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RECURSIVE BACKWARD SCHEME FOR THE SOLUTION OF A BSDE WITH A NON LIPSCHITZ GENERATOR

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On an incomplete financial market, the stocks are modeled as pure jump processes subject to defaults. The exponential utility maximization problem is investigated characterizing the value function in term of Backward Stochastic Differential Equations (BSDEs), driven by pure jump processes. In general, in this setting, there is no unique solution. This is the reason why, the value function is proven to be the limit of a sequence of processes. Each of them is the solution of a Lipschitz BSDE and it corresponds to the value function associated with a subset of bounded admissible strategies. Given a representation of the jump processes driving the model, the aim of this note is to give a recursive backward scheme for the value function of the initial problem.

Keywords: backward stochastic differential equations, default, incomplete market

1. INTRODUCTION

The interest on the problem of the existence and uniqueness of the solutions of Backward stochastic differential equations (BSDEs), in a general setting, has increased quickly in the last years, in particular, in the financial literature. As a matter of fact, BSDEs arise in problems like hedging contingent claims or in the theory of recursive utility; see for instance, Carbone, Ferrario, and Santacroce [6]. The theory of BSDEs has been developed essentially in the Brownian setting; see among others Bielecki, Jeanblanc, and Rutkowski [3] and Jeanblanc, Yor, and Chesney [17]. But this note deals with BSDEs in case of drivers modeled by pure jump processes.

In the literature, financial activity prices have been widely modeled, among other approaches, by using pure jump processes; see, for instance, Jing, Kong, and Liu [18] and references therein.

Anyway, many are the models which can be considered from pure diffusion models to pure-jump models and to some which combine the two. Between others, popular pure-jump models for stock prices include, the Variance-Gamma model, Madan, Carr, and Chang [21], while Carr et al. [7], to allow pure diffusion or pure jumps, introduce the CGMY model, determined by the four parameters C, G, M and Y. C governs the overall level activity, G the rate of the decay of the left tail, M of the right tail and Y the fine structure of the process.

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More, non-Gaussian Ornstein–Uhlenbeck processes are also popular, Barndorff-Nielsen and Shephard [2]. All those papers start from the assumption, common to the diffusion or jump-diffusion setting, that there exists an underlying unobservable process, which can be only observed at some discrete time, fixed or random.

On the contrary, in the approach followed by the present paper, every price change can be recorded together with the timestamp at which it took place, giving rise to a piecewise constant trajectory with only a finite number of jumps in a finite time interval. To better understand the motivation for this kind of modeling, let us think of a market maker, who adjusts his/her quotations only from time to time, when the in stock quantities and the demand–supply considerations make these adjustments to become necessary. Of course, it is not known in advance when and of which amount the price adjustments will be. Processes of that kind have been used, for example, in the field of financial modeling and filtering, between others, by Engle and Russell [12], Centanni and Minozzo [10], Gerardi and Tardelli [15] and Martin, Jasra, and McCoy [24]. Furthermore, regarding microstructure analysis see Bacry et al. [1], and for option pricing see Pringent [25], Frey and Runggaldier [14], and Cartea [8].

Hence, starting from a problem of hedging a defaultable claim, here, the prices are modeled by pure jump processes and represented as semimartingales. This setting brings us to investigate BSDEs driven by pure jump processes.

The model presented is similar to that studied in some previous papers of the same author, see between others Tardelli [27]. The main differences rely in the presence of the default and in the fact that here a Markovian structure is assumed. The Markovianity allows us for a particular representation of the processes involved (see Section 2), and it is a necessary condition to perform the recursive backward scheme, presented in Section 6.

To clarify the link between the solutions of BSDEs driven by pure jump processes and the defaultable hedging problems, note that due to the presence of a pure jump price process, the market is incomplete and a perfect replication is not possible. In Section 3, the idea is to maximize the mean value of an exponential utility function from the terminal wealth, see Hu, Imkeller, and Muller [16] and Mania and Schweizer [22]. By the dynamic programming approach, the value function is characterized as the largest solution to a suitable BSDE with a non-Lipschitz generator.

In order to solve an hedging problem, a problem of utility maximization has to be faced. In this context, the exponential form of the utility function has been widely used, thanks to its nice analytic tractability, which allows for fundamental separation properties when dealing with contingents claims. The procedure provides an easy to handle expression for the associate value process, but it cannot be applied in case of Constant Relative Risk Aversion (CRRA) utility function, and, in particular, the last part of Section 3 fails and a different approach must be considered. On the other hand, in order to solve a pricing problem, a similar procedure of that used in Section 3 could be utilized, since, in this last case, some difficulties can be overcome thanks to the absence of the claim. For a detailed discussion about this topic, see, for instance, Ceci [9], and references therein.

In Section 5, a non-increasing sequence of processes converging to the value function is constructed. Any element of this sequence corresponds to the value function associated with a subset of bounded admissible strategies. Hence, taking into account that the coefficients are bounded, each of these approximating processes is the unique solution of a BSDE with Lipschitz generator, not only the largest one. The procedure is along the lines of Lim and Quenez [20], even if that paper investigates a diffusive model. One of the novelty of this paper is to reach the same result, but for a pure-jump model.

The representation of the jump processes performed in Section 2 join with the uniqueness result given in Section 5 are the essential tools for the construction of the recursive backward scheme for the value function, performed in Section 6, that is the main contribute of this note. The recursive backward scheme is inspired by some of the methods presented in Bouchard and Elie [4].

2. THE MODEL: REPRESENTATION

On a complete real-world probability space, $(\Omega, \mathcal{F}, P)$, endowed with a time window $[0,T]$, there are two non-explosive stochastic point processes, N^1 and N^2, which do not have common jump times. Let $\mathcal{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ be a filtration such that $\mathcal{F}_t$ is the σ-algebra generated by N^1 and N^2 until the time t,

$$\mathcal{F}_t = \sigma\{N^1_s, N^2_s, \quad 0 \le s \le t\}.$$

For $i = 1,2$, the process N^i admits a $(P,\mathcal{F}_t)$-predictable intensity. This means that there exist λ^1_t and λ^2_t, bounded non-negative $(P,\mathcal{F}_t)$-adapted processes, such that, for a real positive constant Λ,

$$0 < \lambda^i_t \le \Lambda < +\infty \tag{1}$$

and $M^i_t := N^i_t - \int_0^t \lambda^i_s\, ds$ is a $(P,\mathcal{F}_t)$-martingale. Thus, the jump times of N^1 and N^2 are $\mathcal{F}$-stopping times.

On the same probability space, let τ be a non-negative random variable modeling the default time, and, as usual, let us assume that, for $t \in [0,T]$, $P(\tau > t) > 0$. Let $N^3_t := \mathbb{I}_{\tau\le t}$ be the default indicating process. Consequently, defining $\mathcal{G} := \{\mathcal{G}_t\}_{t\in[0,T]}$, with

$$\mathcal{G}_t := \sigma\left\{N_s = (N^1_s, N^2_s, N^3_s),\ 0 \le s \le t\right\},$$

τ is a $\mathcal{G}$-stopping time. As usual, let us complete and take the right continuous version of all the filtrations that we consider.

Let τ admit a positive $(P,\mathcal{F}_t)$-predictable intensity. This means that there exists $\{\gamma_t\}_{t\ge0}$ bounded, non-negative, $(P,\mathcal{F}_t)$-adapted process, such that

$$0 < \gamma_t \le \Lambda < +\infty \tag{2}$$

and

$$M^3_t := N^3_t - \int_0^{t\wedge\tau} \gamma_s\, ds = N^3_t - \int_0^t (1-N^3_s)\gamma_s\, ds, \quad t \ge 0$$

is a $(P,\mathcal{G}_t)$-martingale.

