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ON BAIRE-HYPERPLANE SPACES

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In this article we prove that in every infinite dimensional separable Fréchet space there is a dense barrelled subspace which is not the inductive limit of Baire-hyperplane spaces.

The linear spaces we use are defined over the field K of the real or complex numbers. By "space" we mean "separated locally convex topological vector space". Given a space H, H' denotes its topological dual and if A is a bounded closed absolutely convex subset of H, H_A is the normed space over the linear hull of A has as norm the gauge on A. Let $\mathscr{A}$ be the family of all the bounded closed absolutely convex subsets of H. If B is a subspace of H its local closure $\tilde{B}$ is the intersection of all subspaces of H containing B and intersecting H_A, $A \in \mathscr{A}$, in a closed set. We say that H is locally complete if H_A is a Banach space for every $A \in \mathscr{A}$.

Let Ω be a non void open set in the euclidean m-dimensional space $\mathbf{R}^m$. We denote by $\mathscr{D}(\Omega)$ and $\mathscr{D}'(\Omega)$ the spaces of L. Schwartz with the strong topologies. Let M be the set of all points of Ω having rational coordinates. If $b \in M$, T_b is the set formed by the delta of Dirac concentrated in b and its derivatives of all orders. $\mathscr{D}'_0(\Omega)$ is the subspace of $\mathscr{D}'(\Omega)$ generated by $\{T_b: b \in M\}$. As is usual, $C \sim D$ is the difference of C and D. $\mathbf{N}$ denotes the set of the natural numbers.

1. Baire-hyperplane spaces

A space H is a Baire-hyperplane space if every union of a countable family of closed hyperplane of H has void interior.

The following three theorems are of a trivial nature.

Theorem 1. *Every separated quotient of a Baire-hyperplane space is a Baire-hyperplane space.*

Theorem 2. *If a space E contains a dense subspace which is a Baire-hyperplane space then H is a Baire-hyperplane space.*

Theorem 3. *If E is a Baire-hyperplane space and $\mathscr{T}$ is a separated locally convex topology on E coarser than the original topology, then $E[\mathscr{T}]$ is a Baire-hyperplane space.*

Note 1. If E is a space such that its topology is defined by a family of norms there is in $E'[\sigma(E', E)]$ a compact absolutely convex subset A which is total. Then, if $\mathscr{F}$ is a

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topology on E' compatible with the dual pair $\langle E', E\rangle$, Theorem 3 implies that E'_A with the topology induced by $\mathscr{F}$ is a Baire-hyperplane space. According to Theorem 2, $E'[\mathscr{F}]$ is a Baire-hyperplane space.

Theorem 4. *Every metrizable (LF)-space is a Baire-hyperplane space.*

Proof. Let (H_n) be an increasing sequence of subspaces of E with $\cup\{H_n: n \in \mathbb{N}\} = E$. Suppose that there is a topology on H_n, $\mathscr{F}_n$, finer than the original topology such that $H_n[\mathscr{F}_n]$, $n \in \mathbb{N}$, is a Fréchet space and E is the inductive limit of the sequence $(H_n[\mathscr{F}_n])$. The sequence of the closures of H_n in E, $n \in \mathbb{N}$, is increasing and its union in E. Since E is a metrizable barrelled space there is a positive integer p so that $\bar{H}_p = E$, (1). On the other hand, let us suppose that there is a sequence (L_n) of closed hyperplanes of E with E as its union. Then $\cup\{L_n \cap H_p[\mathscr{F}_p]: n \in \mathbb{N}\} = H_p[\mathscr{T}_p]$ and since $H_p[\mathscr{T}_p]$ is a Fréchet space there is a positive integer q such that $L_q \supset Hp[\mathscr{T}_p]$ and therefore $L_q \supset \bar{H}_p = E$ which is a contradiction.

Note 2. In (3, p. 369) a non closed barrelled subspace of l^1 is given. It is easy to show it is not a Baire-hyperplane space. Other examples of normed barrelled spaces which are not Baire-hyperplane spaces are given in (2).

