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Nondegeneracy of the bubble for the critical p-Laplace equation

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We prove the non-degeneracy of the extremals of the Sobolev inequality

$$\int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x \geqslant \mathcal{S}_p \int_{\mathbb{R}^N} |u|^{\frac{Np}{N-p}} \, \mathrm{d}x, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$$

when $1 < p < N$, as solutions of a critical quasilinear equation involving the p-Laplacian.

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1. Introduction and statement of the main result

In this paper we establish the *linear non-degeneracy* of the extremals of the optimal classical Sobolev inequality

$$\mathcal{S}_p \|u\|_{L^{p^*}(\mathbb{R}^N)} \leqslant \|\nabla u\|_{L^p(\mathbb{R}^N)} \text{ for any } u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \tag{1.1}$$

where $p^* := \frac{Np}{N-p}$ and $1 < p < N$.

Aubin [1] and Talenti [22] found the optimal constant and the extremals for inequality (1.1). Indeed, equality is achieved precisely by the functions

$$U_{\delta,\xi}(x) := \delta^{-\frac{N-p}{p}} U\left(\frac{x-\xi}{\delta}\right) \quad \text{where } \delta > 0, \ \xi \in \mathbb{R}^N \tag{1.2}$$

where

$$U(x) = \left(\frac{\alpha_{N,p}}{1 + |x-\xi|^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}} \quad \text{with } \alpha_{N,p} := N^{\frac{1}{p}} \left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}}, \tag{1.3}$$

which solve the critical equation

$$-\Delta_p u = u^{p^*-1} \text{ in } \mathbb{R}^N, \quad u > 0 \text{ in } \mathbb{R}^N, \ u \in \mathcal{D}^{1,p}(\mathbb{R}^N). \tag{1.4}$$

All the solutions to the equation (1.4) are indeed the only ones of (1.2). Caffarelli, Gidas and Spruck proved the claim when $p = 2$. The case $p \neq 2$ has been firstly solved by Guedda and Véron [**14**] in the radial case, where the authors classified all the positive radial solutions and successively by Damascelli, Merchán, Montoro and Sciunzi [**6**] when $\frac{2N}{N+2} \leqslant p < 2$, by Vetóis [**24**] and Damascelli and Ramaswamy [**7**] when $1 < p < 2$ and finally by Sciunzi [**21**] in the remaining cases, namely when $2 < p < N$. We would also mention that Farina, Mercuri and Willem in the recent paper [**11**] proved that the classical Aubin-Talenti functions represent the full set of radial optimizers to the Sobolev inequality (1.1).

Here we are interested in the linear non-degeneracy of the solutions (1.2) to equation (1.4). Let us point out that equation (1.4) is invariant by scaling and by translations. Therefore, if we differentiate the equation

$$-\Delta_p U_{\delta,\xi} = U_{\delta,\xi}^{p^*-1} \text{ in } \mathbb{R}^N$$

with respect to the parameters δ and $\xi_1, \ldots, \xi_N$ at $\delta = 1$ and $\xi = 0$ we see that the functions

$$Z_0(x) := -\partial_\delta U_{\delta,\xi}|_{\delta=1,\xi=0} = \frac{N-p}{p} U + x \cdot \nabla U \tag{1.5}$$

and

$$Z_i(x) := \partial_{\xi_i} U_{\delta,\xi}|_{\delta=1,\xi=0} = -\partial_{x_i} U, \ i = 1, \ldots, N \tag{1.6}$$

annihilate the linearized operator around the function U defined in (1.3), namely they solve the linear equation

$$\begin{aligned} &-\operatorname{div}\left(|\nabla U|^{p-2}\nabla\phi\right) - (p-2)\operatorname{div}\left(|\nabla U|^{p-4}\left(\nabla U, \nabla\phi\right)\nabla U\right) \\ &\quad = (p^*-1)U^{p^*-2}\phi \text{ in } \mathbb{R}^N. \end{aligned} \tag{1.7}$$