This means that τ is a totally inaccessible $\mathcal{G}$-stopping time (see, e.g., Section VI78, Dellacherie and Meyer [11]), while the jump times of N^1 and N^2 are $\mathcal{G}$-totally inaccessible $\mathcal{F}$-stopping time.

Hence, N^1, N^2, and N^3 do not have common jump times. This last condition introduces a difficulty. In order to overcome it, all over this paper, let us assume the so-called Immersion Property or (H)-hypothesis.

DEFINITION 1: *The filtration $\mathcal{F}$ is said to be immerse in the filtration $\mathcal{G}$, or it is said to satisfy the (H)-hypothesis under the measure P, whenever any $(P,\mathcal{F}_t)$-local martingale is also a $(P,\mathcal{G}_t)$-local martingale.*

Therefore, the $(P,\mathcal{F}_t)$-martingales M^1 and M^2 are also $(P,\mathcal{G}_t)$-martingales, see Jeanblanc et al. [17], Bielecki et al. [3], Mansuy and Yor [23].

Proposition 2: *Under all the assumptions given up to now, let us assume that, for* $i = 1, 2, 3$*, the intensity of* N^i *depends just on the value of the process itself in* t *and does not depend on the past values of it, that is,*

$$\lambda^1_t = \lambda^1(N^1_t,\ N^2_t), \quad \lambda^2_t = \lambda^2(N^1_t,\ N^2_t), \quad \gamma_t = \gamma(N^1_t,\ N^2_t).$$

Since $N^3 \in \{0, 1\}$*, setting* $n = (n_1, n_2, n_3)$ *and*

$$\begin{aligned}\mathcal{A}_t F(t,n) = {} & \lambda^1(n_1,n_2)\left[F(t,n_1+1,n_2,n_3) - F(t,n)\right] \\ & + \lambda^2(n_1,n_2)\left[F(t,n_1,n_2+1,n_3) - F(t,n)\right] \\ & + \gamma(n_1,n_2)(1-n_3)\left[F(t,n_1,n_2,n_3+1) - F(t,n)\right],\end{aligned}$$

the process $N = (N^1, N^2, N^3)$ *is Markovian. Furthermore, for a suitable function* F*, its generator is given by the operator*

$$\mathcal{A}F(t,n) = \frac{\partial}{\partial t}F(t,n) + \mathcal{A}_t F(t,n).$$

Proof: For a bounded measurable real-valued function F,

$$F(t,N_t) - F(0,N_0) - \int_0^t \mathcal{A}_t F(s,N_s)\,ds$$

is a $(P, \mathcal{G}_t)$-martingale and the generator $\mathcal{A}_t$ is bounded. By classical discussions, Ethier and Kurtz [13], the Martingale Problem associated with $\mathcal{A}$ and initial condition $N_0 = (0, 0, 0)$ is well posed, and its solution is a Markov process with trajectories in $D_{\mathbb{N}\times\mathbb{N}\times\{0,1\}}[0, T]$. ■

In the real life, often, to distinguish between 'good' and 'bad' trends of a financial market, not necessarily we have to look of all the history. This is the reason why the process N has been assumed to be Markovian.

The last part of this section is devoted to the construction of a suitable representation for the processes involved in this model, as presented in Ethier and Kurtz [13]. As well known, recall that any realization can be used to prove properties related to the law of the processes, but cannot be used to obtain properties of the paths.

To simplify the notation, let us denote

$$\lambda^i(n) := \lambda^i(n_1,n_2), \quad i = 1, 2, \quad \lambda^3(n) := \gamma(n_1,n_2)(1-n_3)$$

and

$$\lambda(n) := \lambda^1(n) + \lambda^2(n) + \lambda^3(n).$$

Setting $y := (y_1, y_2, y_3)$, $dy := (dy_1,\ dy_2,\ dy_3)$, $e_i = (\delta_1(i), \delta_2(i), \delta_3(i))$, where $\delta_j(i) = 1$ if $i = j$ and it is null otherwise, the generator can be written as

$$\mathcal{A}_t F(n) := \lambda(n) \int_{\mathbb{N}^2\times\{0,1\}} \Big[F(y) - F(n)\Big]\ \mu(n,dy),$$

where the measure $\mu(n, dy)$ is defined by

$$\mu(n,dy) := \sum_{i=1,2,3} \frac{\lambda^i(n)}{\lambda(n)}\ \delta_{(y+e_i)}(dy).$$

The structure itself of the measure $\mu(n, dy)$ guarantees that the processes $N^1,\ N^2$, and N^3 cannot have common jump times.

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Hence, let $\{N(k)\}_{k\geq 0} := \{(N^1(k), N^2(k), N^3(k))\}_{k\geq 0}$ be the Markov chain such that the first two components take value in $\mathbb{N}$ and the third one in $\{0,1\}$. The Markov chain is determined by the initial condition $N(0) := \{N^1(0), N^2(0), N^3(0)\}$ and, for $\Gamma \subset \mathbb{N}^2 \times \{0,1\}$, by the transition probabilities

$$P\Big(N(k+1) \in \Gamma \Big| N(k)\Big) = \mu(N(k), \Gamma).$$

Then, let $\{T_i\}_{i\geq 1}$ be a sequence of independent random variables, independent to $\{N(k)\}_{k\geq 0}$, having an exponential law of parameter 1. Let $\{(\rho_1^k, \rho_2^k, \rho_3^k)\}_{k\geq 0}$ be a sequence where, $(\rho_1^0, \rho_2^0, \rho_3^0) = (0,0,0)$, and for $k \geq 1$,

$$\rho_i^k = \frac{T_k}{\lambda^i(N(k-1))}, \quad i = 1,2 \quad \rho_3^k = \begin{cases} \dfrac{T_k}{\lambda^3(N(k-1))} & N^3(k-1) = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Thus, the sequence of the jump times of N_t is

$$t_n = \sum_{k=1}^{n} \rho^k, \quad \text{for } \rho^k = \min\left\{\rho_1^k,\ \rho_2^k,\ \rho_3^k\right\}.$$

Summarizing, the process N_t is represented in term of the Markov chain, defining,

$$N_t := N(k), \quad \text{for } t \in [t_k, t_{k+1}),$$

and, the process counting all the jump times of $\widetilde{N}_t$ is represented as

$$\widetilde{N}_t = \sum_{j\geq 0} j\ \mathbb{I}_{[t_j, t_{j+1})}(t) = \sum_{j\geq 0} \mathbb{I}_{[t_j, +\infty)}(t).$$

3. A DEFAULTABLE HEDGING PROBLEM

Taking into account all this setting, let us consider a financial market with one risky asset and one risk-free asset. The price of the risk-free asset is taken equal to 1 (i.e., the riskless interest rate is supposed to be equal zero). The price S of the stock, discounted with respect to the price of the bond, is modeled as a pure jump process, such that

$$S_t = S_0\ \exp\{Y_t\} \quad S_0 \in \mathbb{R}^+.$$

The logreturn price Y is assumed to be a non-explosive real valued marked point process. Its initial condition is $Y_0 = 0$ and its dynamics is given by assuming

$$Y_t := \sum_{i=1,2,3} \int_0^t \eta_u^i \, dN_u^i.$$

The jump sizes η^1, η^2, and η^3 are $(P, \mathcal{G}_t)$-predictable processes, and for some constant $\underline{\eta}$, $\overline{\eta} \in \mathbb{R}^+$,

$$\underline{\eta} \leq \eta_t^1 \leq \overline{\eta} \quad \text{and} \quad -\overline{\eta} \leq \eta_t^2 \leq -\underline{\eta}. \tag{3}$$

Many authors in this framework assume that $\eta_\tau^3 > -1$. According to (Lim and Quenez [20]), this condition is equivalent to $\eta_t^3 > -1$ for $0 \leq t \leq T$ a.s. In this way, $S_\tau = S_{\tau-}(1 + \eta_\tau^3)$, then

S is still positive after the default τ. However, for our model this condition is not necessary, it is rather obvious that S is positive, since it is an exponential, $S_\tau = S_{\tau-}e^{\eta^3_\tau}$.

