Math. Struct. in Comp. Science (2017), *vol.* 27, *pp.* 651–680.
doi:10.1017/S0960129515000328 First published online 24 August 2015

The parametric continuation monad†

PAUL-ANDRÉ MELLIÈS

CNRS, Laboratoire PPS, UMR 7126, Université Paris Diderot, Sorbonne Paris Cité, F-75205 Paris, France
Email: mellies@pps.univ-paris-diderot.fr

Received September 2012; revised May 2013

Every dialogue category comes equipped with a continuation monad defined by applying the negation functor twice. In this paper, we advocate that this double negation monad should be understood as part of a larger parametric monad (or a lax action) with parameter taken in the opposite of the dialogue category. This alternative point of view has one main conceptual benefit: it reveals that the strength of the continuation monad is the fragment of a more fundamental and symmetric structure – provided by a distributivity law between the parametric continuation monad and the canonical action of the dialogue category over itself. The purpose of this work is to describe the formal properties of this parametric continuation monad and of its distributivity law.

1. Introduction

1.1. *Origins of tensorial logic*

The idea of tensorial logic emerged during my first sabbatical stay at the Research Institute in Mathematical Sciences (RIMS) in Kyoto. There, in the middle of the summer 2006, I realized that the distributivity law of linear logic

$$(A \parr B) \otimes C \quad \longrightarrow \quad A \parr (B \otimes C) \tag{1}$$

could be unified with the tensorial strength of the continuation monad

$$(\neg\neg\, A) \otimes B \quad \longrightarrow \quad \neg\neg\,(A \otimes B) \tag{2}$$

by shifting from linear logic to this more primitive logic of tensor and negation, where negation is not required to be involutive anymore. Accordingly, the very name of 'tensorial logic' came from the observation that the distributivity law

$$\kappa_{X,B,C} \quad : \quad \neg\,(\neg\, B \otimes X) \otimes C \quad \longrightarrow \quad \neg\,(\neg\,(\,B \otimes C\,) \otimes X\,) \tag{3}$$

of tensorial logic can be seen as a refinement of the distributivity law (equation 1) of linear logic and *at the same time* as a parametric version of the tensorial strength (equation 2) with parameter provided by the variable X.

One main purpose of the present article is to clarify the algebraic nature of this primitive and unifying principle of logic (equation 3) starting from the observation

† *Dedicated to Corrado Böhm, on the occasion of his 90th birthday.*

that the family of morphisms κ is canonically defined in every dialogue category. Recall from Melliès (2009) and Melliès-Tabareau (2009) that a dialogue category is a monoidal category $(\mathscr{C}, \otimes, I)$ equipped with an object $\bot$ called its *tensorial pole*, and a pair of natural isomorphisms

$$\begin{array}{llcll} \varphi_{A,B} & : & \mathscr{C}(A \otimes B, \bot) & \cong & \mathscr{C}(B, A \multimap \bot) \\ \psi_{A,B} & : & \mathscr{C}(A \otimes B, \bot) & \cong & \mathscr{C}(A, \bot \multimapinv B) \end{array}$$

providing a representation of the two presheaves

$$A \mapsto \mathscr{C}(A \otimes B, \bot) \quad , \quad B \mapsto \mathscr{C}(A \otimes B, \bot) \quad : \quad \mathscr{C}^{op} \longrightarrow Set.$$

The situation is extremely common in logic and in algebra. A typical illustration is provided by the category of (possibly infinite dimensional) vector spaces on a given field k, with the object $\bot$ defined as the field k itself. Another example is provided by any cartesian closed category $\mathscr{C}$ with a fixed object $\bot$ in it.

When the tensor product $\otimes$ of the dialogue category is symmetric, the objects $A \multimap \bot$ and $\bot \multimapinv A$ are isomorphic, and are thus often identified and written as $\neg A$ for simplicity. It is well known that every dialogue category comes equipped with a monad

$$A \mapsto \bot \multimapinv (A \multimap \bot) \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C} \tag{4}$$

obtained by applying negation twice. This monad is traditionally called the *continuation monad* of the dialogue category in programming language theory because it is related to the continuation-passing style translations used during compilation. Note that the tensorial strength (equation 2) of the continuation monad is defined in any dialogue category $\mathscr{C}$ as the morphism (equation 3) instantiated at the parameter X equal to the tensorial unit I.

Seen from the point of view of tensorial logic, linear logic starts when one decides to force the double-negation monad to coincide with the identity. This step is reflected in categorical terms by the shift from general dialogue categories to the specific case of $*$-autonomous categories. Recall that a $*$-autonomous category is a symmetric dialogue category where the unit

$$A \quad \longrightarrow \quad \neg\neg A$$

of the continuation monad is invertible for every object A. The distributivity law (equation 1) of linear logic is then recovered in any $*$-autonomous category as a special case of the distributivity law (equation 3) of tensorial logic instantiated this time at the parameter $X = \neg A$:

$$\kappa_{\neg A,B,C} \quad : \quad \neg(\neg B \otimes \neg A) \otimes C \quad \longrightarrow \quad \neg(\neg(B \otimes C) \otimes \neg A).$$

The definition of equation (1) is justified by the fact that the multiplicative disjunction $⅋$ of linear logic is defined as

$$A ⅋ B \quad = \quad \neg(\neg B \otimes \neg A) \tag{5}$$

in any $*$-autonomous category.

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

This unification of equations (1) and (2) leads to the methodological question of understanding the algebraic nature of the tensorial principle (equation 3) which underlies both of them. Quite obviously, this mathematical investigation of equation (3) should shed light on equations (1) and (2) and benefit at the same time from what is already known about these two well-studied instances. Typically, the fact that the distributivity law (equation 3) is a parametric refinement of the tensorial strength (equation 2) leads us to decompose it in two independent ingredients, for every dialogue category $\mathscr{C}$:

a. a functor

$$\circledast \quad : \quad (X, A) \quad \mapsto \quad X \circledast A = \bot \circ\!\!- ((A \multimap \bot) \otimes X) \quad : \quad \mathscr{C}^{op} \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

corresponding to a parametric version of the continuation monad, with the object X as parameter,

b. a natural transformation

$$\kappa_{X,A,B} \quad : \quad (X \circledast A) \otimes B \quad \longrightarrow \quad X \circledast (A \otimes B)$$

generalizing the tensorial strength of the continuation monad.

This decomposition of equation (3) into (a.) and (b.) reduces our original problem to understanding in turn the algebraic nature of this specific functor $\circledast$ and of this specific natural transformation κ. As we will see, the exercise is not particularly difficult in itself – although it should be done with great care – but extremely useful, since it reveals the basic two-dimensional structures which regulate the logical discourse, and more specifically its use of negation.

1.2. *Parametric continuation monad*

As explained above, the first aim of this paper is to reconstruct the functor $\circledast$ and more precisely to understand in which sense this functor should be understood as a parametric version of the continuation monad. A preliminary step in this direction is to observe that every dialogue category $\mathscr{C}$ comes equipped with an adjunction

$$\mathscr{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{C}^{op}, \quad (L \dashv R) \tag{6}$$

where L and R denote the expected negation functors:

$$L \quad : \quad a \mapsto a \multimap \bot \qquad\qquad R \quad : \quad b \mapsto \bot \circ\!\!- b.$$

In order to analyse the algebraic nature of this adjunction, it also appears convenient to rename the monoidal categories $\mathscr{C}$ and $\mathscr{C}^{op}$ in the following way:

— the category $\mathscr{A}$ is the new name for $\mathscr{C}$ and its tensor product and unit are noted $\owedge$ and **true** in order to stress the logical interpretation of $\otimes$ and I as a linear conjunction and its neutral element,

— the category $\mathscr{B}$ is the new name for $\mathscr{C}^{op(0,1)}$ whose tensor product and unit are denoted $\ovee$ and **false** in order to stress the logical interpretation of $\otimes$ and I as a linear disjunction and its neutral element.

Here, the notation $\mathscr{C}^{op(0,1)}$ means that the orientation of the morphisms (of dimension 1) is reversed in $\mathscr{C}$ as well as the orientation of the tensor product (of dimension 0). This symmetric formulation of dialogue categories leads to the notion of *dialogue chirality* introduced in our companion paper (Melliès 2015). The interested reader may have a look at the original definition there. However, it will be sufficient in this paper to remember that a dialogue chirality is essentially the same thing as a dialogue category formulated in this two-sided and symmetric fashion. The specific orientation for the disjunction in the category $\mathscr{B}$ is chosen in order to rewrite the formula (5) as follows:

$$A \parr B \quad = \quad R\,(\,LA \ovee LB\,) \tag{7}$$

and thus to interpret $\ovee$ as a primitive variant of $\parr$, with the functors L and R playing the role of coercions (or shifts) interpreted as identity functors in the case of linear logic. At this point, one should remember that just as in the case of any monoidal category, the monoidal structure of $\mathscr{B}$ defines a (weak) left action

$$* \quad = \quad \ovee \quad : \quad \mathscr{B} \times \mathscr{B} \quad \longrightarrow \quad \mathscr{B} \tag{8}$$

of the monoidal category $(\mathscr{B}, \ovee, \textbf{false})$ over itself, seen as a category. Recall that by weak left action of a monoidal category $(\mathscr{M}, \otimes, I)$ on a category $\mathscr{X}$, one means a functor

$$* \quad : \quad \mathscr{M} \times \mathscr{X} \quad \longrightarrow \quad \mathscr{X}$$

equipped with natural isomorphisms

$$\mu_{m,n} \;:\; m \circledast (n * x) \;\longrightarrow\; (m \otimes n) * x \qquad\qquad \mu_I \;:\; x \;\longrightarrow\; I * x$$

satisfying the two expected coherence diagrams:

$$\begin{array}{ccccc} (m\otimes n)*(p*x) & \xleftarrow{\;\mu\;} & m*(n*(p*x)) & \xrightarrow{\;\mu\;} & m*((n\otimes p)*x) \\ \downarrow{\scriptstyle\mu} & & & & \downarrow{\scriptstyle\mu} \\ ((m\otimes n)\otimes p)*x & & \xrightarrow{\qquad\quad\alpha\qquad\quad} & & ((m\otimes n)\otimes p)*x \end{array} \quad ,$$

$$\begin{array}{ccc} m*x & \xrightarrow{\qquad\mu\qquad} & m*(I*x) \\ & {\scriptstyle\rho}\searrow \qquad \swarrow{\scriptstyle\mu} & \\ & (m\otimes I)*x & \end{array} \tag{9}$$

where α and ρ denote the associativity and unit combinators of the monoidal category $\mathscr{M}$. We will establish in Section 2 a general *transfer theorem* which states that

