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ON THE EQUIVALENCE OF SYSTEMS OF DIFFERENT SIZES, WITH APPLICATIONS TO SYSTEM COMPARISONS

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Abstract

The signature of a coherent system is a useful tool in the study and comparison of lifetimes of engineered systems. In order to compare two systems of different sizes with respect to their signatures, the smaller system needs to be represented by an equivalent system of the same size as the larger system. In the paper we show how to construct equivalent systems by adding irrelevant components to the smaller system. This leads to simpler proofs of some current key results, and throws new light on the interpretation of mixed systems. We also present a sufficient condition for equivalence of systems of different sizes when restricting to coherent systems. In cases where for a given system there is no equivalent system of smaller size, we characterize the class of lower-sized systems with a signature vector which stochastically dominates the signature of the larger system. This setup is applied to an optimization problem in reliability economics.

Keywords: Coherent system; system signature; k-out-of-n system; mixed system; reliability polynomial; irrelevant component; cut set; critical path vector; stochastic order; reliability economics

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1. Introduction

Consider a coherent system with n components, as defined in the classical monograph of Barlow and Proschan [1]. (We shall in the following say, for short, that a system with n components is an n-system.) Suppose that the component lifetimes are independent and identically distributed (i.i.d.) with continuous survival function $\bar{F}$ and let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the ordered lifetimes of the n components. Samaniego [8] introduced the *signature vector*, $\mathbf{s} = (s_1, \ldots, s_n)$, of the system, defined by $s_k = \mathbb{P}(T = X_{k:n})$; $k = 1, \ldots, n$. The signature of a system depends only on the system's design and does not depend on the distribution of the component lifetimes. A key result is Samaniego [9, Theorem 3.1], stating that the survival function of the lifetime T of the system can be represented in terms of the signature vector as

$$\bar{F}_T(t) = \mathbb{P}(T > t) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (1 - \bar{F}(t))^j (\bar{F}(t))^{n-j} = h(\bar{F}(t)), \tag{1}$$

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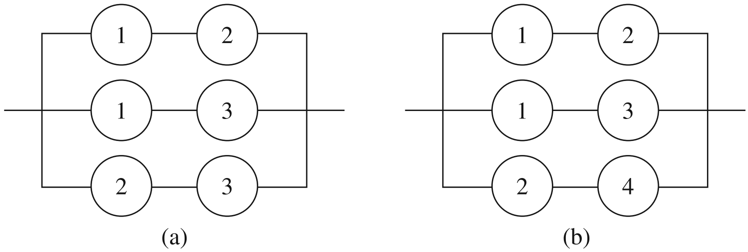

FIGURE 1: Two equivalent systems, with common reliability polynomial $h(p) = 3p^2 - 2p^3$, but with different signature vectors, (0, 1, 0) (a), $(0, \frac{1}{2}, \frac{1}{2}, 0)$ (b).

where

$$h(p) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (1-p)^j p^{n-j}, \qquad 0 \le p \le 1, \tag{2}$$

is the so-called *reliability polynomial* corresponding to the system; see also [1].

As argued in [9, pp. 28–31] it may be convenient to extend the class of n-systems to include so-called *mixed* n-systems, which are stochastic mixtures of coherent n-systems. It is easy to see that (1) continues to hold for mixed systems. Note that while, for a given n, there are finitely many coherent n-systems and therefore also finitely many possible signature vectors corresponding to coherent n-systems, any probability vector $(s_1, \dots, s_n)$ can serve as the signature vector of a mixed system. One possible representation of such a mixed system is the one which assigns, for $k = 1, \dots, n$, probability s_k to a k-out-of-n system, i.e. an n-system which fails upon the kth component failure.

Following Navarro *et al.* [5, Section 2.1], we shall say that two mixed systems with i.i.d. component lifetimes with distribution F are *equivalent* if the lifetime distributions of the systems are identical, given any component distribution F. In view of (1) and (2) we state the following alternative definition.

Definition 1. Two mixed systems are said to be *equivalent* if their reliability polynomials (2) are equal as functions of p, $0 \le p \le 1$.

It is important to note that two systems may be equivalent even if they do not have the same number of components. An example is given by the two systems in Figure 1. These systems have the same reliability polynomial,

$$h(p) = 3p^2 - 2p^3,$$

and are hence equivalent according to Definition 1. Their signature vectors are necessarily different, however, since their dimensions differ.

Thus it would appear that signatures are not well suited for the comparison of lifetime distributions of systems of different sizes. Samaniego [9, p. 32] therefore suggested 'converting' the smaller of two systems into an equivalent system of the same size as the larger one, allowing a meaningful comparison of the systems in terms of their signature vectors. Samaniego [9, Theorem 3.2] shows how to compute the signature vector of a mixed $(n+1)$-system equivalent to a given mixed n-system. A more general result, giving the signature vector of an $(n+r)$-system equivalent to a given n-system, was derived in [5].

The main motivation for this paper is to investigate, from different angles, the property of equivalence between systems. While the proofs of the above cited results on equivalent

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signatures are rather cumbersome, essentially based on the fact that a mixed system is a mixture of k-out-of-n systems, in this paper we suggest a more transparent approach based on the fact that addition of *irrelevant components* to a system always results in an equivalent system. This idea, which appears to be new, is in this paper applied to various settings involving equivalent systems of different sizes.

Recall that an irrelevant component by definition is a component that does not contribute to the functioning of a system [1]. Further, defining a *monotone* system as a system with structure function which is monotone as a function of the component states, it is well known that a monotone system may always be reduced to a coherent system by removing all irrelevant components. We will in this paper demonstrate how one may go the opposite way, by adding irrelevant components, and thereby constructing equivalent systems of arbitrary size larger than the size of the original system.

In Section 2 we use this approach in a constructive manner to give a new and more elementary derivation of the result in [5] on the signature vector of an $(n+r)$-system equivalent to a given n-system. Using the derived formulae, we then discuss briefly in Section 3 the possibility of having equivalent mixed systems of smaller size.

The addition of irrelevant components to make equivalent systems is also the key idea behind the proof of the main result of Section 4. Since system reliability has traditionally been concerned with *coherent* systems, we will in this section have a closer look at the possible existence of equivalent *coherent* systems of different sizes. To the best of the authors' knowledge such an investigation seems to be new in the literature. While adding an irrelevant component to a coherent system makes it noncoherent, we show that under certain conditions we can modify the minimal cut sets of the new system to make it coherent. As a special case we show that any k-out-of-n system with $1 < k < n$ and $n \geq 3$ has an equivalent coherent system of size $n+1$.

Sections 5 and 6 are devoted to applications formulated in a reliability economics framework. In Section 5 we give a characterization of the class of mixed n-systems with signature vectors which stochastically dominate the signature vector of a given coherent or mixed $(n+1)$-system. It is argued that this class is a natural one in the case where there are no smaller-sized equivalent systems to a given one. In Section 6 we consider the problem of optimizing the performance-per-cost function (9) within this set of n-systems, for a given $(n+1)$-system.

Some concluding remarks are given in Section 7.

2. The signature vector of equivalent systems of larger sizes

The purpose of this section is to give a new proof of Theorem 1 below, which is a slight reformulation of the result of [5] cited in the introduction for computation of signature vectors for equivalent systems.