Remark 3: Setting

$$c_u := \sum_{i=1,2,3} (e^{\eta^i_u} - 1)\lambda^i_u,$$

by a standard application of Itô formula, the representation of the price process as a $(P, \mathcal{G}_t)$-semimartingale is

$$S_t = S_0 + \int_0^t S_u c_u \, du + M^S_t,$$

where M^S_t is a $(P, \mathcal{G}_t)$-local martingale represented as

$$M^S_t = \sum_{i=1,2,3} \int_0^t S_{u-}(e^{\eta^i_u} - 1) dM^i_u.$$

Note that (H)-hypothesis is a necessary condition to obtain last formula.

Proposition 4: *For $i = 1, 2, 3$, assuming that $\eta^i_t = \eta^i(N_t)$, then, for $t \in [t_k, t_{k+1})$, $\eta^i_t = \eta^i(N(k))$, and (N_t, Y_t, S_t) is a Markov process.*

The proof is along the line of Proposition 2 and it allows us to write all the processes involved in this model in terms of the Markov chain $\{N(k)\}_{k\geq 0}$ and of the sequence of random variables $\{T_k\}_{k\geq 1}$.

On an incomplete financial market, the investors can trade in a finite time window $[0, T]$. They invest in risky stocks and a riskless bond, assuming also that there exists a default time on the market. According to Bielecki et al. [3], it is enough to formally define a generic defaultable European contingent claim with maturity date T through Definition 5 below.

Definition 5: *Fix a finite horizon date $T > 0$. On a suitable filtered probability space, a defaultable contingent claim with maturity date T consists in a triplet (X, Z, τ):*

(1) The default time τ is a random variable specifying the random time of default, and the default events $\{\tau \leq t\}$, for $t \in [0, T]$. More, let us assume that τ is strictly positive with probability 1.

(2) The promised payoff X represents the random payoff received by the owner of the claim at time of the maturity T, if there was no default prior to or at T. This is a $\mathcal{G}_T$-measurable random variable such that $0 \leq X \leq \overline{B}$, for a positive real constant $\overline{B}$. The actual payoff at T, associated with X, equals X if τ is greater than T.

(3) The $\mathcal{G}$-adapted recovery process Z specifies the recovery payoff, which is the random variable given by the value of Z at τ. This is the quantity received by the owner of a claim at time of default (or at maturity), provided that the default occurred prior to or at maturity date and such that $0 \leq Z \leq \overline{B}$.

In practice hedging of a derivative after default is usually of minor interest, Bielecki et al. [3]. In a model with a single default, hedging after that time reduces to replication of a non-defaultable claim. See, for instance, Tardelli [27].

3.1. Replication

A $(P, \mathcal{G}_t)$-predictable real-valued process θ_t is called a trading strategy, if it is S-integrable and self-financing and such that $\int_0^t \theta_u dS_u$ is uniformly bounded from below, Schachermayer [26]. Let Θ be the class of such strategies. At any $t \in [0, T]$, θ_t corresponds to the number of risky traded assets invested. Since $\mathcal{G}_t$ is a σ-algebra generated by counting processes and $\{t_n\}_{n\geq 0}$ is the sequence of the jump times of N, any $(P, \mathcal{G}_t)$-predictable process has the structure,

$$\theta_t = \sum_{n\geq 0} \theta_n(t_1, t_2, \ldots, t_n, t)\, \mathbb{I}_{(t_n, t_{n+1}]}(t),$$

where θ_n are real-valued measurable functions, (Bremaud [5]). Hence any admissible strategy is completely characterized by the family $\{\theta_n\}_{n\geq 0}$.

For a strategy $\theta \in \Theta$ and for initial capital $w_0 \in \mathbb{R}^+$, constant, the associated wealth process, W_t^θ, is defined as

$$W_t^\theta - w_0 = \int_0^t \theta_u dS_u = \sum_{i=1,2,3} \int_0^t \theta_u S_{u-}(e^{\eta_u^i} - 1) dN_u^i.$$

A hedging problem consists in finding an investment strategy to trade in the available assets, in order to reduce (or avoid) potential losses arising from having to honor the contract. For the purpose of replication of defaultable claims of the form (X, Z, τ), it is sufficient to consider prices stopped at $T \wedge \tau$. Hence, it is natural to define the replication of a defaultable claim in the following way.

Definition 6: *A self-financing strategy $\theta \in \Theta$ replicates a defaultable claim (X, Z, τ), if its wealth process W_t^θ satisfies the hedging conditions*

$$W_T^\theta \mathbb{I}_{T<\tau} = X\mathbb{I}_{T<\tau} \quad \text{and} \quad W_\tau^\theta \mathbb{I}_{T\geq\tau} = Z_\tau \mathbb{I}_{T\geq\tau}.$$

Since in this setting the market is incomplete, perfect replication is not possible. Thus, we have to use a hedging criterion under incompleteness. Many methods are possible. In this note, the choice consists in maximizing the expected exponential utility function of the wealth on the time interval $[0, T \wedge \tau]$, that is, to maximize, for $\theta \in \Theta$,

$$E\left[1 - \exp\left\{-\alpha\left(W_{T\wedge\tau}^\theta - X(1 - N_T^3) - Z_\tau N_T^3\right)\right\}\right],$$

where $\alpha \in \mathbb{R}^+$ is the risk aversion parameter. Hence, we face with an agent with exponential utility function. After receiving the premium, the seller has to hedge to reduce the risk exposure. The expected utility of his final wealth gives him a measure of the quality of a self-financing strategy. At any $t \in [0, T]$, the agent invests the quantity θ_t in the risky traded assets. The investment process θ_t controls the dynamics of the wealth process, W_t^θ. Hence, a stochastic control problem with only final reward arises.

By noting that $\mathbb{I}_{t\leq\tau} = 1 - N_{t-}^3$, then

$$W_{T\wedge\tau}^\theta = w_0 + \int_0^T (1 - N_{u-}^3)\theta_u dS_{u-}.$$

Let w be the amount of capital at time t and let Θ_t be the set of the admissible strategies on $[t, T \wedge \tau]$. Since the process N_t is Markovian, for

$$B_T = X(1 - N_T^3) + Z_\tau N_T^3,$$

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a representation for the associated value process is

$$\mathcal{V}_t(w) = \operatorname*{ess\,sup}_{\theta\in\Theta_t} E\left[1 - \exp\left\{-\alpha\left(W^\theta_{T\wedge\tau} - W^\theta_{t\wedge\tau} + w - B_T\right)\right\}\middle| \mathcal{N}_t\right]$$
$$= 1 - \operatorname*{ess\,inf}_{\theta\in\Theta_t} E\left[\exp\left\{-\alpha\left(w + \int_t^T (1 - N^3_{u-})\theta_u dS_u - B_T\right)\right\}\middle| \mathcal{N}_t\right].$$

As a consequence of the definition of Θ,

$$E\left[\exp\left\{-\alpha\int_t^T (1 - N^3_{u-})\theta_u\, dS_u\right\}\right] < \infty$$

and the value process reduces to $\mathcal{V}_t(w) = 1 - e^{-\alpha w}V_t$, where

$$V_t = \operatorname*{ess\,inf}_{\theta\in\Theta_t} E\left[\exp\left\{-\alpha\left(\int_t^T (1 - N^3_{u-})\theta_u dS_u - B_T\right)\right\}\middle| \mathcal{N}_t\right]. \tag{4}$$

4. BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

Section 4.1 below recalls some properties of the process V_t defined in (4). Even if, in Lim and Quenez [20], diffusion processes are concerned, their procedures to obtain these properties can be easily extended to model with pure jump processes; see, for instance, Tardelli [27] in a non-defaultable case. The generalization to a defaultable model is rather obvious.