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

the weak action (equation 8) may be transported along the adjunction

$$\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} \qquad (\perp) \tag{10}$$

into the lax action defined as

$$\circledast \quad : \quad \mathscr{B} \times \mathscr{A} \xrightarrow{L} \mathscr{B} \times \mathscr{B} \xrightarrow{*} \mathscr{B} \xrightarrow{R} \mathscr{A}.$$

Note that the resulting operation

$$b \circledast a \quad = \quad R\,(\,b \ovee L(a)\,) \tag{11}$$

coincides with the functor we started from. Recall also that a lax action of a monoidal category $(\mathscr{M}, \otimes, I)$ on a category $\mathscr{X}$ is defined as a functor

$$\circledast \quad : \quad \mathscr{M} \times \mathscr{X} \longrightarrow \mathscr{X}$$

equipped with natural morphisms

$$\mu_{m,n} \;:\; m \circledast (n \circledast x) \longrightarrow (m \otimes n) \circledast x \qquad\qquad \mu_I \;:\; x \longrightarrow I \circledast x$$

satisfying the same two coherence diagrams (equation 9) as a weak action. In particular, a weak action is the same thing as a lax action whose morphisms $\mu_{m,n}$ and μ_I are invertible. We will call *parametric monad* in $\mathscr{X}$ with parameters in the monoidal category $(\mathscr{M}, \otimes, I)$ such a lax action on the category $\mathscr{X}$. The terminology is justified by the fact that every such parametric monad $\circledast$ includes a monad in $\mathscr{X}$ defined as $(I \circledast -)$ where I is the unit of the monoidal category $\mathscr{M}$. Typically, starting from the parametric continuation monad (equation 11) defined above, one recovers the continuation monad as

$$\mathbf{false} \circledast a \quad = \quad R\,(\,\mathbf{false} \ovee La\,) \quad \cong \quad R \circ L\,(a).$$

We will see moreover in Section 2 that the coherence diagrams defining a parametric monad are obtained as a direct parametrization of the usual definition of monad.

1.3. *Commutation between monads*

The notion of parametric monad is not only useful in itself: it also leads to a pleasingly symmetric way to think of the notion of tensorial strength. Given a monad T on a monoidal category $(\mathscr{C}, \otimes, I)$, recall that a tensorial strength is defined as a natural family of morphisms

$$\sigma_{A,B} \quad : \quad T(A) \otimes B \longrightarrow T(A \otimes B)$$

regulated by four coherence diagrams. These four diagrams may be organized into two independent series, each of them consisting of two coherence diagrams. The first series of diagrams describes how a single tensor product $\otimes$ interacts with the multiplication and the unit of the monad:

$$
\begin{array}{ccccc}
TT(A)\otimes B & \xrightarrow{\sigma} & T(T(A)\otimes B) & \xrightarrow{\sigma} & TT(A\otimes B) \\
\downarrow{\scriptstyle \mu} & & & & \downarrow{\scriptstyle \mu} \\
T(A)\otimes B & & \xrightarrow{\qquad\sigma\qquad} & & T(A\otimes B)
\end{array}
$$

$$
\begin{array}{ccc}
 & A\otimes B & \\
{\scriptstyle \eta}\swarrow & & \searrow{\scriptstyle \eta} \\
T(A)\otimes B & \xrightarrow{\sigma} & T(A\otimes B).
\end{array}
$$

The second series of diagrams describes how a single monad T interacts with the associativity and unit law of the tensor product:

$$
\begin{array}{ccccc}
(T(A)\otimes B)\otimes C & \xrightarrow{\sigma} & T(A\otimes B)\otimes C & \xrightarrow{\sigma} & T((A\otimes B)\otimes C) \\
\downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle \alpha} \\
T(A)\otimes (B\otimes C) & & \xrightarrow{\qquad\sigma\qquad} & & T(A\otimes (B\otimes C))
\end{array}
$$

$$
\begin{array}{ccc}
 & TA & \\
{\scriptstyle \rho}\swarrow & & \searrow{\scriptstyle \rho} \\
T(A)\otimes I & \xrightarrow{\sigma} & T(A\otimes I)
\end{array}
\qquad ,
$$

where α and ρ are the canonical isomorphism of the monoidal category. As we will see, the apparent dissymmetry between the two series of commutative diagrams hides a symmetry which appears when one thinks

— of the monad T as a parametrized $\mathbb{1}$-monad $\circledast$ on the left, with parameters taken in the trivial monoidal category $\mathbb{1}$ with a single object I and a single morphism,

— of the tensor product $\otimes$ as a parametrized $\mathscr{C}$-monad on the right, with parameters taken in the monoidal category $(\mathscr{C}, \otimes, I)$ and multiplication and unit defined as α and ρ.

Here, by parametric monad on the left, we mean a parametric monad in the usual sense, whereas by parametric $\mathscr{M}$-monad on the right, we mean a parametric $\mathscr{M}^{op(0)}$-monad where $\mathscr{M}^{op(0)}$ denotes the monoidal category $\mathscr{M}$ where the direction of the tensor product has been reversed. This symmetric point of view on tensorial strengths enables to write the tensorial strength as a distributivity law

$$\sigma_{A,B} \quad : \quad (I \circledast A) \otimes B \quad \longrightarrow \quad I \circledast (A \otimes B)$$

between the left and right parametric monads. As we will see, yet another way to think of the category $\mathscr{C}$ equipped with the three data $(\circledast, \otimes, \sigma)$ is to identify it as a

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

lax version of $(\mathbb{1}, \mathscr{C})$-biaction (or bimodule) where the equality

$$(I \circledast A) \otimes B \quad = \quad I \circledast (A \otimes B)$$

has been replaced by a natural transformation σ satisfying the four coherence diagrams recalled above.

1.4. *Plan of the paper*

We introduce the notion of parametric monad in Section 2 and establish an elementary transfer theorem for parametric monads. We construct in Section 3 the parametric continuation monad $\circledast$ of a dialogue category, and deduce from the transfer theorem that it indeed defines such a parametric monad in every dialogue chirality. We introduce in Section 4 the notion of commutator between parametric monads, and show that it generalizes the notion of tensorial strength as well as the notion of distributivity between monads. We conclude the paper in Section 5 by constructing for every dialogue chirality a double negation commutator between the parametric continuation monad and the action $\oslash$ of the category $\mathscr{A}$ over itself.

1.5. *Related works*

Since the main purpose of the paper is to design a bridge between linear logic and the theory of strong monads, the reader is probably advised to read the original papers (Girard 1987, 1995) about linear logic as well as the seminal papers on strong monads (Kock 1970; Moggi 1991) in algebra and in programming language semantics. It should be mentioned that the distributivity law (equation 1) was originally observed in Hu and Joyal (1999) and that it was then extensively studied in Blute *et al*. (1996); Cockett and Seely (1997) in their seminal work on the coherence properties of weakly distributive categories, partially reported in Melliès (2009). As the reader will see, an important part of the present paper is devoted to the idea that lax algebraic structures may be transported along adjunctions. Although the shift from weak to lax structures plays a fundamental role in our work on tensorial logic, the transfer theorem for lax algebras along adjunctions is only a slight variant of similar transfer theorems along equivalences of categories, most specifically the two-categorical account developed in Kelly and Lack (2004) starting from ideas in Bénabou (1963).

1.6. *Other parametrized notions of monad*

The notion of monad (T, μ, η) is well established today, and it is sufficiently important and primitive to be extended and parametrized in various ways, depending on the situation of interest. Let us mention in particular that two other notions of parametrized monad has been recently introduced for different purposes in Atkey (2009) and in Uustalu (2003). Despite the proximity in name, these parametric notions of monad are different, and not immediately related.

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

1.7. *Side remark*

The reader should be aware that there is an element of choice in picking the double-negation monad (equation 4) instead of the other double-negation monad

$$A \mapsto (\bot \multimapinv A) \multimap \bot \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C} \tag{12}$$

also available in any dialogue category, and defined in just the same way as equation (4) except that the order of negations has been interchanged. However, the choice of equation (4) against equation (12) does not really matter because the very notion of dialogue category $\mathscr{C}$ is invariant under the change of orientation

$$\mathscr{C} \quad \mapsto \quad \mathscr{C}^{op(0)} \tag{13}$$

of the tensor product. This change of orientation interchanges the left negation and the right negation. From this follows that the double-negation monad (equation 4) taken in the dialogue category $\mathscr{C}^{op(0)}$ coincides with the double-negation monad (equation 12) taken in the original dialogue category $\mathscr{C}$. In other words, the two choices are simply equivalent modulo (equation 13). As a matter of fact, the only important point to remember is that the strength studied in the present paper permutes the double-negation monad (equation 4) with the right action of the tensor product:

$$\sigma_{A,B} \quad : \quad \big(\bot \multimapinv (A \multimap \bot)\big) \otimes B \quad \longrightarrow \quad \bot \multimapinv \big((A \otimes B) \multimap \bot\big),$$

whereas the other strength permutes its double-negation monad (equation 12) with the left action of the tensor product:

$$A \otimes \big((\bot \multimapinv B) \multimap \bot\big) \quad \longrightarrow \quad \big(\bot \multimapinv (A \otimes B)\big) \multimap \bot.$$

2. Parametric monads

We start by recalling the formal definition of *adjunction* in a 2-category $\mathscr{W}$ introduced in Kelly and Street (1974) and then review a series of basic consequences of the definition. In particular, we establish an elementary *transfer theorem* which states that every parametric $\mathscr{J}$-monad on the $\mathscr{B}$-side of an adjunction $L \dashv R$ is transported to a parametric $\mathscr{J}$-monad on its $\mathscr{A}$-side, this for every monoidal category $\mathscr{J}$.

