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EXAMPLES OF FINITE-DIMENSIONAL POINTED HOPF ALGEBRAS IN CHARACTERISTIC 2†

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Abstract. We present new examples of finite-dimensional Nichols algebras over fields of characteristic 2 from braided vector spaces that are not of diagonal type, admit realizations as Yetter–Drinfeld modules over finite abelian groups, and are analogous to Nichols algebras of finite Gelfand–Kirillov dimension in characteristic 0. New finite-dimensional pointed Hopf algebras over fields of characteristic 2 are obtained by bosonization with group algebras of suitable finite abelian groups.

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1. Introduction. The goal of this paper is to present new examples of finite-dimensional Hopf algebras in characteristic 2, which are pointed, non-commutative, and non-cocommutative. Following the usual guidelines of the lifting method, we focus on finite-dimensional Nichols algebras, then the Hopf algebras are obtained routinely by bosonization. The main result of [**2**] (in characteristic 0) is the classification of the Nichols algebras with finite Gelfand–Kirillov dimension arising from braided vector spaces (V, c) that decompose as:

$$V = V_1 \oplus \cdots \oplus V_t \oplus V_{t+1} \oplus \cdots \oplus V_\theta, \qquad c(V_i \otimes V_j) = V_j \otimes V_i,\ i, j \in \mathbb{I}_\theta,$$

where $V_1, \ldots, V_t$ are blocks (see Section 2.2); $V_{t+1}, \ldots, V_\theta$ are points (i.e. have dimension 1); and the braidings have a specific form, see for example, (3.2), (5.2). This result relies on the classification in [**7**] and assumes a Conjecture treated partially in [**3**], both about Nichols algebras of diagonal type. However in positive characteristic, the

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Table 1. Finite-dimensional Nichols algebra in characteristic 2.

V	$\mathscr{B}(V)$	$\dim K$	$\dim \mathscr{B}(V)$
$\mathfrak{L}_{\wp}(1, 1)$	Proposition 3.6	2^3	2^7
$\mathfrak{L}_{\wp}(1, a),\ a \neq 1$	Proposition 3.7	2^4	2^8
$\mathfrak{P}(\mathbf{q}, \mathbf{a}),\ \mathbf{a} \in (\Bbbk^{\times})^t$	Proposition 5.4	$2^{\lvert\mathcal{A}\rvert}$	$2^{4t+\lvert\mathcal{A}\rvert}$
$\mathfrak{E}_{\wp}(1)$	Proposition 6.2	2^2	2^4
$\mathfrak{E}_{\wp}(\omega),\ \omega \in \mathbb{G}'_3$	Proposition 6.3	3^3	$2^2 3^3$

classification of finite-dimensional Nichols algebras of diagonal type is known only in rank ≤ 4 [9, 10, 11]. Inspired by [6] and by familiar phenomena in Lie theory in positive characteristic, examples of finite-dimensional Nichols algebras in odd characteristic were constructed in [4] by analogy with the Nichols algebras in [2] that have infinite dimension. Here we extend these constructions assuming that the base field $\Bbbk$ is algebraically closed of characteristic 2. There are new features as $1 = -1$ now. For instance in characteristic 0, two main actors are the Jordan and the super Jordan planes. Their restricted versions in characteristic $p > 2$ have dimensions p^2 [6] and $4p^2$ [4], respectively. When char $\Bbbk = 2$, they merge in the restricted Jordan plane that has dimension $16 = 4 \times 2^2$ [6]. Other families of [2] also merge. Finally, the fact that $x_i^2 = 0$ for suitable x_i in the braided vector space brings on more examples with finite dimension. Let us present the main result of this paper.

THEOREM. *If V is a braided vector space as in Table 1, then the dimension of the Nichols algebra $\mathscr{B}(V)$ is finite.*

See Section 2.3.2 for the meaning of K. The braided vector spaces $\mathfrak{L}_{\wp}(1, 1)$ appear to be close to $\mathfrak{L}(-1, \mathscr{G})$ and $\mathfrak{L}_{-1}(-1, \mathscr{G})$ in [4, Table 1], but $\mathscr{B}(\mathfrak{L}_{\wp}(1, a))$, $a \neq 1$ has no finite-dimensional analog in char $\Bbbk = p > 2$. Similarly, the algebras $\mathscr{B}(\mathfrak{P}(\mathbf{q}, \mathbf{a}))$ are finite-dimensional in odd characteristic only when the entries of $\mathbf{a}$ belong to the prime field, in contrast with characteristic 2. Also $\mathfrak{E}_{\wp}(\omega)$ does not appear in the *loc. cit.* Albeit no classification is envisageable yet as the knowledge of diagonal type is still incomplete, we present partial results in Theorems 3.1, 4.1, and 6.1.

After spelling out some preliminaries in Section 2, we devote Sections 3, 4, 5, and 6 to Nichols algebras of one block and one point, one block and several points, several blocks and one point, and one pale block and one point, respectively. Our proofs rely on the splitting technique Section 2.3.2 and the classifications in [9, 10, 11]. Explicit examples of finite-dimensional pointed Hopf algebras are discussed in Sections 3.2, 5.2, and 6.2. More examples by lifting will be presented in a future work.

2. Preliminaries.

2.1. Notations and conventions. We denote the natural numbers by $\mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We set $\mathbb{I}_{k,\ell} = \{n \in \mathbb{N}_0 : k \leq n \leq \ell\}$, $\mathbb{I}_\ell = \mathbb{I}_{1,\ell}$ and $\mathbb{N}_{\geq \ell} = \mathbb{N} \setminus \mathbb{I}_{\ell-1}$, for $k < \ell \in \mathbb{N}_0$. We work over an algebraically closed field $\Bbbk$ of characteristic 2. The group of N-th roots of unity in $\Bbbk$ is denoted by $\mathbb{G}_N$; $\mathbb{G}'_N$ is the subset of the primitive roots of order N and $\mathbb{G}_\infty = \bigcup_{N \in \mathbb{N}} \mathbb{G}_N$.

Throughout H is a Hopf algebra with bijective antipode $\mathcal{S}$. We use the notations $G(H) =$ the group of grouplikes in H, $\mathcal{P}(H) =$ the space of primitive elements, $\widehat{H} = \mathrm{Hom}_{\mathrm{alg}}(H, \Bbbk)$, ${}^H_H\mathcal{YD} =$ the category of Yetter–Drinfeld modules over H; see for example [12, 11.6].

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2.2. Yetter–Drinfeld modules.

2.2.1. Braided vector spaces

A braided vector space V is a pair (V, c) where V is a vector space and $c \in GL(V^{\otimes 2})$ is a solution of the braid equation:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

We are interested in two classes of braided vector spaces. First, (V, c) or simply V is of diagonal type if there exist a basis $(x_i)_{i\in\mathbb{I}_\theta}$ of V and a matrix $\mathbf{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$ such that $q_{ij} \in \Bbbk^\times$ and $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ for all $i, j \in \mathbb{I}_\theta$. We denote in $T(V)$, or any quotient braided Hopf algebra:

$$x_{ij} = (\mathrm{ad}_c\, x_i)\, x_j, \qquad x_{i_1 i_2 \dots i_M} = (\mathrm{ad}_c\, x_{i_1})\, x_{i_2 \dots i_M}, \qquad i, j, i_1, \dots, i_M \in \mathbb{I}, \qquad M \geq 2.$$

Second, let $\epsilon \in \Bbbk^\times$ and $\ell \in \mathbb{N}_{\geq 2}$. A *block* $\mathcal{V}(\epsilon, \ell)$ is a braided vector space with a basis $(x_i)_{i\in\mathbb{I}_\ell}$ such that for $i, j \in \mathbb{I}_\ell, j > 1$:

$$c(x_i \otimes x_1) = \epsilon x_1 \otimes x_i, \qquad c(x_i \otimes x_j) = (\epsilon x_j + x_{j-1}) \otimes x_i. \tag{2.1}$$

For simplicity, a block $\mathcal{V}(\epsilon, 2)$ of dimension 2 is called an ϵ-block.