All this properties allow us to characterize V_t, (4), and, consequently, the value function, as the largest solution to a suitable BSDE (Section 4.2). Note that we are not able to prove an uniqueness result for this BSDE and this justifies the discussion performed in Section 5.

4.1. Properties of V_t

PROPOSITION 7: *The process V_t is positive and bounded. Moreover, since B_T is $\mathcal{N}_T$-measurable $V_T = E[e^{\alpha B_T}|\mathcal{N}_T] = e^{\alpha B_T}$.*

PROOF: Noting that the strategy $\theta \equiv 0$ belongs to Θ_t, then

$$V_t \le E\left[\exp\{\alpha B_T\}\middle|\mathcal{N}_t\right] = E\left[\exp\left\{\alpha(X(1 - N^3_T) + Z_\tau N^3_T)\right\}\middle|\mathcal{N}_t\right] \le e^{\alpha\overline{B}}.$$

Furthermore, by Theorem 2.2 given in Schachermayer [26], there exists an optimal strategy $\theta^* \in \Theta$ such that

$$V_t = \mathbb{E}\left[\exp\left\{-\alpha\left(\int_t^T (1 - N^3_{u-})\theta^*_u dS_u - B_T\right)\right\}\middle|\mathcal{N}_t\right].$$

As a consequence $e^{\alpha\overline{B}} \ge V_t > 0$. ■

PROPOSITION 8: *For each $t \in [0, T\wedge\tau]$, the following results hold true:*

(i) For any $\theta \in \Theta$, the process $V_t \exp\{-\alpha W^\theta_{t\wedge\tau}\}$ is a $(P, \mathcal{G}_t)$-submartingale.

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(ii) V_t *is the largest process* $\mathcal{G}_t$*-adapted verifying (i) such that* $V_T = e^{\alpha B_T}$.

(iii) The process $\theta^* \in \Theta$ *is an optimal strategy if and only if* $V_t \exp\{-\alpha W^{\theta^*}_{t\wedge\tau}\}$ *is a* $(P, \mathcal{G}_t)$*-martingale.*

The proof can be achieved following a procedure analogous to that used in Tardelli [27], for a non-defaultable model.

Proposition 9: *The process* V_t *admits an indistinguishable* $\mathcal{G}$*-adapted cadlag representation.*

Proof: Let $\mathcal{D} = [0, T] \cap \mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rational numbers. Since V_t is a submartingale, by Karatzas and Shreve [19], for any $t \in [0, T)$, there exist the limits $V_{t+} = \lim_{s\in\mathcal{D}, s\downarrow t} V_s$ and $V_{t-} = \lim_{s\in\mathcal{D}, s\uparrow t} V_s$. So that V_{t+}, for each $t \in [0, T]$, is well defined by setting $V_{T+}(\omega) := V_T$ and by the previous limit for $0 \le t < T$.

From the right-continuity of $\mathcal{G}_t$, V_{t+} is $\mathcal{G}$-adapted and a $\mathcal{G}$-submartingale. More, for $\theta \in \Theta$, $e^{-\alpha W^{\theta}_{t\wedge\tau}} V_{t+}$ is $\mathcal{G}$-submartingale. Note also that, for $s \le t$ and for any sequence of rationals $\{t_n\}_{n\ge1}$ converging down to t

$$\mathbb{E}[e^{-\alpha W^{\theta}_{t_n\wedge\tau}} V_{t_n} | \mathcal{G}_s] \ge e^{-\alpha W^{\theta}_{s\wedge\tau}} V_s.$$

By the Lebesgue theorem for conditional expectation, for $n \to +\infty$,

$$\mathbb{E}[e^{-\alpha W^{\theta}_{t\wedge\tau}} V_{t+} | \mathcal{G}_s] \ge e^{-\alpha W^{\theta}_{s\wedge\tau}} V_s.$$

Again, $\mathbb{E}[e^{-\alpha W^{\theta}_{t\wedge\tau}} V_{t+} | \mathcal{G}_{s_n}] \ge e^{-\alpha W^{\theta}_{s_n\wedge\tau}} V_{s_n}$, for any sequence of rationals $\{s_n\}_{n\ge1}$ converging down to s. More, by the Lebesgue theorem for conditional expectation, for $n \to +\infty$, by the right-continuity of $\mathcal{G}$

$$\mathbb{E}[e^{-\alpha W^{\theta}_{t\wedge\tau}} V_{t+} | \mathcal{G}_s] \ge e^{-\alpha W^{\theta}_{s\wedge\tau}} V_{s+},$$

which gives the submartingale property of the process $e^{-\alpha W^{\theta}_{t\wedge\tau}} V_{t+}$. For $\theta = 0$, for $s = t$ and by the right-continuity of $\mathcal{G}$, $V_{t+} = \mathbb{E}[V_{t+} | \mathcal{G}_t] \ge V_t$ a.s. On the other hand, since V_t is the largest process $\mathcal{G}$-adapted, verifying (i) and, for each $\theta \in \Theta$, $e^{-\alpha W^{\theta}_{t\wedge\tau}} V_{t+}$ is a $\mathcal{G}$-submartingale, then for each $t \in [0, T]$, $V_{t+} \le V_t$ a.s., which implies that $V_{t+} = V_t$ a.s. ■

4.2. Backward Stochastic Differential Equations

As a consequence of Propositions 7 and 8, for a vanishing strategy, V_t is a bounded and strictly positive $(P, \mathcal{G}_t)$-submartingale. Then its Doob–Meyer decomposition join with the classical representation of martingales gives us

$$V_t = V_0 + \sum_{i=1,2,3} \left[\int_0^t R^i_u \, dM^i_u + \int_0^t R^i_u \lambda^i_u du \right]$$

where R^1_u, R^2_u, and R^3_u measurable $(P, \mathcal{G}_t)$-predictable processes such that

$$E\left\{ \sum_{i=1,2,3} \int_0^t \left(R^i_u\right)^2 \lambda^i_u \, du \right\} < +\infty,$$

$\sum_{i=1,2,3} \int_0^t R^i_u \, dM^i_u$ is a square integrable $(P, \mathcal{G}_t)$-martingale and $\sum_{i=1,2,3} R^i_u \lambda^i_u$ is increasing and $(P, \mathcal{G}_t)$-predictable process.

Theorem 10: *Setting* $Q_u(\theta) = \sum_{i=1,2,3} (V_u + R_u^i) K_u^\theta(\eta^i)\lambda_u^i$ *and*

$$K_u^\theta(x) = \exp\left\{-\alpha(1-N_{u-}^3)\theta_u S_{u-}\left(e^x - 1\right)\right\} - 1, \tag{5}$$

(a) *for* $t \in [0, T\wedge\tau]$, *the process* $(V_t, R_t^1, R_t^2, R_t^3)$ *verifies the BSDE*

$$V_t = e^{\alpha B_T} - \sum_{i=1,2,3}\int_t^T R_u^i dM_u^i + \int_t^T \operatorname{ess}\inf_{\theta\in\Theta_u} Q_u(\theta)\, du,$$

(b) V_t *is the largest solution to this BSDE and* (R_t^1, R_t^2, R_t^3) *is uniquely determined by the martingale representation theorem.*

Proof: Regarding assertion (a), recalling the definition of W^θ and setting $C_t^\theta := e^{-\alpha(W_{t\wedge\tau}^\theta - w_0)}$, Itô formula provides

$$C_t^\theta = 1 + \sum_{i=1,2,3}\left[\int_0^t C_{u-}^\theta K_u^\theta(\eta^i)\, dM_u^i + \int_0^t C_u^\theta K_u^\theta(\eta^i)\lambda_u^i\, du\right].$$