2.1. *Formal adjunctions*

Recall that an adjunction in a 2-category $\mathscr{W}$ consists of a pair $\mathscr{A}$, $\mathscr{B}$ of zero-dimensional cells, of a pair

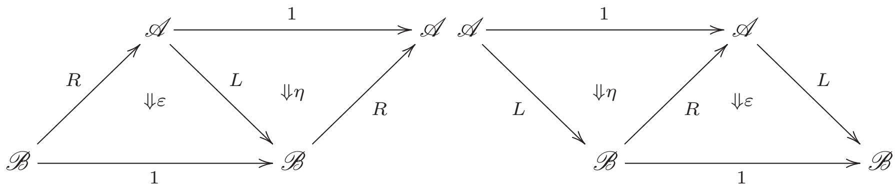
$$L : \mathscr{A} \longrightarrow \mathscr{B} \qquad\qquad R : \mathscr{B} \longrightarrow \mathscr{A}$$

of one-dimensional cells, and of a pair

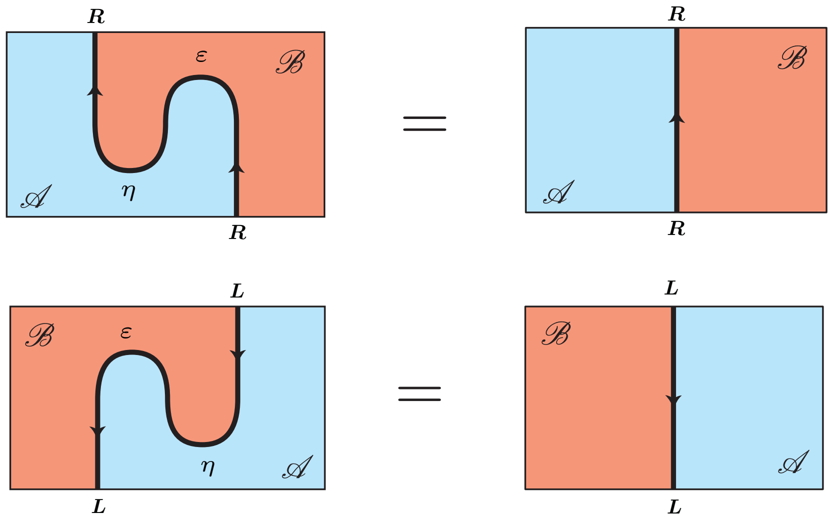
$$\eta : 1_{\mathscr{A}} \Rightarrow R \circ L \qquad\qquad \varepsilon : L \circ R \Rightarrow 1_{\mathscr{B}}$$

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

of two-dimensional cells. One requires moreover that the two-dimensional cells obtained by pasting:



coincide with the identity on the one-dimensional cells L and R, respectively. In that case, one writes $L \dashv R$ and one says that the 1-cell L is left adjoint to the 1-cell R, and conversely, that the 1-cell R is right adjoint to the 1-cell L. These equations may be depicted in string diagrams in the following way, with the 0-cell $\mathscr{A}$ coloured blue (or light grey) and the 0-cell $\mathscr{B}$ coloured red (or dark grey).



By convention, the black string representing the functors L and R is oriented downwards when it depicts the functor L and upwards when it depicts the functor R. This specific orientation is justified by the connection with dialogue games exhibited in our companion paper (Melliès 2012) where the functor R corresponds to the Opponent moves of a dialogue game, and the functor L corresponds to the Player moves. In that case, the orientation of L and R reflects the flow of information and control in the proof.

2.2. *Formal monads*

A monad in a 2-category $\mathscr{W}$ is defined as a 0-cell $\mathscr{A}$ together with a 1-cell

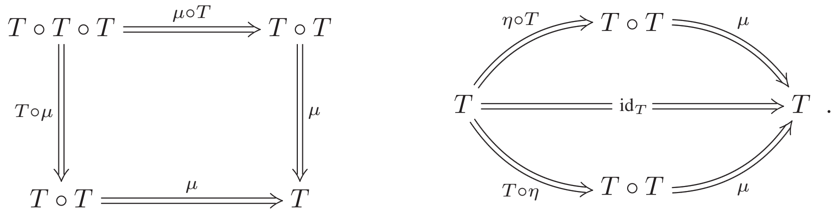
$$T \quad : \quad \mathscr{A} \longrightarrow \mathscr{A}$$

together with a pair of 2-cells

$$\eta \;:\; 1_{\mathscr{A}} \longrightarrow T \qquad\qquad \mu \;:\; T \circ T \longrightarrow T$$

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

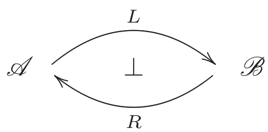
making the two diagrams below commute:



A comonad in a 2-category $\mathscr{W}$ is defined as a monad in the 2-category $\mathscr{W}^{op^{(2)}}$ obtained by reversing the orientation of the 2-cells in the 2-category $\mathscr{W}$.

2.3. *The external adjunction*

Suppose given a formal adjunction



(14)

in a 2-category $\mathscr{W}$. It is well known that every such adjunction induces a monad $R \circ L$ on the 0-cell $\mathscr{A}$ and a comonad $L \circ R$ on the 0-cell $\mathscr{B}$. Less known is the fact that this monad is part of a much broader structure, originally noticed by Jean Bénabou, see Bénabou (1963) for details, as well as the more recent account in Kelly and Lack (2004). We describe this broader structure now. Let

$$\mathrm{End}(\mathscr{A}) \quad = \quad \mathscr{W}(\mathscr{A}, \mathscr{A})$$

denote the hom-category of the 0-cells $\mathscr{A}$

— with objects the 1-cells from the 0-cell $\mathscr{A}$ to itself,

— with morphisms the 2-cells between these 1-cells.

The category $\mathrm{End}(\mathscr{A})$ is strict monoidal, with composition $\circ$ as tensor product, and with the identity 1-cell $1_{\mathscr{A}}$ as tensor unit. Note that a monoid in this category $\mathrm{End}(\mathscr{A})$ is the same thing as a monad in $\mathscr{W}$ on the 0-cell $\mathscr{A}$. Similarly, a comonoid in the category $\mathrm{End}(\mathscr{B})$ is the same thing as a comonad in $\mathscr{W}$ on the 0-cell $\mathscr{B}$. Now, the main observation is that the two 1-cells L and R induce in turn two functors

$$\begin{array}{rcccc} [L,R] & : & \mathrm{End}(\mathscr{A}) & \rightarrow & \mathrm{End}(\mathscr{B}) \\ & & F & \mapsto & L \circ F \circ R \end{array}$$

$$\begin{array}{rcccc} [R,L] & : & \mathrm{End}(\mathscr{B}) & \rightarrow & \mathrm{End}(\mathscr{A}) \\ & & G & \mapsto & R \circ G \circ L \end{array}$$

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

defined by pre- and postcomposition. The two functors are moreover involved in an adjunction

$$\mathrm{End}(\mathscr{A}) \underset{[R,L]}{\overset{[L,R]}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{End}(\mathscr{B}) \tag{15}$$

between the hom-categories. This adjunction is called the *external adjunction* associated to the formal adjunction $L \dashv R$. By external, one means that it is an adjunction between categories $\mathrm{End}(\mathscr{A})$ and $\mathrm{End}(\mathscr{B})$ reflecting to the outside the formal adjunction $L \dashv R$ living inside the 2-category $\mathscr{W}$. The unit and counit of the external adjunction are defined in the expected way:

$$[\eta]_F = \eta \circ F \circ \eta{:}F \Rightarrow R \circ L \circ F \circ R \circ L,$$
$$[\epsilon]_G = \varepsilon \circ G \circ \varepsilon{:}L \circ R \circ G \circ L \circ R \Rightarrow G.$$

Looking at it more closely, the adjunction simply says that there exists a one-to-one correspondence between the 2-cells

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{F} & \mathscr{A} \\ {\scriptstyle R}\uparrow & \Downarrow & \downarrow{\scriptstyle L} \\ \mathscr{B} & \xrightarrow[G]{} & \mathscr{B} \end{array}$$

and the 2-cells

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{F} & \mathscr{A} \\ {\scriptstyle L}\downarrow & \Downarrow & \uparrow{\scriptstyle R} \\ \mathscr{B} & \xrightarrow[G]{} & \mathscr{B} \end{array}$$

in the 2-category $\mathscr{W}$, and that this correspondence is natural wrt. the action on F in $\mathrm{End}(\mathscr{A})$ and wrt. the action on G in $\mathrm{End}(\mathscr{B})$. The adjunction may be also seen as an avatar of what Kelly and Street like to call 'mate 2-cells' in their two-categorical theory of adjunctions, see Kelly and Street (1974) for details.

2.4. *Lax monoidal functors*

Recall that a lax monoidal functor between monoidal categories $(\mathscr{M}, \otimes, I)$ and $(\mathscr{N}, \otimes, I)$ is defined as a functor

$$F \quad : \quad \mathscr{M} \quad \longrightarrow \quad \mathscr{N}$$

equipped with two natural transformations

$$m_{A,B} \quad : \quad FA \otimes FB \quad \longrightarrow \quad F(A \otimes B) \qquad\qquad m_I \quad : \quad I \quad \longrightarrow \quad F(I)$$

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

making the three diagrams below commute:

$$
\begin{array}{ccccc}
(FA \otimes FB) \otimes FC & \xrightarrow{m} & F(A \otimes B) \otimes FC & \xrightarrow{m} & F((A \otimes B) \otimes C) \\
\downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle \alpha} \\
FA \otimes (FB \otimes FC) & \xrightarrow{m} & FA \otimes F(B \otimes C) & \xrightarrow{m} & F(A \otimes (B \otimes C))
\end{array}
\tag{16}
$$

$$
\begin{array}{ccc}
 & FA & \\
{\scriptstyle m_I} \swarrow & & \searrow {\scriptstyle \lambda} \\
FI \otimes FA & \xrightarrow{m} & F(I \otimes A)
\end{array}
\qquad
\begin{array}{ccc}
 & FA & \\
{\scriptstyle m_I} \swarrow & & \searrow {\scriptstyle \rho} \\
FA \otimes FI & \xrightarrow{m} & F(A \otimes I)
\end{array}
\tag{17}
$$

for all objects A, B, C of the category $\mathscr{M}$, where α, λ and ρ denote the canonical morphisms of the monoidal categories. An important property of lax monoidal functors is that they compose, and in fact define a 2-category with

— monoidal categories as 0-cells,
— lax monoidal functors as 1-cells,
— monoidal natural transformations as 2-cells.

