dv.$$

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We say that F preserves the volume if $V(t) = V(0)$ in a neighbourhood of 0 and in this case we have $\int_M f\,dv = 0$. Conversely, for all smooth functions f so that $\int_M f\,dv = 0$, there exists a preserving volume variation so that $f = \langle \partial F/\partial t(0, x), \nu\rangle$. It is well known that M has constant mean curvature if and only if the immersion ϕ is a critical point of the area functional (that is, $A'(0) = 0$) for all variations which preserve the volume.

An immersion with constant mean curvature is called stable if $A''(0) \geqslant 0$ for all variations preserving the volume. Then $A''(0)$ depends only on f and

$$\begin{aligned} A''(0) &= \int_M |df|^2\,dv - \int_M (\mathrm{Ric}^N(\nu,\nu) + |B|^2) f^2\,dv \\ &= \int_M f\Delta f - \int_M (\mathrm{Ric}^N(\nu,\nu) + |B|^2) f^2\,dv \\ &= \int_M fJf\,dv, \end{aligned}$$

where Ric^N is the Ricci curvature of N with respect to the metric $\overline{g}$ and J is the so-called Jacobi operator defined by $Jf = \Delta f - (\mathrm{Ric}^N(\nu,\nu) + |B|^2)f$. It is well known that ϕ is a stable constant mean curvature immersion if and only if $A'(0) \geqslant 0$ for any smooth function f so that $\int_M f\,dv = 0$ or equivalently if J is a nonnegative operator. (See [2] and [3] for more details about the notion of stability.)

Barbosa and do Carmo [2] proved that the only stable closed hypersurfaces of constant mean curvature (CMC hypersurfaces) of the Euclidean space are the round spheres. This result was extended later by Barbosa *et al.* [3] for spheres and hyperbolic spaces.

In [4], Grosjean with the first author considered the stability of CMC hypersurfaces in Riemannian manifolds with (nonconstant) bounded sectional curvature. After proving a pinching result for the first eigenvalue of the Laplacian, they were able to show that a closed stable CMC hypersurface of a Riemannian manifold with bounded sectional curvature and contained in a geodesic ball of sufficiently small radius is *close* to a geodesic sphere. Here, *close* means diffeomorphic and almost isometric to a geodesic sphere of appropriate radius (depending upon the mean curvature).

The aim of this short note is to show that the assumption of being stable can be relaxed to almost stable in the result of [4]. By *almost stable*, we mean that the Jacobi operator J is not supposed to be nonnegative but is greater than some small negative constant, that is,

$$\int_M fJf\,dv \geqslant -n\varepsilon \int_M h^2 f^2\,dv \tag{1.1}$$

for any smooth function f so that $\int_M f\,dv = 0$, where $h = \sqrt{\|H\|_\infty^2 + \delta}$. Note that h^2 appears in the right-hand side of the almost stability condition for homogeneity reasons.

In the remainder of this paper, we assume that the sectional curvature of $(N, \overline{g})$ satisfies $\mu \leqslant \mathrm{Sect}_N \leqslant \delta$ for $\mu \leqslant \delta$ two real constants. Before stating the main result of

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this note, we introduce the function

$$s_\delta(r) = \begin{cases} \dfrac{1}{\sqrt{\delta}} \sin \sqrt{\delta} r & \text{if } \delta > 0, \\ r & \text{if } \delta = 0, \\ \dfrac{1}{\sqrt{|\delta|}} \sinh \sqrt{|\delta|} r & \text{if } \delta < 0 \end{cases}$$

and the set $\mathcal{H}_V(n, N)$, which is the space of all Riemannian compact, connected and oriented n-dimensional Riemannian manifolds without boundary isometrically immersed into $(N^{n+1}, \overline{g})$ and which satisfy the volume hypothesis

$$V(M) \leqslant \begin{cases} \dfrac{c\omega_n}{\delta^{n/2}} & \text{if } \delta > 0, \\ c\omega_n i(N)^n & \text{if } \delta \leqslant 0 \end{cases}$$

for some constant c. This condition on the volume is required to apply the result of [4] (see Theorem 2.1 below) about the pinching of the first eigenvalue of the Laplacian. The condition comes from the extrinsic Sobolev inequality of Hoffman and Spruck [6], which is used in the proof of that pinching result.

