Probability in the Engineering and Informational Sciences, **18**, 2004, 561–571. Printed in the U.S.A.

A NOTE ON BOUNDS AND MONOTONICITY OF SPATIAL STATIONARY COX SHOT NOISES

Naoto Miyoshi
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
Tokyo 152-8552, Japan
E-mail: miyoshi@is.titech.ac.jp

We consider shot-noise and max-shot-noise processes driven by spatial stationary Cox (doubly stochastic Poisson) processes. We derive their upper and lower bounds in terms of the increasing convex order, which is known as the order relation to compare the variability of random variables. Furthermore, under some regularity assumption of the random intensity fields of Cox processes, we show the monotonicity result which implies that more variable shot patterns lead to more variable shot noises. These are direct applications of the results obtained for so-called Ross-type conjectures in queuing theory.

1. INTRODUCTION

Let $\{(X_n, Z_n)\}_{n\in\mathbb{N}}$ denote a marked point process on $\mathbb{R}^d \times K$, $d \in \mathbb{N}$, defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, where $(K, \mathcal{K})$ is some measurable mark space and $\mathbb{N} = \{1, 2, \ldots\}$. We consider shot-noise process $\{V(s)\}_{s\in\mathbb{R}^d}$ defined by

$$V(s) = \sum_{n\in\mathbb{N}} h(s - X_n, Z_n), \qquad s \in \mathbb{R}^d \tag{1}$$

and max-shot-noise process $\{U(s)\}_{s\in\mathbb{R}^d}$ defined by

$$U(s) = \max_{n\in\mathbb{N}}\{h(s - X_n, Z_n)\}, \qquad s \in \mathbb{R}^d, \tag{2}$$

 0269-9648/04 $16.00

https://doi.org/10.1017/S026996480418409X Published online by Cambridge University Press

where h: $\mathbb{R}^d \times K \to \mathbb{R}$ is called a response function and is measurable and finite almost everywhere (see, e.g., [7]). The mark process $\{Z_n\}_{n\in\mathbb{N}}$ is a family of independent and identically distributed (i.i.d.) random elements and also independent of the point process $\{X_n\}_{n\in\mathbb{N}}$. The stability condition for $|V(s)| < \infty$, P-a.s., for each $s \in \mathbb{R}^d$, is investigated in Westcott [16, Thm. 1] (see also [4,6]), and we assume that such a condition is fulfilled as far as (1) is concerned.

The random fields like (1) and (2) are often used as basic models in optics, meteorology, astronomy, and other fields (see, e.g., [2,15] and references therein). For example, if K is the set of compact subsets of $\mathbb{R}^d$ and $h(s,z) = 1_{\{s\in z\}}$, the indicator of event $\{s \in z\}$, then (2) gives the indicator function if the position s is in a germ-grain model $U_{n\in N}\{X_n + y : y \in Z_n\}$. However, it should be noted that their characteristics are explicitly evaluated only in exceptional cases such as when $\{X_n\}_{n\in\mathbb{N}}$ is Poisson (see, e.g., [15]). In this article, we consider the case where $\{X_n\}_{n\in\mathbb{N}}$ is a stationary Cox (doubly stochastic Poisson) process and h is nonnegative on $\mathbb{R}^d \times K$ and semicontinuous in the first variable, and we investigate bounds and monotonicity of (1) and (2) in terms of some stochastic order. The stochastic order considered here is called increasing convex (icx) order and is known as the order relation to compare the variability of random variables (see, e.g., 10]). Our result shows that the upper bound is realized when $\{X_n\}_{n\in\mathbb{N}}$ is a mixed Poisson process with the same marginal distribution of random intensity and that, under some positive dependence assumption of the random intensity field, the lower bound is realized when $\{X_n\}_{n\in\mathbb{N}}$ is a homogeneous Poisson process with the same mean intensity. Furthermore, the monotonicity result states that under some regularity assumption of the random intensity field, more variable shot patterns lead to more variable noises. These are direct applications of the results obtained for so-called Ross-type conjectures in queuing theory (see, e.g., [1,3,8,11–13]) and provide an example that queuing theory applies into other fields.

This article is organized as follows. As a preliminary, in the next section, we give the definitions and some properties of the stochastic orders used. In Section 3, we derive the upper and lower bounds of (1) and (2) in terms of increasing convex order. The bounds of their Palm versions are also presented in a similar way. In Section 4, we consider the shot-noise processes (1) and (2) where the Cox point process has the random intensity fields $\{\lambda_c(s)\}_{s\in\mathbb{R}^d}$, $c > 0$, defined by $\lambda_c(s) = \lambda(cs)$ for $s \in \mathbb{R}^d$. We show that under some regularity assumption of $\{\lambda(s)\}_{s\in\mathbb{R}^d}$, (1) and (2) and their Palm versions are decreasing in c in terms of the increasing convex order. It is also noted that the bounds in Section 3 are given as two extremal cases of the monotonicity result.