Theorem 5. *Every product of Baire-hyperplane spaces is a Baire-hyperplane space.*

Proof. Let $\{E_i; i \in I\}$ be a family of Baire-hyperplane spaces and let E be its topological product. If we suppose that E is not a Baire-hyperplane space let (L_n) be a sequence of closed hyperplanes of E covering E. For each $i \in I$, let $\mathscr{L}_i$ be the family of those elements of (L_n) not containing E_i. $\mathscr{L}_i$ does not cover E_i since E_i is a Baire-hyperplane space and therefore a vector $x_i \in H_i$ can be found which is not in any member of $\mathscr{L}_i$. Call F_i the linear hull of $\{x_i\}$. Every element of the family $\mathscr{L} = \{\mathscr{L}_i; i \in I\}$ does not contain $F = \Pi\{F_i : i \in I\}$, but F is a Baire space and thus $\mathscr{L}$ cannot cover F and therefore $\mathscr{M} = \{L_1, L_2, \ldots, L_n, \ldots\} \sim L$ is not void which means that every $M \in \mathscr{M}$ contains E_i, $i \in I$, and, therefore $M \supset E$ which is a contradiction.

2. Bornological barrelled spaces

Proposition 1. *Let H be a locally complete space and let S and T be subspaces of E such that $\bar{T} \subset S$. If $\mathscr{F}$ denotes the associated ultrabornological topology to S then T is $\mathscr{F}$-dense in $\bar{T}$.*

Proof. Let T^* be the $\mathscr{F}$-closure in S of T and write T_1 for $T^* \cap \bar{T}$. We show that $\bar{T} \subset T^*$. Given any bounded closed absolutely convex subset A of H we have that H_A is a Banach space and $H_A \cap \bar{T}$ is a closed subspace of H_A coinciding with $H_{A \cap \bar{T}}$ which is therefore a Banach space and its topology is finer than the topology induced by $\mathscr{F}$ and thus $H_{A\cap\bar{T}} \cap T^*$, which coincides with $H_{A\cap T_1}$, is closed in H_A. Therefore T_1 coincides with $\bar{T}$ and the conclusion follows.

Next we consider a locally complete space E which has a bi-orthogonal sequence (x_n, u_n) such that the linear hull G of (x_n) is dense in E. If $x \in E$ we write

$$\Delta(x) = \{n \in \mathbb{N}: u_n(x) \neq 0\}.$$

Suppose that there is a $x_0 \in E$ such that $\Delta(x_0)$ is infinite and call $\mathcal{J}$ the filter on $\mathbb{N}$ such that $F \in \mathcal{J}$ if and only if $F \cap \Delta(x_0)$ has a finite complement in $\Delta(x_0)$ and let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ finer than $\mathcal{J}$. For any $U \in \mathcal{U}$ we write $L(U)$ to denote the local closure of the linear hull of $\{x_n: n \in \mathbb{N} \sim U\}$. Since $L(U_1) \cup L(U_2) \subset L(U_1 \cap U_2)$ for every pair U_1, $U_2 \in \mathcal{U}$ it follows that $L = \cup \{L(U): U \in \mathcal{U}\}$ is a subspace of E containing G. We write F for the linear hull of $L \cup \{x_0\}$. In what follows $\mathcal{F}$ denotes the associated ultrabornological topology to L.

Proposition 2. *The neighbourhoods of the origin in $L[\mathcal{F}]$ absorb the bounded sets of G.*

Proof. Let V be an absolutely convex neighbourhood of the origin in $L[\mathcal{T}]$ and suppose the existence in G of a bounded closed absolutely convex set Q such that Q is not absorbed by V. Take $y_1 \in Q \sim V$. Suppose we have already constructed the elements $y_1, y_2, \ldots, y_q$ of Q such that

$$y_p \notin pV, \; y_p = \sum_{n \in N(p)} \lambda_n x_n, \; \lambda_n \in K, \qquad p = 1, 2, \ldots, q,$$

where

$$N(1) = 1, 2, \ldots, n_1$$
$$N(2) = n_1 + 1, n_1 + 2, \ldots, n_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$N(q) = n_{q-1} + 1, n_{q-1} + 2, \ldots, n_q.$$