Although we will not really use this notion in the paper, we find useful to recall that a monoidal natural transformation

$$\theta \quad : \quad (F, m) \quad \Rightarrow \quad (G, m) \quad : \quad (\mathscr{M}, \otimes, I) \quad \longrightarrow \quad (\mathscr{N}, \otimes, I)$$

between lax monoidal functors (F, m) and (G, n) is defined as a natural transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad \mathscr{M} \quad \longrightarrow \quad \mathscr{N}$$

between the underlying functors, making the diagrams commute

$$
\begin{array}{ccc}
F(A) \otimes F(B) & \xrightarrow{\theta \otimes \theta} & G(A) \otimes G(B) \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\
F(A \otimes B) & \xrightarrow{\theta} & G(A \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
 & I & \\
{\scriptstyle m_I} \swarrow & & \searrow {\scriptstyle n_I} \\
F(I) & \xrightarrow{\theta} & G(I)
\end{array}
\quad .
$$

A lax monoidal functor is called weak when the coercions m_I and $m_{A,B}$ are isomorphisms for all objects A, B of the category $\mathscr{C}$.

2.5. *Parametric monads*

The notion of lax monoidal functor was introduced by Jean Bénabou, who was guided by his important observation that a monoid in a monoidal category $(\mathscr{M}, \otimes, I)$ is the same thing as a lax monoidal functor

$$\mathbb{1} \quad \longrightarrow \quad (\mathscr{M}, \otimes, I)$$

from the monoidal category $\mathbb{1}$ with a single object and a single morphism. This specific formulation of monoids provides a nice conceptual explanation for the fact that every lax monoidal functor transports monoids to monoids.

Now, it is not difficult to see that a formal monad (T, μ, η) on a 0-cell $\mathscr{A}$ of a 2-category $\mathscr{W}$ is the same thing as a monoid in the monoidal category $\mathrm{End}(\mathscr{A}) = \mathscr{W}(\mathscr{A}, \mathscr{A})$. From this follows that a formal monad is the same thing as a lax monoidal functor

$$\mathbb{1} \quad \longrightarrow \quad \mathrm{End}(\mathscr{A}).$$

The discussion justifies considering a parametric notion of monad, parametrized by a monoidal category $(\mathscr{J}, \otimes, \boldsymbol{e})$ in the following way.

Definition 1 (parametric monad). A parametric $\mathscr{J}$-monad on a 0-cell $\mathscr{A}$ of a 2-category $\mathscr{W}$ is defined as a lax monoidal functor

$$(T, \mu) \quad : \quad \mathscr{J} \quad \longrightarrow \quad \mathrm{End}(\mathscr{A}).$$

The monoidal category $\mathscr{J}$ is called the parameter category of the $\mathscr{J}$-monad; and an object j of the category $\mathscr{J}$ is called a parameter.

The definition is a straightforward application of Bénabou's ideas and we do not claim any originality for it. It is worth mentioning here that in the case of $\mathscr{W} = \boldsymbol{Cat}$, a parametric monad is the same thing as a lax action of the monoidal category $\mathscr{J}$ on the category $\mathscr{A}$. By lax action, one simply means a lax algebra of the 2-monad

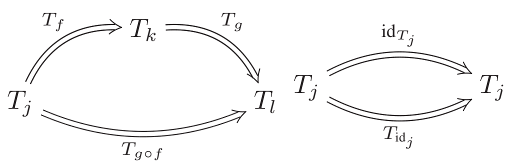
$$\mathscr{J} \times - \quad : \quad \boldsymbol{Cat} \quad \longrightarrow \quad \boldsymbol{Cat}$$

on the 2-category of categories. Note also that we use for convenience the greek letter μ rather than the latin letter m in order to denote the coercion maps of the lax monoidal functor T.

At this point, it seems reasonable to give an equivalent and fully explicit description of the notion of parametric monad. A parametric $\mathscr{J}$-monad (T, μ) consists of

— a 1-cell $T_j : \mathscr{A} \longrightarrow \mathscr{A}$ for every parameter j and a 2-cell $T_f : T_j \Rightarrow T_k$ for every morphism $f : j \longrightarrow k$ between such parameters,
— a 2-cell $\mu_{\boldsymbol{e}} : 1_{\mathscr{A}} \Rightarrow T_{\boldsymbol{e}}$ called the unit of the parametric monad,
— a 2-cell $\mu_{j,k} : T_j \circ T_k \Rightarrow T_{j \otimes k}$ called the (j, k)-component of the multiplication of the parametric monad, for every pair of parameters j and k.

These data are moreover required to make a series of coherence diagrams commute in the category $\mathrm{End}(\mathscr{A})$. First, the diagrams



which express the functoriality of T. Then, the diagrams

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

$$\begin{array}{ccc} T_j \circ T_k & \xRightarrow{T_f \circ T_g} & T_{j'} \circ T_{k'} \\ {\scriptstyle \mu_{j,k}} \Downarrow & & \Downarrow {\scriptstyle \mu_{j',k'}} \\ T_{j \otimes k} & \xRightarrow{T_{f \otimes g}} & T_{j' \otimes k'} \end{array}$$

which express the naturality of μ. Finally, the diagrams

$$\begin{array}{ccccc} T_j \circ T_k \circ T_l & & \xRightarrow{\mu_{j,k} \circ T_l} & & T_{j \otimes k} \circ T_l \\ {\scriptstyle T_j \circ \mu_{k,l}} \Downarrow & & & & \Downarrow {\scriptstyle \mu_{j \otimes k, l}} \\ T_j \circ T_{k \otimes l} & \xRightarrow{\mu_{j, k \otimes l}} & T_{j \otimes (k \otimes l)} & \xRightarrow{\alpha} & T_{(j \otimes k) \otimes l} \end{array}$$

and

$$\begin{array}{ccccc} & \overset{\mu_{\boldsymbol{e}} \circ T_j}{\nearrow} & T_{\boldsymbol{e}} \circ T_j & \overset{\mu_{\boldsymbol{e},j}}{\searrow} & \\ T_j & & \xRightarrow{\mathrm{id}_{T_j}} & & T_j \\ & \underset{T_j \circ \mu_{\boldsymbol{e}}}{\searrow} & T_j \circ T_{\boldsymbol{e}} & \underset{\mu_{j,\boldsymbol{e}}}{\nearrow} & \end{array}$$

which express the monoidality of μ. These diagrams should commute for all indices j, j', k, k', l and all morphisms f, g, h of the parameter category $\mathscr{J}$.

Remark. Note that every parametric $\mathscr{J}$-monad T comes equipped with a morphism $A \to T_{\boldsymbol{e}}\, A$ where $\boldsymbol{e}$ is the unit of the monoidal category $\mathscr{J}$. On the other hand, the reader should be careful that is (at least in general) no morphism $A \to T_j A$ for an object j different of the unit $\boldsymbol{e}$ in the category $\mathscr{J}$.

2.5.1. *Parametric comonads.* There is also a notion of parametric comonad (K, δ) indexed by a monoidal category $(\mathscr{J}, \otimes, I)$ in a 2-category $\mathscr{W}$ defined by duality as a parametric $\mathscr{J}^{op(1)}$-monad in the 2-category $\mathscr{W}^{op(2)}$. Here, the category $\mathscr{J}^{op(1)}$ is obtained by reversing the orientation of the morphisms of the category $\mathscr{J}^{op(1)}$ and the 2-category $\mathscr{W}^{op(2)}$ by reversing the orientation of the 2-cells. Note in particular that

$$\mathscr{W}^{op(2)}(\mathscr{A}, \mathscr{A}) \quad = \quad \mathscr{W}(\mathscr{A}, \mathscr{A})^{op(1)}$$

and thus that a parametric $\mathscr{J}$-comonad is the same thing as an oplax (rather than lax) monoidal functor

$$(K, \delta) \quad : \quad \mathscr{J} \quad \longrightarrow \quad \mathrm{End}(\mathscr{A}) = \mathscr{W}(\mathscr{A}, \mathscr{A})$$

with the expected notion of oplax monoidal functor between monoidal categories.

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

2.6. *The transfer theorem*

At this point, we are ready to establish our transfer theorem for parametric monads. To that purpose, we start by considering a formal adjunction (equation 14) in a 2-category $\mathscr{W}$, together with the external adjunction (equation 15) resulting from it. The transfer theorem is based on the key observation that

Proposition 1. The right adjoint functor

$$[R, L] \quad : \quad \mathrm{End}(\mathscr{B}) \quad \longrightarrow \quad \mathrm{End}(\mathscr{A})$$

defines a lax monoidal functor.

In order to establish the property, we need to define a 2-cell

$$m_1 \quad : \quad 1_{\mathscr{A}} \quad \Rightarrow \quad R \circ 1_{\mathscr{B}} \circ L$$

as well as a family of 2-cells

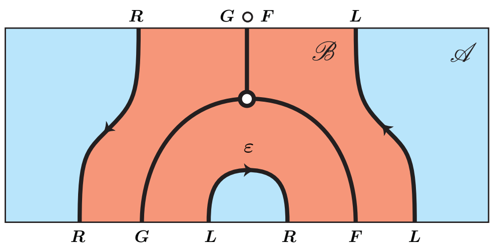
$$m_{G,F} \quad : \quad (R \circ G \circ L) \circ (R \circ F \circ L) \quad \Rightarrow \quad R \circ (G \circ F) \circ L$$

indexed by the 1-cells

$$F, G \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{B}$$

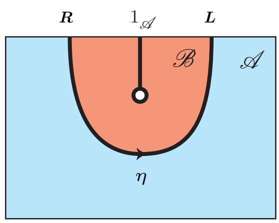
and making a series of coherence diagrams commute in the category $\mathrm{End}(\mathscr{A})$. The 2-cells m_1 and $m_{G,F}$ are defined in the expected way, using the unit η and the counit ε of the formal adjunction $L \dashv R$, respectively. The construction may be depicted in the language of string diagrams. The 2-cell $m_{G,F}$ is depicted as

$m_{G,F}=$



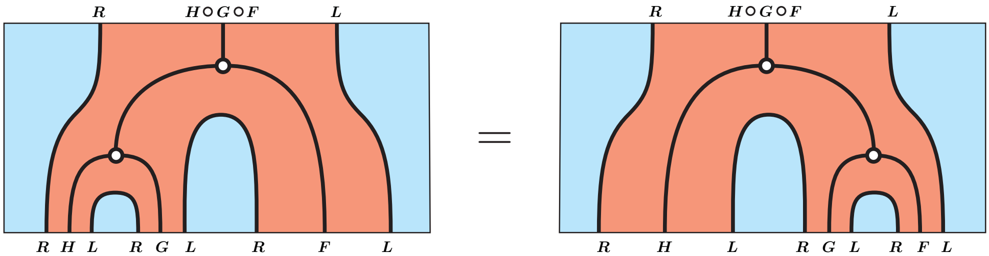
while the 2-cell m_I is depicted as

$m_I=$

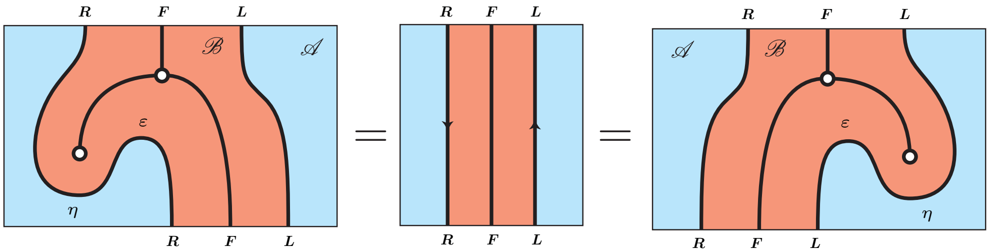


The proof that the coherence diagrams required of a lax monoidal functor commute works exactly in the same way as the proof that the endofunctor $R \circ L$ defines a monoid in the category $\mathrm{End}(\mathscr{A})$. The first coherence diagram is reflected by the pictorial equality

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press



and the second coherence diagrams involving the unit m_I are depicted as the diagrammatic equalities below:



At this point, the transfer theorem below follows from Proposition (1) and the fact that lax monoidal functors compose.