The product formula with V_t, by using its Doob–Meyer decomposition, gives

$$C_t^\theta\ V_t = V_0 + \sum_{i=1,2,3}\int_0^t C_{s-}^\theta\left[R_s^i + (V_{s-} + R_s^i)K_s^\theta(\eta^i)\right]\, dM_s^i$$
$$+ \int_0^t C_s^\theta\left[Q_s(\theta) + \sum_{i=1,2,3} R_s^i\lambda_s^i\right] ds.$$

Since $C_t^\theta\ V_t = V_t e^{-\alpha(W_{t\wedge\tau}^\theta - w_0)}$ has to be a $(P,\mathcal{G}_t)$-submartingale, the bounded variation term has to be increasing in t, for any strategy. Furthermore, Proposition 8 (iii), if θ is the optimal strategy, $C_t^\theta\ V_t$ is a $(P,\mathcal{G}_t)$-martingale, which implies that its bounded variation term is null. Consequently,

$$\sum_{i=1,2,3} R_s^i\lambda_s^i = -\operatorname{ess}\inf_{\theta\in\Theta_s} Q_s(\theta),$$

where, as usual, $\operatorname{ess}\inf_{\theta\in\Theta_u} Q_u(\theta) := \sup_{a\in\mathbb{R}}\{a : P(\theta\in\Theta_u : Q_u(\theta) < a) = 0\}$.

The assertion (b) is a consequence of Proposition 8 (ii), by noting that

$$V_t = V_T - \sum_{i=1,2,3}\int_t^T R_u^i\, dM_u^i - \sum_{i=1,2,3}\int_t^T R_u^i\lambda_u^i du.$$

■

5. CONVERGING SEQUENCE: UNIQUENESS RESULT

Taking into account that the recursive backward scheme requires the uniqueness for the solution of the BSDE, a sequence $\{V_t^k\}_{k\geq 1}$ is constructed such that

(1) The sequence $\{V_t^k\}_{k\geq 1}$ is non-increasing and it converges to V_t.
(2) For each k, V_t^k is the unique solution to a BSDE with Lipschitz generator.

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From now, let us assume that, for $t \in [0,T]$,

$$\eta_t^3 \quad \text{is uniformly bounded.} \tag{6}$$

Definition 11: *For a positive integer k, let us define*

$$\Theta^k = \{\theta \in \Theta : |\theta_u S_u| \le k,\ \forall u \in [0,T], a.s.\}\,.$$

More, let Θ_t^k be the set of the strategies in Θ^k on the time interval $[t, T \wedge \tau]$.

Analogously with the definition of V_t, (4), for a positive integer k, let

$$V_t^k = \operatorname{ess}\inf_{\theta\in\Theta_t^k} E\left[\exp\left\{-\alpha(W^\theta_{T\wedge\tau} - W^\theta_{t\wedge\tau} - B_T)\right\}\Big|\ N_t\right]. \tag{7}$$

Remark 12: Since the argument of the expectation in the definition of V_t^k has finite mean value and Θ^k is a bounded set, then the Lebesgue theorem applies and, consequently, there exists an optimal strategy $\theta_k^* \in \Theta^k$.

All the results of Propositions 7 and 8 hold for V_t^k, which means that V_t^k is bounded and positive and it is a $(P, \mathcal{G}_t)$-submartingale. Hence,

$$V_t^k = V_0^k + \sum_{i=1,2,3}\int_0^t R_u^{i,k}\, dM_u^i + \int_0^t A_u^{V^k}\, du,$$

where $\sum_{i=1,2,3}\int_0^t R_u^{i,k}\, dM_u^i$ is a square integrable $(P, \mathcal{G}_t)$-martingale and $R_u^{1,k}$, $R_u^{2,k}$, $R_u^{3,k}$ are measurable $(P, \mathcal{G}_t)$-predictable processes such that

$$E\left\{\int_0^t \sum_{i=1,2,3}\left(R_u^{i,k}\right)^2 \lambda_u^i\, du\right\} < +\infty.$$

Then, $A_t^{V^k}$ is an increasing $(P, \mathcal{G}_t)$-predictable process such that $A_0^{V^k} = 0$ and

$$A_u^{V^k} = \sum_{i=1,2,3} R_u^{i,k}\lambda_u^i.$$

Proposition 13: *For each $k \in \mathbb{N}$, the process $(V_t^k, R_t^{1,k}, R_t^{2,k}, R_t^{3,k})$ is the unique positive cadlag $\mathcal{G}$-adapted solution of the following BSDE with Lipschitz continuous generator:*

$$V_t^k = e^{\alpha B_T} - \sum_{i=1,2,3}\int_t^T R_u^{i,k} dM_u^i + \int_t^T \operatorname{ess}\inf_{\theta\in\Theta_u^k} Q_u^k(\theta)\, du, \tag{8}$$

where, defining $K_u^\theta(x)$ as in (5),

$$Q_u^k(\theta) = \sum_{i=1,2,3}\left(V_u^k + R_u^{i,k}\right) K_u^\theta(\eta^i)\lambda_u^i.$$

Proof: By (7) and Remark 12, there exists an optimal strategy $\theta_k^* \in \Theta^k$, such that

$$V_t^k = E\left[\exp\left\{-\alpha(W^{\theta_k^*}_{T\wedge\tau} - W^{\theta_k^*}_{t\wedge\tau} - B_T)\right\}\Big|\ N_t\right]. \tag{9}$$

The process (V_t^k, R_t^k), $R_t^k = (R_t^{1,k}, R_t^{2,k}, R_t^{3,k})$, is a solution of (8) and the procedure to get it is the same used in Theorem 10.

On the other hand, note that the generator of the BSDE (8) is given by

$$f(t, V_t^k, R_t^k) := \text{ess} \inf_{\theta \in \Theta_t^k} Q_t^k(\theta) = \text{ess} \inf_{\theta \in \Theta_t^k} \sum_{i=1,2,3} \left[\left(V_t^k + R_t^{i,k} \right) K_t^\theta(\eta^i) \lambda_t^i \right].$$

Taking into account (3), (6) and Definition 11, $K_t^\theta(x)$ is bounded. More, recalling that $\lambda_t^3 := \gamma_t(1 - N_t^3)$, by (1) and (2), for each $i = 1, 2, 3$, λ_t^i is also bounded. Consequently, given two solutions of (8), (V_t^k, R_t^k) and $(\tilde{V}_t^k, \tilde{R}_t^k)$, there exists a real constant L such that

$$|f(t, V_t^k, R_t^k) - f(t, \tilde{V}_u^k, \tilde{R}_u^k)| \leq L \cdot \left(\left| V_t^k - \tilde{V}_t^k \right| + \sum_{i=1,2,3} \left| R_t^{i,k} - \tilde{R}_t^{i,k} \right| \right).$$

Thus, the Lipschitz property for the generator f allows us to apply the results of uniqueness, given in Carbone et al. [6], Section 2. This means that the value function given by (9) is the solution to the BSDE (8). ■

THEOREM 14: *Since η_t^3 is uniformly bounded $V_t = \lim_{k \to +\infty} V_t^k$ a.s., $\forall t \in [0, T]$.*

For the sake of completeness, the proof is in Appendix, even if this result could be achieved generalizing the proof given in Lim and Quenez [20].

6. A RECURSIVE BACKWARD SCHEME

All the discussions performed in Section 5 allow us to consider a function V_t which is the unique solution to a suitable BSDE. Recall that the uniqueness is an essential tool to get, in this section, a recursive backward scheme for V_t, which is the main contribute of the paper.