2. PRELIMINARIES

In this section, we give the definitions and useful properties of some stochastic orders used in the article. A good reference for this section is a recent monograph by Müller and Stoyan [10]. First, we give the definitions of some classes of functions related to the stochastic orders. Throughout this article, we use "increasing" and "decreasing" in the nonstrict sense.

Definition 1:

(i) A function $f: \mathbb{R}^k \to \mathbb{R}$ is said to be supermodular if for all x and $y \in \mathbb{R}^k$,

$$f(x) + f(y) \le f(x \wedge y) + f(x \vee y),$$

where $x \wedge y$ and $x \vee y$ denote componentwise minimum and maximum, respectively.

(ii) A function $f: \mathbb{R}^k \to \mathbb{R}$ is said to be directionally convex (dcx) if for all x_1, x_2, $y \in \mathbb{R}^k$ with $x_1 \le x_2$ and $y \ge 0$,

$$f(x_1 + y) - f(x_1) \le f(x_2 + y) - f(x_2).$$

Note that a function is dcx if and only if it is supermodular and componentwise convex and note also that usual convexity neither implies nor is implied by directional convexity (see, e.g., [13]). Useful properties of increasing and dcx (idcx) functions, which often appear in stochastic models, are as follows (see [7,11]):

Lemma 1:

(i) Let $\{S_n^{(i)}\}_{n\in\mathbb{N}}$, $i = 1,\ldots,k$, denote k independent sequences of i.i.d. nonnegative random variables. If $f: \mathbb{R}^k \to \mathbb{R}$ is idcx, then $\phi: \mathbb{Z}_+^k \to \mathbb{R}$, defined by

$$\phi(n_1,\ldots,n_k) = \mathrm{E}\, f\left(\sum_{j=1}^{n_1} S_j^{(1)},\ldots,\sum_{j=1}^{n_k} S_j^{(k)}\right),$$

is idcx, where $\sum_{j=1}^{0}(\cdot) = 0$ conventionally, and $\psi: \mathbb{Z}_+^k \to \mathbb{R}$, defined by

$$\psi(n_1,\ldots,n_k) = \mathrm{E}\, f\left(\max_{j=1,\ldots,n_1}\{S_j^{(1)}\},\ldots,\max_{j=1,\ldots,n_k}\{S_j^{(k)}\}\right),$$

is also idcx, where we take $\max_{j\in\varnothing}(\cdot) = 0$ since each $S_j^{(i)}$ is nonnegative.

(ii) Let N_i; $i = 1,\ldots,k$, denote k mutually independent Poisson random variables, where the mean of N_i is λ_i, $i = 1,\ldots,k$. If $\phi: \mathbb{Z}_+^k \to \mathbb{R}$ is idcx, then $g: \mathbb{R}_+^k \to \mathbb{R}$, defined by

$$\begin{aligned} g(\lambda_1,\ldots,\lambda_k) &= \mathrm{E}\,\phi(N_1,\ldots,N_k) \\ &= \sum_{(n_1,\ldots,n_k)\in\mathbb{Z}_+^k} \phi(n_1,\ldots,n_k)\,\frac{\lambda_1^{n_1}\ldots\lambda_k^{n_k}}{n_1!\ldots n_k!}\,e^{-(\lambda_1+\cdots+\lambda_k)}, \end{aligned}$$

is also idcx.

Proof: The proof of the first part of (i) and that of (ii) are found in [11, Lemmas 4 and 3] and [8, Lemmas 2.17 and 2.18]. The second part of (i) seems new but is proved similar to the first part by replacing the sums with the maxima since $x_1 + \cdots + x_n$ and $\max\{x_1,\ldots,x_n\}$ are both increasing and convex in x_i, $i = 1,\ldots,n$. ∎

https://doi.org/10.1017/S026996480418409X Published online by Cambridge University Press

Note that in Lemma 1(i), $\{S_n^{(i)}\}_{n\in\mathbb{N}}$ and $\{S_n^{(j)}\}_{n\in\mathbb{N}}$, $i \neq j$, are mutually independent, but they are not necessarily identical.

The following are the important stochastic orders used throughout this article:

Definition 2:

(i) For two $\mathbb{R}$-valued random variables X and Y, we say that X is smaller than Y in the increasing convex (icx) order and write $X \le_{\mathrm{icx}} Y$ if $\mathrm{E}\, f(X) \le \mathrm{E}\, f(Y)$ for all increasing and convex functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the expectations exist.

(ii) For two $\mathbb{R}^k$-valued random vectors X and Y, we say that X is smaller than Y in the supermodular order and write $X \le_{\mathrm{sm}} Y$ if

$$\mathrm{E}\, f(X) \le \mathrm{E}\, f(Y), \tag{3}$$

for all supermodular functions $f\colon \mathbb{R}^k \to \mathbb{R}$ such that the expectations exist.