Proposition 2 (transfer theorem). Every parametric $\mathscr{J}$-monad (T, μ) in the 0-cell $\mathscr{B}$ induces a parametric $\mathscr{J}$-monad in the 0-cell $\mathscr{A}$.

The parametric $\mathscr{J}$-monad on the 0-cell $\mathscr{A}$ is simply defined by composing the two lax monoidal functors

$$\mathscr{J} \xrightarrow{\quad T \quad} \mathrm{End}(\mathscr{B}) \xrightarrow{\quad [R,L] \quad} \mathrm{End}(\mathscr{A}).$$

The composite functor is lax monoidal and thus defines a parametric $\mathscr{J}$-monad on the 0-cell $\mathscr{A}$. This parametric monad is called the *transferred* parametric monad. Observe that the fact that every formal adjunction $L \dashv R$ defines a formal monad on the 0-cell $\mathscr{A}$ may be seen as a consequence of the transfer theorem. Indeed, the monad on $\mathscr{A}$ is obtained by transferring the identity monad on $\mathscr{B}$ along the adjunction.

3. The parametric continuation monad

As explained in the introduction, the purpose of the transfer theorem is to shed light on the algebraic structure of the continuation monad defined in any dialogue category $(\mathscr{C}, \otimes, I)$. The key idea to deduce the operation (11) defined as

$$b \circledast a \quad = \quad R\,(\,b \otimes L(a)\,)$$

from the action $* = \otimes$ of the monoidal category $(\mathscr{B}, \otimes, \mathbf{false})$ on itself. Here, it is worth recalling that $(\mathscr{B}, \otimes, \mathbf{false})$ is just another name for the opposite $\mathscr{C}^{op(0,1)}$ of the

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

original dialogue category $\mathscr{C}$. So, if we think of $\mathscr{A} = \mathscr{C}$ as a category of formulas and *proofs*, it is natural to think of $\mathscr{B}$ as a category of formulas and *refutations*. Accordingly, if we think of the tensor product $\otimes = \owedge$ as a *conjunction* in the original dialogue category $\mathscr{A}$, it is natural to think of the tensor product $\otimes^{op(0)} = \ovee$ of the category $\mathscr{B}$ as a *disjunction*.

The important point is that the action of the monoidal category $\mathscr{B}$ on itself may be seen as a parametric $\mathscr{B}$-monad S on the category $\mathscr{B}$, defined by the family of functors

$$S_b \quad : \quad b' \quad \mapsto \quad b \ovee b' \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{B}, \tag{18}$$

where the 2-category $\mathscr{W}$ is taken in that case equal to the 2-category ***Cat*** of categories, functors and natural transformations. Alternatively, the parametric monad S may be formulated as the weak monoidal functor

$$S \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathrm{End}(\mathscr{B})$$

obtained by currifying the tensor product

$$\ovee \quad : \quad \mathscr{B} \times \mathscr{B} \quad \longrightarrow \quad \mathscr{B}.$$

At this point, the transfer theorem established in Proposition 2 enables us to conclude that:

Proposition 3. In every dialogue chirality, the functor

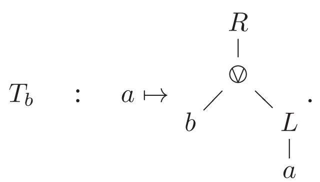
$$\circledast \quad : \quad \mathscr{B} \times \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

defines a parametric monad

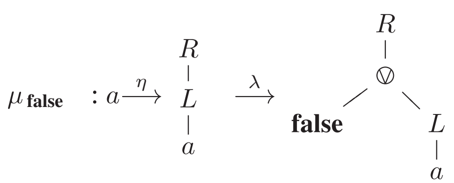
$$T_b \quad : \quad a \quad \mapsto \quad b \circledast a = R\,(\,b \ovee L(a)\,)$$

on the category $\mathscr{A}$, parametrized by the monoidal category $(\mathscr{B}, \ovee, \mathbf{false})$.

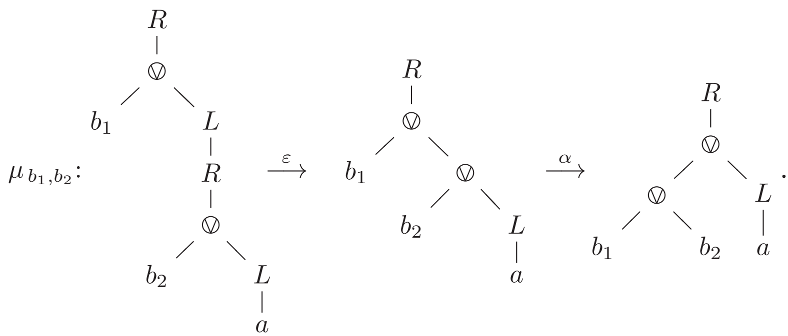
For convenience and readability, we like to write the functor T_b using the following tree notation:



Using this notation, the natural transformations $\mu_{\mathbf{false}}$ and μ_{b_1,b_2} which equip the parametric monad (T, μ) are defined as follows:



https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press



Although this may be easily checked directly, the fact that (T, μ) satisfies the coherence properties of a parametric monad is a consequence of our general transfer theorem, applied to the parametric monad S defined in equation (18) and to the adjunction $L \dashv R$.

Remark. The construction of the parametric monad T is not specific to dialogue categories. In particular, it would work for any monoidal category $\mathscr{A} = \mathscr{C}$ equipped with an adjunction $L \dashv R$ with its opposite category $\mathscr{B} = \mathscr{C}^{op(0,1)}$. Even more generally, for any category $\mathscr{A}$ equipped with an adjunction $L \dashv R$ with a monoidal category $(\mathscr{B}, \varovee, \mathbf{false})$.

4. Commutators between parametric monads

At this point, we are ready to introduce the notion of *commutator* between parametric monads, and to establish at the same time that every dialogue chirality is equipped with such a structure. As we will see, the notion of commutator unifies and generalizes the celebrated notions of *tensorial strength* on the one hand, and of *distributivity law* between two monads on the other hand.

4.1. *definitiona*

We suppose given a 0-cell $\mathscr{C}$ in a 2-category $\mathscr{W}$ equipped with a parametric $\mathscr{J}$-monad

$$T = \bullet \quad : \quad \mathscr{J} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

and a parametric $\mathscr{M}^{op(0)}$-monad

$$S = \circ \quad : \quad \mathscr{M}^{op(0)} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

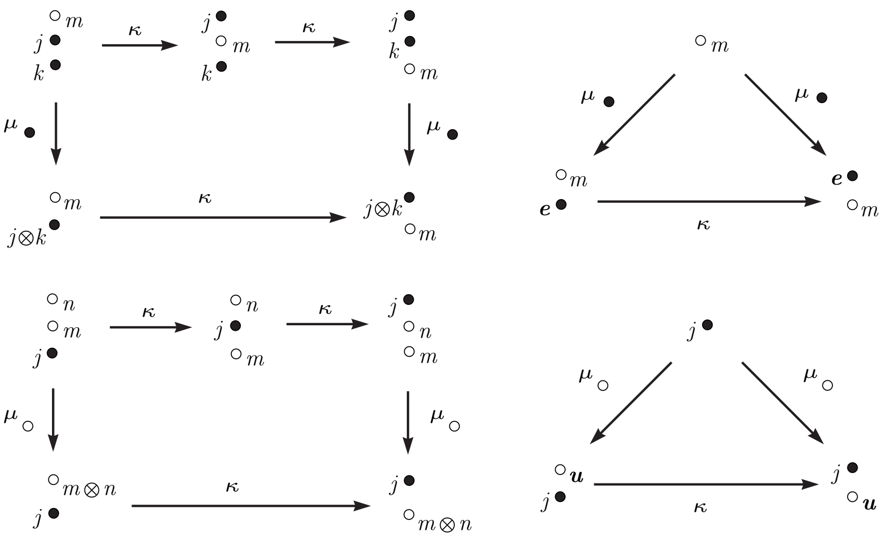
with parameters taken in the monoidal categories $(\mathscr{J}, \otimes, \boldsymbol{e})$ and $(\mathscr{M}, \otimes, \boldsymbol{u})$.

Definition 2 (commutator). A commutator between two parametric monads $T = \bullet$ and $S = \circ$ is defined as a natural transformation

$$\kappa \quad : \quad S\,T \quad \Rightarrow \quad T\,S \quad : \quad \mathscr{J} \times \mathscr{M}^{op(0)} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

making the four diagrams below commute

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press



for all objects i, j of the category $\mathscr{J}$ and all objects m, n of the category $\mathscr{M}$.

Remark. In the particular case when $\mathscr{W} = \boldsymbol{Cat}$, a commutator may be alternatively formulated as a natural transformation

$$\kappa \quad : \quad (- \bullet -) \circ - \quad \Rightarrow \quad - \bullet (- \circ -) \quad : \quad \mathscr{J} \times \mathscr{C} \times \mathscr{M} \quad \longrightarrow \quad \mathscr{C}$$

with components

$$\kappa_{j,m,A} \quad : \quad (j \bullet A) \circ m \quad \longrightarrow \quad j \bullet (A \circ m)$$

parametrized by the objects j of the category $\mathscr{J}$, m of the category $\mathscr{M}$ and A of the category $\mathscr{C}$.