2.2.2. Realizations

Any Yetter–Drinfeld module V bears a structure of braided vector space by $c(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}$, $v, w \in V$ where $\delta(v) = v_{(-1)} \otimes v_{(0)}$. The braided vector spaces above appear as Yetter–Drinfeld modules in different ways called *realizations*. Let Γ be an abelian group and let $\widehat{\Gamma}$ be the group of characters of Γ. The Yetter–Drinfeld modules over the group algebra $\Bbbk\Gamma$ are the Γ-graded Γ-modules, the Γ-grading being denoted by $V = \oplus_{g\in\Gamma} V_g$; thus $h \cdot V_g = V_g$ for $g, h \in \Gamma$. If $g \in \Gamma$ and $\chi \in \widehat{\Gamma}$, then the one-dimensional vector space $\Bbbk_g^\chi$, with action and coaction given by g and χ, is in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$. Given $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ with a basis $(v_i)_{i\in I}$ where v_i is homogeneous of degree g_i, there are skew derivations ∂_i, $i \in I$, of $T(V)$ such that:

$$\partial_i(v_j) = \delta_{ij}, \ \ i, j \in I, \qquad \partial_i(xy) = \partial_i(x)(g_i \cdot y) + x\partial_i(y), \ \ x, y \in T(V). \tag{2.2}$$

More generally, a *YD-pair* for H is a pair $(g, \chi) \in G(H) \times \widehat{H}$ such that:

$$\chi(h)\, g = \chi(h_{(2)})h_{(1)}\, g\, \mathcal{S}(h_{(3)}), \qquad h \in H. \tag{2.3}$$

Let $\Bbbk_g^\chi$ be a one-dimensional vector space with H-action and H-coaction given by χ and g, respectively; then (2.3) says that $\Bbbk_g^\chi \in {}^H_H\mathcal{YD}$. Thus, a realization of V of diagonal type with matrix $\mathbf{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$ is just a collection $(g_1, \chi_1), \dots, (g_\theta, \chi_\theta)$ such that $q_{ij} = \chi_j(g_i)$ for all $i, j \in \mathbb{I}_\theta$.

2.2.3. Realizations of ϵ-blocks

For $\chi \in \widehat{H}$, the space of (χ, χ)-derivations is

$$\mathrm{Der}_{\chi,\chi}(H, \Bbbk) = \{\eta \in H^* : \eta(h\ell) = \chi(h)\eta(\ell) + \chi(\ell)\eta(h)\ \forall h, \ell \in H\}.$$

The realizations of ϵ-blocks are given by the notion of *YD-triple* for H [4]; this is a collection (g, χ, η) where (g, χ), is a YD-pair for H, $\eta \in \mathrm{Der}_{\chi,\chi}(H, \Bbbk)$, $\chi(g) = \epsilon$, $\eta(g) = 1$ and

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$$\eta(h)g = \eta(h_{(2)})h_{(1)}g\mathcal{S}(h_{(3)}), \qquad h \in H. \tag{2.4}$$

Given a YD-triple (g, χ, η), we define $\mathcal{V}_g(\chi, \eta) \in {}^H_H\mathcal{YD}$ as the vector space with a basis $(x_i)_{i\in\mathbb{I}_2}$, whose H-action and H-coaction are given by:

$$h \cdot x_1 = \chi(h)x_1, \qquad h \cdot x_2 = \chi(h)x_2 + \eta(h)x_1, \qquad \delta(x_i) = g \otimes x_i, \qquad h \in H,\ i \in \mathbb{I}_2.$$

Then, $\mathcal{V}_g(\chi, \eta) \simeq \mathcal{V}(\epsilon, 2)$ as braided vector spaces.

EXAMPLE 2.1. Let $\epsilon = 1$ and $\Gamma = \langle g\rangle$ be a cyclic group of order N. Let $\mathcal{V}$ be the vector space with a basis $(x_i)_{i\in\mathbb{I}_2}$ with grading $\deg x_i = g$, $i \in \mathbb{I}_2$. Then the assignment $g \longmapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ defines a representation of Γ (hence, a structure of Yetter–Drinfeld module over $\Bbbk\Gamma$) if and only if N is even. Thus if $\dim H < \infty$ and H admits a YD-triple (for $\epsilon = 1$), then $\dim H$ is even.

2.3. Nichols algebras. Let $V \in {}^H_H\mathcal{YD}$. The Nichols algebra of V is the unique graded connected Hopf algebra $\mathscr{B}(V) = \oplus_{n\geq 0}\mathscr{B}^n(V)$ in ${}^H_H\mathcal{YD}$ such that $V \simeq \mathscr{B}^1(V) = \mathcal{P}(\mathscr{B}(V))$ generates $\mathscr{B}(V)$ as algebra. See [1] for an exposition. The algebra and coalgebra underlying $\mathscr{B}(V)$ depend only on the braiding. If $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ is as in Section 2.2, then the ∂_i's induce skew-derivations on $\mathscr{B}(V)$. Then, $w \in \mathscr{B}^k(V)$, $k \geq 1$, is 0 if and only if $\partial_i(w) = 0$ in $\mathscr{B}(V)$ for all $i \in I$.

2.3.1. The restricted Jordan plane

This is the Nichols algebra of a 1-block.

THEOREM 2.2 ([6]). *The algebra $\mathscr{B}(\mathcal{V}(1, 2))$ is presented by generators x_1, x_2 and relations:*

$$x_1^2, \qquad x_2^4, \qquad x_2^2x_1 + x_1x_2^2 + x_1x_2x_1, \qquad x_1x_2x_1x_2 + x_2x_1x_2x_1. \tag{2.5}$$

Let $x_{21} := x_1x_2 + x_2x_1$. Then, $\dim \mathscr{B}(\mathcal{V}(1, 2)) = 16$ *since $\mathscr{B}(\mathcal{V})$ has a basis:*

$$\{x_1^{m_1}x_{21}^{m_2}x_2^{n} : m_1, m_2 \in \mathbb{I}_{0,1},\ n \in \mathbb{I}_{0,3}\}. \qquad \square$$

2.3.2. The splitting technique

Let $V = V_1 \oplus V_2$ be a direct sum of objects in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$. Then, $\mathscr{B}(V) \simeq K\#\mathscr{B}(V_1)$ where $K = \mathscr{B}(V)^{\operatorname{co}\mathscr{B}(V_1)}$. By [8, Proposition 8.6], K is the Nichols algebra of

$$K^1 = \operatorname{ad}_c \mathscr{B}(V_1)(V_2). \tag{2.6}$$

Here, $K^1 \in {}^{\mathscr{B}(V_1)\#\Bbbk\Gamma}_{\mathscr{B}(V_1)\#\Bbbk\Gamma}\mathcal{YD}$ with the adjoint action and the coaction given by

$$\delta = (\pi_{\mathscr{B}(V_1)\#\Bbbk\Gamma} \otimes \operatorname{id})\Delta_{\mathscr{B}(V)\#\Bbbk\Gamma}. \tag{2.7}$$

3. One block and one point. Let $(q_{ij})_{i,j\in\mathbb{I}_2}$, $q_{ij} \in \Bbbk^\times$, $a \in \Bbbk$. In this section, we assume that

$$q_{11} = 1, \qquad q_{12}q_{21} = 1. \tag{3.1}$$

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Sometimes, we use $\wp = q_{12} = q_{21}^{-1}$. Let $\mathfrak{L}_{\wp}(q_{22}, a)$ be the braided vector space with basis $(x_i)_{i\in\mathbb{I}_3}$ and braiding given by

$$(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_3} = \begin{pmatrix} x_1 \otimes x_1 & (x_2 + x_1) \otimes x_1 & q_{12}x_3 \otimes x_1 \\ x_1 \otimes x_2 & (x_2 + x_1) \otimes x_2 & q_{12}x_3 \otimes x_2 \\ q_{21}x_1 \otimes x_3 & q_{21}(x_2 + ax_1) \otimes x_3 & q_{22}x_3 \otimes x_3 \end{pmatrix}. \tag{3.2}$$