The main result of this note is the following theorem.

Theorem 1.1. *Let $(N^{n+1}, \overline{g})$ be an $(n+1)$-dimensional Riemannian manifold whose sectional curvature Sect_N satisfies $\mu \leqslant \mathrm{Sect}_N \leqslant \delta$ and $i(N) \geqslant \pi/\sqrt{\delta}$ if $\delta > 0$ and let $M \in \mathcal{H}_V(n, N)$. Assume that $\phi(M)$ lies in a convex ball of radius*

$$\min\Big(\frac{\pi}{8\sqrt{\delta}}, \frac{i(N)}{2}\Big) \quad \text{if } \delta > 0 \quad \text{and} \quad \frac{i(N)}{2} \quad \text{if } \delta \leq 0.$$

Let $\varepsilon < \frac{1}{12}$, $q > n$ and $A > 0$. Assume that $V(M)^{1/n}\|B\|_q \leqslant A$ for $\delta \geqslant 0$ and that $\max(H/h, V(M)^{1/n}\|B\|_q) \leqslant A$ for $\delta < 0$. Then there exist positive constants $\alpha := \alpha(q, n)$, $K := K(n, q, A)$ and $R_0(\delta, \mu, \varepsilon)$ such that if ϕ is of constant mean curvature H and almost stable in the sense of (1.1), $\varepsilon^\alpha < 1/K$ and $\phi(M)$ is contained in a convex ball of radius $R_0(\delta, \mu, \varepsilon)$, then M is diffeomorphic and $K\varepsilon^\alpha$-quasi-isometric to $S(p, s_\delta^{-1}(1/h))$, that is, there exists a diffeomorphism F from M into $S(p, s_\delta^{-1}(1/h))$ so that

$$||dF_x(u)|^2 - 1| \leqslant K\varepsilon^\alpha$$

for any $x \in M$, $u \in T_xM$ and $|u| = 1$.

In the case of the Euclidean space, without the assumption of being contained in a small ball, almost stability implies that the hypersurface is a geodesic hypersphere (see [8]). As a corollary of Theorem 1.1, we give an analogue of the result of [8] for spheres and hyperbolic spaces.

Corollary 1.2. *Let $(N^{n+1}, \overline{g})$ be the $(n+1)$-dimensional Riemannian space form of constant sectional curvature δ and $M \in \mathcal{H}_V(n, N)$. Suppose that $\phi(M)$ lies in a convex ball of radius $\pi/(8\sqrt{\delta})$ if $\delta > 0$, where $q > n$ and $A > 0$. Assume that $V(M)^{1/n}\|B\|_q \leqslant A$*

for $\delta \geqslant 0$ *and* $\max(H/h, V(M)^{1/n}\|B\|_q) \leqslant A$ *for* $\delta < 0$. *Then there exists a positive constant* ε_0 *depending only on* n, q *and* A *such that if* ϕ *is of constant mean curvature* H *and almost stable in the sense of (1.1) for* $\varepsilon \leqslant \varepsilon_0$, *then* $\phi(M)$ *is a geodesic hypersphere of radius* $s_\delta^{-1}(1/h)$.

2. Proof of the results

2.1. Proof of Theorem 1.1. Let $M \in \mathcal{H}_V(n, N)$ and denote by ϕ the isometric immersion of (M, g) into (N, h). Moreover, let us assume that ϕ has constant mean curvature and is almost stable in the sense of (1.1) for some positive ε. Let f be an eigenfunction associated with the first eigenvalue $\lambda_1(M)$ of the Laplacian on (M, g). Since $\int_M f\,dv = 0$, condition (1.1) gives

$$\lambda_1(M)\int_M f^2\,dv - \int_M (\mathrm{Ric}^N(\nu,\nu) + nH^2 + |\tau|^2) f^2\,dv \geqslant -\varepsilon n h^2 \int_M f^2\,dv,$$