Theorem 1. *Let* $\mathbf{s}$ *be the signature vector of a mixed or coherent system of order* n*. Then for any positive integer* r *there is an equivalent system of order* $n+r$ *with signature vector* $\mathbf{s}^*$ *with entries*

$$s_k^* = \frac{n}{n+r}\frac{1}{\binom{n+r-1}{k-1}}\sum_{i=\max(1,k-r)}^{\min(k,n)}\binom{n-1}{i-1}\binom{r}{k-i}s_i \quad \text{for } k = 1, 2, \ldots, n+r.$$

Proof. Let $X_1, \ldots, X_n$ be the lifetimes of the components of the n-system, assumed to be i.i.d. with distribution F. Let $Y_1, \ldots, Y_r$ be an independent set of i.i.d. variables from F (representing r irrelevant components).

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First, place all the $n+r$ variables $X_1, \dots, X_n, Y_1, \dots, Y_r$ in increasing order as follows:

$$X^*_{1:\,n+r} < X^*_{2:\,n+r} < \cdots < X^*_{n+r:\,n+r}.$$

Then for $j = 1, 2, \dots, n+r$, define

$$U_j = \begin{cases} 1 & \text{if } X^*_{j:\,n+r} \text{ originates from an } X, \\ 0 & \text{if } X^*_{j:\,n+r} \text{ originates from a } Y. \end{cases}$$

Let T be the lifetime of the n-system defined above, and let this also be the lifetime of the $(n+r)$-system obtained by adding the r irrelevant components with lifetimes Y_i.

Then, since $T = X^*_{k:\,n+r}$ is impossible when $U_k = 0$, we may write

$$\begin{aligned} s^*_k &\equiv \mathbb{P}(T = X^*_{k:\,n+r}) \\ &= \sum_{i=1}^{k} \mathbb{P}\bigg(T = X^*_{k:\,n+r} \,\bigg|\, U_k = 1, \sum_{m=1}^{k-1} U_m = i-1\bigg)\mathbb{P}\bigg(U_k = 1, \sum_{m=1}^{k-1} U_m = i-1\bigg) \\ &= \sum_{i=1}^{k} \mathbb{P}(T = X_{i:\,n})\mathbb{P}\bigg(\sum_{m=1}^{k-1} U_m = i-1\bigg)\mathbb{P}\bigg(U_k = 1 \,\bigg|\, \sum_{m=1}^{k-1} U_m = i-1\bigg). \end{aligned}$$

Here we have used the independence of the event $\{T = X_{i:\,n}\}$ and the $\{U_1, \dots, U_{n+r}\}$. This holds since the former event depends on the permutation of the indices $1, \dots, n$ in the ordering of the X_i, while the latter set depends on the values of the X_i and Y_i that are actually observed. This is similar to Randles and Wolfe [7, Lemma 8.3.11].

We then continue, noting that $\{U_1, \dots, U_{n+r}\}$ are distributed as independent draws without replacement from an urn containing n X and r Y, giving result 1 to an X and 0 to a Y. Thus, in particular, $\sum_{m=1}^{k-1} U_m$ is hypergeometrically distributed. From this, we obtain

$$s^*_k = \sum_{i=1}^{k} s_i \frac{\binom{n}{i-1}\binom{r}{k-i}}{\binom{n+r}{k-1}} \frac{n-i+1}{n+r-k+1} = \frac{n}{n+r} \frac{1}{\binom{n+r-1}{k-1}} \sum_{i=1}^{k} \binom{n-1}{i-1}\binom{r}{k-i} s_i.$$

Finally, noting that $\binom{r}{k-i}$ is 0 if $i < k-r$ and that s_i is defined for $i \le n$, we can redefine the limits of the summing variable as in the statement of the theorem. □

Example 1. Let $n = 3$ and $r = 2$. Then the theorem gives that a 3-system with signature vector $\boldsymbol{s} = (s_1, s_2, s_3)$ is equivalent to a 5-system with signature vector

$$\boldsymbol{s}^* = \bigg(\frac{3s_1}{5}, \frac{3s_1 + 3s_2}{10}, \frac{s_1 + 4s_2 + s_3}{10}, \frac{3s_2 + 3s_3}{10}, \frac{3s_3}{5}\bigg).$$

We now present a corollary to Theorem 1 which is formulated in terms of cumulative signature vectors. More precisely, for an n-system and an equivalent $(n+1)$-system, with signatures $\boldsymbol{s}$ and $\boldsymbol{s}^*$, respectively, we introduce the cumulative signature vectors, respectively, $\boldsymbol{b}$ and $\boldsymbol{b}^*$ given by entries $b_j = \sum_{i=1}^{j} s_i$ for $j = 1, \dots, n$ and $b^*_j = \sum_{i=1}^{j} s^*_i$ for $j = 1, \dots, n+1$. The proof is straightforward using $r = 1$ in the theorem.

Corollary 1. *Let* $\boldsymbol{s}$ *be the signature vector of an* n*-system and let* $\boldsymbol{b}$ *be the corresponding cumulative signature vector. Then an equivalent coherent or mixed system with* $n+1$ *components*

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has the cumulative signature vector $\boldsymbol{b}^*$ *given by*

$$b_j^* = \begin{cases} b_{j-1} + \dfrac{n-j+1}{n+1} s_j & \text{for } j = 1, \dots, n, \\ 1 & \text{for } j = n+1, \end{cases}$$

where $b_0 = 0$.

3. When is there an equivalent system of lower order?

In this section we shall consider the problem of whether there exists a smaller sized system that is equivalent to a given system. This question is of evident practical importance, as the smaller system, consisting of components with identical lifetime distributions, may be constructed at a lower cost.

Thus, consider a coherent or mixed $(n+1)$-system with signature vector $\boldsymbol{s}^* = (s_1^*, \dots, s_{n+1}^*)$. The question we pose is whether there is an equivalent n-system, and if so, what is its signature vector $\boldsymbol{s} = (s_1, \dots, s_n)$.

By Definition 1, it may be natural to start by computing the reliability polynomial $h^*(p)$ corresponding to $\boldsymbol{s}^*$. If h^* has degree $n+1$, i.e. the coefficient of the term p^{n+1} is nonzero, then it is clear that there cannot be an equivalent system of lower size.

In [9, Chapter 6] an explicit recipe for computation of the coefficients of the reliability polynomial from the signature vector is given. Considering the coefficient of p^{n+1}, we hence obtain the necessary criterion for existence of an equivalent n-system,

$$\sum_{j=1}^{n+1} (-1)^{j-1} \binom{n+1}{j} b_j^* = 0, \tag{3}$$

where $\boldsymbol{b}^* = (b_1^*, \dots, b_{n+1}^*)$ is the cumulative signature vector as defined in the previous section. The following example shows, however, that (3) is not sufficient for the existence of an equivalent n-system.

Example 2. Suppose that $n = 5$ and let a mixed 6-system have cumulative signature vector

$$\boldsymbol{b}^* = \big(0, \tfrac{4}{10}, \tfrac{5}{10}, \tfrac{6}{10}, 1, 1\big).$$

A computation shows that (3) holds, while it will be seen below that there is no equivalent 5-system.