If G_q denotes the linear hull of

$$\{x_n: n \in \mathbb{N}, \; n \notin N(p), p = 1, 2, \ldots, q\}$$

then $G_q \cap G_Q$ is a closed finite codimensional subspace of G_Q. If X is a topological complement of $G_q \cap G_Q$ in G_Q, call Q_1 and Q_2 the projections of Q over X and $G_q \cap G_Q$, respectively. Obviously, V absorbs Q_1 which is a bounded set in a finite dimensional space. Since Q_2 is a bounded set in $G_q \cap G_Q$ there is a $\lambda > 0$ such that

$$Q_2 \subset \lambda(Q \cap G_q).$$

V cannot absorb $Q \cap G_q$ because otherwise there is a $\mu > 0$ such that

$$Q_1 \subset \mu V, \; Q \cap G_q \subset \mu V$$

and therefore

$$Q \subset Q_1 + Q_2 \subset Q_1 + \lambda(Q \cap G_q) \subset \mu V + \lambda\mu V = \mu(1 + \lambda)V,$$

which is a contradiction. Thus, there is an element y_{q+1} in $Q \cap G_q$ so that

$$y_{q+1} \notin (q + 1)V$$

which can be written

$$y_{q+1} = \sum_{n \in N(q+1)} \lambda_n x_n, \lambda_n \in K,$$

where

$$N(q+1) = \{n_q + 1, n_q + 2, \ldots, n_{q+1}\}.$$

The elements of the sequence $(N(q))$ are a partition of $\mathbb{N}$. We write

$$M_1 = \cup \{N(2q-1): q = 1, 2, \ldots\}$$
$$M_2 = \cup \{N(2q): q = 1, 2, \ldots\}.$$

Obviously, $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = N$ and since $\mathcal{U}$ is an ultrafilter it follows that, for instance, M_1 belongs to $\mathcal{U}$. The sequence (y_{2q}) belongs to $L(M_1)$ and it is bounded. Writing D for the closed absolutely convex hull of (y_{2q}) in E we have that $E_D \cap L(M_1)$ is a Banach space since E_D is a Banach space and therefore $V \cap E_{D \cap L(M_1)}$ is a neighbourhood of the origin in $E_{D \cap L(M_1)}$ (remember that V is a $\mathcal{F}$-neighbourhood of the origin in L) and thus V absorbs the elements of the sequence (y_{2q}) which is in contradiction with $y_{2q} \notin (2q)V$.

Proposition 3. *If G is a bornological space then L is an ultrabornological space.*

Proof. Let J be the canonical injection of G into $L[\mathcal{F}]$. The former proposition implies that J is continuous since if V is an absolutely convex $\mathcal{T}$-neighbourhood of the origin in L, then $J^{-1}(V) = V \cap G$ absorbs the bounded sets of G and since G is bornological $J^{-1}(V)$ is a neighbourhood of the origin in G. If $\widehat{L[\mathcal{F}]}$ is the completion of $L[\mathcal{T}]$, J can be extended to a continuous linear mapping φ from L into $\widehat{L[\mathcal{T}]}$. The conclusion follows if we show that φ is the identity from L in $L[\mathcal{T}]$. But this is the case since if $z \in L$ there is $U \in \mathcal{U}$ such that $z \in L(U)$. According to Proposition 1, $L(U) \cap G$ is $\mathcal{F}$-dense in $L(U)$ and therefore there is a net

$$\{z_i: i \in I, \geq\}$$

in $L(U) \cap G$ $\mathcal{F}$-converging to z which obviously also converges to z in L and thus

$$\lim \{\varphi(z_i): i \in L, \geq\} = \varphi(z) = \lim \{J(z_i): i \in I, \geq\} = \lim \{z_i: i \in I, \geq\} = z.$$

Note 3. In (2) ultrafilters on $\mathbb{N}$ are used also to construct examples of metrizable ultrabarrelled spaces which are not Baire-hyperplane spaces.

Proposition 4. *L is an hyperplane of F.*

Proof. Given any x belongs to L there is an element $U \in \mathcal{U}$ such that $x \in L(U)$ and therefore $\Delta(x) \notin \mathcal{U}$. On the other hand $\Delta(x_0) \in \mathcal{U}$ and thus $x_0 \notin L$. Since F is the linear hull of $L \cup \{x_0\}$, L is an hyperplane of F.