where τ is the traceless part of the second fundamental form B. Since $\mu \leqslant \mathrm{Sect}^N$,

$$\begin{aligned}\lambda_1(M) &\geqslant n(H^2 + \mu - \varepsilon h^2)\\ &\geqslant nh^2 + n(\mu - \delta - \varepsilon h^2)\\ &\geqslant nh^2\Big(1 + \frac{\mu - \delta - \varepsilon h^2}{h^2}\Big)\end{aligned}$$

and so

$$nh^2 \leqslant \lambda_1(M)\Big(1 + \frac{1}{\frac{h^2}{\delta - \mu + \varepsilon h^2} - 1}\Big).$$

Set $R_1(\delta, \mu, \varepsilon) = s_\delta^{-1}(\sqrt{(\frac{1}{2} - \varepsilon)/(\delta - \mu)(1 + 1/2\varepsilon)})$. We recall that the extrinsic radius of M is defined as the radius of the smallest ball containing $\phi(M)$. For compact hypersurfaces of a Riemannian manifold of sectional curvature bounded from above by δ, there is a well-known lower bound for the extrinsic radius $R(M)$ given by

$$s_\delta(R(M)) \geqslant \frac{1}{h}$$

(see, for example, [1]). If we assume that $\phi(M)$ is contained in a ball of radius R_1,

$$h^2 \geqslant \frac{1}{s_\delta^2(R(M))} \geqslant \frac{1}{s_\delta^2(R_1)} = \frac{(\delta - \mu)(\frac{1}{\varepsilon} + 1)}{\frac{1}{2} - \varepsilon},$$

since s_δ is an increasing function. From this, we deduce easily that

$$\frac{1}{\frac{h^2}{\delta - \mu + \varepsilon h^2} - 1} \leqslant 2\varepsilon$$

and so

$$nh^2 \leqslant \lambda_1(M)(1 + 2\varepsilon).$$

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That is, $(\Lambda_{2\varepsilon})$ holds, where we denote by (Λ_η) the pinching condition

$$nh^2 \leqslant \lambda_1(M)(1+\eta) \qquad (\Lambda_\eta)$$

associated with the upper bound for the first eigenvalue of the Laplacian proved by Heintze [5], namely

$$\lambda_1(M) \leqslant nh^2.$$

Now, we recall the result proved by Grosjean and the first author in [4].

Theorem 2.1 [4]. *Let $(N^{n+1}, \overline{g})$ be an $(n+1)$-dimensional Riemannian manifold whose sectional curvature Sect_N satisfies $\mu \leqslant \mathrm{Sect}_N \leqslant \delta$ and $i(N) \geqslant \pi/\sqrt{\delta}$ if $\delta > 0$. Let $M \in \mathcal{H}_V(n,N)$. Assume that $\phi(M)$ lies in a convex ball of radius*

$$\min\Big(\frac{\pi}{8\sqrt{\delta}}, \frac{i(N)}{2}\Big) \quad \text{if } \delta > 0 \quad \text{and} \quad \frac{i(N)}{2} \quad \text{if } \delta \leq 0.$$

Let p_0 be the centre of mass of M. Let $\eta < 1/6$, $q > n$ and $A > 0$ and assume that $\max(V(M)^{1/n}\|H\|_\infty, V(M)^{1/n}\|B\|_q) \leqslant A$ for $\delta \geqslant 0$ and

$$\max(V(M)^{1/n}\|H\|_\infty, \|H\|_\infty/h, V(M)^{1/n}\|B\|_q) \leqslant A$$

for $\delta < 0$. Then there exist positive constants $C := C(n,q,A)$ and $\alpha := \alpha(q,n)$ such that if (Λ_η) holds, $\eta^\alpha < 1/C$ and $\phi(M)$ is contained in the ball $B(p_0, s_\delta^{-1}(\sqrt{\eta/(\delta-\mu)}))$, then M is diffeomorphic and $C\eta^\alpha$-quasi-isometric to $S(p, s_\delta^{-1}(1/h))$.