(iii) For two $\mathbb{R}^k$-valued random vectors X and Y, we say that X is smaller than Y in the increasing directionally convex (idcx) order and write $X \le_{\mathrm{idcx}} Y$ if (3) holds for all idcx functions $f\colon \mathbb{R}^k \to \mathbb{R}$ such that the expectations exist.

(iv) For two $\mathbb{R}$-valued random fields $\{X(s)\}_{s\in\mathbb{R}^d}$ and $\{Y(s)\}_{s\in\mathbb{R}^d}$, we say that $\{X(s)\}_{s\in\mathbb{R}^d}$ is smaller than $\{Y(s)\}_{s\in\mathbb{R}^d}$ in the supermodular [idcx resp.] order and write

$$\{X(s)\}_{s\in\mathbb{R}^d} \le_{\mathrm{sm}} [\le_{idcx} \text{ resp.}]\, \{Y(s)\}_{s\in\mathbb{R}^d},$$

if for any $k \in \mathbb{N}$ and all $s_1, \dots, s_k \in \mathbb{R}^d$, $(X(s_1), \dots, X(s_k)) \le_{\mathrm{sm}} [\le_{\mathrm{idcx}}$ resp.$]$ $(Y(s_1), \dots, Y(s_k))$.

Because each idcx function is supermodular, we have that $X \le_{\mathrm{sm}} Y$ implies $X \le_{\mathrm{idcx}} Y$. Both supermodular and idcx orders are known as the order relations to compare the strength of positive dependence in random vectors (see, e.g., [10, Chap. 3]). One of the most famous results, called Lorentz's inequality, is as follows (see, e.g., [10, Thm. 3.9.8]):

Lemma 2 (Lorentz's Inequality): *Let $X_1, \dots, X_k$ be $\mathbb{R}$-valued random variables and let F_i denote the marginal distribution of X_i, $i = 1, \dots, k$. Then, for a random variable U uniformly distributed on $[0,1)$,*

$$(X_1, \dots, X_k) \le_{\mathrm{sm}} (F_1^{-1}(U), \dots, F_k^{-1}(U)),$$

where $F_i^{-1}(u) = \inf\{x \in \mathbb{R} : F_i(x) \ge u\}$, $u \in [0,1)$, $i = 1, \dots, k$.

In Lemma 2, the right-hand side is known as the random vector, which has the strongest positive dependence among ones with marginal distributions $(F_1, \dots, F_k)$. The next lemma, which is given by [8, Lemma 3.3] (see also [9, Lemma 3]) for stochastic processes on the real line, is often used in the following sections.

LEMMA 3: *Suppose that two $\mathbb{R}$-valued random fields $\{X(s)\}_{s\in\mathbb{R}^d}$ and $\{Y(s)\}_{s\in\mathbb{R}^d}$ are a.s. Riemann integrable. If $\{X(s)\}_{s\in\mathbb{R}^d} \leq_{\mathrm{idcx}} \{Y(s)\}_{s\in\mathbb{R}^d}$, then*

$$\left(\int_{I_1} X(s)\,ds,\ldots,\int_{I_k} X(s)\,ds\right) \leq_{\mathrm{idcx}} \left(\int_{I_1} Y(s)\,ds,\ldots,\int_{I_k} Y(s)\,ds\right),$$

for any $k\in\mathbb{N}$ and any disjoint and bounded $I_1,\ldots,I_k\in\mathcal{B}(\mathbb{R}^d)$.

Before concluding this section we give the definition of a notion describing the positive dependence of random vectors and random fields (see [10, Def. 3.10.9] and [8, Def. 3.7]):

DEFINITION 3:

(i) An $\mathbb{R}^k$-valued random vector $(X_1,\ldots,X_k)$ is said to be conditionally increasing if $\mathrm{E}[\,f(X_i)|X_j = x_j, j\in J]$ is increasing in x_j, $j\in J$, for all increasing function f, any $i\in\{1,\ldots,k\}$, and any subset $J\subset\{1,\ldots,k\}$.
(ii) An $\mathbb{R}$-valued random field $\{X(s)\}_{s\in\mathbb{R}^d}$ is said to be conditionally increasing if for any $k\in\mathbb{N}$ and all $s_1,\ldots,s_k\in\mathbb{R}^d$, $(X(s_1),\ldots,X(s_k))$ is conditionally increasing.

Combining Theorems 3.6 and 3.8 in [8], we have the following:

LEMMA 4: *Let $X=(X_1,\ldots,X_k)$ be conditionally increasing and let $Y=(Y_1,\ldots,Y_k)$ be mutually independent random variables with $Y_i\leq_{\mathrm{icx}} X_i$ for all $i=1,\ldots,k$. Then, $Y\leq_{\mathrm{idcx}} X$.*