Let $V_1 = \langle x_1, x_2\rangle \simeq \mathcal{V}(1, 2)$ (the block) and $V_2 = \langle x_3\rangle$ (the point); then $\mathfrak{L}_{\wp}(q_{22}, a) = V_1 \oplus V_2$. For simplicity, $V = \mathfrak{L}_{\wp}(q_{22}, a)$. Let $\Gamma = \mathbb{Z}^2$ with canonical basis g_1, g_2. Observe that (V, c) can be realized in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ via:

$$\begin{aligned} &g_1 \cdot x_1 = x_1, && g_1 \cdot x_2 = x_1 + x_2, && g_1 \cdot x_3 = q_{12}x_3, \\ &g_2 \cdot x_1 = q_{21}x_1, && g_2 \cdot x_2 = q_{21}(x_2 + ax_1), && g_2 \cdot x_3 = q_{22}x_3, \\ &\deg x_1 = g_1, && \deg x_2 = g_1, && \deg x_3 = g_2. \end{aligned} \tag{3.3}$$

If $a = 0$, then $\mathscr{B}(\mathfrak{L}_{\wp}(q_{22}, 0)) \simeq \mathscr{B}(V_1)\underline{\otimes}\mathscr{B}(V_2)$, where $\underline{\otimes}$ is the braided tensor product. Since $\dim \mathscr{B}(V_1) = 2^4$, $\dim \mathscr{B}(\mathfrak{L}_{\wp}(q_{22}, 0)) < \infty \iff \dim \mathscr{B}(\Bbbk x_3) < \infty \iff q_{22} \in \mathbb{G}_\infty$. Thus, we can assume that $a \in \Bbbk^\times$.

Our main goal in this section is to prove the following result.

THEOREM 3.1. *Assume* (3.1) *and that* $a \neq 0$. *Then,* $\dim \mathscr{B}(\mathfrak{L}_{\wp}(q_{22}, a)) < \infty$ *if and only if* $q_{22} = 1$. *Precisely,* $\dim \mathscr{B}(\mathfrak{L}_{\wp}(1, a)) = \begin{cases} 2^7 & \text{if } a = 1, \\ 2^8 & \text{if } a \in \Bbbk \setminus \{0, 1\}. \end{cases}$

We shall apply the splitting technique cf. Section 2.3.2. To describe K^1, we set

$$z_n := (\operatorname{ad}_c x_2)^n x_3, \qquad n \in \mathbb{N}_0. \tag{3.4}$$

We establish first a series of useful formulae.

LEMMA 3.2. *The following formulae hold in* $\mathscr{B}(V)$ *for all* $n \in \mathbb{N}_0$*:*

$$g_1 \cdot z_n = q_{12}z_n, \qquad x_1z_n = q_{12}z_nx_1, \qquad x_{21}z_n = q_{12}^2 z_nx_{21}, \tag{3.5}$$

$$g_2 \cdot z_n = q_{21}^n q_{22}z_n, \qquad x_2z_n = q_{12}z_nx_2 + z_{n+1}. \tag{3.6}$$

Proof. Note that (3.5) holds for $n = 0$. Indeed, $g_1 \cdot z_0 = g_1 \cdot x_3 = q_{12}z_0$ and using derivations is easy to check that $x_1z_0 = q_{12}z_0x_1$ and $x_{21}z_0 = q_{12}^2 z_0x_{21}$. Now suppose that (3.5) holds for n. Then, $z_{n+1} = (\operatorname{ad}_c x_2)^{n+1}x_3 = (\operatorname{ad}_c x_2)z_n = x_2z_n + (g_1 \cdot z_n)x_2 = x_2z_n + q_{12}z_nx_2$. So we compute

$$\begin{aligned} g_1 \cdot z_{n+1} &= g_1 \cdot (x_2z_n + q_{12}z_nx_2) = q_{12}(x_1 + x_2)z_n + q_{12}^2 z_n(x_1 + x_2) \\ &= q_{12}[(x_2z_n + q_{12}z_nx_2) + (x_1z_n + q_{12}z_nx_1)] = q_{12}z_{n+1}. \end{aligned}$$

Similarly,

$$\begin{aligned} x_1z_{n+1} &= x_1(x_2z_n + q_{12}z_nx_2) = (x_{21} + x_2x_1)z_n + q_{12}^2 z_nx_1x_2 \\ &= q_{12}^2 z_nx_{21} + q_{12}x_2z_nx_1 + q_{12}^2 z_n(x_{21} + x_2x_1) \\ &= q_{12}(x_2z_n + q_{12}z_nx_2)x_1 = q_{12}z_{n+1}x_1. \end{aligned}$$

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Also, since $x_{21}x_2 = x_2x_{21} + x_{21}x_1$ we have that

$$\begin{aligned} x_{21}z_{n+1} &= x_{21}(x_2z_n + q_{12}z_nx_2) = x_{21}x_2z_n + q_{12}^3 z_nx_{21}x_2 \\ &= x_2x_{21}z_n + x_{21}x_1z_n + q_{12}^3 z_nx_{21}x_2 \\ &= q_{12}^2 x_2z_nx_{21} + q_{12}x_{21}z_nx_1 + q_{12}^3 z_n(x_2x_{21} + x_{21}x_1) \\ &= q_{12}^2 x_2z_nx_{21} + q_{12}^3 z_nx_{21}x_1 + q_{12}^3 z_nx_2x_{21} + q_{12}^3 z_nx_{21}x_1 \\ &= q_{12}^2 (x_2z_n + q_{12}z_nx_2)x_{21} \\ &= q_{12}^2 z_{n+1}x_{21}. \end{aligned}$$

Finally, the first equation in (3.6) follows by induction. For $n = 0$, $g_2 \cdot z_0 = q_{22}z_0$. Suppose that $g_2 \cdot z_n = q_{21}^n q_{22}z_n$. Then,

$$\begin{aligned} g_2 \cdot z_{n+1} &= g_2 \cdot (x_2z_n + q_{12}z_nx_2) \\ &= q_{21}(x_2 + ax_1)(q_{21}^n q_{22}z_n) + q_{12}(q_{21}^n q_{22}z_n)q_{21}(x_2 + ax_1) \\ &= q_{21}^{n+1} q_{22}(x_2z_n + q_{12}z_nx_2) + aq_{21}^{n+1} q_{22}(x_1z_n + q_{12}z_nx_1) \\ &= q_{21}^{n+1} q_{22}z_{n+1}. \end{aligned}$$

□

We define

$$\begin{aligned} &\mu_0 = 1, && \mu_1 = a, && \mu_2 = a, && \mu_3 = a(a+1), \\ &y_0 = 1, && y_1 = x_1, && y_2 = x_{21}, && y_3 = x_1x_{21}. \end{aligned}$$

LEMMA 3.3. *For all $k \in \mathbb{N}_0$, $\partial_1(z_k) = \partial_2(z_k) = 0$, and*

$$\partial_3(z_k) = \mu_k y_k, \quad k \in \mathbb{I}_{0,3}, \qquad \partial_3(z_k) = 0, \quad k \geq 4.$$