We say that U is non-degenerate if the kernel of the associated linearized operator (1.7) is spanned only by the functions Z_i's defined in (1.5) and (1.6). This property is true when $p = 2$ as it was established by Rey in [**20**]. Our main result extends the non-degeneracy of the solution U to any $p \in (1, N)$ in the weighted Sobolev space $\mathcal{D}^{1,2}_*(\mathbb{R}^N)$, which is defined as the completion of $C^1_c(\mathbb{R}^N)$ with respect to the norm

$$\|\phi\| := \left(\int_{\mathbb{R}^N} |\nabla U|^{p-2}|\nabla\phi|^2 \,\mathrm{d}x\right)^{1/2}.$$

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THEOREM 1.1. *The solution*

$$U(x) = \left(\frac{\alpha_{N,p}}{1+|x|^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}} \quad \textit{with } \alpha_{N,p} := N^{\frac{1}{p}}\left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}}$$

of equation (1.4) *is non-degenerate in the sense that all the solutions of the equation* (1.7) *in the space* $\mathcal{D}^{1,2}_*(\mathbb{R}^N)$ *are linear combination of the functions*

$$Z_0(x) = \frac{N-p}{p}U + x\cdot\nabla U, \quad Z_1(x) = \partial_{x_1}U(x), \ldots, Z_N(x) = \partial_{x_N}U(x).$$

The structure of the linearized equation (1.7) strongly suggests to introduce the space $\mathcal{D}^{1,2}_*(\mathbb{R}^N)$. A similar weighted Sobolev space approach has been extensively used since the paper by Damascelli and Sciunzi [**8**], where the authors introduced it to study a linearized operator on a bounded domain. Here the situation is much more delicate due to the unboundness of the domain. Section 2 is devoted to prove some properties of $\mathcal{D}^{1,2}_*(\mathbb{R}^N)$ which are essential to get theorem 1.1 whose proof is carried out in §3.

Quasilinear equations with critical growth involving the p-Laplace operator have been widely studied in recent years using a variational framework, starting from the quasilinear version of the classical Brezis–Nirenberg problem (see [**2**]) studied by Guedda and Véron in [**15**]. In particular, we would like to focus on the problem of the existence of sign-changing solutions to the critical equation

$$-\Delta_p u = |u|^{p^*-2}u \text{ in } \Omega, \tag{1.8}$$

where Ω is either the whole space $\mathbb{R}^N$ or a bounded smooth domain in $\mathbb{R}^N$ in which case we assume homogeneous Dirichlet boundary conditions. As far as we know the only result concerning existence of sign-changing solutions to (1.8) in the whole space is due to Clapp and Lopez Rios in [**4**], where they prove that (1.8) has a certain finite number (depending on the dimension N) of non-radial sign-changing solutions. On the other hand if $p = 2$ del Pino, Musso, Pacard and Pistoia in [**9**, **10**] used the Lyapunov–Schmidt procedure to build infinitely many sign-changing solutions which look like a positive bubble crowned by an arbitrary large number of negative bubbles arranged on a regular polygon. It would be interesting to check if it is possible to build this kind of solutions in the quasilinear case. When Ω is a bounded domain, the existence of solutions is a more delicate issue. Indeed if Ω is starshaped the problem does not have any solutions because of a Pohozaev identity obtained by Guedda and Véron in [**15**]. The existence of a positive solution has been proved by Mercuri, Sciunzi and Squassina in [**17**] when the domain has a small hole, in the same spirit of Coron's result [**5**] when $p = 2$. The existence of a sign-changing solution has been obtained by Mercuri and Pacella in [**16**] when the domain Ω has either a small hole and little symmetry or a hole of any size and more symmetry. On the other hand, if $p = 2$ and Ω has a small hole, Musso and Pistoia in [**18**] (see also [**12**, **13**]) used the Lyapunov–Schmidt procedure to built sign-changing solutions which look like the superposition of bubbles with alternating sign whose number becomes arbitrary large as the size of the hole approaches zero. It is natural to ask if this kind of solutions do exist also in the quasilinear case.

In both cases the understanding of the linear non-degeneracy of the bubble is the first step in the application of the Lyapunov–Schmidt procedure.

Finally, to give a larger perspective to p-Laplacian problems, we would also quote the monograph by Véron [23].