Remark. The question of extending Beck's theorem (Beck 1969) from distributivity laws between monads to general commutators between parametric monads is interesting, but outside the scope of this paper, and we thus prefer to leave it for later work. Let us simply observe at this stage that the existence of a commutator between a left $\mathscr{J}$-monad S and a right $\mathscr{M}$-monad T enables one to construct a $\mathscr{J} \times \mathscr{M}^{op(0)}$-monad noted $T \circ S$ on the 0-cell $\mathscr{A}$ on which the two monads S and T act in the 2-category $\mathscr{W}$. The parametric monad $T \circ S$ is defined as the family of 1-cells

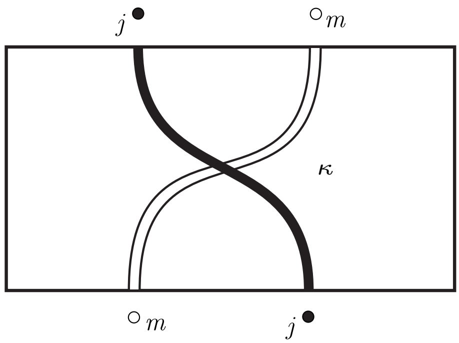
$$(T \circ S)_{(j,m)} \quad = \quad T_m \circ S_j$$

parametrized by the objects (j, m) of the category $\mathscr{J} \times \mathscr{M}^{op(0)}$. The commutator between S and T is used in the definition of the multiplicative structure μ of the parametric monad $T \circ S$. This observation justifies to think of a commutator as a lax notion of bimodule (or biaction).

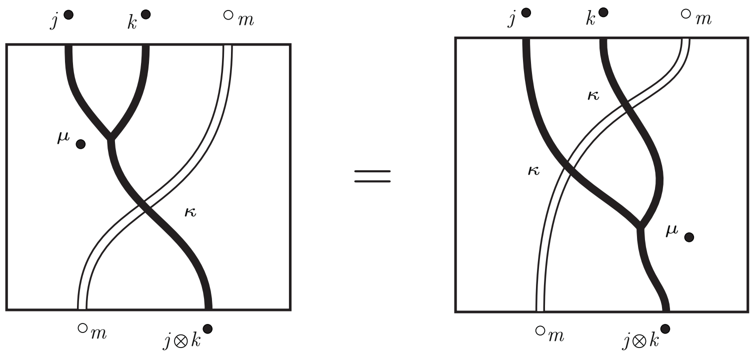
https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

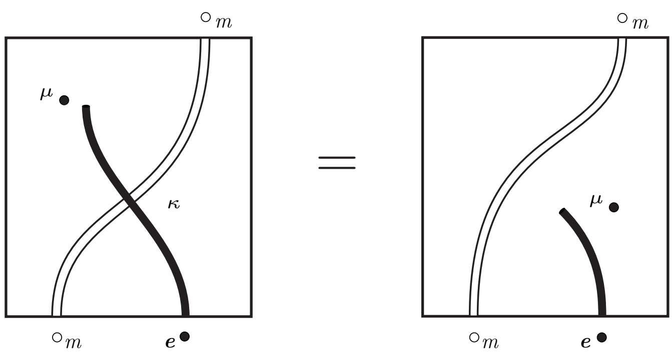
4.2. *Commutators in string diagrams*

The notion of commutator may be depicted in string diagrams as follows. The basic idea is to depict the commutator κ itself as a braiding



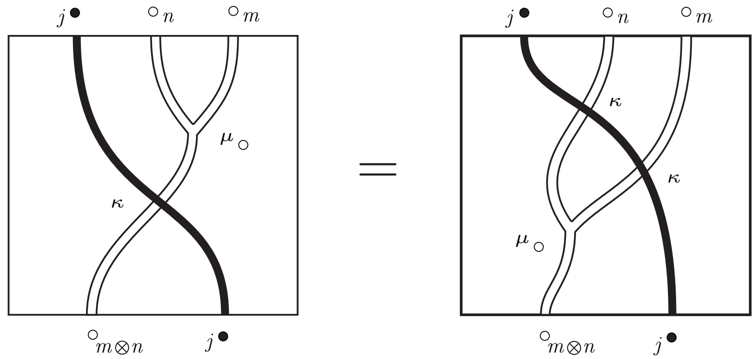
commuting the string representing the action $\bullet$ over the string representing $\circ$. This notation enables to depict the coherence diagrams of the commutator κ as a series of topologically intuitive equations, permuting the multiplication and unit of each parametric monad under or over the string representing the other parametric monad. Typically, the first series of equations in the definition of a commutator 'permutes' the operations $\mu_\bullet$ over the string representing the action $\circ$

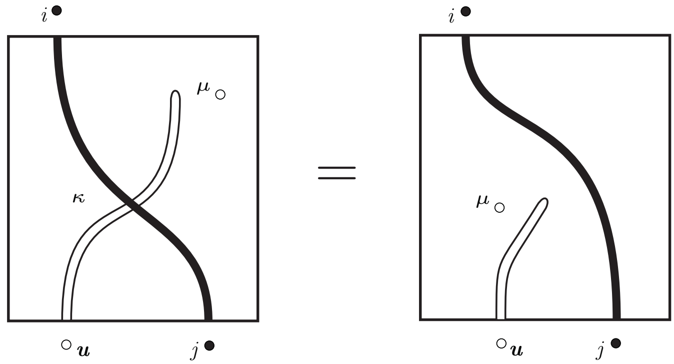




while the second series of equations 'permutes' the operations $\mu_\circ$ under the string representing the action $\bullet$

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press





Remark. The notion of commutator may be easily adapted to the case of a parametric comonad commuting with a parametric comonad, or of a parametric monad commuting with a parametric comonad, with their associated string diagrams.

4.3. *Illustrations*

We show that the notion of commutator is sufficiently general to recover two well known and apparently disjoint notions of commutation with a monad. The first example is provided by the notion of tensorial strength recalled in the introduction. It is essentially immediate that

Proposition 4. A tensorial strength

$$\sigma_{A,B} \quad : \quad T(A) \otimes B \quad \longrightarrow \quad T(A \otimes B)$$

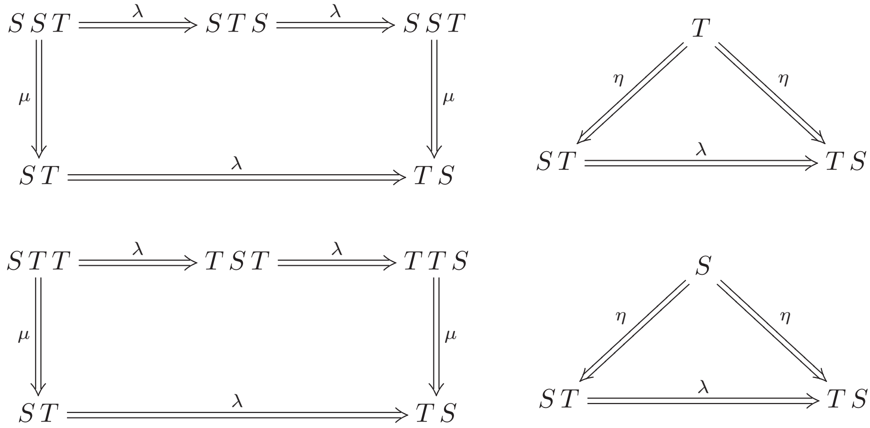
is the same thing as a commutator between a monad T and the action of the monoidal category $(\mathscr{C}, \otimes, I)$ over itself.

The parametrization is given in that case by $\mathscr{J} = \mathbb{1}$ and $\mathscr{M} = \mathscr{C}$. The second example is provided by the notion of a distributivity law between two monads S and T. Recall that such a distributivity law is defined as a natural transformation

$$\lambda \quad : \quad S\,T \quad \Rightarrow \quad T\,S$$

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

making the four coherence diagrams



commute. Once again, it is essentially straightforward that

Proposition 5. A distributivity law between two monads S and T is the same thing a commutator between them.

The parametrization is given in that case by $\mathscr{J} = \mathbb{1}$ and $\mathscr{M} = \mathbb{1}$.

5. The double negation commutator

In this final section, we conclude the paper and show that in every dialogue chirality $(\mathscr{A}, \mathscr{B})$, the strength

$$\sigma_{a_1,a_2} \quad : \quad RL(a_1) \oslash a_2 \quad \longrightarrow \quad RL(a_1 \oslash a_2)$$

of the continuation monad $T = R \circ L$ is the emerged fragment of a much wider structure, provided by a commutator

$$\kappa_{\,b,a_1,a_2} \quad : \quad (b \circledast a_1) \oslash a_2 \quad \longrightarrow \quad b \circledast (a_1 \oslash a_2)$$

between the parametric continuation monad $T = \circledast$ and the action $S = \oslash$ of the monoidal category $(\mathscr{A}, \oslash, \mathbf{true})$ over itself. Quite obviously, the strength σ_{a_1,a_2} is recovered by instantiating the commutator κ_{b,a_1,a_2} at the specific instance $b = \mathbf{false}$. In order to construct the commutator κ in every dialogue chirality, we find convenient to introduce first the notion of transjunction which provides a pleasant and illuminating shortcut to the construction.

5.1. *Formal transjunctions*

The notion of formal transjunction in a 2-category $\mathscr{W}$ refines the notion of formal adjunction recalled in Section 2.1 to a situation where the 0-cells $\mathscr{A}$ and $\mathscr{B}$ are themselves replaced by formal adjunctions.