In order to get a reasonable recursive backward scheme and to avoid too many technicalities, let us assume that the recovery process coincides with a $\mathcal{G}_T$-measurable random variable Z, that is, $Z_\tau = Z$, then

$$B_T = X(1 - N_T^3) + Z N_T^3.$$

Taking into account that, for $i \geq 1$, $\theta_{t_i} = \theta_{i-1}(t_1, t_2, \ldots, t_{i-1}, t_i)$ and that

$$W_{T \wedge \tau}^\theta - W_{t \wedge \tau}^\theta = \sum_{\tilde{N}_t \leq i \leq \tilde{N}_T - 1} (1 - N_{t_i}^3) \theta_{t_i} \left(S_{t_{i+1}} - S_{t_i} \right),$$

then, for a fix positive integer k, a representation of the value function is obtained by setting

$$V_t^k = \sum_{\tilde{N}_t \leq i \leq \tilde{N}_T - 1} V_{t_i}^k \, \mathbb{I}_{[t_i, t_{i+1})}(t).$$

Hence, $V_{t_{N_T}}^k = V_T = e^{\alpha B_T}$, while, for $j = 0, \ldots, \tilde{N}_T - 1$,

$$V_{t_j}^k = e^{\alpha B_T} - \sum_{i=1,2,3} \int_{t_j}^T R_u^{i,k} dM_u^i + \int_{t_j}^T \text{ess} \inf_{\theta \in \Theta_r^k} Q_u^k(\theta) \, du.$$

PROPOSITION 15: *For $t \in [t_j, t_{j+1})$*

$$R_t^{i,k} = R_{t_j}^{i,k}, \quad i = 1, 2, 3, \quad \text{and} \quad V_t^k = V_{t_j}^k.$$

PROOF: By the Doob–Meyer decomposition and by the classical representation of martingales,

$$V_{t_j}^k = V_0^k + \sum_{i=1,2,3} \int_0^{t_j} R_u^{i,k} dM_u^i + \int_0^{t_j} A_u^{V^k} du.$$

Consequently, as before, for $A_u^{V^k} = -\operatorname{ess\,inf}_{\theta \in \Theta_u^k} Q_u^k(\theta)$,

$$V_{t_j}^k = V_{t_{j+1}}^k - \sum_{i=1,2,3} \int_{t_j}^{t_{j+1}} R_u^{i,k} dM_u^i + \int_{t_j}^{t_{j+1}} \operatorname*{ess\,inf}_{\theta \in \Theta_u^k} Q_u^k(\theta)\, du.$$

Again, by the Doob–Meyer decomposition,

$$V_{t_j}^k = V_{t_{j+1}}^k - \sum_{i=1,2,3} \int_{t_j}^{t_{j+1}} R_u^{i,k}\, dN_u^i.$$

Thus, for $t \in [0, T]$,

$$V_t^k = \sum_{\tilde{N}_t \le j \le \tilde{N}_T - 1} V_{t_j}^k\ \mathbb{I}_{[t_j, t_{j+1})}(t).$$

On the other hand, for $i = 1, 2, 3$, setting $\Delta N_{t_j}^i = N_{t_{j+1}}^i - N_{t_j}^i$ and, in a similar way, $\Delta M_{t_j}^i$, we get that

$$V_{t_{j+1}}^k - V_{t_j}^k = \sum_{i=1,2,3} R_{t_j}^{i,k} \Delta N_{t_j}^i = \sum_{i=1,2,3} \left[\int_{t_j}^{t_{j+1}} R_{t_j}^{i,k}\, dM_u^i - \int_{t_j}^{t_{j+1}} R_{t_j}^{i,k}\ \lambda_u^i\, du \right].$$

Since λ_u^1, λ_u^2, and λ_u^3 just depend on N, then, by the uniqueness of the Doob–Meyer decomposition, the thesis is achieved. ∎

Setting $H_{t_j}^k(V, R, N) := -\operatorname{ess\,inf}_{\theta \in \Theta_{t_j}^k} Q_{t_j}^k(\theta) = A_{t_j}^{V^k}$ and $\Delta t_j = t_{j+1} - t_j$, the previous proposition allows us to claim that

$$V_{t_j}^k = V_{t_{j+1}}^k - \sum_{i=1,2,3} R_{t_j}^{i,k} \Delta M_{t_j}^i - H_{t_j}^k(V, R, N)\ \Delta t_j \tag{10}$$

where $H_{t_j}^k(V, R, N) = \sum_{i=1,2,3} R_{t_j}^{i,k} \lambda_{t_j}^i$.

At this point, we need some technical preliminaries.

LEMMA 16: *Setting* $\Lambda_j := \lambda^1(N(j)) \vee \lambda^2(N(j)) \vee \lambda^3(N(j))$, *then*

$$E\left[\Delta t_j | N_{t_j}\right] = \frac{1}{\Lambda_j}, \quad E\left[(\Delta t_j)^2 | N_{t_j}\right] = \frac{2}{\Lambda_j^2}, \tag{11}$$

and, for $i = 1, 2, 3$,

$$E\left[\Delta N_{t_j}^i | N_{t_j}\right] = \frac{\lambda^i(N(j))}{\Lambda_j}, \quad E\left[\Delta t_j \Delta N_{t_j}^i | N_{t_j}\right] = \frac{\lambda^i(N(j))}{\Lambda_j^2}.$$

PROOF: These results can be achieved by taking into account that all the quantities involved can be represented in term of the sequences of the jump times $\{t_j\}_{j \ge 0}$ and in term of the Markov chain $\{N(k)\}_{k \ge 0}$, which are independent. Recalling the construction performed to get the representation of the process N and that $\{T_n\}_{n\geq 1}$ is a sequence of random variables having exponential law of parameter 1, then

$$\begin{aligned}E\left[\Delta t_j|N_{t_j}\right] &= E\left[t_{j+1}-t_j|N_{t_j}\right] = E\left[\rho^{j+1}|N_{t_j}\right]\\ &= E\left[\min\left\{\frac{T_{j+1}}{\lambda^1(N(j))},\frac{T_{j+1}}{\lambda^2(N(j))},\frac{T_{j+1}}{\lambda^3(N(j))}\right\}|N_{t_j}\right]\\ &= E\left[\frac{T_{j+1}}{\Lambda_j}|N_{t_j}\right] = \frac{1}{\Lambda_j}E\left[T_{j+1}\right] = \frac{1}{\Lambda_j}.\end{aligned}$$

Analogously

$$E\left[(\Delta t_j)^2|N_{t_j}\right] = E\left[\frac{T_{j+1}^2}{\Lambda_j^2}|N_{t_j}\right] = \frac{1}{\Lambda_j^2}E\left[T_{j+1}^2\right] = \frac{2}{\Lambda_j^2}.$$

Furthermore, note that, for $i=1,2,3$,

$$E\left[\Delta N_{t_j}^i|N_{t_j}\right] = E\left[\int_{t_j}^{t_{j+1}}\lambda_s^i\,ds|N_{t_j}\right] = \lambda_{t_j}^i E\left[\Delta t_j|N_{t_j}\right] = \frac{\lambda^i(N(j))}{\Lambda_j}$$

and that, since $\{T_j\}_{j\geq 1}$ and the Markov chain $\{N(k)\}_{k\geq 0}$ are independent

$$\begin{aligned}E\left[\Delta t_j\Delta N_{t_j}^i|N_{t_j}\right] &= E\left[\Delta t_j\Delta N_{t_j}^i|\mathcal{G}_{t_j}\right] = \frac{1}{\Lambda_j}E\left[T_{j+1}\Delta N_{t_j}^i|\mathcal{G}_{t_j}\right]\\ &= \frac{1}{\Lambda_j}E\left[T_{j+1}\Delta N^i(j)|\mathcal{G}_{t_j}\right]\\ &= \frac{1}{\Lambda_j}E\left[T_{j+1}|\mathcal{G}_{t_j}\right]E\left[\Delta N^i(j)|\mathcal{G}_{t_j}\right]\\ &= \frac{1}{\Lambda_j}E\left[T_{j+1}\right]E\left[\Delta N_{t_j}^i|N_{t_j}\right] = \frac{\lambda^i(N(j))}{\Lambda_j^2}.\end{aligned}$$

■

The last part of this section is inspired by some of the methods presented in (Bouchard and Elie [4]).