In view of this counter example, we will instead seek to 'invert' Theorem 1 by looking for solutions for $\boldsymbol{s} = (s_1, \dots, s_n)$ of the equations given there for $r = 1$, given the signature vector $\boldsymbol{s}^*$ of the $(n+1)$-system.

Using Corollary 1, we will hence seek to solve the equations

$$\begin{aligned} \frac{n}{n+1} s_1 &= b_1^*, \\ s_1 + \frac{n-1}{n+1} s_2 &= b_2^*, \\ s_1 + s_2 + \frac{n-2}{n+1} s_3 &= b_3^*, \\ &\vdots \end{aligned}$$

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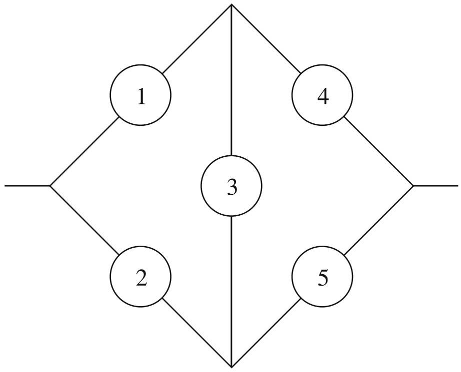

FIGURE 2: The bridge system.

$$s_1 + s_2 + \cdots + s_{n-1} + \frac{1}{n+1} s_n = b_n^*,$$
$$s_1 + s_2 + \cdots + s_n = 1. \tag{4}$$

The first n equations of (4) clearly have a unique solution for $s_1, \ldots, s_n$. However, it may happen that these s_i do not sum to 1, in which case the $(n+1)$th equation will not be satisfied. Further, even if $\sum_{i=1}^{n} s_i = 1$, it may happen that there are some negative s_i in the solution, again leading to the conclusion that there is no equivalent system. Examples of these possibilities are given in [3].

Somewhat unexpectedly, we are able to show (Proposition 1) that (3) holds if and only if $\sum_{i=1}^{n} s_i = 1$ when $(s_1, \ldots, s_n)$ is the solution of the first n equations of (4). A proof is provided in [3].

Proposition 1. *For an $(n+1)$-system, the reliability polynomial $h^*(p)$ is of degree n if and only if the solutions $(s_1, \ldots, s_n)$ to the first n equations of (4) sum to 1.*

Example 2. (*Continued.*) By solving the first $n(= 5)$ equations of (4), we obtain the solution

$$\boldsymbol{s} = \left(0, \tfrac{3}{5}, -\tfrac{1}{5}, \tfrac{3}{5}, 0\right)',$$

This is not a legitimate signature vector, and hence there is no 5-system equivalent to the given 6-system. Still, the solution sums to 1, as it should by Proposition 1.

Example 3. (*The bridge system.*) The bridge system (see Figure 2) is a standard example in textbooks of system reliability. This is a 5-system with signature vector $\boldsymbol{s}^* = (0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)$; see [9, Example 4.1]. The solution (s_1, s_2, s_3, s_4) of (4) is $(0, \frac{1}{3}, \frac{7}{6}, -\frac{5}{2})$, which neither sums to 1 nor is nonnegative. Thus, there is no 4-system equivalent to the bridge system, which would also be clear from computation of the reliability polynomial which is of degree 5. The result is hence consistent with Proposition 1. (We shall, however, see in the next section that there is a *coherent* 6-system which is equivalent to the bridge system).

4. Equivalence of coherent systems of different sizes

In this section we restrict our attention to coherent systems. The question we pose is, for a given coherent n-system, does there exist an equivalent *coherent* $(n+1)$-system?

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It is clear that for any coherent n-system, an equivalent $(n+1)$-system can be obtained by addition of an irrelevant component, with a lifetime which is independent of the lifetimes of the other components, and having the same distribution. While such a derived system is not coherent, we shall see that under certain conditions it may be modified to become a coherent $(n+1)$-system equivalent to the original coherent n-system.

As a simple example, consider the 2-out-of-3-system shown in Figure 1(a). This system has minimal path sets $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$. We have already noted that this 2-out-of-3 system is equivalent to the 4-system shown in Figure 1(b). The 4-system is here simply obtained by replacing component 3 in the minimal path set $\{2, 3\}$ by a new component 4. Alternatively, the system in Figure 1(b) could be obtained by replacing the minimal cut set $\{1, 3\}$ of the 2-out-of-3 system by the set $\{1, 4\}$.

In a similar way, we may, by direct inspection, obtain coherent 5-systems which are equivalent to, respectively, 2-out-of-4 and 3-out-of-4-systems, simply by modifying one of the minimal path or minimal cut sets of the respective 4-system. These findings suggest the general result for k-out-of-n systems given below. The proof is based on the following lemma which is essentially given in Boland [2], by noting that Boland's proof still holds for monotone systems which may contain irrelevant components. Recall that a *cut set* is a subset of the component set $\{1, 2, \dots, n\}$ such that the system fails if all components in the set have failed. Thus, a cut set necessarily contains at least one minimal cut set, but need not itself be minimal.

Lemma 1. (Boland [2].) *Consider a monotone n-system with cumulative signature vector $\boldsymbol{b} = (b_1, \dots, b_n)$. Then for $i = 1, \dots, n$,*

$$b_i = \frac{\#\ \textit{cut sets of size } i \textit{ for the system}}{\binom{n}{i}}.$$

Theorem 2. *For any k-out-of-n system, where $n \geq 3$ and $1 < k < n$, there exists an equivalent coherent $(n+1)$-system.*

Proof. The cut sets of a k-out-of-n system are all subsets of $\{1, 2, \dots, n\}$ of size k or larger. Consider now the equivalent $(n+1)$-system obtained by adding an irrelevant component, $n+1$, to the n components. The new system constructed this way has the following cut sets.

Size k. All the $\binom{n}{k}$ subsets of $\{1, 2, \dots, n\}$ of size k.

Size $k+1$. All the $\binom{n}{k}$ subsets of $\{1, 2, \dots, n\}$ of size k, with the component $n+1$ added to them, plus all the $\binom{n}{k+1}$ subsets of $\{1, 2, \dots, n\}$ of size $k+1$.

Size $r \in \{k+2, \dots, n+1\}$. All the $\binom{n+1}{r}$ subsets of $\{1, 2, \dots, n, n+1\}$ of size r.

The minimal cut sets of this new system are exactly the cut sets of size k as given above. Hence the system is not coherent, since the union of the minimal cut sets is a strict subset of $\{1, 2, \dots, n, n+1\}$; see [1, Exercise 5(a), p. 15]. The modification where the minimal cut set $\{n-k+1, n-k+2, \dots, n-1, n\}$ is replaced by $\{n-k+1, n-k+2, \dots, n-1, n+1\}$, and the other minimal cut sets are unchanged, will however be coherent. The assumption that $1 < k < n$ is crucial here (the result of the theorem does in fact not hold if $k = 1$ or $k = n$). We are done if we can prove that this system is equivalent to the incoherent $(n+1)$-system constructed above. To see this, consider the cut sets of the constructed coherent $(n+1)$-system. One can readily verify that these are as follows.

Size k. The $\binom{n}{k} - 1$ unmodified subsets of $\{1, 2, \dots, n\}$ plus the modified cut set $\{n-k+1, n-k+2, \dots, n-1, n+1\}$.