2. A suitable weighted Sobolev space

First of all, let us point out the following fact.

LEMMA 2.1. (i) *If $p \in (1,2)$ there exists $C > 0$ such that*

$$\left(\int_{\mathbb{R}^N} |\phi|^{\frac{Np}{N-p}}\,\mathrm{d}x\right)^{\frac{N-p}{p}} \leqslant C\left(\int_{\mathbb{R}^N} |\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \text{ for any } \phi \in C^1_c(\mathbb{R}^N).$$

(ii) *If $p \in (2,N)$ for any $R > 0$ there exists $C(R) > 0$ such that*

$$\left(\int_{\mathbb{R}^N\setminus B_R(0)} |x|^{-\frac{Np+p-2N}{p-1}}|\phi|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \leqslant C(R)\left(\int_{\mathbb{R}^N} |\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \text{ for any } \phi \in C^1_c(\mathbb{R}^N).$$

Proof. To get (i) it is useful to recall the Caffarelli–Kohn–Nirenberg inequality (see [3]): if $r, q \geqslant 1$, $\frac{1}{r} + \frac{\gamma}{N} = \frac{1}{q} + \frac{\alpha-1}{N} > 0$ then for any $\phi \in C^1_c(\mathbb{R}^N)$

$$\left\||x|^\gamma \phi\right\|_{L^r(\mathbb{R}^N)} \leqslant c\left\||x|^\alpha |\nabla\phi|\right\|_{L^q(\mathbb{R}^N)}. \tag{2.1}$$

Then we apply (2.1) with $\gamma = 0$, $r = \frac{Np}{N-p}$, $q = 2$ and $\alpha = \frac{N(2-p)}{2p}$

$$\left(\int_{\mathbb{R}^N} |\phi|^{\frac{Np}{N-p}}\right)^{\frac{(N-p)}{Np}} \leqslant c\left(\int_{\mathbb{R}^N} |x|^{\frac{N(2-p)}{p}}|\nabla\phi|^2\right)^{1/2} \leqslant c\left(\int_{\mathbb{R}^N} |\nabla U|^{p-2}|\nabla\phi|^2\right)^{\frac{1}{2}}.$$

and the claim follows since if $p < 2$ there exists a constant c such that

$$|x|^{\frac{N(2-p)}{p}} \leqslant c\frac{|x|^{\frac{p-2}{p-1}}}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{N(p-2)}{p}}} \text{ for any } x \in \mathbb{R}. \tag{2.2}$$

To get (ii) it is useful to recall the weighted Hardy–Sobolev inequality (see for example Lemma 2.3 in [7]):

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if $q \geqslant 1$, $s > q - N$ and $R \geqslant 0$ then

$$\int_{\mathbb{R}^N \setminus B_R(0)} |x|^{s-q}|\varphi|^q \leqslant c(N,q,s) \int_{\mathbb{R}^N \setminus B_R(0)} |x|^s |\nabla\varphi|^q \,\mathrm{d}x \quad \text{for any } \varphi \in C_c^1(\mathbb{R}^N). \tag{2.3}$$

Then we apply (2.3) with $s = -\frac{(N-1)(p-2)}{p-1}$ and $q = 2$ (note that $s - q > -N$ because $p < N$)

$$\begin{aligned}\int_{\mathbb{R}^N \setminus B_R(0)} |x|^{-\frac{(N-1)(p-2)}{p-1}-2}|\varphi|^2 &\leqslant c \int_{\mathbb{R}^N \setminus B_R(0)} |x|^{-\frac{(N-1)(p-2)}{p-1}} |\nabla\varphi|^2 \\ &\leqslant c(R) \int_{\mathbb{R}^N} |\nabla U|^{p-2}|\nabla\phi|^2.\end{aligned}$$

and the claim follows since if $p > 2$ for any $R > 0$ there exists $c(R)$ such that

$$|\nabla U(x)| \geqslant c|x|^{\frac{N-1}{p-1}} \text{ if } |x| \geqslant R.$$

□

Lemma 2.1 allows us to define the Hilbert space $\mathcal{D}_*^{1,2}(\mathbb{R}^N)$, which is defined as the completion of $C_c^1(\mathbb{R}^N)$ with respect to the norm $\|\phi\| := (\int_{\mathbb{R}^N} |\nabla U|^{p-2}|\nabla\phi|^2 \,\mathrm{d}x)^{1/2}$ induced by the scalar product

$$\langle \phi, \psi \rangle := \int_{\mathbb{R}^N} |\nabla U|^{p-2}(\nabla\phi, \nabla\psi) \,\mathrm{d}x.$$