Proposition 5. *F is not an inductive limit of Baire-hyperplane spaces.*

Proof. Suppose that there is a family $\{(F_i, \varphi_i): i \in I\}$ such that F_i is a Baire-hyperplane space, φ_i is a linear continuous mapping from F_i into F and F is the locally convex hull of the family. Let u be a linear form on F such that $u^{-1}(0) = L$. Since u is not continuous there is an index $h \in I$ such that $\mu \circ \varphi_h$ is not continuous on F_h and thus $\varphi_h^{-1}(L)$ is a dense hyperplane of F_h. Take a net $\{y_j; j \in J, \geq\}$ in $\varphi_h^{-1}(L)$ converging in F_h to an element $y_0 \notin \varphi_h^{-1}(L)$ and select a sequence (t_n) such that if $\varphi_h(y_0) = z_0$ and $\varphi_h(t_n) = z_n$ then

$$\lim_{n\to\infty} u_m(z_n) = u_m(z_0), \ m = 1, 2, \ldots$$

Write $z_0 = \lambda x_0 + x$, $\lambda \neq 0$, $x \in L$ and then

$$\Delta(z_0) \supset \Delta(\lambda x_0) \sim \Delta(x) = \Delta(x_0) \sim \Delta(x)$$

and since $\Delta(x_0) \in \mathcal{U}$ and $\Delta(x) \notin \mathcal{U}$ it follows that $\Delta(z_0) \in \mathcal{U}$. Write

$$Q_1 = \Delta(z_1) \cap \Delta(z_0)$$

$$Q_m = [\Delta(z_m) \sim \bigcup_{p=1}^{m-1} \Delta(z_p)] \cap \Delta(z_0), \qquad m = 2, 3, \ldots$$

If $n \in \Delta(z_0)$ the sequence $(u_n(z_p))_{p=1}^{\infty}$ converges to $u_n(z_0)$ and therefore there is a positive integer p_0 such that

$$u_n(z_p) \neq 0, \text{ for } p \geq p_0.$$

If $u_n(z_r)$ is the first element of $(u_n(z_p))_{p=1}^{\infty}$ which is non-zero then $n \in Q_r$ and therefore

$$\Delta(z_0) = \bigcup \{Q_p: p = 1, 2, \ldots\}.$$

Since

$$Q_p \subset \Delta(z_p) \notin \mathcal{U},$$

given a finite subset M of $\mathbb{N}$ the union

$$\bigcup \{Q_p: p \in M\} \notin \mathcal{U}$$

and thus

$$\bigcup \{Q_p: p \in M\} \neq \Delta(z_0)$$

and therefore there is a strictly increasing sequence of positive integers $(n(q))_{q=1}^{\infty}$ such that

$$Q_{n(q)} \neq \emptyset, \ q = 1, 2, \ldots$$

hence we can suppose that the sequence (z_n) has been selected so that $Q_n \neq \emptyset$, $n = 1, 2, \ldots$.

For any pair of positive integers n, m construct a continuous linear form ψ_{nm} on F_h such that

$$\psi_{nm}(t) = u_n(z_0)u_m(\varphi_h(t)) - u_n(\varphi_h(t))u_m(z_0), \ t \in F_h.$$

Let $\{f_n: n \in P \subset \mathbb{N}\}$ be the set of all the non-null elements of $\{\psi_{nm}: n, m \in \mathbb{N}\}$. If p, q are positive integers, $p < q$, and if $n \in Q_p$, $m \in Q_q$, then $m \notin \Delta(z_p)$ and $n \in \Delta(z_p)$ and therefore $u_m(z_p)=0$ and $u_n(z_p)\neq 0$. Then

$$\begin{aligned}\psi_{nm}(t_p)&=u_n(z_0)u_m(\varphi_h(t_p))-u_n(\varphi_h(t_p))u_m(z_0)=u_n(z_0)u_m(z_p)-u_n(z_p)u_m(z_0)\\&=-u_n(z_p)u_m(z_0)\neq 0,\end{aligned}\tag{1}$$

hence $\{f_n : n\in P\subset \mathbb{N}\}\neq \emptyset$. Since F_h is a Baire-hyperplane space the family $\{f_n^{-1}(0)\colon n\in P\}$ of closed hyperplanes of F_h does not cover F_h and therefore there is $v\in F_h$ with $f_n(v)\neq 0$ for every $n\in P$.

The set $\Delta(\varphi_h(v))$ belongs to $\mathcal{U}$. Indeed, suppose that $\Delta(z_0)\not\subset\Delta(\varphi_h(v))$, which is the difficult case. Then there is $q\in\Delta(z_0)$ so that $u_q(\varphi_h(v))=0$. Take a positive integer k such that $q\in Q_k$. If $p\in N$, $p\neq k$, let m be an element of Q_p. Then, according to (1), we can take $s\in P$ such that $\psi_{qm}=f_s$ and therefore

$$0\neq f_s(v)=\psi_{qm}(v)=u_q(z_0)u_m(\varphi_h(v))-u_q(\varphi_h(v))u_m(z_0)=u_q(z_0)u_m(\varphi_h(v)),$$

and therefore $u_m(\varphi_h(v))\neq 0$ and thus

$$\Delta(\varphi_h(v))\supset\Delta(z_0)\sim Q_k,$$

hence $(\varphi_h(v))\in\mathcal{U}$.