Size $k+1$. All the $\binom{n}{k}-1$ unmodified subsets of $\{1, 2, \dots, n\}$ of size k, with the component $n+1$ added to them, plus the modified set $\{n-k+1, n-k+2, \dots, n-1, n+1\}$ with the component n added to it. In addition, all the $\binom{n}{k+1}$ subsets of $\{1, 2, \dots, n\}$ of size $k+1$.

Size $r \in \{k+2, \dots, n+1\}$. All the $\binom{n+1}{r}$ subsets of $\{1, 2, \dots, n, n+1\}$ of size r.

It is seen that the number of cut sets of each size are unchanged when we modify the system. Hence, by Lemma 1, the cumulative signature vector $\boldsymbol{b}$ is not changed and hence the systems are equivalent. □

We shall next see that Theorem 2 can be extended to more general classes of coherent n-systems for which equivalent coherent $(n+1)$-systems can be constructed. For example, consider the bridge system shown in Figure 2. The minimal *path* sets of this system are $\{1, 4\}$, $\{2, 5\}$, $\{1, 3, 5\}$, and $\{2, 3, 4\}$. Consider the system as a network where the source node is to the left and the target node is to the right in the figure. Component 3 is then the 'bridge' in the network and it turns out to have a role as a twoway-relevant component. More precisely, in the minimal path set $\{1, 3, 5\}$, component 3 is relevant 'downwards', while in the minimal path set $\{2, 3, 4\}$, it is relevant 'upwards'. Thus, if we replace component 3 with a *directed* component, then this component will be relevant both if directed upwards and if directed downwards. All the other components have just one relevant direction. If we, for example, replace one of the components 1, 2, 4, 5 with a directed component, it will be relevant only if the direction is from the source node to the target node.

The above suggests that, based on the undirected bridge system, we can construct a 6-system where we replace component 3 by two components, $3'$ and $3''$, say, where $3'$ is directed downwards and $3''$ is directed upwards. In this case, $3'$ and $3''$ are said to be connected in anti-parallel. It is easy to check that the new 6-system is equivalent to the original bridge system with five components. Furthermore, the new system is also coherent.

This type of construction turns out to be generally valid for undirected two-terminal network systems containing at least one twoway-relevant component. An equivalent coherent $(n+1)$-system can then be constructed by replacing a twoway-relevant component by two directed components connected in anti-parallel. This is the motivating idea behind Theorem 3 below.

We find it convenient below to represent coherent systems in terms of their minimal cut sets. Since the minimal cut sets of a coherent system are exactly the minimal path sets of the dual system (see [1, p. 12]), all stated assumptions and results that involve minimal cut sets have equivalent versions for path sets.

In order to simplify the exposition we let, without loss of generality, the two components of special interest be the ones numbered as n and $n+1$.

Theorem 3. *Let Φ be a coherent n-system for which component n is contained in at least two minimal cut sets. Let Φ^* be the $(n+1)$-system obtained from Φ by replacing component n by $n+1$ in at least one, but not all, of these minimal cut sets of Φ. Then the system Φ^* is coherent. Further, Φ and Φ^* are equivalent provided any cut set K of Φ^* is of one of the following three types, where we define $K' = K \setminus \{n, n+1\}$ (the set of elements of K different from n and $n+1$).*

Type A: K' is not a cut set; $K' \cup \{n\}$ is a cut set; $K' \cup \{n+1\}$ is not a cut set.

Type B: K' is not a cut set; $K' \cup \{n\}$ is not a cut set; $K' \cup \{n+1\}$ is a cut set.

Type C: K' is a cut set.

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Proof. It is clear that Φ^* is coherent, since Φ is coherent and any component $1, 2, \dots, n+1$ is a member of at least one minimal cut set of Φ^*. We next construct a noncoherent $(n+1)$-system equivalent to Φ by adding an irrelevant component, named $n+1$, to Φ. In the following this system is denoted by Ψ. The cut sets of Ψ are all cut sets of Φ plus the sets obtained by adding component $n+1$ to each of these sets. We shall prove that the systems Φ^* and Ψ are equivalent by showing that they have the same number of cut sets of each size. The equivalence will then follow from Lemma 1. It is clearly enough to show that each cut set of Φ^* can be mapped to a cut set of Ψ of the same size, in a way such that no two cut sets of system Φ^* are mapped to the same cut set of system Ψ, and that the map is *onto*.

Thus, consider a cut set K of system Φ^*. We distinguish between the following disjoint cases.

Case 1: $n \notin K, n+1 \notin K$. The set K is then of type C and is clearly also a cut set for system Ψ. Thus, K is mapped to itself.

Case 2: $n \in K, n+1 \notin K$. Now K is either of type A or type C, but is in any case a cut set of Ψ. Thus, again K is mapped to itself.

Case 3: $n \notin K, n+1 \in K$. Here K is either of type B or type C. If it is of type C then it is clearly also a cut set of Ψ and K is mapped to itself.

If K is of type B then it is not a cut set of Ψ. This is because by the definition of type B, K must contain at least one minimal cut set of Φ^* which includes $n+1$, and K contains no other minimal cut set. Then K will be a cut set of Ψ, however, if component $n+1$ is replaced by component n in K, thereby changing K to $K'' = K' \cup \{n\}$. To see that K'' is indeed a cut set of Ψ, recall that K contains at least one minimal cut set of Φ^* which includes $n+1$. By changing $n+1$ to n, this minimal cut set becomes instead a minimal cut set of the system Ψ, and hence K'' must be a cut set of Ψ. We hence map K to K''. To ensure that our mapping of cut sets is 1-1, we then need to check that K'' is not one of the sets that is obtained in case 2 above. This is however not so since, by the definition of type B, K'' is not a cut set of Φ^*.

Case 4: $n \in K, n+1 \in K$. Now K can be any of the types A, B or C. If it is of type C then K' is a cut set of Φ^* and it is clear that K then is also a cut set of system Ψ. Thus, K is mapped to itself.

If K is of type A then it is also a cut set of Ψ, so again K is mapped to itself. To see this, note that by the definition of type A, K must contain at least one minimal cut set of Φ^* which includes n. But this set is also a minimal cut set of Ψ, and hence K is a cut set of Ψ as well.

Finally, if K is of type B then K contains at least one minimal cut set of Φ^* which includes $n+1$. But then K is also a cut set for system Ψ, since the present K contains n, and by our construction of Φ^*, n replaces $n+1$ when going from minimal cut sets of Ψ^* to minimal cut sets of Ψ. Thus, K is again mapped to itself.

We have thus shown that the cut sets K of Φ^* are mapped in a 1-1 fashion to the cut sets of Ψ. It remains to prove that the mapping is *onto* the collection of cut sets of Ψ. Thus, suppose for contradiction that there is a cut set L of Ψ which is not mapped from a cut set of Φ^* in the way considered above.

Assume first that L is a cut set of Φ^*. Then, by the proof of the 1-1 property, it would be mapped to itself, which is impossible, unless $n+1 \in L$, $n \notin L$, and L is of type B. But as stated in case 3 of the proof, L is not a cut set of Ψ, which gives a contradiction.