Now, we can look for a weak solution $\phi \in \mathcal{D}_*^{1,2}(\mathbb{R}^N)$ to the linear equation (1.7), namely

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla U|^{p-2}(\nabla\phi, \nabla\psi), \mathrm{d}x + (p-2)\int_{\mathbb{R}^N} |\nabla U|^{p-4} \left(\nabla U, \nabla\phi\right) \left(\nabla U, \nabla\psi\right) \,\mathrm{d}x \\ &\quad = \frac{Np - N + p}{N - p} \int_{\mathbb{R}^N} U^{\frac{Np-2N+2p}{N-p}} \phi\psi \,\mathrm{d}x \ \text{ for any } \psi \in \mathcal{D}_*^{1,2}(\mathbb{R}^N).\end{aligned} \tag{2.4}$$

All the integrals involved in (2.4) are finite. Indeed, the integrals in the L.H.S. can be easily estimated using Hölder inequality and Cauchy–Schwarz inequality. The finiteness of the integral in the R.H.S. is more delicate and follows by the continuous embedding of the weighted space $\mathcal{D}_*^{1,2}(\mathbb{R}^N)$ into the weighted space

$$L_*^2(\mathbb{R}^N) = \left\{\phi \ : \ \int_{\mathbb{R}^N} U^{\frac{Np-2N+2p}{N-p}} \phi^2 \,\mathrm{d}x < +\infty\right\}, \tag{2.5}$$

which is stated in the following result.

Proposition 2.2. *There exists $C > 0$ such that*

$$\int_{\mathbb{R}^N} |\nabla U|^{p-2}|\nabla\phi|^2 \,\mathrm{d}x \geqslant C \int_{\mathbb{R}^N} U^{\frac{Np}{N-p}-2}\phi^2 \,\mathrm{d}x \quad \textit{for any } \phi \in \mathcal{D}_*^{1,2}(\mathbb{R}^N). \tag{2.6}$$

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Proof. We will prove (2.6) for any $\phi \in C_c^1(\mathbb{R}^N)$. The statement will follow by a density argument. Throughout the proof c will denote a constant (possibly depending on the parameters) which may change from line to line. Although we will not estimate the constants explicitly, it will be clear from the arguments that our claims hold.

It is useful to remind that

$$U(x) = c\frac{1}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{N(p-2)+2p}{p}}} \quad \text{and} \quad |\nabla U(x)|^{p-2} = c\frac{|x|^{\frac{p-2}{p-1}}}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{N(p-2)}{p}}}.$$

We distinguish three cases.

- *The case* $\frac{2N}{N+2} < p < 2$.

  We remark that since $\frac{2N}{N+2} < p$ Hölder's inequality implies

$$\int_{\mathbb{R}^N} \frac{1}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{N(p-2)+2p}{p}}}|\phi|^2 \leqslant \left(\int_{\mathbb{R}^N} \frac{1}{\left(1+|x|^{\frac{p}{p-1}}\right)^{N}}\right)^{\frac{N(p-2)+2p}{Np}} \left(\int_{\mathbb{R}^N} |\phi|^{\frac{Np}{N-p}}\right)^{\frac{2(N-p)}{Np}}.$$

  Now, we apply Caffarelli–Kohn–Nirenberg's inequality (2.1) (with $\gamma = 0$, $r = \frac{Np}{N-p}$, $q = 2$ and $\alpha = \frac{N(2-p)}{2p}$) and we get

$$\left(\int_{\mathbb{R}^N} |\phi|^{\frac{Np}{N-p}}\right)^{\frac{(N-p)}{Np}} \leqslant c\left(\int_{\mathbb{R}^N} |x|^{\frac{N(2-p)}{p}}|\nabla\phi|^2\right)^{1/2}.$$

  The claim follows because of (2.2).