Proof. Clearly, $\partial_1(z_0) = \partial_2(z_0) = 0$, $\partial_3(z_0) = 1$. Recursively, $\partial_1(z_k) = 0$ for all k. If $\partial_2(z_k) = 0$, then $\partial_2(z_{k+1}) = \partial_2(x_2z_k + q_{12}z_kx_2) = g_1 \cdot z_k + q_{12}z_k \overset{(3.5)}{=} 0$. Next,

$$\begin{aligned} \partial_3(z_1) &= \partial_3(x_2x_3 + q_{12}x_3x_2) = x_2 + q_{12}(q_{21}(x_2 + ax_1)) = ax_1 = \mu_1 y_1, \\ \partial_3(z_2) &= \partial_3(x_2z_1 + q_{12}z_1x_2) = ax_2x_1 + q_{12}ax_1q_{21}(x_2 + ax_1) = ax_{21} = \mu_2 y_2, \\ \partial_3(z_3) &= \partial_3(x_2z_2 + q_{12}z_2x_2) = ax_2x_{21} + q_{12}ax_{21}q_{21}(x_2 + ax_1) \\ &= ax_2(x_2x_1 + x_1x_2) + a(x_2x_1 + x_1x_2)(x_2 + ax_1) \\ &= ax_2^2x_1 + ax_1x_2^2 + a^2x_1x_2x_1 \overset{(2.5)}{=} (a + a^2)x_1x_2x_1 \\ &= (a + a^2)x_1x_{21} = \mu_3 y_3, \\ \partial_3(z_4) &= \partial_3(x_2z_3 + q_{12}z_3x_2) \\ &= (a + a^2)x_2x_1x_2x_1 + q_{12}(a + a^2)x_1x_2x_1(g_2 \cdot x_2) \\ &= (a + a^2)x_2x_1x_2x_1 + q_{12}(a + a^2)x_1x_2x_1(q_{21}(x_2 + ax_1)) \\ &= (a + a^2)(x_2x_1x_2x_1 + x_1x_2x_1x_2) \overset{(2.5)}{=} 0. \end{aligned}$$

□

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LEMMA 3.4. *Let* $B_1 := \{z_i : i \in \mathbb{I}_{0,2}\}$ *and* $B_2 := \{z_i : i \in \mathbb{I}_{0,3}\}$. *If* $a = 1$ *(resp.* $a \neq 1$*), then* B_1 *(resp.* B_2*) is a basis of* K^1.

Proof. Notice that $(\mathrm{ad}_c\, x_1)z_n = 0$ and $(\mathrm{ad}_c\, x_{21})z_n = 0$. By Theorem 2.2 and Lemma 3.3, if $a = 1$ (resp. $a \neq 1$), then B_1 (resp. B_2) generates K^1. Since the elements of B_i ($i \in \mathbb{I}_{0,2}$) are homogeneous of distinct degrees and are nonzero, it follows that B_i ($i \in \mathbb{I}_{0,2}$) is a linearly independent set. □

Let $i \in \mathbb{N}_0$. We define recursively the scalars $\nu_{i,j}$, for $j > i$, by

$$\nu_{i,i} = 1, \qquad \nu_{i,j} = (a + (j-1))\, \nu_{i,j-1}.$$

LEMMA 3.5. *The coaction* (2.7) *on* z_i, $i \in \mathbb{I}_{0,3}$, *is given, (for* $n = 0, 1$*) by*

$$\delta(z_{2n}) = \sum_{k=1}^{n} \nu_{k,n} x_1 x_{21}^{n-k} g_1^{2k-1} g_2 \otimes z_{2k-1} + \sum_{k=0}^{n} \nu_{k,n} x_{21}^{n-k} g_1^{2k} g_2 \otimes z_{2k},$$

$$\delta(z_{2n+1}) = \sum_{k=0}^{n} \nu_{k,n+1} x_1 x_{21}^{n-k} g_1^{2k} g_2 \otimes z_{2k} + \sum_{k=0}^{n} \nu_{k+1,n+1} x_{21}^{n-k} g_1^{2k+1} g_2 \otimes z_{2k+1}.$$

Proof. Similar to the proof of [**2**, Lemma 4.2.5]. □

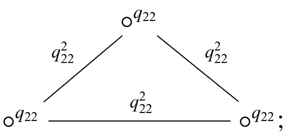
Lemma 3.5 implies that K^1 is of diagonal type with braiding given by

$$c(z_i \otimes z_j) = q_{21}^{j-i} q_{22} z_j \otimes z_i, \qquad \forall\ i, j. \tag{3.7}$$

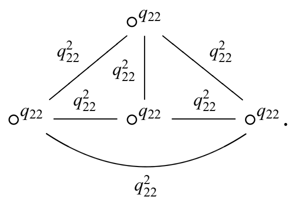
Now we are ready for to prove the main result of this Section.

Proof of Theorem 3.1. If $q_{22} = 1$, then the Dynkin diagram of K^1 is totally disconnected with vertices labeled with 1. Thus, if $a = 1$, then $\dim \mathscr{B}(K^1) = 2^3$ and $\dim \mathscr{B}(\mathfrak{L}_{\wp}(1, 1)) = 2^7$; if $a \neq 1$, then $\dim \mathscr{B}(K^1) = 2^4$ and $\dim \mathscr{B}(\mathfrak{L}_{\wp}(1, a)) = 2^8$. If $q_{22} \neq 1$, then the Dynkin diagram of K^1 is

$a = 1$:



$a \neq 1$:



By inspection of the lists in [**10**, **11**], we conclude that $\dim \mathscr{B}(K^1) = \infty$. □

3.1. The presentation by generators and relations. Let c be the braiding of K^1 as in (3.7). Then $q_{22} = 1$ if and only if $c^2 = \mathrm{id}$. Hence, for $a = 1$ (resp. $a \neq 1$), $\mathscr{B}(K^1)$ is the algebra generated by z_0, z_1, z_2 (resp. z_0, z_1, z_2, z_3) with relations:

$$z_i^2 = 0, \qquad z_i z_j = q_{21}^{j-i} z_j z_i, \qquad i \neq j.$$

Thus, we have the following results.

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PROPOSITION 3.6. *The algebra $\mathscr{B}(\mathfrak{L}_{\wp}(1,1))$ is presented by generators x_1, x_2, x_3 with defining relations* (2.5) *and*

$$x_1 z_j = q_{12}\, z_j x_1, \quad z_{j+1} = x_2 z_j + q_{12} z_j x_2, \qquad j \in \mathbb{N}_0, \tag{3.8}$$

$$z_i z_j = q_{21}^{j-i} z_j z_i, \quad z_j^2 = 0, \quad z_k = 0, \qquad i,j \in \mathbb{I}_{0,2}, \quad k \geq 3. \tag{3.9}$$

The dimension of $\mathscr{B}(\mathfrak{L}_{\wp}(1,1))$ is 2^7, since it has a PBW-basis:

$$\{x_1^{m_1} x_{21}^{m_2} x_2^{m_3} z_2^{n_2} z_1^{n_1} z_0^{n_0} \ : \ m_1, m_2, n_i \in \mathbb{I}_{0,1},\ m_3 \in \mathbb{I}_{0,3}\}.$$

□

PROPOSITION 3.7. *The algebra $\mathscr{B}(\mathfrak{L}_{\wp}(1,a))$, $a \neq 1$, is presented by generators x_1, x_2, x_3 with defining relations* (2.5) *and*

$$x_1 z_j = \wp\, z_j x_1, \quad z_{j+1} = x_2 z_j + \wp z_j x_2, \qquad j \in \mathbb{N}_0, \tag{3.10}$$

$$z_i z_j = \wp^{i-j} z_j z_i, \quad z_j^2 = 0, \quad z_k = 0, \qquad i,j \in \mathbb{I}_{0,3}, \quad k \geq 4. \tag{3.11}$$

The dimension of $\mathscr{B}(\mathfrak{L}_{\wp}(1,a))$ is 2^8, since it has a PBW-basis:

$$\{x_1^{m_1} x_{21}^{m_2} x_2^{m_3} z_3^{n_3} z_2^{n_2} z_1^{n_1} z_0^{n_0} \ : \ m_1, m_2, n_i \in \mathbb{I}_{0,1},\ m_3 \in \mathbb{I}_{0,3}\}.$$