The only possibility is hence that L is not a cut set of Φ^*. Since it is a cut set of Ψ, it must hence contain at least one minimal cut set of the original Φ, where n was changed to $n+1$

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in the creation of Φ^*, and contain no other minimal cut set. Further, L cannot contain $n+1$. But then if n was changed to $n+1$ in L, the set would be a cut set of Ψ^*, or more precisely, $L \setminus \{n\} \cup \{n+1\}$ is a cut set of Φ^* of type B. But then it is seen from case 3 in the proof that L was already obtained by the mapping.

Thus the proof is complete. □

Remark 1. To see that the conditions in Theorem 3 are not always met, consider the 3-system Φ with minimal cut sets $\{1, 3\}$, $\{2, 3\}$. Then change the second minimal cut set to $\{2, 4\}$, thus defining the system Φ^* with minimal cut sets $\{1, 3\}$, $\{2, 4\}$, and 3 and 4 playing the roles of n and $n+1$, respectively. Now $K = \{1, 2, 3, 4\}$ is a cut set of Φ^*, while $K' = K \setminus \{3, 4\} = \{1, 2\}$ is not. But since both $K' \cup \{3\}$ and $K' \cup \{4\}$ are cut sets, K is not of any of the types A, B, C in Theorem 3.

Remark 2. Conditions for the equivalence of the systems Φ and Φ^* which are equivalent to the conditions of Theorem 3 can be given in terms of the structure function of the system Φ^*. Thus, let the structure function be given by $\Phi^*(y_1, \ldots, y_{n+1})$, which equals 1 if the system functions when the component states are $(y_1, \ldots, y_{n+1})$, and equals 0 if it is failed. Here y_i equals 1 if component i is working and 0 otherwise.

Define the following sets:

$$A = \{(y_1, \ldots, y_{n-1}) \mid \Phi^*(y_1, \ldots, y_{n-1}, 1, 0) > \Phi^*(y_1, \ldots, y_{n-1}, 0, 1)\},$$
$$B = \{(y_1, \ldots, y_{n-1}) \mid \Phi^*(y_1, \ldots, y_{n-1}, 0, 1) > \Phi^*(y_1, \ldots, y_{n-1}, 1, 0)\},$$
$$C = \{(y_1, \ldots, y_{n-1}) \mid \Phi^*(y_1, \ldots, y_{n-1}, 1, 1) = \Phi^*(y_1, \ldots, y_{n-1}, 0, 0)\}.$$

Now if $A \cup B \cup C = \{0, 1\}^{n-1}$, the conditions of Theorem 3 will be satisfied, and vice versa. A proof is given in [3].

By the definition of *a critical path vector* in Barlow and Proschan [1, pp. 13–14], it follows that A is the set of $(y_1, \ldots, y_{n-1})$ such that $(y_1, \ldots, y_{n-1}, 1, y_{n+1})$ is a critical path vector for component n (i.e. if the 1 in place n is changed to 0, then Φ^* changes from 1 to 0), for whatever y_{n+1}, and such that $(y_1, \ldots, y_{n-1}, y_n, 1)$ is *not* a critical path vector for component $n+1$, for any y_n.

A similar interpretation can be given for the set B, if components n and $n+1$ are interchanged in the above explanation for A.

Finally, C is the set of $(y_1, \ldots, y_{n-1})$ such that the state of the system is not influenced by the states of components n and $n+1$. Hence, no critical path vectors for n or $n+1$ can be formed from a vector $(y_1, \ldots, y_{n-1})$ in C.

Remark 3. It is easy to verify that Theorem 2 is a consequence of Theorem 3. In fact, if Φ is a k-out-of-n system, we can define Φ^* by changing component n to $n+1$ in exactly one minimal cut set of the k-out-of-n system, for example the set $\{n-k+1, \ldots, n-1, n\}$ (as we also did in the proof of Theorem 2). It is then straightforward to verify that the assumptions of Theorem 3 are satisfied.

Example 4. Let us reconsider the bridge system (Figure 2), which we used as a motivation for Theorem 3. The minimal cut sets are $\{1, 2\}$, $\{4, 5\}$, $\{1, 3, 5\}$, and $\{2, 3, 4\}$. Introduce a new component, 6, which replaces component 3 in the last minimal cut set. We claim that the conditions of Remark 2 are satisfied when respectively 3 and 6 play the roles of n and $n+1$ in the theorem. To see this, note that in the modified system, called Φ^* in the theorem, the vector $(0, 1, 1, 1, 0, y_6)$ is the only critical path vector for state 3, for any value of y_6, so

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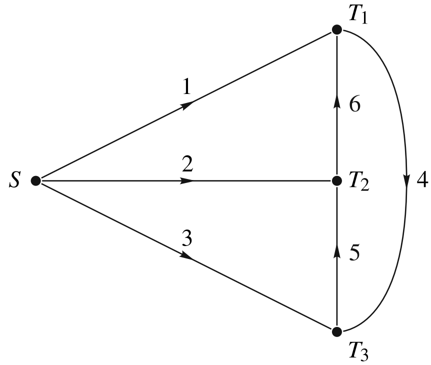

FIGURE 3: Directed network system with source node S and terminal nodes T_1, T_2, T_3.

that $A = \{(0, 1, 1, 0)\}$ gives the set of states for components 1, 2, 4, and 5 under this condition. By symmetry in the problem, $B = \{(1, 0, 0, 1)\}$ gives the states of components 1, 2, 4, and 5 for which component 6 is critical, which happens independently of the state of component 3. Finally, the set C contains the remaining 14 state vectors of components 1, 2, 4, and 5, and it is seen that the system state is not influenced by the states of components 3 and 6 for these state vectors. For example, for the state vector $(1, 1, 1, 0)$, components 1, 2, and 4 are working while 5 is failed. In this case the system is working whatever the state of components 3 and 6.

We close the section by giving an example where there is given a coherent $(n + 1)$-system, and we ask whether there exists an equivalent *coherent* n-system. In Section 3 we considered this question for general *mixed* systems. We are, however, not able to give a complete answer to the question when restricting to coherent systems.

The example shows, in particular, that the conditions of Theorem 3 for a coherent $(n + 1)$-system are not necessary for the existence of an equivalent coherent n-system. Somewhat surprisingly, however, we are in this example able to find an equivalent coherent $(n+1)$-system for which the conditions of Theorem 3 hold. A natural question is whether this is a general fact, i.e. that any coherent $(n + 1)$-system which has an equivalent coherent n-system, also has an equivalent coherent $(n + 1)$-system for which the conditions of Theorem 3 hold. Again, we do not have a general answer to this.

Example 5. Consider the 6-system Φ^* with the following seven minimal path sets: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 5\}$, $\{2, 3, 6\}$, $\{1, 4, 5\}$, $\{2, 4, 6\}$, $\{3, 5, 6\}$. This system can be represented as the directed network system depicted in Figure 3, with one source node, S, and three terminal nodes, T_1, T_2, T_3, where the system is said to be working if the source S can send signals through the network to all three terminal nodes.

The special feature of this system is that there is no pair of components that satisfies the condition in Theorem 3, i.e. can play the roles of n and $n + 1$. To check this, one can start by looking for pairs of components included in at least one common minimal path set. (As explained earlier, Theorem 3 can be reformulated to involve path sets instead of cut sets). Such component pairs will obviously not satisfy the condition on Φ^* in the theorem, and it is seen that almost all component pairs can be excluded precisely for this reason. The only pairs we are left with as possible candidates are $(1, 6)$, $(2, 5)$, and $(3, 4)$. Critical path vectors for component 6 depend, however, on the condition of component 1, so the pair $(1, 6)$ is not usable. The same applies to the two remaining pairs.