Write

$$\varphi_h(v)=\rho z_0+w,\qquad\qquad \rho\neq 0,\ w\in L.$$

Since $\Delta(\omega)\notin\mathcal{U}$ there are two different $p,q\in\mathbb{N}$ such that

$$(\Delta(z_0)\sim\Delta(w))\cap Q_p\neq\emptyset,\ (\Delta(z_0)\sim\Delta(z))\cap Q_q\neq\emptyset.$$

Take $n\in Q_p$ and $m\in Q_q$. Then

$$\begin{aligned}u_n(\varphi_h(v))&=\rho u_n(z_0)+u_n(w)=\rho u_n(z_0)\\u_m(\varphi_h(v))&=\rho u_m(z_0)+u_m(w)=\rho u_m(z_0)\end{aligned}$$

and therefore

$$0\neq\psi_{nm}(v)=u_n(z_0)u_m(\varphi_h(v))-u_n(\varphi_h(v))u_m(z_0)=u_n(z_0)\rho u_m(z_0)-\rho u_n(z_0)u_m(z_0)=0,$$

which is a contradiction.

Proposition 6. *If a barrelled space H admits a bi-orthogonal sequence (x_n,u_n) such that the linear hull M of (x_n) is not closed in H then there is an element $x\in H$ so that the sequence $(u_n(x))$ has an infinity of non vanishing terms.*

Proof. If Z denotes the subspace of H orthogonal to $\{u_1,u_2,\ldots,u_n,\ldots\}$ the sequence $(u_n(z))$, for any $z\in H$, has a finite number of non vanishing terms if and only if $z\in M+Z$. Suppose that H coincides with $M+Z$. If φ denotes the canonical mapping from H onto H/Z and ψ stands for the restriction of φ to M then ψ is an injective continuous mapping from M onto H/Z. Since H/Z is a barrelled space of countable dimension it is provided with the finest locally convex topology and therefore also M which implies its completeness but M was supposed to be non closed in E.

Note 4. If E is a space with a dense subspace G of infinite countable dimension, a theorem of Klee (4, p. 148) can be applied to obtain a bi-orthogonal system (x_n, u_n) for E so that the linear hull of (x_n) is G. If E is locally complete and G is bornological and different from E there is an element $x_0 \in E$ such that $\Delta(x_0)$ is infinite, according to Proposition 6, and the spaces L and F can be constructed with the properties obtained in Propositions 3, 4, 5.

Theorem 6. *Let E be a separable Fréchet space of infinite dimension. Then there are two dense subspaces L and F of E so that*

1. *L is an hyperplane of F.*
2. *L is ultrabornological.*
3. *F is a barrelled space which is not an inductive limit of Baire-hyperplane space.*

Proof. Since E is a separable Fréchet space of infinite dimension there is an infinite countable dimensional dense subspace which is different from E. According to Note 4 the theorem follows.

Theorem 7. *Let E be the inductive limit of a sequence of infinite dimensional separable Fréchet spaces. If E is locally complete there are two dense subspaces L and F of E so that*

1. *L is an hyperplane of F.*
2. *L is ultrabornological.*
3. *F is a bornological barrelled space which is not an inductive limit of Baire-hyperplane spaces.*

Proof. Let (E_n) be an increasing sequence of subspaces of E with union E. Let $\mathscr{F}_n$ be a topology on E_n, finer than the original topology, such that $E_n[\mathscr{F}_n]$ is an infinite dimensional separable Fréchet space. Take in $E_n[\mathscr{F}_n]$ a dense subspace G_n of countable dimension. Then the linear hull G of $\cup\{G_n: n = 1, 2, \ldots\}$ is dense in E and different from E. On the other hand, it is easy to check that G is the inductive limit of the sequence of metrizable spaces (G_n) and thus G is bornological. We apply Note 4 and we finish by showing that F is bornological. Indeed, if we suppose that $F \cap E_n$ is provided with the topology induced by $\mathscr{F}_n$ it is easy to show that F is the inductive limit of the sequence of metrizable spaces $(F \cap E_n)$.