□

3.2. Realizations. Let (g_1, χ_1, η) be a YD-triple and (g_2, χ_2) a YD-pair for H, see Section 2.2.3. Let (V, c) be a braided vector space with braiding (3.2). Then $\mathcal{V}_{g_1}(\chi_1, \eta) \oplus \Bbbk_{g_2}^{\chi_2} \in {}^H_H\mathcal{YD}$ is a *principal realization* of (V, c) over H if

$$q_{ij} = \chi_j(g_i), \qquad\qquad i,j \in \mathbb{I}_2; \qquad\qquad a = q_{21}^{-1}\eta(g_2).$$

Thus, $(V,c) \simeq \mathcal{V}_{g_1}(\chi_1, \eta) \oplus \Bbbk_{g_2}^{\chi_2}$ as braided vector space. Hence, if H is finite-dimensional and $(V,c) \simeq \mathfrak{L}_{\wp}(1,a)$, $a \neq 0$, then $\mathscr{B}\big(\mathcal{V}_{g_1}(\chi_1, \eta) \oplus \Bbbk_{g_2}^{\chi_2}\big)\#H$ is a finite-dimensional Hopf algebra. Observe that the existence of a YD-triple for H finite-dimensional is not granted, for instance, $\wp = q_{12}$ should be a root of 1, otherwise there is no such triple. Suppose that $\operatorname{ord}\wp = M \in \mathbb{N}$. Notice that M is odd because $\operatorname{char}\Bbbk = 2$. Here are some explicit examples of finite-dimensional pointed Hopf algebras like this: take $\Gamma = \langle g_1\rangle \times \langle g_2\rangle$ where both g_1 and g_2 have order $2M$. Then, (V,c) is realized in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ with structure as in (3.3) and $\dim \mathscr{B}(V)\#\Bbbk\Gamma = 2^9 M^2$ (if $a = 1$) or $2^{10}M^2$ (if $a \neq 1$).

4. One block and several points. Let $\theta \in \mathbb{N}_{\geq 3}$, $\mathbb{I}_\theta^\dagger = \mathbb{I}_\theta \cup \{\frac{3}{2}\}$; as usual $\lfloor x \rfloor$ is the integral part of $x \in \mathbb{R}$. We fix a matrix $\mathbf{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$ with entries in $\Bbbk^\times$ and $\mathsf{a} = (1, a_2, \dots, a_\theta) \in \Bbbk^\theta$. We assume that

$$q_{11} = 1, \qquad\quad q_{1j}q_{j1} = 1, \ \text{ for all } j \in \mathbb{I}_{2,\theta}, \qquad\quad \mathsf{a} \neq (1, 0, \dots, 0). \tag{4.1}$$

Let (V,c) be the braided vector space of dimension $\theta + 1$, with a basis $(x_i)_{i\in\mathbb{I}_\theta^\dagger}$ and braiding given by

$$c(x_i \otimes x_j) = \begin{cases} q_{\lfloor i \rfloor j} x_j \otimes x_i, & i \in \mathbb{I}_\theta^\dagger,\ j \in \mathbb{I}_\theta; \\ q_{\lfloor i \rfloor 1}(x_{\frac{3}{2}} + a_{\lfloor i \rfloor} x_1) \otimes x_i, & i \in \mathbb{I}_\theta^\dagger,\ j = \frac{3}{2}. \end{cases} \tag{4.2}$$

Then, $V = V_1 \oplus V_2$ where $V_1 = \langle x_1, x_{\frac{3}{2}}\rangle \simeq \mathcal{V}(1,2)$ (the block) and $V_2 = \langle x_2, \dots, x_\theta\rangle$ (the points). If $\Gamma = \mathbb{Z}^\theta$ with basis $(g_h)_{h\in\mathbb{I}_\theta}$, then V can be realized in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ as in (3.3). Here is the main result of this section.

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THEOREM 4.1. *Assume* (4.1). *Then* $\dim \mathscr{B}(V) = \infty$.

We shall use the material from the previous section with $\frac{3}{2}$ replacing 2 for instance $x_{\frac{3}{2}1} = x_{\frac{3}{2}}x_1 + x_1x_{\frac{3}{2}}$. We shall apply the splitting technique cf. Section 2.3.2. To describe K^1, we introduce the elements:

$$z_{i,n} := \left(ad_c x_{\frac{3}{2}}\right)^n x_i, \qquad i \in \mathbb{I}_{2,\theta}, \qquad n \in \mathbb{N}_0. \tag{4.3}$$

Let $i \in \mathbb{I}_{2,\theta}$, $n \in \mathbb{N}_0$. By Lemma 3.2, we have that

$$g_1 \cdot z_{i,n} = q_{1i}z_{i,n}, \qquad z_{i,n+1} = x_{\frac{3}{2}}z_{i,n} + q_{1i}z_{i,n}x_{\frac{3}{2}}, \qquad x_1z_{i,n} = q_{1,i}z_{i,n}x_1. \tag{4.4}$$

Consequently,

$$g_h \cdot z_{i,n} = q_{h1}^n q_{hi} z_{i,n}, \qquad h \in \mathbb{I}_{2,\theta}. \tag{4.5}$$

In fact, $g_h \cdot z_{i,0} = g_h \cdot x_i = q_{hi}x_i$. Suppose that $g_h \cdot z_{i,n} = q_{h1}^n q_{hi} z_{i,n}$. Thus,

$$\begin{aligned} g_h \cdot z_{i,n+1} &= g_h \cdot \left(x_{\frac{3}{2}}z_{i,n} + q_{1i}z_{i,n}x_{\frac{3}{2}}\right) \\ &= q_{h1}\left(x_{\frac{3}{2}} + a_hx_1\right) q_{h1}^n q_{hi}z_{i,n} + q_{1i}q_{h1}^{n+1}q_{hi}z_{i,n}\left(x_{\frac{3}{2}} + a_hx_1\right) \\ &= q_{h1}^{n+1}q_{hi}\left(x_{\frac{3}{2}}z_{i,n} + q_{1i}z_{i,n}x_{\frac{3}{2}}\right) = q_{h1}^{n+1}q_{hi}z_{i,n+1}. \end{aligned}$$

As in Lemma 3.3, we define for $i \in \mathbb{I}_{2,\theta}$,

$$\begin{aligned} &\mu_0^{(i)} = 1, && \mu_1^{(i)} = a_i, && \mu_2^{(i)} = a_i, && \mu_3^{(i)} = a_i(a_i+1), \\ &y_0 = 1, && y_1 = x_1, && y_2 = x_{\frac{3}{2}1}, && y_3 = x_1x_{\frac{3}{2}1}. \end{aligned}$$

Hence, $\partial_h(z_{i,n}) = 0$ for $i \in \mathbb{I}_{2,\theta}$, $n \in \mathbb{N}_0$, $i \neq h \in \mathbb{I}_\theta^\dagger$ and

$$\partial_i(z_{i,n}) = \mu_n^{(i)}y_n, \quad n \in \mathbb{I}_{0,3}, \qquad \partial_i(z_{i,n}) = 0, \quad n \geq 4.$$

For $i \in \mathbb{I}_{2,\theta}$, we define

$$J_i = \begin{cases} \{(i,0)\}, & a_i = 0, \\ \{(i,0),(i,1),(i,2)\}, & a_i = 1, \\ \{(i,0),(i,1),(i,2),(i,3)\}, & a_i \notin \{0, 1\}, \end{cases} \qquad J = \bigcup_{i \in \mathbb{I}_{2,\theta}} J_i.$$

LEMMA 4.2. *The family $B = (z_{i,n})_{(i,n)\in J}$ is a basis of the braided vector space K^1, which is of diagonal type with braiding:*

$$c(z_{i,m} \otimes z_{j,n}) = q_{i1}^n q_{1j}^m q_{ij} z_{j,n} \otimes z_{i,m}, \qquad (i,m),(j,n) \in J. \tag{4.6}$$