We then calculate the reliability polynomial of the system, $h(p) = 7p^3 - 9p^4 + 3p^5$. This polynomial is of degree 5 (which is a direct consequence of the fact that this is a directed cyclic network system), giving hope that there is an equivalent 5-system. It turns out, in fact, that there are four different coherent 5-systems equivalent to this system, namely systems 50–53 from the complete list of coherent 5-systems in Navarro and Rubio [4].

Finally, performing a search among 6-systems with reliability polynomials of order 5, using the file containing all 16,145 coherent 6-systems, referred to by Navarro and Rubio [4], we find that the 6-system with minimal path sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, and $\{2, 3, 6\}$ is equivalent to Φ^*, and furthermore satisfies the conditions of Theorem 3 with n and $n + 1$ given as, respectively, components 4 and 6.

5. The set of n-systems that stochastically dominate a given $(n + 1)$-system

We have seen that, for coherent systems as well as for mixed systems, it is not always possible to find equivalent systems of lower sizes. Still, for a given system, there may be reasons to look for interesting lower sized systems, for example due to the possible lower cost of building a smaller system. If there are no equivalent systems of lower size, one may instead look for smaller systems which in some sense perform approximately as well as the given one. In this section we study the class of mixed n-systems with signature vectors which stochastically dominate the signature of a given coherent or mixed $(n + 1)$-system.

Let an $(n+1)$-system with signature vector $\boldsymbol{s}^*$ and corresponding cumulative signature vector $\boldsymbol{b}^*$ be given. Suppose there is an n-system with signature $\boldsymbol{s} = (s_1, \dots, s_n)$ that stochastically dominates the given $(n + 1)$-system in the sense that the extension $\tilde{\boldsymbol{s}}$ of $\boldsymbol{s}$ to dimension $(n + 1)$, as described by Theorem 1, stochastically dominates $\boldsymbol{s}^*$ in the sense that the corresponding cumulative signature vectors satisfy

$$\tilde{b}_j \leq b_j^* \quad \text{for } j = 1, \dots, n. \tag{5}$$

(It should be noted that, by [9, Theorem 4.2], stochastic domination of signature vectors implies stochastic domination of the corresponding *system lifetimes* under i.i.d. component lifetimes).

The following theorem characterizes the signatures of the n-systems that dominate a given $(n + 1)$-system in this way.

Theorem 4. *Let there be given an $(n + 1)$-system with signature vector $\mathbf{s}^*$ and cumulative signature vector $\mathbf{b}^*$, satisfying $s_{n+1}^* \leq n/(n + 1)$. Then there is a nonempty convex set of signature vectors $\mathbf{s}$ of n-systems which stochastically dominate the given $(n+1)$-system, where each such vector satisfies the n inequalities*

$$\begin{aligned}
\frac{n}{n+1}s_1 &\leq b_1^*,\\
s_1 + \frac{n-1}{n+1}s_2 &\leq b_2^*,\\
s_1 + s_2 + \frac{n-2}{n+1}s_3 &\leq b_3^*,\\
&\vdots\\
s_1 + s_2 + \cdots + s_{n-2} + \frac{2}{n+1}s_{n-1} &\leq b_{n-1}^*,\\
s_1 + s_2 + \cdots + s_{n-1} &\leq \frac{n+1}{n}b_n^* - \frac{1}{n}.
\end{aligned} \tag{6}$$

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Proof. The $n-1$ first inequalities are the corresponding ones from (5), where we use Corollary 1 to represent the $\tilde{b}_j$. The nth inequality in (5) can be written as

$$s_1 + s_2 + \cdots + s_{n-1} + \frac{1}{n+1} s_n \le b_n^*$$

so that substitution of $s_n = 1 - s_1 - s_2 - \cdots - s_{n-1}$ gives the last inequality in (6). Furthermore, since a nonnegative solution for $\boldsymbol{s}$ can only exist if all the right-hand sides of the inequalities are nonnegative, the last inequality of (6) implies that an n-system that stochastically dominates an $(n+1)$-system with signature vector $\boldsymbol{s}^*$ can only exist if $b_n^* \ge 1/(n+1)$ or, equivalently, $s_{n+1}^* \le n/(n+1)$. (Informally this can be stated, 'unless the $(n+1)$-system is a parallel system, or close to being so, we can find a better n-system'.) This completes the proof. □

As a first application, suppose we would like to build an n-system which is at least as good as a given $(n+1)$-system, at minimum cost. ('At least as good' here means with respect to stochastic ordering of signature vectors.) Let the expected cost of a system with signature vector $\boldsymbol{s}$ be $\sum_{i=1}^{n} c_i s_i$, where $0 \le c_1 \le c_2 \le \cdots \le c_n$. (A discussion and motivation for this cost function is given in Section 6.) Then write

$$\sum_{i=1}^{n} c_i s_i = \sum_{i=1}^{n-1} c_i s_i + c_n(1 - s_1 - \cdots - s_{n-1}) = c_n - \sum_{i=1}^{n-1} (c_n - c_i) s_i.$$

Thus, the problem of minimizing the cost of the n-system over the convex set of signature vectors satisfying (6) is equivalent to maximizing the linear combination $\sum_{i=1}^{n-1}(c_n - c_i)s_i$. By the theory of linear programming (see, e.g. Nering and Tucker [6]) this maximum will occur at an extreme point of the convex set defined in Theorem 4, where we need to add the conditions $s_1 \ge 0, \ldots, s_{n-1} \ge 0$.

Example 6. Let $n = 3$ and let a coherent $(n+1)$-system have minimal cut sets $\{1\}, \{2, 3, 4\}$. The signature vector is then $\boldsymbol{s}^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$, while $\boldsymbol{b}^* = (\frac{1}{4}, \frac{1}{2}, 1, 1)$. Putting $n = 3$ in Theorem 4, we obtain the inequalities

$$\tfrac{3}{4} s_1 \le \tfrac{1}{4}, \qquad s_1 + \tfrac{2}{4} s_2 \le \tfrac{1}{2}, \qquad s_1 + s_2 \le \tfrac{4}{3} - \tfrac{1}{3} = 1.$$

Adding the conditions $s_1 \ge 0$, $s_2 \ge 0$, it can be seen that the extreme points of the resulting convex set of (s_1, s_2) are

$$(0, 0), \ \left(\tfrac{1}{3}, 0\right), \ \left(\tfrac{1}{3}, \tfrac{1}{3}\right), \ (0, 1). \tag{7}$$

so that the extreme points of the set of stochastically dominating signature vectors of size 3 are $(0, 0, 1)$, $(\frac{1}{3}, 0, \frac{2}{3})$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(0, 1, 0)$. The first and last of these are signatures of coherent systems, while the other two are not. The signature $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is, on the other hand, the signature of a 3-system equivalent to a single component, i.e. a 1-system. It is intuitively clear that this system is stochastically better than the given system, since the latter has a minimal cut set *in addition to* the set $\{1\}$, which makes it more frail. Which of these systems might be considered best if system costs were taken into account remains undetermined at this stage. We examine such questions in the next section.