3. BOUNDS

In this section, we derive the upper and lower bounds of (1) and (2) and also those of their Palm versions, in terms of the increasing convex order. For notational convenience, we take $s=0$ in (1) and (2) due to the stationarity, suppress the symbol 0, and change the sign of $\{X_n\}_{n\in\mathbb{N}}$; that is, we consider

$$V = \sum_{n\in\mathbb{N}} h(X_n, Z_n), \tag{4}$$

$$U = \max_{n\in\mathbb{N}}\{h(X_n, Z_n)\}. \tag{5}$$

We consider $\{X_n\}_{n\in N}$ a stationary Cox process and let $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ denote the stationary random intensity field of $\{X_n\}_{n\in N}$. The corresponding random measure Λ is given by $\Lambda(ds)=\lambda(s)\,ds$. Then, lettting N denote the random counting measure which counts the points $\{X_n\}_{n\in\mathbb{N}}$ on $\mathbb{R}^{\mathrm{d}}$, we have that, for any bounded $I\in\mathcal{B}(\mathbb{R}^d)$,

$$\mathrm{P}(N(I)=n) = \mathrm{E}\left[\frac{\Lambda(I)^n}{n!}\,e^{-\Lambda(I)}\right], \qquad n=0,1,2,\ldots.$$

https://doi.org/10.1017/S026996480418409X Published online by Cambridge University Press

We assume that $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ has positive and finite mean $\bar{\lambda} = \mathrm{E}\,\lambda(0)$ and that $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ is a.s. Riemann integrable.

Two special cases of $\{X_n\}_{n\in\mathbb{N}}$ are the homogeneous Poisson process with constant intensity $\bar{\lambda}$ and the mixed Poisson process with random, but constant on $\mathbb{R}^d$, intensity $\tilde{\lambda} =_{\mathrm{st}} \lambda(0)$, where "$=_{\mathrm{st}}$" denotes equivalence in distribution. We compare (4) and (5) with the ones that have these special cases of Cox point processes and the identical mark process $\{Z_n\}_{n\in\mathbb{N}}$.

THEOREM 1: *Let V_{mix} and U_{mix} denote respectively (4) and (5) where the point process $\{X_n\}_{n\in\mathbb{N}}$ is the mixed Poisson process with random intensity $\tilde{\lambda} =_{\mathrm{st}} \lambda(0)$. Then,*

$$V \le_{\mathrm{icx}} V_{\mathrm{mix}}, \tag{6}$$

$$U \le_{\mathrm{icx}} U_{\mathrm{mix}}. \tag{7}$$

Furthermore, let V_{hom} and U_{hom} denote respectively (4) and (5) where the $\{X_n\}_{n\in\mathbb{N}}$ is the homogeneous Poisson process with intensity $\bar{\lambda}$. If $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ is conditionally increasing, then

$$V_{\mathrm{hom}} \le_{\mathrm{icx}} V, \tag{8}$$

$$U_{\mathrm{hom}} \le_{\mathrm{icx}} U. \tag{9}$$

PROOF: First, we show (6) and (8). For a positive integer k, let $I_k = \{I_{k,j}; j = 1, \dots, (k2^{k+1})^d\}$ denote a partition of $[-k,k]^d$ such that each $I_{k,j}$ is a cube of side length $1/2^k$ with $\bigcup_j I_{k,j} = [-k,k]^d$ and $I_{k,j} \cap I_{k,i} = \emptyset$, $j \ne i$. The first step is to show that for any increasing and convex f, there exists a family of idcx functions $\{g^{(k)} : \mathbb{R}_+^{\nu(k)} \to \mathbb{R}_+\}_{k\in\mathbb{N}}$ such that

$$\mathrm{E}\, f(V) = \lim_{k\to\infty} \mathrm{E}\, g^{(k)}(\Lambda(I_{k,1}), \dots, \Lambda(I_{k,\nu(k)})), \tag{10}$$

where $\nu(k) = (k2^{k+1})^d$. Now, for any subset $I \in \mathcal{B}(\mathbb{R}^d)$, we define $\underline{h}_I : K \to \mathbb{R}_+$ by

$$\underline{h}_I(z) = \inf_{t\in I} h(t,z), \qquad z \in K.$$

Note that $\underline{h}_I$ is measurable on $(K, \mathcal{K})$ for any fixed $I \in \mathcal{B}(\mathbb{R}^d)$. Thus, defining $h^{(k)}$: $\mathbb{R}^d \times K \to \mathbb{R}_+$ by

$$h^{(k)}(s,z) = \sum_{j=1}^{\nu(k)} \underline{h}_{I_{k,j}}(z) 1_{I_{k,j}}(s), \qquad (s,z) \in \mathbb{R}^d \times K, \tag{11}$$

we have $h^{(k)}(s,z) \uparrow h(s,z)$ as $k \to \infty$ a.e. on $\mathbb{R} \times K$. Therefore, the random sequence $\{V^{(k)}\}_{k\in\mathbb{N}}$, given by

$$V^{(k)} = \sum_{n\in\mathbb{N}} h^{(k)}(X_n, Z_n),$$

satisfies $V^{(k)} \uparrow V$ a.s. as $k \to \infty$. Since f is continuously increasing, by the monotone convergence theorem we have $\mathrm{E}\, f(V) = \lim_{k\to\infty} \mathrm{E}\, f(V^{(k)})$. Now, for $I \in \mathcal{B}(\mathbb{R}^d)$, let $M(I) = \sum_{n\in\mathbb{N}} \underline{h}_I(Z_n) 1_I(X_n)$. Then, $V^{(k)}$ is expressed by