Proof. Arguing as in Lemma 3.4, we see that B is a basis. We compute the coaction (2.7) on $z_{j,n}$ as in Lemma 3.5 and then (4.6) follows. □

Proof of Theorem 4.1. It is enough to show that $\dim \mathscr{B}(K^1) = \infty$. By (4.6), we may assume that the Dynkin diagram of the matrix $(q_{ij})_{i,j\in\mathbb{I}_{2,\theta}}$ is connected. We then may assume that $\theta = 3$ by taking a suitable subdiagram; thus $\widetilde{q}_{23} := q_{23}q_{32} \neq 0$. We distinguish then three cases. First assume $\mathsf{a} = (1, a, 0)$ with $a \neq 0$. By Theorem 3.1 applied to $V_1 \oplus \langle x_2 \rangle$,

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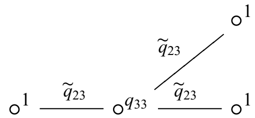
$q_{22} = 1$. By Lemma 4.2, K^1 is of diagonal type. If $a = 1$, then its Dynkin diagram is



which does not appear in the list in [11]. If $a \neq 1$, then the diagram above appears a subdiagram. The case $\mathsf{a} = (1, 0, b)$ with $b \neq 0$ is similar. As well, if $\mathsf{a} = (1, a, b)$ with $a, b \neq 0$, then the diagram above also appears a subdiagram of that of K^1. $\square$

5. Several blocks and one point. Let $t \geq 2$ and $\theta = t + 1$. As in [2, 4] we use the notation:

$$\mathbb{I}_k^{\ddagger} = \left\{k, k + \frac{1}{2}\right\}, \qquad k \in \mathbb{I}_t; \qquad \mathbb{I}^{\ddagger} = \mathbb{I}_1^{\ddagger} \cup \dots \cup \mathbb{I}_t^{\ddagger} \cup \{\theta\}.$$

We fix a matrix $\mathbf{q} = (q_{ij})_{i,j \in \mathbb{I}_\theta}$ with entries in $\Bbbk^\times$ and $\mathbf{a} = (a_1, \dots, a_t) \in \Bbbk^t$. We assume that

$$q_{ii} = 1, \qquad q_{ij}q_{ji} = 1, \qquad \text{for all } i \neq j \in \mathbb{I}_\theta; \qquad a_j \neq 0, \qquad j \in \mathbb{I}_t. \tag{5.1}$$

Let $\mathfrak{P}(\mathbf{q}, \mathbf{a})$ be the braided vector space with basis $(x_i)_{i \in \mathbb{I}^\ddagger}$ and braiding:

$$c(x_i \otimes x_j) = \begin{cases} q_{\lfloor i \rfloor \lfloor j \rfloor}\, x_j \otimes x_i, & \lfloor i \rfloor \leq t,\ \lfloor i \rfloor \neq \lfloor j \rfloor, \\ x_j \otimes x_i, & \lfloor i \rfloor = j \leq t, \\ (x_j + x_{\lfloor j \rfloor}) \otimes x_i, & \lfloor i \rfloor \leq t,\ j = \lfloor i \rfloor + \frac{1}{2}, \\ q_{\theta j}\, x_j \otimes x_\theta, & i = \theta,\ j \in \mathbb{I}_\theta, \\ q_{\theta \lfloor j \rfloor}\, (x_j + a_{\lfloor j \rfloor} x_{\lfloor j \rfloor}) \otimes x_\theta, & i = \theta,\ j \notin \mathbb{I}_\theta. \end{cases} \tag{5.2}$$

Let $V_1 = W_1 \oplus \dots \oplus W_t$ where $W_k = \langle x_k, x_{k+\frac{1}{2}} \rangle \simeq \mathcal{V}(1, 2)$ (the blocks); and let $V_2 = \langle x_\theta \rangle$ (the point). Then, $\mathfrak{P}(\mathbf{q}, \mathbf{a}) = V_1 \oplus V_2$. If $\Gamma = \mathbb{Z}^\theta$ with basis $(g_i)_{i \in \mathbb{I}_\theta}$, then there is an action of Γ on V determined by

$$c(x_i \otimes x_j) = g_i \cdot x_j \otimes x_i, \qquad i \in \mathbb{I}_\theta,\ j \in \mathbb{I}^\ddagger. \tag{5.3}$$

Thus, V is realized in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ with the grading $\deg(x_i) = g_{\lfloor i \rfloor}$, $i \in \mathbb{I}^\ddagger$.

Here is the main result of this section; see (5.9) for the explicit formula of the dimension.

THEOREM 5.1. *Assume* (5.1). *Then* $\dim \mathscr{B}(\mathfrak{P}(\mathbf{q}, \mathbf{a})) < \infty$.

Let $j \in \mathbb{I}_t$. We set $x_{j+\frac{1}{2}j} = x_{j+\frac{1}{2}}x_j + x_j x_{j+\frac{1}{2}}$ and define

$$\mu_0^{(j)} = 1, \quad \mu_1^{(j)} = a_j, \quad \mu_2^{(j)} = a_j, \quad \mu_3^{(j)} = a_j(a_j + 1), \quad \mu_n^{(j)} = 0 \ \text{ if } n \geq 4,$$
$$y_{j,0} = 1, \quad y_{j,1} = x_j, \quad y_{j,2} = x_{j+\frac{1}{2}j}, \quad y_{j,3} = x_j x_{j+\frac{1}{2}j}, \quad y_{j,n} = 0 \ \text{ if } n \geq 4.$$

To apply the splitting technique, see Section 2.3.2, we introduce the elements:

$$\text{ш}_{\mathbf{n}} := \left(\operatorname{ad}_c x_{\frac{3}{2}}\right)^{n_1} \dots \left(\operatorname{ad}_c x_{t+\frac{1}{2}}\right)^{n_t} x_\theta, \qquad \mathbf{n} = (n_1, \dots, n_t) \in \mathbb{N}_0^t. \tag{5.4}$$

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We start establishing some useful formulas.

LEMMA 5.2. *Let $j \in \mathbb{I}_t$ and $\mathbf{n} = (n_1, \dots, n_t) \in \mathbb{N}_0^t$. Then*

$$\operatorname{ad}_c x_j(\text{Ш}_{\mathbf{n}}) = \operatorname{ad}_c x_{j+\frac{1}{2}j}(\text{Ш}_{\mathbf{n}}) = 0, \tag{5.5}$$

$$\operatorname{ad}_c x_{j+\frac{1}{2}}(\text{Ш}_{\mathbf{n}}) = \prod_{i<j} q_{ji}^{n_i} \text{Ш}_{\mathbf{n}+\mathbf{e}_j}, \tag{5.6}$$

$$g_j \cdot \text{Ш}_{\mathbf{n}} = q_{j\theta} \prod_{i=1}^{t} q_{ji}^{n_i}, \quad g_\theta \cdot \text{Ш}_{\mathbf{n}} = \prod_{i=1}^{t} q_{\theta i}^{n_i} \text{Ш}_{\mathbf{n}} \tag{5.7}$$

$$\partial_j(\text{Ш}_{\mathbf{n}}) = \partial_{j+1}(\text{Ш}_{\mathbf{n}}) = 0, \qquad \partial_\theta(\text{Ш}_{\mathbf{n}}) = \prod_{i=1}^{t} \mu_{n_i}^{(i)} y_{1,n_1} \dots y_{t,n_t}. \tag{5.8}$$

Proof. Similar to the proof of [2, Lemma 7.2.3]. □

Let us set

$$b_j := 2, \text{ if } a_j = 1, \qquad b_j := 3, \text{ if } a_j \neq 1, \qquad \text{and } \mathbf{b} = (b_1, \dots, b_t) \in \mathbb{N}^t.$$

Arguing as in [2, Section 7.2], we conclude from Lemma 5.2:

LEMMA 5.3. *Let $\mathcal{A} = \{\mathbf{n} \in \mathbb{N}_0^t : \mathbf{n} \leq \mathbf{b}\}$ ordered lexicographically.*

(i) *The elements $(\text{Ш}_{\mathbf{n}})_{\mathbf{n}\in\mathcal{A}}$ form a basis of K^1.*

(ii) *The coaction (2.7) on $\text{Ш}_{\mathbf{n}}$ is given by*

$$\delta(\text{Ш}_{\mathbf{n}}) = \sum_{0\leq\mathbf{k}\leq\mathbf{n}} \nu_{\mathbf{k}}^{\mathbf{n}} y_{1,n_1-k_1} \dots y_{t,n_t-k_t} g_1^{k_1} \dots g_t^{k_t} g_\theta \otimes \text{Ш}_{\mathbf{k}}$$

for some scalars $\nu_{\mathbf{k}}^{\mathbf{n}}$, $0 \leq \mathbf{k} \leq \mathbf{n}$, with $\nu_{\mathbf{n}}^{\mathbf{n}} = 1$.