6. Application to reliability economics

Samaniego [9, Chapter 7] considered the problem of optimizing the performance of a system under given cost constraints. Here the performance of a system with signature vector $\boldsymbol{s} = (s_1, \ldots, s_n)$ is represented as a linear function of the signatures, $\sum_{i=1}^{n} h_i s_i$. A motivation for this choice of performance measure is that both the expected lifetime of the system and

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its reliability function can be written in this way. More precisely, $\mathbb{E}T = \sum_{i=1}^{n} s_i \mathbb{E}X_{i\,:\,n}$ and $\mathbb{P}(T > t_0) = \sum_{i=1}^{n} s_i \mathbb{P}(X_{i\,:\,n} > t_0)$. An important property of these measures is that their values are invariant among equivalent systems.

Similarly, in [9, Chapter 7] it was assumed that the expected cost of building a system with signature vector $\boldsymbol{s}$ is $\sum_{i=1}^{n} c_i s_i$, where c_i is interpreted as the cost of building an i-out-of-n system, $i = 1, \ldots, n$. In order to determine the values of c_i, Samaniego [9, p. 95] suggested the so-called *salvage model*. Here, it is assumed that the cost c_i of an i-out-of-n system can be written as

$$c_i = C + nA - (n - i)B \quad \text{for } i = 1, \ldots, n, \tag{8}$$

where C is the initial fixed cost of manufacturing the system, A is the cost of an individual component, and B is the salvage value of a used but working component which is removed after system failure.

In our subsequent application it is of interest to compare costs of systems of different sizes. The following Proposition, based on the salvage model, is useful.

Proposition 2. *Suppose that the costs of n-systems and $(n + 1)$-systems are given by the salvage model (8) with the same constants A and B, but with possibly different constants C, respectively C and C^* for the n-systems and $(n + 1)$-systems. Then the cost reduction by using an i-out-of-n system compared to the equivalent $(n + 1)$-system is*

$$C^* - C + (A - B) + \frac{iB}{n + 1}.$$

Proof. Consider an i-out-of-n-system with expected cost given by (8). To obtain an equivalent $(n + 1)$-system, add an irrelevant component to the original n-system, with a lifetime independent of the components of the n-system and with the same lifetime distribution. The cost of this extra component is A units, but the component can be salvaged for B units if it does not fail before the i-out-of-n system fails. The probability of the latter event equals the probability that in the simultaneous ordering of the lifetimes $X_1, \ldots, X_n$ of the original components and the lifetime Y of the irrelevant component, Y is not among the i smallest. This probability is $1 - i/(n + 1)$. Thus, using the equivalent $(n + 1)$-system obtained this way adds a cost $A - B(1 - i/(n + 1)) = A - B + i/(n + 1)B$ to the original system. Adding $C^* - C$, which is the difference in fixed costs, proves the proposition. □

Thus, there are reasons to assume that the expected cost of equivalent systems is reduced when reducing the number of components. The salvage model (8) provides an explicit way of expressing this reduction, if we assume that the fixed costs A, B are independent of n. Since, naturally, $A > B > 0$, the n-system hence has lower expected cost, unless C is too large in comparison with C^* (normally one would assume that $C = C^*$ for the fixed costs).

For a given performance vector $(h_1, \ldots, h_n)$ and a cost vector $(c_1, \ldots, c_n)$, [9, Chapter 7] defined the following measure of the relative value of performance versus cost for a mixed n-system with signature vector $\boldsymbol{s}$:

$$m_r(\boldsymbol{s}, \boldsymbol{h}, \boldsymbol{c}) = \frac{\sum_{i=1}^{n} h_i s_i}{(\sum_{i=1}^{n} c_i s_i)^r}. \tag{9}$$

As explained in [9, p. 97], the power parameter $r > 0$ serves as a calibration parameter, determining the weight to be put on cost relative to performance in the criterion function (9). Thus, $r = 1$ is the natural choice if equal weight is put on performance and cost.

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The optimality problem considered in [9, Chapter 7] is the problem of maximizing the performance-per-cost criterion (9) with respect to the signature vector $\boldsymbol{s}$ among all mixed systems of the given size.

In the following we consider a different problem, namely whether one may increase performance-per-cost by building smaller systems, i.e. systems with fewer components. The motivation is that a smaller system, equivalent to a larger one, is expected to have a lower cost than the larger system, while performing exactly as well as the larger system. However, as there may not be equivalent systems of smaller size, we will instead search for smaller systems in the class of systems dominating the given one, as characterized in Theorem 4.

Consider the following situation. Suppose that there is given an $(n+1)$-system with signature vector $\boldsymbol{s}^*$. Let there also be given a cost vector $\boldsymbol{c}^* = (c_1^*, \dots, c_{n+1}^*)$, where c_i^* defines the expected cost of an i-out-of-$(n+1)$-systems for $i = 1, \dots, n+1$. Suppose similarly that there is given a performance vector $\boldsymbol{h}^* = (h_1^*, \dots, h_{n+1}^*)$. For the given system, the value corresponding to the criterion function (9) is, hence,

$$A^* = m_r(\boldsymbol{s}^*, \boldsymbol{h}^*, \boldsymbol{c}^*) = \frac{\sum_{i=1}^{n+1} h_i^* s_i^*}{(\sum_{i=1}^{n+1} c_i^* s_i^*)^r}. \tag{10}$$

Now, if the given $(n+1)$-system has an equivalent n-system with criterion function (9), then their criterion functions will have the same numerator, and hence the criterion function for the n-system will be the smaller of the two if and only if the n-system has a lower expected cost.

Suppose next that we start with an $(n+1)$-system for which there is no equivalent n-system. There may be reasons to search among the class of n-systems that stochastically dominate the given $(n+1)$-system. One motivation is that these systems may still have a lower cost than the $(n+1)$-system, and they will also have a better performance.

Theorem 4 defines the convex set of signature vectors of all n-systems that stochastically dominate the given $(n+1)$-system. We assume below that the required condition on s_{n+1}^* given in the theorem is satisfied, and we let R denote the convex set of $(s_1, \dots, s_{n-1})$ defined by Theorem 4. We now seek to maximize the criterion function m_r from (9) over this set. Since $s_n = 1 - s_1 - \cdots - s_{n-1}$ we may write (9) as

$$m_r(\boldsymbol{s}, \boldsymbol{h}, \boldsymbol{c}) = \frac{h_n - \sum_{i=1}^{n-1}(h_n - h_i)s_i}{(c_n - \sum_{i=1}^{n-1}(c_n - c_i)s_i)^r} \equiv \frac{h_n - \sum_{i=1}^{n-1} \tilde{h}_i s_i}{(c_n - \sum_{i=1}^{n-1} \tilde{c}_i s_i)^r}, \tag{11}$$

where $\tilde{c}_i = c_n - c_i$ and $\tilde{h}_i = h_n - h_i$ for $i = 1, \dots, n-1$.