Let $\varepsilon < \frac{1}{12}$ and $\eta = 2\varepsilon < \frac{1}{6}$. We set

$$R_0(\delta,\mu,\varepsilon) = \min\Big\{ s_\delta^{-1}\Big(\sqrt{\frac{2\varepsilon}{\delta-\mu}}\Big), R_1(\delta,\mu,\varepsilon)\Big\}$$

and $K(n,q,A) = 2^{\alpha(n,q)}C(n,q,A)$, where α, C and R are the constants given by Theorem 2.1 and R_1 is defined at the beginning of the proof.

If we assume that $\phi(M)$ is contained in a ball of radius $R_0(\delta,\mu,\varepsilon)$, by the definition of $R_0(\delta,\mu,\varepsilon)$, we have $R_0(\delta,\mu,\varepsilon) \leqslant R_1(\delta,\mu,\varepsilon)$ and so the above computation shows that $(\Lambda_{2\varepsilon})$ holds. Moreover, $\phi(M)$ is contained in a ball of radius $s_\delta^{-1}(\sqrt{2\varepsilon/(\delta-\mu)})$. In addition, if we assume that $\varepsilon^\alpha < 1/K$, then, from the definition of K, we see that $(2\varepsilon)^\alpha < 1/C$ and Theorem 2.1 (applied with $\eta = 2\varepsilon$) shows that M is diffeomorphic and $C\eta^\alpha$-quasi-isometric to $S(p, s_\delta^{-1}(1/h)$. Since $C\eta^\alpha = K\varepsilon^\alpha$, by the definition of K, Theorem 1.1 is proven.

2.2. Proof of Corollary 1.2. We assume here that N is the $(n+1)$-dimensional Riemannian space form of constant sectional curvature δ. Let $M \in \mathcal{H}_V(n,N)$ so that $\phi(M)$ lies in a convex ball of radius $\pi/(8\sqrt{\delta})$ if $\delta > 0$, where $q > n$ and $A > 0$. Moreover, let us assume that $V(M)^{1/n}\|B\|_q \leqslant A$ for $\delta \geqslant 0$ and that $\max(H/h, V(M)^{1/n}\|B\|_q) \leqslant A$ for $\delta < 0$. Since N is of constant sectional curvature δ, we are in the case where $\delta = \mu$. From the definition of $R_0(\delta,\mu,\varepsilon)$ in the proof of Theorem 1.1, in the case $\delta = \mu$, it follows that $\delta = \mu = +\infty$. Let $\alpha := \alpha(q,n)$ and $K := K(n,q,A)$ be the constants of Theorem 1.1.

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We set $\varepsilon_0 = \min\{\frac{1}{12}, 1/K^{1/\alpha}\}$. Note that ε_0 depends only on n, q and A. Now, if M is of constant mean curvature H and almost stable in the sense of (1.1) with $\varepsilon \leqslant \varepsilon_0$, then, applying Theorem 1.1, M is diffeomorphic to $S(p, s_\delta^{-1}(1/h))$. The diffeomorphism F between M and $S(p, s_\delta^{-1}(1/h))$ is explicitly given in the proof of Theorem 2.1 (see [4]). Namely,

$$\begin{array}{rcccc} F & : & M & \longrightarrow & S(p, s_\delta^{-1}(1/h)) \\ & & x & \longmapsto & \exp_p(s_\delta^{-1}(1/h)Y/|Y|), \end{array}$$

where $Y = \exp_p^{-1}(\phi(x))$. Hence, F is of the form $F = G \circ \phi$, where ϕ is the immersion of M into N. But, since F is a diffeomorphism, then ϕ is necessarily injective and so ϕ is an embedding. In conclusion, M is embedded either into a hyperbolic space or into an open half-sphere (because it is contained in a ball of radius $\pi/(8\sqrt{\delta})$ if $\delta > 0$) and so, by the Alexandrov theorem (see [7]), $\phi(M)$ is a geodesic sphere of radius $s_\delta^{-1}(1/h)$. This concludes the proof of the corollary.

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