$$V^{(k)} = M(I_{k,1}) + \cdots + M(I_{k,\nu(k)}).$$

Clearly, $g(x_1, \dots, x_k) = f(x_1 + \cdots + x_k)$ is idcx for any increasing and convex f. Therefore, since $\{\underline{h}_I(Z_n)\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. variables for fixed $I \in \mathcal{B}(\mathbb{R}^d)$, by applying the first part of Lemma 1(i) (conditioning on $N(I_{k,1}), \dots, N(I_{k,\nu(k)})$) and then by Lemma 1(ii) (conditioning on $\Lambda(I_{k,1}), \dots, \Lambda(I_{k,\nu(k)})$), we have the form of (10).

Let $\{\tilde{\lambda}(s)\}_{s\in\mathbb{R}^d}$ be such that $\tilde{\lambda}(s) = \tilde{\lambda} =_{\mathrm{st}} \lambda(0)$ for all $s \in \mathbb{R}^d$. Then, by Lemma 2 (Lorentz's inequality), we have $\{\lambda(s)\}_{s\in\mathbb{R}^d} \le_{\mathrm{sm}} \{\tilde{\lambda}(s)\}_{s\in\mathbb{R}^d}$, and therefore by Lemma 3, $(\Lambda(I_{k,1}), \dots, \Lambda(I_{k,\nu(k)})) \le_{\mathrm{idcx}} (\tilde{\lambda}/2^{kd}, \dots, \tilde{\lambda}/2^{kd})$, where we use that the Lebesgue measure of $I_{k,j}$ is given as $|I_{k,j}| = 1/2^{kd}$ for $j = 1, \dots, \nu(k)$. Hence, from (10),

$$\mathrm{E}\, f(V) \le \lim_{k\to\infty} \mathrm{E}\, g^{(k)}(\tilde{\lambda}/2^{kd}, \dots, \tilde{\lambda}/2^{kd}) = \mathrm{E}\, f(V_{\mathrm{mix}}),$$

which completes the proof of (6). On the other hand, let $\{\bar{\lambda}(s)\}_{s\in\mathbb{R}^d}$ be such that $\bar{\lambda}(s) = \bar{\lambda} = \mathrm{E}\, \lambda(0)$ for all $s \in \mathbb{R}^d$. Since $\bar{\lambda} \le_{\mathrm{icx}} \lambda(0)$ clearly by Jensen's inequality, we have by Lemma 4 that $\{\bar{\lambda}(s)\}_{s\in\mathbb{R}^d} \le_{\mathrm{idcx}} \{\lambda(s)\}_{s\in\mathbb{R}^d}$ if $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ is conditionally increasing. Hence, similar to the above,

$$\mathrm{E}\, f(V) \ge \lim_{k\to\infty} g^{(k)}(\bar{\lambda}/2^{kd}, \dots, \bar{\lambda}/2^{kd}) = \mathrm{E}\, f(V_{\mathrm{hom}}),$$

which completes the proof of (8).

Next, we show (7) and (9). Using $h^{(k)}$ in (11), we define the sequence $\{U^{(k)}\}_{k\in\mathbb{N}}$ by

$$U^{(k)} = \max_{n\in\mathbb{N}}\{h^{(k)}(X_n, Z_n)\}.$$

Then, $U^{(k)} \uparrow U$ a.s. as $k \to \infty$ and, therefore, $\mathrm{E}\, f(U) = \lim_{k\to\infty} \mathrm{E}\, f(U^{(k)})$ by the monotone convergence theorem. Since $\underline{h}_I$ is nonnegative and $1_{I_{k,j}}(s)$ takes value one at most one $j \in \{1, \dots, \nu(k)\}$ for a given $s \in \mathbb{R}^d$, we can rewrite (11) as $h^{(k)}(s, z) = \max_{j=1,\dots,\nu(k)}\{\underline{h}_{I_{k,j}}(z) 1_{I_{k,j}}(s)\}$. Thus, defining $L(I) = \max_{n\in\mathbb{N}}\{\underline{h}_I(Z_n) 1_I(X_n)\}$ for $I \in \mathcal{B}(\mathbb{R}^d)$, we have

$$U^{(k)} = \max\{L(I_{k,1}), \dots, L(I_{k,\nu(k)})\}.$$

Finally, since $g(x_1, \dots, x_k) = f(\max\{x_1, \dots, x_k\})$ is idcx for any increasing and convex f, we can show (7) and (9) similar to the above argument but using the second part of Lemma 1(i) instead of the first part. ■

https://doi.org/10.1017/S026996480418409X Published online by Cambridge University Press