We claim that the maximum of (11) occurs on the boundary of the convex set R. To see this, assume for contradiction that the maximum occurs at an interior point $\hat{\boldsymbol{s}}$ of R. At this point, consider the two hyperplanes of $(s_1, \dots, s_{n-1})$ for which, respectively, the linear functions $\sum_{i=1}^{n-1} \tilde{h}_i s_i$ and $\sum_{i=1}^{n-1} \tilde{c}_i s_i$ have the same value as in the optimum point $\hat{\boldsymbol{s}}$. Let N be an open neighborhood of $\hat{\boldsymbol{s}}$ which is included in R, and consider the intersection N_0 of N and the above hyperplane defined by the $\tilde{c}_i$. On this set, the denominator of (11) is constant. The set N_0 will, however, contain points on both sides of the hyperplane defined by the $\tilde{h}_i$, meaning that $\sum_{i=1}^{n-1} \tilde{h}_i s_i$ on N_0 will have values both larger and smaller than its value at $\hat{\boldsymbol{s}}$. But then (11) will in N_0 take values larger than the value at $\hat{\boldsymbol{s}}$, which gives a contradiction since the maximum is assumed to be at $\hat{\boldsymbol{s}}$. Thus the maximum point of (11) is a boundary point of R.

The above argument clearly holds for all r. We now argue that for $r = 1$, the maximum value of (11) must be at an *extreme point* of R. Suppose for contradiction that the maximum is at a boundary point of R which is not an extreme point. Let the maximum value of (11) be A.

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Then (11) equals A in a hyperplane in the space of $s_1, \dots, s_{n-1}$ (the hyperplane will depend on A). But since the hyperplane contains a point on the boundary of R which is not an extreme point, the hyperplane will intersect the interior of R. Hence the interior of R will also contain an optimum point of (11). But this is impossible by what we have already seen for general r, so we have a contradiction. This shows that the maximum of (11) for $r = 1$ must be at an extreme point.

While it is shown in [9, Chapter 7] that, for $r \neq 1$, the maximum of m_r on the full simplex of signature vectors $\boldsymbol{s} = (s_1, \dots, s_n)$ is attained for an $\boldsymbol{s}$ with at most two positive elements, it will be seen in an example below that the maximum of (11) restricted to the set R may well occur at boundary points with more than two positive entries.

Example 6. (*Continued.*) We consider the convex set of signatures for 3-systems which stochastically dominate the given 4-system. Suppose that costs are given by a salvage model where the constants A, B, C are equal for 3- and 4-systems, with values $C = \frac{1}{10}$, $A = \frac{3}{5}$, $B = \frac{1}{2}$. This gives the cost values for the 4-system and the 3-system given by, respectively, $\boldsymbol{c}^* = (1, \frac{3}{2}, 2, \frac{5}{2})$ and $\boldsymbol{c} = (\frac{9}{10}, \frac{7}{5}, \frac{19}{10})$.

Let the component lifetimes X_i be exponential with expected value 1. It is well known that $\mathbb{E}X_{i:n} = 1/n + 1/(n-1) + \cdots + 1/(n-i+1)$ for $i = 1, \dots, n$, so we have the performance vectors for the 4-system and 3-system given by, respectively, $\boldsymbol{h}^* = (\frac{1}{4}, \frac{7}{12}, \frac{13}{12}, \frac{25}{12})$ and $\boldsymbol{h} = (\frac{1}{3}, \frac{5}{6}, \frac{11}{6})$.

It follows that A^* from (10) equals

$$A^* = \frac{6 \cdot 8^{r-1}}{13^r}.$$

Since we consider the set of 3-systems which stochastically dominate the given system, the function to maximize is

$$m_r(s_1, s_2) = \frac{11/6 - (3/2)s_1 - s_2}{(19/10 - s_1 - (1/2)s_2)^r}$$

on the convex set R with extreme points given in (7): $(0, 0)$, $(\frac{1}{3}, 0)$, $(\frac{1}{3}, \frac{1}{3})$, $(0, 1)$.

A computer search was used to find the optimal point in R for any $r > 0$. In accordance with the above theoretical discussion, the optimal point was always found on the boundary of R. More precisely, the result was that for, approximately, $r \leq 1.55$, the maximum is obtained at $(s_1, s_2, s_3) = (0, 0, 1)$, i.e. a parallel system of 3 components. For $1.55 \leq r \leq 1.77$, the optimum point changes continuously from $(0, 0, 1)$ to $(\frac{1}{3}, 0, \frac{2}{3})$ along the path $(p, 0, 1-p)$ for $0 \leq p \leq \frac{1}{3}$. Next, for $1.77 \leq r \leq 2.35$ the optimum is at the single point $(\frac{1}{3}, 0, \frac{2}{3})$. When r increases further, $2.35 \leq r \leq 2.80$, the optimum changes continuously from $(\frac{1}{3}, 0, \frac{2}{3})$ to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ along the path $(\frac{1}{3}, p, \frac{2}{3} - p)$ for $0 \leq p \leq \frac{1}{3}$, while for $r \geq 2.80$ the optimum is steady at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The optimum value of the criterion function is for each $r > 0$ larger than the corresponding A^* for the original $(n+1)$-system. We finally note as a curiosity that the optimal 3-system for large r, i.e. $r \geq 2.80$, is the system which is equivalent to a system consisting of one single component. Since large r favors low-cost systems, this should be reasonable. On the other hand, small r is supposed to favor high-performance systems. This is seen in the present example from the fact that the optimum for $r \leq 1.55$ is attained for the parallel system with three components.

7. Concluding remarks

This paper is concerned with the existence of equivalent systems of different sizes, where equivalence means having the same system lifetime distribution under i.i.d. componentlifetimes.

Within the class of mixed systems one can always find equivalent mixed systems of larger sizes, but not necessarily if the size is decreased. An obvious necessary condition for an $(n+1)$-system to have an equivalent n-system, is that the reliability polynomial of the former is of degree n. This, however, is not sufficient in general for the existence of equivalent mixed systems of lower size, as was shown by a counter example in Section 3. We have, however, not been able to find a counterexample when restricting to coherent systems. In fact, a complete search among all possible coherent 5-systems showed that to each system with reliability polynomials of order 4 corresponds an equivalent coherent system of order 4. Similarly, all 4-systems with reliability polynomials of order 3 are equivalent to coherent 3-systems (while there is no 3-system with a reliability polynomial of order 2). A corresponding search among the possible 6-systems has not been performed, but we note that our Example 5 shows a case of a coherent 6-system with reliability polynomial of order 5, and for which there is an equivalent coherent 5-system.

In this paper we have mostly studied the problem of finding pairs of equivalent systems of sizes that differ by *one* component. Referring to our motivating example in Section 4 involving twoway-relevant components, it may be possible to derive equivalence results also for coherent systems that differ in size by more than one component by considering directed systems containing more than one twoway-relevant component. We have not pursued such a task. It is notable, however, that the 2-out-of-3 system with minimal cut sets $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ is equivalent to the coherent 5-system with minimal cut sets $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3, 5\}$. (This is in fact the only coherent 5-system which has reliability polynomial of order 3).

We have seen that, for coherent systems as well as for mixed systems, it is not always possible to find equivalent systems of lower sizes. Still, for a given system, there may be reasons to look for interesting lower sized systems, for example due to the possible lower cost of building a smaller system. In Sections 5 and 6 we have studied the class of (mixed) n-systems with signature vector which stochastically dominates the signature of a given coherent or mixed $(n + 1)$-system, motivated by certain optimization problems in reliability economics.

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