Proposition 7. *I´˙ H be a subspace of $\mathscr{D}'(\Omega)$ such that it contains $\mathscr{D}'_0(\Omega)$. Then H is bornological.*

Proof. If B is a bounded set of H there is a compact absolutely convex set A in $\mathscr{D}'(\Omega)$ such that B is a precompact set of $\mathscr{D}'(\Omega)_A$ since $\mathscr{D}'(\Omega)$ is a nuclear complete space. Then B is a precompact set of $\mathscr{D}'(\Omega)_{A\cap H}$ and therefore there is in $\mathscr{D}'(\Omega)_{A\cap H}$ a sequence converging to the origin with closed absolutely convex hull containing B (3, p. 273). Then the topology $\mathscr{F}$ on $\mathscr{D}(\Omega)$ of the uniform convergence of the sequences of H converging to the origin in the sense of Mackey coincides with the topology on $\mathscr{D}(\Omega)$ of the uniform convergence of the bounded sets of H.

Let $\{\varphi_i: i \in I, \geqslant\}$ be a Cauchy net in $\mathscr{D}(\Omega)$ for the topology $\mathscr{F}$. Let M be the set of

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all the points of Ω with rational coordinates. If $b \in M$, $T_m(b)$ is the set of the delta of Dirac δ_b concentrated in b and all its derivatives $\delta_b^{(m)}$ with order $|m|$ less or equal than n. Let

$$T_n = \cup\{T_n(b): b \in M\}.$$

Then T_n is a bounded set of H and thus given $\epsilon > 0$ and $p = (p_1, p_2, \dots, p_m)$ where p_r is a non negative integer, $r = 1, 2, \dots, m$, there is an $i_0 \in I$ such that

$$|\langle \varphi_i - \varphi_j, \delta_b^{(p)} \rangle| = |\varphi_i^{(p)}(b) - \varphi_j^{(p)}(b)| \leq \epsilon, i, j \geq i_0, \forall b \in M$$

and since φ_i, $i \in I$, is a function on Ω with values in K having continuous derivatives of all orders we have that

$$|\varphi_i^{(p)}(x) - \varphi_j^{(p)}(x)| \leq \epsilon, \forall x \in \Omega,$$

and therefore there is a function φ on Ω with values in K having continuous derivatives of all orders such that

$$\lim\{\varphi_i^{(p)}(x): i \in T, \geq\} = \varphi^{(p)}(x)$$

uniformly on Ω. Suppose that φ has non compact support in Ω. Then there is a sequence (x_n) in Ω not contained in any compact of Ω such that $\varphi(x_n) \neq 0$, $n = 1, 2, \dots$. The subset of H

$$\left\{\frac{1}{\varphi(x_n)} \delta_{x_n}: n = 1, 2, \dots\right\}$$

is bounded and therefore given a positive ϵ less than 1 there is an index $k \in I$ such that

$$\left|\left\langle \varphi_k - \varphi, \frac{1}{\varphi(x_n)} \delta_{x_n} \right\rangle\right| = \left|\frac{1}{\varphi(x_n)} \varphi_k(x_n) - 1\right| < \epsilon, n = 1, 2, \dots$$

Since the support D of φ_k is compact there is a positive integer q such that $x_q \notin D$ and thus

$$\left|\frac{1}{\varphi(x_q)} \varphi_k(x_q) - 1\right| = 1 < \epsilon < 1$$

which is a contradiction and therefore $\varphi \in \mathscr{D}(\Omega)$ and thus $\mathscr{D}(\Omega)[\mathscr{F}]$ is complete. Since H is a Mackey space, (5), and according to a theorem of Köthe, (3, p. 386), H is bornological.

Theorem 8. *There are two dense subspaces L and F in $\mathscr{D}'[\Omega]$ such that*
1. *L is a hyperplane of F.*
2. *L is ultrabornological.*
3. *F is a bornological barrelled space which is not an inductive limit of Baire-hyperplane spaces.*

Proof. $\mathscr{D}_0'(\Omega)$ has countable dimension, is dense in $\mathscr{D}(\Omega)$ and $\mathscr{D}_0'(\Omega) \neq \mathscr{D}(\Omega)$. On the other hand, $\mathscr{D}(\Omega)$ is complete. Then, according to Note 4, the theorem follows.

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