In the remainder of this section, we consider the Palm versions of (4) and (5), which are interpreted as those according to the conditional distribution given a point at the origin. Recall that the Palm version of a stationary Cox process driven by stationary random measure Λ is given by the Cox process driven by the random measure Λ° plus a point added at the origin, where Λ° is also the Palm version of Λ (see, e.g., [5, Thm. 2 in Sect. 2.4]). Since the Palm probability P_Λ^0 with respect to the stationary random measure Λ satisfies $\mathrm{P}_\Lambda^0(A) = \mathrm{E}[\lambda(0) 1_A]/\bar{\lambda}$, $A \in \mathcal{F}$, we can define the Palm version $\{\lambda^\circ(s)\}_{s\in\mathbb{R}^d}$ of the stationary process $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ by the finite-dimensional distributions:

$$\mathrm{P}(\lambda^\circ(s_1) \in C_1, \dots, \lambda^\circ(s_k) \in C_k) = \frac{1}{\bar{\lambda}} \mathrm{E}[\lambda(0) 1_{\{\lambda(s_1)\in C_1,\dots,\lambda(s_k)\in C_k\}}], \tag{12}$$

for any $k \in \mathbb{N}$, $s_1, \dots, s_k \in \mathbb{R}^d$, and $C_1, \dots, C_k \in \mathcal{B}(\mathbb{R}_+)$. The Palm version Λ° of the random measure Λ is then given by $\Lambda^\circ(ds) = \lambda^\circ(s)\, ds$. Therefore, the Palm versions of V in (4) and U in (5) are respectively given by

$$V^\circ = h(0, Z_0) + \sum_{n\in\mathbb{N}} h(X_n^\circ, Z_n), \tag{13}$$

$$U^\circ = \max\Big\{h(0, Z_0), \max_{n\in\mathbb{N}} h(X_n^\circ, Z_n)\Big\}, \tag{14}$$

where $\{X_n^\circ\}_{n\in\mathbb{N}}$ denotes the Cox process with driving random measure Λ°, and Z_0 denotes a random element on $(K, \mathcal{K})$ with the same distribution as Z_n, $n = 1, 2, \dots$, whereas Z_0 is independent of $\{(X_n^\circ, Z_n)\}_{n\in\mathbb{N}}$. Note that these Palm versions are not stationary in probability measure P but are so in the respective Palm probability measures.

We compare the Palm versions (13) and (14) with their special cases. Note that the Palm version of a homogeneous Poisson process is the same homogeneous Poisson process plus a point added at the origin. Also, the Palm version of the mixed Poisson process with random intensity $\tilde{\lambda}$ is the mixed Poisson process with random intensity $\tilde{\lambda}^\circ$ plus a point at the origin, where $\mathrm{P}(\tilde{\lambda}^\circ \in C) = \mathrm{E}[\tilde{\lambda} 1_{\{\tilde{\lambda}\in C\}}]/\bar{\lambda}$. To obtain the Palm version of Theorem 1, we use the following lemma, the case of $d = 1$, which is implicitly used in [9].

LEMMA 5: *If two $\mathbb{R}$-valued random fields $\{X(s)\}_{s\in\mathbb{R}^d}$ and $\{Y(s)\}_{s\in\mathbb{R}^d}$ are a.s. Riemann integrable and $\{X(s)\}_{s\in\mathbb{R}^d} \le_{\mathrm{idcx}} \{Y(s)\}_{s\in\mathbb{R}^d}$, then for any idcx f: $\mathbb{R}^k \to \mathbb{R}$,*

$$\mathrm{E}\left[X(0) f\left(\int_{I_1} X(s)\, ds, \dots, \int_{I_k} X(s)\, ds\right)\right] \le \mathrm{E}\left[Y(0) f\left(\int_{I_1} Y(s)\, ds, \dots, \int_{I_k} Y(s)\, ds\right)\right],$$

for any disjoint and bounded $I_1, \dots, I_k \in \mathcal{B}(\mathbb{R}^d)$, provided that the expectations exist.

https://doi.org/10.1017/S026996480418409X Published online by Cambridge University Press

PROOF: The proof is similar to that of Lemma 3 because we can easily check that if $f\colon \mathbb{R}^k \to \mathbb{R}$ is idcx, then $g\colon \mathbb{R}^{k+1} \to \mathbb{R}$ defined by $g(x_0, x_1, \ldots, x_k) = x_0 f(x_1, \ldots, x_k)$ is also idcx (see [8, Lemma 3.3] and [9, Lemma 3]). ■