(iii) *The braided vector space K^1 is of diagonal type with respect to the basis $(\text{Ш}_{\mathbf{n}})_{\mathbf{n}\in\mathcal{A}}$ with matrix braiding $(p_{\mathbf{m},\mathbf{n}})_{\mathbf{m},\mathbf{n}\in\mathcal{A}}$, where*

$$p_{\mathbf{m},\mathbf{n}} = \prod_{i,j=1}^{t} q_{ij}^{m_i n_j} q_{i\theta}^{m_i} q_{\theta j}^{n_j}.$$

Hence, the corresponding generalized Dynkin diagram has labels:

$$p_{\mathbf{m},\mathbf{m}} = 1 \qquad p_{\mathbf{m},\mathbf{n}} p_{\mathbf{n},\mathbf{m}} = 1, \qquad \mathbf{m} \neq \mathbf{n}.$$

Proof of Theorem 5.1. By Lemma 5.3, $\dim \mathscr{B}(K^1) = 2^{|\mathcal{A}|}$. Now the blocks W_i and W_j, $i \neq j$, commute in the braided sense by definition, therefore $\mathscr{B}(V_1) \simeq \mathscr{B}(W_1) \underline{\otimes} \mathscr{B}(W_2) \dots \underline{\otimes} \mathscr{B}(W_1)$. Hence,

$$\dim \mathscr{B}(\mathfrak{P}(\mathbf{q}, \mathbf{a})) = 2^{4t+|\mathcal{A}|}. \tag{5.9}$$

□

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5.1. The presentation by generators and relations.

PROPOSITION 5.4. *The algebra $\mathscr{B}(\mathfrak{P}(\mathbf{q}, \mathbf{a}))$ is presented by generators x_i, $i \in \mathbb{I}^{\ddagger}$, and relations:*

$$x_i^2 = 0, \qquad x_{i+\frac{1}{2}}^4 = 0, \qquad i \in \mathbb{I}_t, \tag{5.10}$$

$$x_{i+\frac{1}{2}}^2 x_i + x_i x_{i+\frac{1}{2}}^2 + x_i x_{i+\frac{1}{2}} x_i = 0, \qquad i \in \mathbb{I}_t, \tag{5.11}$$

$$x_i x_{i+\frac{1}{2}} x_i x_{i+\frac{1}{2}} + x_{i+\frac{1}{2}} x_i x_{i+\frac{1}{2}} x_i = 0, \qquad i \in \mathbb{I}_t, \tag{5.12}$$

$$x_i x_j = q_{\lfloor i \rfloor \lfloor j \rfloor} x_j x_i, \qquad \lfloor i \rfloor \neq \lfloor j \rfloor \in \mathbb{I}_t, \tag{5.13}$$

$$x_i x_\theta = q_{i\theta} x_\theta x_i, \qquad i \in \mathbb{I}_t, \tag{5.14}$$

$$\left(\operatorname{ad}_c x_{i+\frac{1}{2}}\right)^{1+b_i} (x_\theta) = 0, \qquad i \in \mathbb{I}_t, \tag{5.15}$$

$$Ш_{\mathbf{m}} Ш_{\mathbf{n}} = p_{\mathbf{m},\mathbf{n}} Ш_{\mathbf{n}} Ш_{\mathbf{m}}, \qquad \mathbf{m} \neq \mathbf{n} \in \mathcal{A}, \tag{5.16}$$

$$Ш_{\mathbf{n}}^2 = 0, \qquad \mathbf{n} \in \mathcal{A}. \tag{5.17}$$

A basis of $\mathscr{B}(\mathfrak{P}(\mathbf{q}, \mathbf{a}))$ is given by

$$B = \left\{ y_{1,m_1} x_{\frac{3}{2}}^{m_2} \dots y_{t,m_{2t-1}} x_{t+\frac{1}{2}}^{m_{2t}} \prod_{\mathbf{n} \in \mathcal{A}} Ш_{\mathbf{n}}^{b_{\mathbf{n}}} \; : \; 0 \leq b_{\mathbf{n}} < 2, \;\; 0 \leq m_i < 4 \right\}.$$

Hence, $\dim \mathscr{B}(\mathfrak{P}(\mathbf{q}, \mathbf{a})) = 2^{4t+|\mathcal{A}|}$. □

5.2. Realizations. Let H be a Hopf algebra, (g_i, χ_i, η_i), $i \in \mathbb{I}_t$, a family of YD-triples and (g_θ, χ_θ) a YD-pair for H, see Section 2.2.3. Let (V, c) be a braided vector space with braiding (5.2). Then,

$$\mathcal{V} := \left(\oplus_{i \in \mathbb{I}_t} \mathcal{V}_{g_i}(\chi_i, \eta_i) \right) \oplus \Bbbk_{g_\theta}^{\chi_\theta} \in {}^H_H\mathcal{YD} \tag{5.18}$$

is a *principal realization* of (V, c) over H if

$$q_{ij} = \chi_j(g_i), \qquad i, j \in \mathbb{I}_\theta; \qquad a_j = q_{j1}^{-1} \eta_j(g_j), \; j \in \mathbb{I}_t.$$

Consequently, if H is finite-dimensional, then so is $\mathscr{B}(\mathcal{V})\#H$. But the existence of such H requires that all q_{ij}'s are roots of 1. In this case, let $\Gamma = (\mathbb{Z}/N)^\theta$ where N is even and divisible by $\operatorname{ord} q_{ij}$ for all i, j. Then, (V, c) is realized in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ with action (5.3). Thus, $\mathscr{B}(\mathfrak{P}(\mathbf{q}, \mathbf{a}))\#\Bbbk\Gamma$ is a pointed Hopf algebra of dimension $2^{4t+|\mathcal{A}|} N^\theta$.

6. One pale block and one point. An indecomposable Yetter–Drinfeld module which is decomposable as braided vector space is called a *pale block* [5]; the simplest examples were studied in [2, 4]. We extend the analysis there to characteristic 2.