- *The case* $1 < p \leqslant \frac{2N}{N+2}$.

  In this case $\frac{N(p-2)+2p}{p} \leqslant 0$ and so

$$\left(1+|x|^{\frac{p}{p-1}}\right)^{-\frac{N(p-2)+2p}{p}} \leqslant c\left(1+|x|^{-\frac{N(p-2)+2p}{(p-1)}}\right).$$

  Then

$$\int_{\mathbb{R}^N} \frac{1}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{N(p-2)+2p}{p}}}|\phi|^2 \leqslant c\int_{\mathbb{R}^N} |\phi|^2 + c\int_{\mathbb{R}^N} |x|^{-\frac{N(p-2)+2p}{(p-1)}}|\phi|^2.$$

  Now, we apply Caffarelli–Kohn–Nirenberg's inequality and we get (with $\gamma = 0$, $r = q = 2$ and $\alpha = 2$)

$$\left(\int_{\mathbb{R}^N} |\phi|^2\right)^{1/2} \leqslant c\left(\int_{\mathbb{R}^N} |x|^2|\nabla\phi|^2\right)^{1/2}.$$

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and also (with $\gamma = -\frac{N(p-2)+2p}{2(p-1)}$, $r = q = 2$ and $\alpha = 1 - \frac{N(p-2)+2p}{2(p-1)} = \frac{-2-Np+2N}{2(p-1)}$)

$$\left(\int_{\mathbb{R}^N} |x|^{-\frac{N(p-2)+2p}{(p-1)}} |\phi|^2\right)^{1/2} \leqslant c\left(\int_{\mathbb{R}^N} |x|^{\frac{-2-Np+2N}{(p-1)}} |\nabla\phi|^2\right)^{1/2}.$$

Then

$$\int_{\mathbb{R}^N} \frac{1}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{N(p-2)+2p}{p}}} |\phi|^2 \leqslant c\int_{\mathbb{R}^N} \left(|x|^2 + |x|^{\frac{-2-Np+2N}{(p-1)}}\right) |\nabla\phi|^2.$$

The claim follows because if $p \leqslant \frac{2N}{N+2}$ it is easy to check that there exists a constant c such that

$$|x|^2 + |x|^{\frac{-2-Np+2N}{(p-1)}} \leqslant c\frac{|x|^{\frac{p-2}{p-1}}}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{N(p-2)}{p}}} \text{ for any } x \in \mathbb{R}.$$

- *The case* $p > 2$.

  The proof in this case is much more delicate because the weight $|\nabla U|^{p-2}$ has different decay as $|x| \to 0$ or $|x| \to \infty$. Let $m \geqslant 1$ be a fixed integer. We can write

$$\int_{\mathbb{R}^N} U^{p^*-2}\phi^2\,\mathrm{d}x = \underbrace{\int_{\mathbb{R}^N\setminus B_{2^m}(0)} U^{p^*-2}\phi^2\,\mathrm{d}x}_{(I)} + \underbrace{\int_{B_{2^m}(0)} U^{p^*-2}\phi^2\,\mathrm{d}x}_{(II)}.,$$

  where $B_{2^m}(0)$ is the ball centred at the origin with radius 2^m.

First, we estimate (I). We remark that there exists constants $c_1, \dots, c_4$ such that

$$\frac{c_1}{|x|^{\frac{N-p}{p-1}}} \leqslant U(x) \leqslant \frac{c_2}{|x|^{\frac{N-p}{p-1}}} \quad \text{and} \quad \frac{c_3}{|x|^{\frac{N-1}{p-1}}} \leqslant |\nabla U| \leqslant \frac{c_4}{|x|^{\frac{N-1}{p-1}}} \text{ if } |x| \geqslant 2^m. \tag{2.7}$$