COROLLARY 1: *Let V°_{mix} and U°_{mix} denote respectively (13) and (14) where $\{X^\circ_n\}_{n\in\mathbb{N}}$ is the mixed Poisson process with random intensity $\tilde{\lambda}^\circ =_{\mathrm{st}} \lambda^\circ(0)$. Then,*

$$V^\circ \le_{\mathrm{icx}} V^\circ_{\mathrm{mix}},$$

$$U^\circ \le_{\mathrm{icx}} U^\circ_{\mathrm{mix}}.$$

Furthermore, let V°_{hom} and U°_{hom} denote respectively (13) and (14) where $\{X^\circ_n\}_{n\in\mathbb{N}}$ is the homogeneous Poisson process with intensity $\bar{\lambda}$. If $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ is conditionally increasing, then

$$V^\circ_{\mathrm{hom}} \le_{\mathrm{icx}} V^\circ,$$

$$U^\circ_{\mathrm{hom}} \le_{\mathrm{icx}} U^\circ.$$

PROOF: In a method similar to the proof of Theorem 1 and using (12), we have

$$\begin{aligned} \mathrm{E}\, f(V^\circ) &= \lim_{k\to\infty} \mathrm{E}\, g^{(k)}(\Lambda^\circ(I_{k,1}), \ldots, \Lambda^\circ(I_{k,\nu(k)})) \\ &= \lim_{k\to\infty} \frac{1}{\bar{\lambda}} \mathrm{E}[\lambda(0) g^{(k)}(\Lambda(I_{k,1}), \ldots, \Lambda(I_{k,\nu(k)}))]. \end{aligned} \tag{15}$$

Hence, the result follows using Lemma 5 instead of Lemma 3. ■

4. MONOTONICITY

In this section, we introduce a regularity property for stationary and isotropic (i.e., motion-invariant; see, e.g., [15]) random fields, which serves as the assumption for the monotonicity results (see [9] for the version of stationary stochastic processes on the line).

DEFINITION 4: *A stationary and isotropic random field $\{X(s)\}_{s\in\mathbb{R}^d}$ is said to be $\le_{\mathrm{sm}}$-regular [$\le_{\mathrm{idcx}}$-regular resp.] if for any $k \in \mathbb{N}$ and $s_1, \ldots, s_k, t_1, \ldots, t_k \in \mathbb{R}^d$ such that $\|s_i - s_j\| \le \|t_i - t_j\|$ for all $i, j = 1, \ldots, k$,*

$$(X(t_1), \ldots, X(t_k)) \le_{\mathrm{sm}} [\le_{\mathrm{idcx}} \text{resp.}]\, (X(s_1), \ldots, X(s_k)).$$

Note that if a random field $\{X(s)\}_{s\in\mathbb{R}^d}$ is $\le_{\mathrm{sm}}$-regular, it is also $\le_{\mathrm{idcx}}$-regular. The above-described regularity properties define the strength of positive dependence in random fields. The relation between the regularity property here and the conditionally increasing property in Section 2 needs further research. The regularity property in this section seems to require more than the conditionally increasing since the influence of the distance on the dependence is not concerned in the latter. However, in the case where $d = 1$, the one-dimensional case, and $\{X(s)\}_{s\in\mathbb{R}}$ is Markovian, the conditionally increasing property reduces to doubly stochastic monotonicity,

https://doi.org/10.1017/S026996480418409X Published online by Cambridge University Press

defined in [9] (see also [1]), and it is shown in [9] that stationary and doubly stochastic monotone Markov processes are $\le_{\mathrm{sm}}$-regular.

Example 1: Consider a stationary and isotropic Gaussian field $\{X(s)\}_{s\in\mathbb{R}^d}$. Due to stationarity and isotropy, the covariance function is just a function of the distance and not of the direction; that is, $\mathrm{Cov}[X(s_i), X(s_j)] = C(\|s_i - s_j\|)$ for $s_i, s_j \in \mathbb{R}^d$. Therefore, by Theorem 3.13.5 of [10], we have that $\{X(s)\}_{s\in\mathbb{R}^d}$ is $\le_{\mathrm{sm}}$-regular if and only if the covariance function C is decreasing.

LEMMA 6: *Let a stationary and isotropic random field $\{X(s)\}_{s\in\mathbb{R}^d}$ be $\le_{\mathrm{sm}}$-regular [$\le_{\mathrm{idcx}}$-regular resp.]. Then, the random field $\{X_c(s)\}_{s\in\mathbb{R}^d}$ defined by $X_c(s) = X(cs)$, $s \in \mathbb{R}^d$, is $\le_{\mathrm{sm}}$-decreasing [$\le_{\mathrm{idcx}}$-decreasing resp.] in $c > 0$.*

In Lemma 6, $\{X_c(s)\}_{s\in\mathbb{R}^d}$ fluctuates more actively as c increases while the mean value remains the same because of the stationarity. As an example of the $\le_{\mathrm{idcx}}$-regular random intensity field of Cox process, we can choose $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ defined by $\lambda(s) = \max\{X(s), 0\}$, $s \in \mathbb{R}^d$, for a stationary and isotropic Gaussian field $\{X(s)\}_{s\in\mathbb{R}^d}$ with decreasing covariance function since $f(g_1(\cdot), \dots, g_k(\cdot))$ is idcx for idcx $f\colon \mathbb{R}^k \to \mathbb{R}$ and increasing and convex $g_1, \dots, g_k\colon \mathbb{R} \to \mathbb{R}$.