Proposition 17: *For* $j=0,\ldots,\tilde{N}_T-1$,

$$V_{t_j}^k = E\left[V_{t_{j+1}}^k\middle|N_{t_j}\right] - \frac{1}{\Lambda_j}\,H_{t_j}^k(V,R,N). \tag{12}$$

Proof: Taking the conditional expectation with respect to $\mathcal{G}_{t_j}$ of both sides of (10), since $R_{t_j}^{1,k}$, $R_{t_j}^{2,k}$, $R_{t_j}^{3,k}$, and $H_{t_j}^k(V,R,N)$ are $\mathcal{G}_{t_j}$-measurable and

$$E\left[\sum_{i=1,2,3}R_{t_j}^{i,k}\Delta M_{t_j}^i\middle|\mathcal{G}_{t_j}\right] = 0,$$

then $V_{t_j}^k = E[V_{t_{j+1}}^k|N_{t_j}] - H_{t_j}^k(V,R,N)E[\Delta t_j|N_{t_j}]$. By (11) the claim follows. ■

Thus, V_t^k is uniquely determined by a backward scheme once we have a recursive scheme for $R_{t_j}^{i,k}$, $i=1,2,3$, and therefore for $H_{t_j}^k$.

PROPOSITION 18: *For* $i = 1, 2, 3$,

$$\frac{\lambda^i(N_{t_j})}{\Lambda_j} H^k_{t_j}(V,R,N) = -\Lambda_j E\left[V^k_{t_{j+1}} \Delta N^i_{t_j} | N_{t_j}\right] + \lambda^i(N_{t_j}) E\left[V^k_{t_{j+1}} T_{j+1} | N_{t_j}\right] + R^{i,k}_{t_j} \lambda^i(N_{t_j}).$$

PROOF: For $i = 1, 2, 3$

$$\Delta M^i_{t_j} = M^i_{t_{j+1}} - M^i_{t_j} = N^i_{t_{j+1}} - N^i_{t_j} - \int_{t_j}^{t_{j+1}} \lambda^i_s \, ds = \Delta N^i_{t_j} - \lambda^i_{t_j} \Delta t_j.$$

Taking again (10) and pre-multiplying by $\Delta M^h_{t_j}$, $h = 1, 2, 3$, the expectation conditionally to $\mathcal{G}_{t_j}$ on both sides provides

$$V^k_{t_j} E\left[\Delta M^h_{t_j} | N_{t_j}\right] = E\left[V^k_{t_{j+1}} \Delta M^h_{t_j} | N_{t_j}\right] - \sum_{i=1,2,3} R^{i,k}_{t_j} E\left[\Delta M^i_{t_j} \Delta M^h_{t_j} | N_{t_j}\right] - H^k_{t_j}(V,R,N) \; E\left[\Delta t_j \Delta M^h_{t_j} | N_{t_j}\right].$$

Let us take into account successively that $E[\Delta M^h_{t_j} | N_{t_j}] = 0$ and that

$$E\left[V^k_{t_{j+1}} \Delta M^h_{t_j} | N_{t_j}\right] = E\left[V^k_{t_{j+1}} \Delta N^h_{t_j} | N_{t_j}\right] - \lambda^h_{t_j} E\left[V^k_{t_{j+1}} \Delta t_j | N_{t_j}\right].$$

Recalling that N^1, N^2, and N^3 do not have common jump times and that the jump sizes are equal to 1,

$$\begin{aligned} E\left[(\Delta M^h_{t_j})^2 | N_{t_j}\right] &= E\left[(\Delta N^h_{t_j} - \lambda^h_{t_j} \Delta t_j)^2 | N_{t_j}\right] \\ &= E\left[\Delta N^h_{t_j} + (\lambda^h_{t_j})^2 (\Delta t_j)^2 - 2\lambda^h_{t_j} \Delta t_j \Delta N^h_{t_j} | N_{t_j}\right] \\ &= \frac{\lambda^h_{t_j}}{\Lambda_j} + (\lambda^h_{t_j})^2 \frac{2}{\Lambda_j^2} - 2\lambda^h_{t_j} \frac{\lambda^h_{t_j}}{\Lambda_j^2} = \frac{\lambda^h_{t_j}}{\Lambda_j} = \frac{\lambda^h(N_{t_j})}{\Lambda_j}. \end{aligned}$$

For $i \neq h$,

$$E\left[\Delta M^h_{t_j} \Delta M^i_{t_j} | N_{t_j}\right] = E\left[-\lambda^h_{t_j} \Delta t_j \Delta N^i_{t_j} - \lambda^i_{t_j} \Delta t_j \Delta N^h_{t_j} + \lambda^h_{t_j} \lambda^i_{t_j} (\Delta t_j)^2 | N_{t_j}\right] = 0$$

and

$$E\left[\Delta t_j \Delta M^h_{t_j} | N_{t_j}\right] = E\left[\Delta t_j \Delta N^h_{t_j} | N_{t_j}\right] - \lambda^h_{t_j} E\left[(\Delta t_j)^2 | N_{t_j}\right] = -\frac{\lambda^h(N_{t_j})}{\Lambda_j^2}.$$

■

PROPOSITION 19: *For* $j = 0, \ldots, \tilde{N}_T - 1$,

$$\frac{H^k_{t_j}(V,R,N)}{\Lambda_j} = \frac{\Lambda_j E\left[V^k_{t_{j+1}} | N_{t_j}\right] - \lambda(N_{t_j}) E\left[V^k_{t_{j+1}} T_{j+1} | N_{t_j}\right]}{\Lambda_j - \lambda(N_{t_j})}. \tag{13}$$

PROOF: Adding the three conditions reached in Proposition 18 and recalling that $\lambda = \lambda^1 + \lambda^2 + \lambda^3$ and that $H^k_{t_j}(V,R,N) = \sum_{i=1,2,3} R^{i,k}_{t_j} \lambda^i_{t_j}$, we have the claim. ■

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Finally, the main result of this note is given by Theorem 20 below.

THEOREM 20: *The value function is uniquely determined by the recursive backward scheme*

$$V^k_{t_{\tilde{N}_T}} = E\Big[e^{\alpha B_T}\Big|\ N_T\Big] = e^{\alpha B_T}$$

$$V^k_{t_j} = \frac{\lambda(N_{t_j})}{\lambda(N_{t_j}) - \Lambda_j} E\Big[V^k_{t_{j+1}}(1 - T_{j+1})\Big| N_{t_j}\Big] \quad j = \tilde{N}_T - 1, \dots, 0.$$

PROOF: Substituting (13) into (12), we get

$$\begin{aligned} V^k_{t_j} &= E\Big[V^k_{t_{j+1}}\Big| N_{t_j}\Big] + \frac{\Lambda_j E\left[V^k_{t_{j+1}}|N_{t_j}\right] - \lambda(N_{t_j}) E\left[V^k_{t_{j+1}} T_{j+1}|N_{t_j}\right]}{\lambda(N_{t_j}) - \Lambda_j} \\ &= \frac{\lambda(N_{t_j})}{\lambda(N_{t_j}) - \Lambda_j} E\Big[V^k_{t_{j+1}}\Big| N_{t_j}\Big] - \frac{\lambda(N_{t_j})}{\lambda(N_{t_j}) - \Lambda_j} E\left[V^k_{t_{j+1}} T_{j+1}|N_{t_j}\right]. \end{aligned}$$

■

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APPENDIX

The following propositions allow us to get the proof of Theorem 14. All over this section, fix a finite time window $[0, T]$ and $t \in [0, T]$.