Therefore

$$\int_{\mathbb{R}^N\setminus B_{2^m}(0)} U^{p^*-2}\phi^2\,\mathrm{d}x$$

(we use (2.7))

$$\leqslant c\int_{\mathbb{R}^N\setminus B_{2^m}(0)} \frac{1}{|x|^{\frac{N-p}{p-1}(p^*-2)}}\phi^2\,\mathrm{d}x$$

(we set $s = -\frac{N-1}{p-1}(p-2) > 2-N$ and $\beta = (p^*-2)\frac{N-p}{p-1} + s - 2 > 0$)

$$= c\int_{\mathbb{R}^N\setminus B_{2^m}(0)} \frac{1}{|x|^{\beta}}|x|^{s-2}\phi^2\,\mathrm{d}x$$

(we use (2.3) with $q = 2$ and $R = 2^m$)

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$$\leqslant c\int_{\mathbb{R}^N\setminus B_{2^m}(0)}|x|^s|\nabla\phi|^2\,\mathrm{d}x$$

(we use (2.7))

$$\leqslant c\int_{\mathbb{R}^N\setminus B_{2^m}(0)}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x$$

and so

$$\int_{\mathbb{R}^N\setminus B_{2^m(0)}}U^{p^*-2}\phi^2\,\mathrm{d}x\leqslant c\int_{\mathbb{R}^N}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x \tag{2.8}$$

Now, we estimate (II). Firstly, given $A:=\{x\in\mathbb{R}^N:1<|x|\leqslant 2\}$, it is useful to recall the standard interpolation inequality (see for example [**19**])

$$\int_A|u-u_A|^m\,\mathrm{d}x\leqslant c\left(\int_A|\nabla u|^r\,\mathrm{d}x\right)^{\frac{m}{r}} \tag{2.9}$$

where

$$\frac{1}{m}=\frac{1}{r}-\frac{1}{N}>0 \text{ and } u_A:=\frac{1}{|A|}\int_A u\,\mathrm{d}x.$$

Therefore, if $r=2$, $m=\frac{2N}{N-2}>0$ by Hölder inequality we immediately deduce

$$\int_A|u-u_A|^2\,\mathrm{d}x\leqslant\left(\int_A 1^{\frac{m}{m-2}}\,\mathrm{d}x\right)^{\frac{m-2}{m}}\left(\int_A|u-u_A|^m\,\mathrm{d}x\right)^{\frac{2}{m}}\leqslant c|A|^{\frac{2}{N}}\int_A|\nabla u|^2\,\mathrm{d}x.$$

Moreover, if $\lambda>0$ and $\lambda A:=\{\lambda x:x\in A\}$, a simply scaling gives

$$\int_{\lambda A}|u-u_{\lambda A}|^2\,\mathrm{d}x\leqslant c|A|^{\frac{2}{N}}\lambda^2\int_{\lambda A}|\nabla u|^2\,\mathrm{d}x \tag{2.10}$$

Now, let us introduce a sequence of disjoint annuli $A_k:=\{x\in\mathbb{R}^N:2^k<|x|\leqslant 2^{k+1}\}$ which covers the ball $B_{2^m}(0)$, namely $A_k\cap A_h=\emptyset$ for $h\neq k$ and

$$B_{2^m}(0)=\bigcup_{k=-\infty}^{m-1}A_k,$$

so that

$$\int_{B_{2^m}(0)}U^{p^*-2}\phi^2\,\mathrm{d}x=\sum_{k=-\infty}^{m}\int_{A_k}U^{p^*-2}\phi^2\,\mathrm{d}x. \tag{2.11}$$

We are going to estimate each term in the sum of R.H.S. of (2.11), taking into account that

$$\inf_{A_k}|\nabla U|^{p-2}\geqslant c\frac{2^{k\frac{p-2}{p-1}}}{(1+2^{k\frac{p}{p-1}})^{\frac{N}{p}(p-2)}}. \tag{2.12}$$