Let $(q_{ij})_{i,j \in \mathbb{I}_2}$ be a matrix with nonzero entries; we assume that $q_{11} = 1$ and $q_{12} q_{21} = 1$; we set $\wp = q_{12} = q_{21}^{-1}$. Let $V = \mathfrak{E}_\wp(q_{22})$ be the braided vector space of dimension 3 with basis $(x_i)_{i \in \mathbb{I}_3}$ and braiding given by

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$$(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_3} = \begin{pmatrix} x_1 \otimes x_1 & x_2 \otimes x_1 & q_{12}x_3 \otimes x_1 \\ x_1 \otimes x_2 & x_2 \otimes x_2 & q_{12}x_3 \otimes x_2 \\ q_{21}x_1 \otimes x_3 & q_{21}(x_2 + x_1) \otimes x_3 & q_{22}x_3 \otimes x_3 \end{pmatrix}. \tag{6.1}$$

Let $V_1 = \langle x_1, x_2\rangle$ (the pale block), $V_2 = \langle x_3\rangle$ (the point), and $\Gamma = \mathbb{Z}^2$ with a basis g_1, g_2. Notice that $\mathscr{B}(V_1)$ is a truncated symmetric algebra of dimension 4. We realize V in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ by $\deg x_1 = \deg x_2 = g_1$, $\deg x_3 = g_2$:

$$\begin{aligned} &g_1 \cdot x_1 = x_1, && g_1 \cdot x_2 = x_2, && g_1 \cdot x_3 = q_{12}x_3, \\ &g_2 \cdot x_1 = q_{21}x_1, && g_2 \cdot x_2 = q_{21}(x_2 + x_1), && g_2 \cdot x_3 = q_{22}x_3. \end{aligned} \tag{6.2}$$

THEOREM 6.1. *The Nichols algebra $\mathscr{B}(\mathfrak{C}_{\wp}(q_{22}))$ is finite-dimensional if and only if $q_{22} = 1$ or $q_{22} = \omega$, with $\omega \in \mathbb{G}_3'$.*

To apply the splitting technique, see Section 2.3.2, we introduce the elements:

$$\text{ш}_{m,n} = (\operatorname{ad}_c x_1)^m(\operatorname{ad}_c x_2)^n x_3, \qquad w_m = \text{ш}_{m,0}, \qquad z_n = \text{ш}_{0,n}, \qquad m, n \in \mathbb{N}_0.$$

By direct computation,

$$g_1 \cdot \text{ш}_{m,n} = q_{12}\text{ш}_{m,n}, \qquad g_2 \cdot w_m = q_{21}^m q_{22} w_m, \tag{6.3}$$

$$z_{n+1} = x_2 z_n + q_{12} z_n x_2, \qquad \text{ш}_{m+1,n} = x_1 \text{ш}_{m,n} + q_{12}\text{ш}_{m,n} x_1, \tag{6.4}$$

$$\partial_1(\text{ш}_{m,n}) = \partial_2(\text{ш}_{m,n}) = 0, \qquad \partial_3(w_m) = 0, \ \text{ for all } m > 0. \tag{6.5}$$

Since x_1 and x_2 commute, $\text{ш}_{m,n} = (\operatorname{ad}_c x_2)^n(\text{ш}_{m,0}) = (\operatorname{ad}_c x_2)^n(w_m)$. By (6.5) $w_m = 0$ and thus $\text{ш}_{m,n} = 0$, for all $m > 0$. Hence, $\{z_n : n \in \mathbb{N}_0\}$ generates K^1. It is easy to check that

$$g_2 \cdot z_n = q_{21}^n q_{22} z_n, \qquad \partial_3(z_n) = x_1^n, \qquad n \in \mathbb{N}_0. \tag{6.6}$$

As $x_1^2 = 0$, we conclude that $\{z_0, z_1\}$ is a basis of K^1. The coaction is given by $\delta(z_0) = g_2 \otimes z_0$ and $\delta(z_1) = x_1 g_2 \otimes z_0 + g_1 g_2 \otimes z_1$. From (6.6) follows that K^1 is a braided vector space of diagonal type with braiding:

$$c(z_i \otimes z_j) = q_{21}^{j-i} q_{22} z_j \otimes z_i, \qquad i, j \in \mathbb{I}_{0,1}.$$

Proof of Theorem 6.1. If $q_{22} = 1$, then the Dynkin diagram of K^1 is totally disconnected with vertices labelled with q_{22}. In this case, $z_0^2 = z_1^2 = 0$, $\dim \mathscr{B}(K^1) = 4$ and so $\dim \mathscr{B}(\mathfrak{C}_{\wp}(1)) = 2^4$. If $q_{22} \neq 1$, the Dynkin diagram of K^1 is

$$\overset{q_{22}}{\circ} \xrightarrow{\ q_{22}^2\ } \overset{q_{22}}{\circ}.$$

By inspection in the list of [9], $\dim \mathscr{B}(K^1) < \infty$ if and only if $q_{22} = \omega$, with $\omega \in \mathbb{G}_3'$. In this case, $\dim \mathscr{B}(K^1) = 3^3$ and so $\dim \mathscr{B}(\mathfrak{C}_{\wp}(\omega)) = 2^2 3^3$. □

6.1. The presentation by generators and relations.

PROPOSITION 6.2. *The algebra $\mathscr{B}(\mathfrak{C}_{\wp}(1))$ is presented by generators x_1, x_2, x_3 with defining relations:*

$$x_1^2 = 0, \qquad x_2^2 = 0, \qquad x_1 x_2 = x_2 x_1, \tag{6.7}$$

$$x_1 x_3 = \wp x_3 x_1, \qquad z_1 = x_2 x_3 + \wp x_3 x_2 \tag{6.8}$$

$$x_3^2 = 0, \qquad z_1^2 = 0. \tag{6.9}$$

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The dimension of $\mathscr{B}(\mathfrak{E}_{\wp}(1))$ *is* 2^4*, since it has a PBW-basis:*

$$\left\{x_1^{m_1}x_2^{m_2}z_1^{n_1}x_3^{n_0} \,:\, m_i, n_i \in \mathbb{I}_{0,1}\right\}. \qquad \square$$

PROPOSITION 6.3. *Let* $z_{01} := \operatorname{ad}_c x_3(z_1)$*. The algebra* $\mathscr{B}(\mathfrak{E}_{\wp}(\omega))$ *is presented by generators* x_1, x_2, x_3 *with defining relations:*

$$x_1^2 = 0, \qquad x_2^2 = 0, \qquad x_1x_2 = x_2x_1, \tag{6.10}$$

$$x_1x_3 = \wp x_3x_1, \quad z_1 = x_2x_3 + \wp x_3x_2 \tag{6.11}$$

$$x_3^3 = 0, \qquad z_1^3 = 0. \tag{6.12}$$

$$z_{01}^3 = 0, \qquad (\operatorname{ad}_c x_3)^2(z_1) = 0. \tag{6.13}$$

The dimension of $\mathscr{B}(\mathfrak{E}_{\wp}(\omega))$ *is* $2^2 3^3$*, since it has a PBW-basis:*

$$\left\{x_1^{m_1}x_2^{m_2}z_1^{n_2}z_{01}^{n_1}x_3^{n_0} \,:\, m_i \in \mathbb{I}_{0,1},\ n_i \in \mathbb{I}_{0,2}\right\}. \qquad \square$$

6.2. Realizations. Assume that $\wp$ is a root of 1 of odd order M. Take $\Gamma = \langle g_1\rangle \times \langle g_2\rangle$ where g_1 has order M and g_2 has order $2M$. We realize $\mathfrak{E}_{\wp}(1)$ in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ by $\deg x_1 = \deg x_2 = g_1$, $\deg x_3 = g_2$ and action (6.2). Then, $\mathscr{B}\big(\mathfrak{E}_{\wp}(1)\big)\#\Bbbk\Gamma$ is a pointed Hopf algebra of dimension 2^5M^2.

Also, let $\Upsilon = \langle h_1\rangle \times \langle h_2\rangle$ where h_1 has order M and h_2 have order $P := \operatorname{lcm}(6, M)$. We realize $\mathfrak{E}_{\wp}(\omega)$ in ${}^{\Bbbk\Upsilon}_{\Bbbk\Upsilon}\mathcal{YD}$ by $\deg x_1 = \deg x_2 = h_1$, $\deg x_3 = h_2$ and action as in (6.2) with h_i's instead of the g_i's. Then, $\mathscr{B}\big(\mathfrak{E}_{\wp}(\omega)\big)\#\Bbbk\Upsilon$ is a pointed Hopf algebra of dimension $2^3 3^3 MP$.

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