THEOREM 2: *Given a stationary and isotropic random field $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ with mean $\bar\lambda$, we define $\{\lambda_c(s)\}_{s\in\mathbb{R}^d}$, $c > 0$, by $\lambda_c(s) = \lambda(cs)$ for all $s \in \mathbb{R}^d$. Let V_c, U_c, V_c°, and U_c° denote respectively shot noise (4), max shot noise (5), and their Palm versions (13) and (14) where the point process $\{X_n\}_{n\in\mathbb{N}}$ is the Cox process driven by the random intensity field $\{\lambda_c(s)\}_{s\in\mathbb{R}^d}$. If $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ is $\le_{\mathrm{idcx}}$-regular, then V_c, U_c, V_c°, and U_c° are all $\le_{\mathrm{icx}}$-decreasing in c (>0).*

PROOF: The proof is immediate from the proof of Theorem 1, applying Lemmas 3 and 6. For the Palm versions, we use (15) and Lemma 5 instead of Lemma 3. ■

The bounds obtained in the previous section are considered the extremal cases of Theorem 2 in the sense that the limit as $c \to 0$ reduces to the case of the mixed Poisson process and the limit as $c \to \infty$ reduces to the case of the homogeneous Poisson process; that is, for any bounded subset $I \in \mathcal{B}(\mathbb{R}^d)$,

$$\lim_{c\to 0} \Lambda_c(I) = \lim_{c\to 0} \int_I \lambda(cs)\, ds = \lambda(0)|I|, \quad \text{a.s.,}$$

and if $\{\lambda(s)\}_{s\in\mathbb{R}^d}$ is ergodic,

$$\lim_{c\to\infty} \Lambda_c(I) = \lim_{c\to\infty} \int_I \lambda(cs)\, ds = \lim_{c\to\infty} \frac{|I|}{c|I|} \int_{cI} \lambda(s)\, ds = \bar\lambda|I|, \quad \text{a.s.,}$$

where $\Lambda_c(ds) = \lambda_c(s)\, ds$ and $cI = \{cs \in \mathbb{R}^d : s \in I\}$, $c > 0$.

Acknowledgment

The author wishes to thank Tomasz Rolski for stimulating the author's interest in the Ross-type problems.

References

1. Bäuerle N. & Rolski, T. (1998). A monotonicity result for the workload in Markov-modulated queues. *Journal of Applied Probability* 35: 741–747.
2. Bondesson, L. (1988). Shot-noise processes and distributions. In S. Kotz, N.L. Johnson, and C.B. Read (eds.), *Encyclopedia of Statistical Science, Volume 8*. New York: Wiley, pp. 448–452.
3. Chang, C.-S., Chao, X.L., & Pinedo, M. (1991). Monotonicity results for queues with doubly stochastic Poisson arrivals: Ross's conjecture. *Advances in Applied Probability* 23: 210–228.
4. Daley, D.J. (1971). The definition of a multi-dimensional generalization of shot noise. *Journal of Applied Probability* 8: 128–135.
5. Grandell, J. (1976). *Doubly stochastic Poisson processes*. New York: Springer-Verlag.
6. Heinrich, L. & Schmidt, V. (1985). Normal convergence of multidimensional shot noise and rates of this convergence. *Journal of Applied Probability* 17: 709–730.
7. Heinrich, L. & Molchanov, I.S. (1994). Some limit theorems for external and union shot-noises. *Mathematische Nachrichten* 168: 139–159.
8. Meester, L.E. & Shanthikumar, J.G. (1993). Regularity of stochastic processes: A theory based on directional convexity. *Probability in the Engineering and Informational Sciences* 7: 343–360.
9. Miyoshi, N. & Rolski, T. (2004). Ross type conjectures on monotonicity of queues. *Australian & New Zealand Journal of Statistics* 46: 121–131.
10. Müller, A. & Stoyan, D. (2002). *Comparison methods for stochastic models and risks*. New York: Wiley.
11. Rolski, T. (1989). Upper bounds for single server queues with doubly stochastic Poisson arrivals. *Mathematics of Operations Research* 11: 442–450.
12. Rolski, T. (1989). Queues with nonstationary inputs. *Queueing Systems: Theory and Applications* 5: 113–130.
13. Ross, S.M. (1978). Average delay in queues with non-stationary Poisson arrivals. *Journal of Applied Probability* 15: 602–609.
14. Shaked, M. & Shanthikumar, J.G. (1990). Parametric stochastic convexity and concavity of stochastic processes. *Annals of the Institute of Statistical Mathematics* 42: 509–531.
15. Stoyan, D., Kendall, W.S., & Mecke, J. (1995). *Stochastic geometry and its applications*, 2nd ed. New York: Wiley.
16. Westcott, M. (1976). On the existence of a generalized shot-noise process. In E.J. Williams (ed.), *Studies in Probability and Statistics: Papers in Honour of Edwin J.G. Pitman*. Amsterdam: North-Holland, pp. 73–88.