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

Definition 3 (transjunction). Suppose given a pair of formal adjunctions

$$\mathscr{A}_1 \underset{R_1}{\overset{L_1}{\rightleftarrows}} \mathscr{B}_1 \quad (\bot) \qquad\qquad \mathscr{A}_2 \underset{R_2}{\overset{L_2}{\rightleftarrows}} \mathscr{B}_2 \quad (\bot)$$

whose units and counits are denoted η_1, η_2 and $\varepsilon_1, \varepsilon_2$ respectively. A formal transjunction $F \dashv\!\!\!\!\! - \, G$ between a pair of 1-cells

$$F \;:\; \mathscr{A}_1 \;\to\; \mathscr{A}_2 \qquad\qquad G \;:\; \mathscr{B}_2 \;\to\; \mathscr{B}_1$$

across the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is defined as a pair of natural transformations

$$\textbf{axiom} : L_1 \Rightarrow G \circ L_2 \circ F \qquad\qquad \textbf{cut} : F \circ R_1 \circ G \Rightarrow R_2$$

making the two diagrams

$$\begin{array}{ccc} F \circ R_1 \circ L_1 & \xrightarrow{\ \textbf{axiom}\ } & F \circ R_1 \circ G \circ L_2 \circ F \\ \eta_1 \Uparrow & (a) & \Downarrow \textbf{cut} \\ F & \xrightarrow{\ \eta_2\ } & R_2 \circ L_2 \circ F \end{array} \qquad \begin{array}{ccc} G \circ L_2 \circ F \circ R_1 \circ G & \xrightarrow{\ \textbf{cut}\ } & G \circ L_2 \circ R_2 \\ \textbf{axiom} \Uparrow & (b) & \Downarrow \varepsilon_2 \\ L_1 \circ R_1 \circ G & \xrightarrow{\ \varepsilon_1\ } & G \end{array}$$

commute.

The notion of transjunction is ultimately justified by the following observation, which holds in every 2-category $\mathscr{W}$.

Proposition 6. A transjunction $F \dashv\!\!\!\!\! - \, G$ accross the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is the same thing as a formal adjunction $L_2 \circ F \dashv R_1 \circ G$.

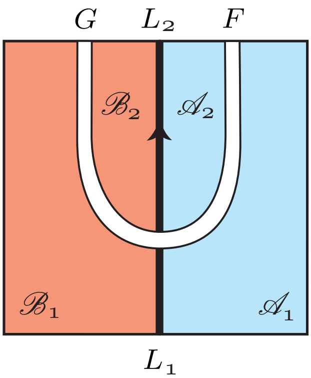
5.1.1. *Side remark.* Given a pair of 1-cells $F : \mathscr{C}_1 \to \mathscr{C}_2$ and $G : \mathscr{C}_2 \to \mathscr{C}_1$ of the 2-category $\mathscr{W}$, a formal adjunction $F \dashv G$ is the same thing as a formal transjunction $F \dashv\!\!\!\!\! - \, G$ accross the identity 1-cells $L_1 = R_1 = id_{\mathscr{C}_1}$ and $L_2 = R_2 = id_{\mathscr{C}_2}$ where $\mathscr{A}_1 = \mathscr{B}_1 = \mathscr{C}_1$ and $\mathscr{A}_2 = \mathscr{B}_2 = \mathscr{C}_2$.

5.2. *Transjunctions in string diagrams*

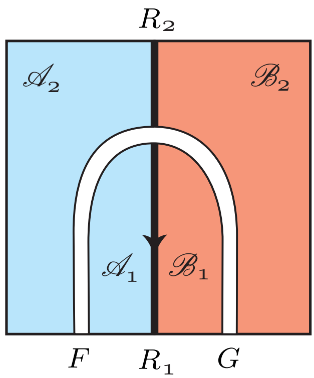
These various equations between 2-cells may be alternatively depicted as string diagrams living in the ambient 2-category $\mathscr{W}$. First of all, the generators **axiom** and **cut** of a transjunction $F \dashv\!\!\!\!\! - \, G$ accross the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ are depicted as

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

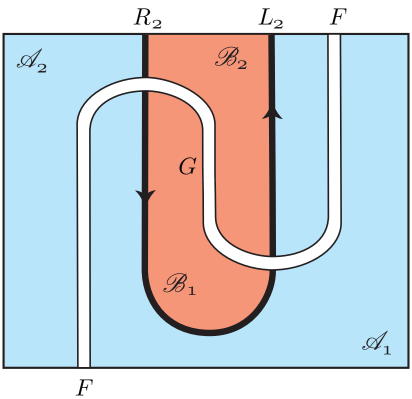
axiom =



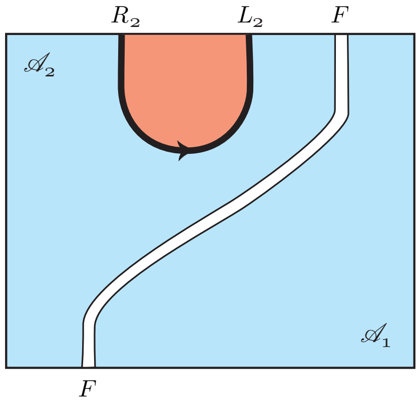
cut =

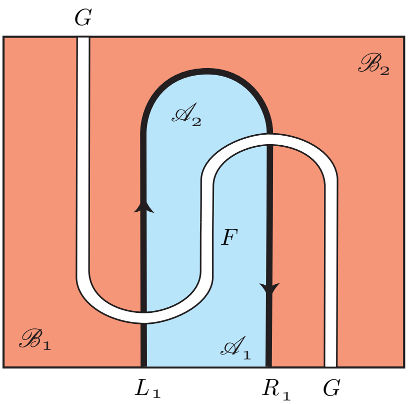


Then, the two coherence equations (a) and (b) of Definition 3 are depicted as follows:

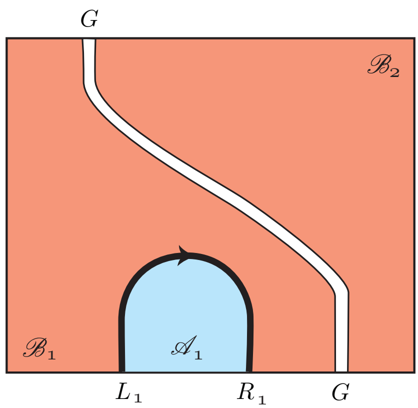


$\overset{(a)}{=}$





$\overset{(b)}{=}$



Note that one recovers in this way a pair of equations akin to the cut-axiom rule of proof-nets in linear logic. One main difference with linear logic is that the background in tensorial logic (and in transjunctions) is polychromic rather than monochromic – with the 0-cells $\mathscr{A}_i$ in blue (or light grey) and $\mathscr{B}_i$ in red (or dark grey) separated by the oriented boundary defined by the 1-cells L_i and R_i for $i = 1, 2$.

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

5.3. *Transjunction homomorphism*

It is also useful to consider a notion of homorphism between transjunctions, which gives rise to a category of transjunctions.

Definition 4 (homomorphism). A homomorphism

$$(f, g) \quad : \quad F \mathrel{-\!\!\!3} G \quad \longrightarrow \quad F' \mathrel{-\!\!\!3} G'$$

between two transjunctions $F \mathrel{-\!\!\!3} G$ and $F' \mathrel{-\!\!\!3} G'$ accross the same adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is defined as a pair of natural transformations

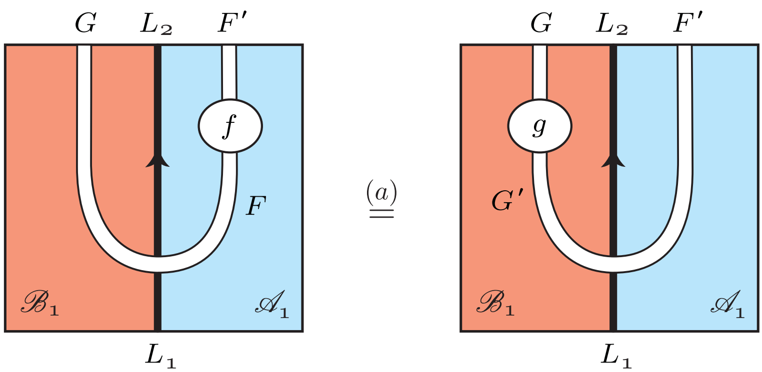
$$f \quad : \quad F \quad \Rightarrow \quad F' \qquad\qquad g \quad : \quad G' \quad \Rightarrow \quad G$$

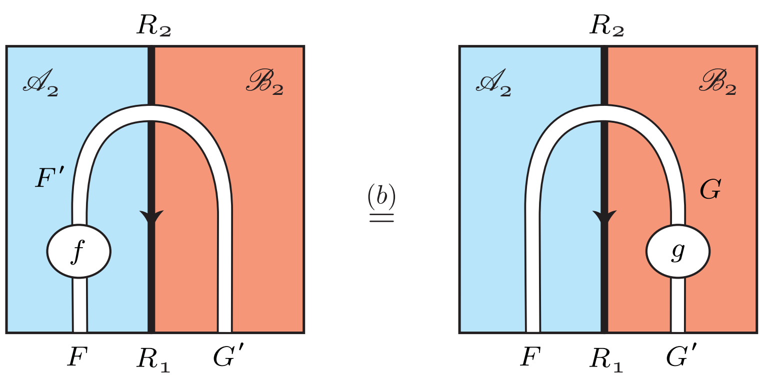
making the two diagrams

$$\begin{array}{ccc} G \circ L_2 \circ F & \xRightarrow{\;f\;} & G \circ L_2 \circ F' \\ \text{axiom} \Uparrow & (a) & \Uparrow g \\ L_1 & \xRightarrow{\;\text{axiom'}\;} & G' \circ L_2 \circ F' \end{array} \qquad \begin{array}{ccc} F' \circ R_1 \circ G' & \xRightarrow{\;\text{cut'}\;} & R_2 \\ f \Uparrow & (b) & \Uparrow \text{cut} \\ F \circ R_1 \circ G' & \xRightarrow{\;g\;} & F \circ R_1 \circ G \end{array}$$

commute.

Pictorially, such a homomorphism (f, g) is a pair of natural transformations $f : F \Rightarrow F'$ and $g : G' \Rightarrow G$ satisfying the pictorial equalities below:





https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

5.4. *The continuation commutator*

At this final stage of the paper, we are ready to establish that

Proposition 7. Every dialogue chirality is equipped with a commutator

$$\kappa_{b,a,m} \quad : \quad (b \circledast a) \owedge m \quad \longrightarrow \quad b \circledast (a \owedge m) \tag{19}$$

between

— the parametric monad $T = \circledast$ acting on the category $\mathscr{A}$, with parameters taken in the monoidal category $(\mathscr{B}, \ovee, \mathbf{false})$,
— the monoidal action $S = \owedge$ of the monoidal category $\mathscr{A}$ over itself.