PROPOSITION 21: *There exists a cadlag $\mathcal{G}$-submartingale V_t^*, such that*

$$\lim_{k \to +\infty} V_t^k = V_t^* \geq V_t \quad a.s.$$

PROOF: By definition, the set of strategies are such that $\Theta_t^k \subset \Theta_t$, hence, for each $k \in \mathbb{N}$, $V_t^k \geq V_t > 0$ a.s. Moreover, $\Theta_t^k \subset \Theta_t^{k+1}$, which implies that $\{V_t^k\}_{k \in \mathbb{N}}$ is a non-increasing sequence also lower bounded. Thus, there exists

$$\widetilde{V}_t := \lim_{k \to +\infty} \downarrow V_t^k \geq V_t \quad \text{a.s.},$$

which is an adapted process, not necessarily cadlag.

Fix $0 \leq s \leq t \leq T$ and recall that, by Proposition 8, V_t^k is a $\mathcal{G}$-submartingale, then, for each $k \in \mathbb{N}$,

$$\mathbb{E}[V_t^k | \mathcal{G}_s] \geq V_s^k \geq \widetilde{V}_s \geq V_s \quad \text{a.s.}$$

Thus, by monotone convergence theorem for conditional expectation

$$\mathbb{E}[\widetilde{V}_t | \mathcal{G}_s] \geq \widetilde{V}_s \quad \text{a.s.},$$

which implies that the process $\widetilde{V}$ is a $\mathcal{G}$-submartingale.

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Again, setting $\mathcal{D} = [0,T] \cap \mathbb{Q}$, as in the proof of Proposition 9, by Karatzas and Shreve [19], the process

$$\widetilde{V}_{t+} := \lim_{s \in \mathcal{D}, s \downarrow t} \widetilde{V}_t, \quad \text{and} \quad \widetilde{V}_{T+} = \widetilde{V}_T = V_T^k = \mathbb{E}\left[e^{\alpha B_T} \middle| \mathcal{G}_T\right]$$

is well defined and it is a $\mathcal{G}$-submartingale. Since the filtration $\mathcal{G}$ is right-continuous

$$\widetilde{V}_t \leq \mathbb{E}[\widetilde{V}_{t+}|\mathcal{G}_t] = \mathbb{E}[\widetilde{V}_{t+}|\mathcal{G}_{t+}] = \widetilde{V}_{t+} \quad \text{a.s.}$$

and $V_t \leq \widetilde{V}_t \leq \widetilde{V}_{t+}$ a.s. Setting $\widetilde{V}_{t+} = V_t^*$, the thesis is achieved. ■

PROPOSITION 22: *For each $\theta \in \Theta^*$, where Θ^* is the set of the essentially bounded admissible strategies, $e^{-\alpha(W^\theta_{t\wedge\tau} - w_0)} V_t^*$ is a $\mathcal{G}$-submartingale.*

PROOF: First, let us prove that the process $e^{-\alpha(W^\theta_{t\wedge\tau} - w_0)} \widetilde{V}_t$ is a $\mathcal{G}$-submartingale, for each θ admissible bounded strategy, namely $\theta \in \Theta$ and θ bounded.

Indeed, if θ is a bounded strategy, there exists $n \in \mathbb{N}$ such that θ is uniformly bounded by n and for each $k \geq n$, $\theta \in \Theta^k$. Thus, $e^{-\alpha(W^\theta_{t\wedge\tau} - w_0)} V_t^k$ is a $\mathcal{G}$-submartingale. This implies that, by the monotone convergence theorem for conditional expectation, $e^{-\alpha(W^\theta_{t\wedge\tau} - w_0)} \widetilde{V}_t$ is also a $\mathcal{G}$-submartingale.

Next, we prove the claim for each strategy $\theta \in \Theta^*$. Recall that $\theta \in \Theta^*$, if $\operatorname{ess\,sup} \theta := \inf_{a \in \mathbb{R}}\{a : P(\omega : \theta(\omega) > a) = 0\} < +\infty$.

To this end, we write down the Doob–Meyer decomposition of the cadlag $\mathcal{G}$-submartingale V^*,

$$V_t^* = V_0^* + \sum_{i=1,2,3} \left[\int_0^t R_r^{i*}\, dM_r^i + \int_0^t R_r^{i*} \lambda_r^i\, dr\right].$$

As a consequence, for a bounded strategy θ, we deduce the Doob–Meyer decomposition of $e^{-\alpha(W^\theta_{t\wedge\tau} - w_0)} V_t^*$. Recalling Proposition 8 (i), $\forall \theta \in \Theta^*$, $e^{-\alpha(W^\theta_{t\wedge\tau} - w_0)} V_t^*$ is a $\mathcal{G}$-submartingale and its Doob–Meyer decomposition is given by

$$e^{-\alpha(W^\theta_{t\wedge\tau} - w_0)} V_t^* = \int_0^t dM_s^{V^*\theta} + \int_0^t A_s^{V^*\theta}\, ds,$$

where

$$dM_s^{V^*\theta} = e^{-\alpha(W^\theta_{s\wedge\tau} - w_0)} \sum_{i=1,2,3} \left\{\left[R_s^{i*} + (V_{s-}^* + R_s^{i*}) K_s^\theta(\eta^i)\right] dM_s^i\right\}, \tag{14}$$

$$A_s^{V^*\theta} = e^{-\alpha(W^\theta_{s\wedge\tau} - w_0)} \sum_{i=1,2,3} \left\{\left(V_s^* + R_s^{i*}\right) K_s^\theta(\eta^i)\lambda_s^i + R_s^{i*}\lambda_s^i\right\}, \tag{15}$$

$A_0^{V^*\theta} = 0$ and $M_0^{V^*\theta} = V_0^*$. For each $\theta \in \Theta^*$, since $e^{-\alpha(W^\theta_{t\wedge\tau} - w_0)} V_t^*$ is a $\mathcal{G}$-submartingale, then $A_t^{V^*\theta} \geq 0$ a.s. and hence,

$$A_s^{V^*\theta} \geq -\operatorname{ess}\inf_{\theta \in \Theta^*} \left\{\sum_{i=1,2,3} \left(V_t^* + R_t^{i*}\right) K_t^\theta(\eta^i)\lambda_t^i\right\}.$$

■

PROPOSITION 23: *For each $t \in [0,T]$, then $V_t \geq V_t^*$ a.s.*

PROOF: Setting $Q_t^*(\theta) := \sum_{i=1,2,3}(V_t^* + R_t^{i*})K_t^\theta(\eta^i)\lambda_t^i$, then

$$\operatorname{ess}\inf_{\theta\in\Theta^*} Q_s^*(\theta) = \operatorname{ess}\inf_{\theta\in\Theta} Q_s^*(\theta).$$

This implies that $\forall\theta\in\Theta$ (not necessarily bounded), given $A_t^{V^*\theta}$ and $M_t^{V^*\theta}$ as in (15) and (14), $e^{-\alpha(W^\theta_{t\wedge\tau}-w_0)}V_t^* = M_t^{V^*\theta} + A_t^{V^*\theta}$. Since $dA_t^{V^*\theta} \geq 0$ a.s., we have two consequences. First,

$$M_t^{V^*\theta} \leq e^{-\alpha(W^\theta_{t\wedge\tau}-w_0)}V_t^*,$$

which in turn implies that $M_t^{V^*\theta}$ is a local $\mathcal{G}$-martingale bounded from above, namely $M_t^{V^*\theta}$ is a $\mathcal{G}$-submartingale. More, since the process $A_t^{V^*\theta}$ is non-decreasing, $e^{-\alpha(W^\theta_{t\wedge\tau}-w_0)}V_t^*$ is a $\mathcal{G}$-submartingale for all $\theta\in\Theta$. Finally, V_t^* is cadlag $\mathcal{G}$-adapted and $V_T^* = \mathbb{E}[e^{\alpha B_T}|\mathcal{G}_T]$ and by noting that V_t is the largest process satisfying these properties, $V_t^* \leq V_t$ a.s. ∎

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