The construction of the commutator κ is based on the observation that every dialogue chirality is equipped with a family of adjunctions

$$L(- \owedge m) \quad \dashv \quad R(- \ovee m^*) \tag{20}$$

parametrized by the objects m of the category $\mathscr{A}$. Here, the functor

$$(-)^* \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}^{op(0,1)}$$

denotes the *change of frame* consisting in transporting an object m in the category $\mathscr{A} = \mathscr{C}$ to the same object m^* seen this time in the opposite category $\mathscr{B} = \mathscr{C}^{op(0,1)}$. Note that this adjunction (equation 19) formulated in the style of dialogue chiralities corresponds in the language of dialogue categories to the adjunction

$$(- \otimes A) \multimap \bot \quad \dashv \quad \bot \multimapinv (A \otimes -)$$

which generalizes the adjunction (equation 6) and identifies it as the particular instance where $A = I$ is the tensorial unit of the dialogue category. Each adjunction (equation 20) may be alternatively seen as a transjunction

$$(- \owedge m) \quad \mathrel{-\!\!\!3} \quad (- \ovee m^*)$$

accross the adjunction $L \dashv R$, presented by the natural transformation

$$\mathbf{axiom}[m] \quad : \quad L(a) \quad \longrightarrow \quad L(a \owedge m) \ovee m^*$$

$$\mathbf{cut}[m] \quad : \quad R(b \ovee m^*) \owedge m \quad \longrightarrow \quad R(b)$$

parametrized by the objects m, a of the category $\mathscr{A}$ and the objects b of the category $\mathscr{B}$. At this point, the morphism

$$\kappa_{b,a,m} \quad : \quad R(b \ovee L(a)) \owedge m \quad \longrightarrow \quad R(b \ovee L(a \owedge m))$$

is simply obtained by composing these two combinators and the associativity law of $\mathscr{B}$ in an appropriate fashion:

$$\begin{array}{ccc}
R(b \ovee L(a)) \owedge m & & R(b \ovee L(a \owedge m)) \\
\Big\downarrow {\scriptstyle \mathbf{axiom}[m]} & & \Big\uparrow {\scriptstyle \mathbf{cut}[m]} \\
R(b \ovee (L(a \owedge m) \ovee m^*)) \owedge m & \xrightarrow{\;associativity\;} & R((b \ovee L(a \owedge m)) \ovee m^*) \owedge m
\end{array} .$$

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

It is not difficult to check that the resulting natural transformation κ satisfies all the coherence diagrams of Section 4.1 and thus defines a commutator between the parametric continuation monad $T = \circledast$ and the action $S = \oslash$ of the monoidal category $\mathscr{A}$ over itself. This concludes the proof of Proposition 7.

6. Conclusion

The present paper is part of a series of articles (Melliès 2012, 2013a,b, 2015) whose general purpose is to provide an algebraic and type-theoretic status to game semantics, based on the study of tensorial negation in dialogue categories. An interesting outcome of this reconstruction of game semantics is to identify the commutator

$$\kappa_{b,a_1,a_2} \quad : \quad R(b \otimes La_1) \oslash a_2 \quad \longrightarrow \quad R(b \otimes L(a_1 \oslash a_2))$$

as the basic building block of *views* in innocent strategies, see Melliès (2012) for details. This means that the commutator κ is not just a parametric version of the strength of the continutation monad. It is also the algebraic principle which secretly underlies the notion of Böhm tree. Accordingly, the commutator κ appears as the central ingredient in our adaptation to dialogue categories of the combinatorial presentation of $*$-autonomous categories elaborated in Blute *et al*. (1996) and Cockett and Seely (1997).

Acknowledgments

This work has been partly supported by the ANR Project RECRE.

Appendix A. A general 2-categorical transfer theorem

In this Appendix, we would like to show that the transfer theorem (Proposition 2) established in Section 2.6 is a particular case of a more general 2-categorical property – at least when $\mathscr{W}$ coincides with the 2-category ***Cat*** of categories, functors and natural transformations. In that case, the 2-functor

$$T \quad : \quad \mathscr{X} \quad \mapsto \quad \mathscr{J} \times \mathscr{X} \quad : \quad \boldsymbol{Cat} \quad \longrightarrow \quad \boldsymbol{Cat} \tag{21}$$

defines a weak 2-monad for every monoidal category $(\mathscr{J}, \otimes, I)$ and a parametric $\mathscr{J}$-monad on a category $\mathscr{C}$ is the same thing as a lax T-algebra

$$* \quad : \quad \mathscr{J} \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}.$$

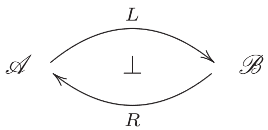
Recall that a lax T-algebra for a weak 2-monad (T, m, e) in a 2-category $\mathscr{W}$ is a 1-cell

$$* \quad : \quad T\mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

together with a pair of 2-cells

making the expected coherence diagrams commute. Now, a general transfer theorem established in Melliès (2006) states that given a formal adjunction



in a 2-category $\mathscr{W}$ equipped with a weak 2-monad T, every lax T-algebra

$$* \quad : \quad T\mathscr{B} \longrightarrow \mathscr{B}$$

on the 0-cell $\mathscr{B}$ induces a lax T-algebra structure on the 0-cell $\mathscr{A}$, defined as follows:

$$\circledast \quad : \quad T\mathscr{A} \xrightarrow{TL} T\mathscr{B} \xrightarrow{*} \mathscr{B} \xrightarrow{R} \mathscr{A}.$$

In the particular case when $\mathscr{W} = \boldsymbol{Cat}$, one recovers our original transfer theorem (Proposition 2) by applying the result to the weak 2-monad (equation 21). The general transfer theorem may be also applied to the 2-monad

$$T \quad : \quad \boldsymbol{Cat} \longrightarrow \boldsymbol{Cat}$$

which transports every category $\mathscr{C}$ to its free monoidal category $T\mathscr{C}$. In that case, the transfer theorem applied to a dialogue chirality establishes that the monoidal structure $(\mathscr{B}, \oslash, \mathbf{false})$ induces a *lax monoidal structure* on the category $\mathscr{A}$, provided by the family of n-ary disjunctions

$$[\, A_1 \parr \cdots \parr A_n \,] \quad = \quad R\,(\, LA_1 \oslash \cdots \oslash LA_n \,).$$

This algebraic construction is important because it provides a way to adapt to tensorial logic the familiar definition (7) of the $\parr$ connective in linear logic. The interested reader will find in Leinster (2004) a precise definition of the notion of lax monoidal category. The key idea is to replace the binary disjunction of linear logic by a family of n-ary disjunctions. The reason for moving to a family of connectives is that the tensorial version of binary disjunction is not associative – in the sense that the two objects

$$[\,[\, A \parr B \,] \parr C \,] \qquad\qquad [\, A \parr [\, B \parr C \,]\,]$$

are in general not isomorphic in a dialogue category, and more generally in a lax monoidal category. However, the family of n-ary disjunctions is itself associative, but in a more subtle and oriented fashion. For instance, there are canonical proofs of tensorial logic connecting the two clusters of binary disjunctions above with the

https://doi.org/10.1017/S0960129515000328 Published online by Cambridge University Press

ternary disjunction:

$$[\,[\,A \parr B\,] \parr C\,] \quad\longrightarrow\quad [\,A \parr B \parr C\,] \quad\longleftarrow\quad [\,A \parr [\,B \parr C\,]\,].$$

However, these canonical associativity maps are not invertible in general. This purely algebraic reconstruction of the linear disjunction $\parr$ in the dialogue category $\mathscr{A} = \mathscr{C}$ clarifiies in what sense it is derived by *deformation* – one should probably say by *adjunction* in that case – from the disjunction $\oslash$ living in the opposite category $\mathscr{B} = \mathscr{C}^{op(0,1)}$.

References

Atkey, R. (2009). Parameterised notions of computation. *Journal of Functional Programming* **19** (3 & 4) 335–376.

Beck, J. (1969). Distributive laws. *Lecture Notes in Mathematics* **80** 119–140.

Bénabou, J. (1963). Catégories avec multiplication. *C. R. Académie des Sciences, Paris* **256** 1887–1890.

Blute, R., Cockett, R., Seely, R. and Trimble, T. (1996). Natural deduction and coherence for weakly distributive categories. *Journal of Pure and Applied Algebra* **113** 229–296.

Cockett, R. and Seely, R. (1997). Weakly distributive categories. *Journal of Pure and Applied Algebra* **114** (2) 133–173.

Girard, J-Y. (1987) Linear logic. *Theoretical Computer Science, London Mathematical* **50** (1) 1–102.

Girard, J-Y. (1995). Linear logic, its syntax and semantics. In: Girard, J.-Y., Lafont, Y. and Regnier, L. (eds.) *Advances in Linear Logic,* London Mathematical Society Lecture Notes Series volume 222, Cambridge University Press.

Hu, H. and Joyal, A. (1999) Coherence completions of categories. *Theoretical Computer Science* 01/1999; **27**(2) 153–184.

Kelly, M. and Lack, S. (2004). Monoidal functors generated by adjunctions, with applications to transport of structure. *Fields Institute Communications* **43** 319–340.

Kelly, M. and Street, R. (1974). Review of the elements of 2-categories. In: Proceedings of the Sydney Category Theory Seminar 1972/73. *LNM* **420**, Springer 75–103.

Kock, A. (1970). On double dualization monads. *Mathematica Scandinavica* **27** 151–165.

Leinster, T. (2004). *Higher Operads, Higher Categories,* London Mathematical Society Lecture Note Series volume 298, Cambridge University Press.

Moggi, E. (1991). Notions of computation and monads. *Information and Computation* **93** (1) 55–92.

Melliès, P.-A. (2006). A transfer theorem along left adjoints. Unpublished manuscript.

Melliès, P.-A. (2009). Categorical semantics of linear logic. Published in *Interactive models of computation and program behaviour*. Panoramas et Synthèses 27, Société Mathématique de France.

Melliès, P.-A. (2012). Game semantics in string diagrams. In: *Proceedings of the Annual Symposium on Logic in Computer Science*.

Melliès, P.-A. (2013a). A micrological study of negation. Submitted. Manuscript available on the author's webpage.

Melliès, P.-A. (2013b). A micrological study of helical negation. Submitted. Manuscript available on the author's webpage.

Melliès, P.-A. (2015). Dialogue categories and chiralities. In: *Publications of the Research Institute in Mathematical Sciences*. To appear.

Melliès, P.-A. and Tabareau, N. (2009). Resource modalities in tensorial logic. *Annals of Pure and Applied Logic* **161** (5) 632–653.

Uustalu, T. (2003) Generalizing substitution. *Theoretical Informatics and Applications* **37** (4) 315–336.