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We have

$$\begin{aligned}
&\int_{A_k} U^{p^*-2}\phi^2\,\mathrm{d}x \\
&\quad (\text{since } p \geqslant 2,\ \sup_{x\in\mathbb{R}^N} |x|^{\frac{p}{p-1}} U^{p^*-2} = L) \\
&\quad\leqslant L\int_{A_k} |x|^{-\frac{p}{p-1}}\phi^2\,\mathrm{d}x \\
&\quad\leqslant c\int_{A_k} |x|^{-\frac{p}{p-1}}|\phi-\phi_{A_k}|^2\,\mathrm{d}x + c\int_{A_k}|x|^{-\frac{p}{p-1}}|\phi_{A_k}|^2\,\mathrm{d}x \\
&\quad\leqslant c\int_{A_k} |x|^{-\frac{p}{p-1}}|\phi-\phi_{A_k}|^2\,\mathrm{d}x + c2^{k\left(-\frac{p}{p-1}+N\right)}|\phi_{A_k}|^2
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
\int_{A_k} |x|^{-\frac{p}{p-1}}|\phi-\phi_{A_k}|^2\,\mathrm{d}x &\leqslant c2^{-k\frac{p}{p-1}}\int_{A_k}|\phi-\phi_{A_k}|^2\,\mathrm{d}x \\
&\quad (\text{we use (2.10) with } \lambda = 2^k) \\
&\leqslant c2^{k\left(-\frac{p}{p-1}+2\right)}\int_{A_k}|\nabla\phi|^2\,\mathrm{d}x \\
&\quad (\text{we use (2.12)}\ ) \\
&\leqslant c2^{k\left(-\frac{p}{p-1}+2\right)}\frac{(1+2^{k\frac{p}{p-1}})^{\frac{N}{p}(p-2)}}{2^{k\frac{p-2}{p-1}}}\int_{A_k}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x \\
&= c(1+2^{k\frac{p}{p-1}})^{\frac{N}{p}(p-2)}\int_{A_k}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x.
\end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14) we get

$$\int_{A_k} U^{p^*-2}\phi^2\,\mathrm{d}x \leqslant c(1+2^{k\frac{p}{p-1}})^{\frac{N}{p}(p-2)}\int_{A_k}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x + c2^{k\left(-\frac{p}{p-1}+N\right)}|\phi_{A_k}|^2$$

and summing upon k

$$\begin{aligned}
&\int_{B_{2^m}} U^{p^*-2}\phi^2\,\mathrm{d}x \\
&\quad = \sum_{k=-\infty}^{m-1}\int_{A_k} U^{p^*-2}\phi^2\,\mathrm{d}x \\
&\quad \leqslant c\sum_{k=-\infty}^{m-1}\left[(1+2^{k\frac{p}{p-1}})^{\frac{N}{p}(p-2)}\int_{A_k}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x\right] \\
&\qquad + c\sum_{k=-\infty}^{m-1} 2^{k\left(-\frac{p}{p-1}+N\right)}|\phi_{A_k}|^2
\end{aligned}$$

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$$
\begin{aligned}
&\leqslant c\sum_{k=-\infty}^{m-1}\int_{A_k}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x+c\sum_{k=-\infty}^{m-1}\left[2^{kN\frac{p-2}{p-1}}\int_{A_k}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x\right]\\
&\quad+c\sum_{k=-\infty}^{m-1}2^{k\left(-\frac{p}{p-1}+N\right)}|\phi_{A_k}|^2\\
&\leqslant c\int_{B_{2^m}}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x+c\underbrace{\sum_{k=-\infty}^{m-1}2^{kN\frac{p-2}{p-1}}}_{<+\infty}\int_{B_{2^m}}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x+\\
&\quad+c\sum_{k=-\infty}^{m-1}2^{k\left(-\frac{p}{p-1}+N\right)}|\phi_{A_k}|^2\\
&\leqslant c\int_{\mathbb{R}^N}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x+c\sum_{k=-\infty}^{m-1}2^{k\left(-\frac{p}{p-1}+N\right)}|\phi_{A_k}|^2 \qquad (2.15)
\end{aligned}
$$

It remains to estimate the last term of (2.15), namely $\sum_{k=-\infty}^{m-1}2^{k(-\frac{p}{p-1}+N)}|\phi_{A_k}|^2$. Now

$$
\begin{aligned}
&\int_{A_k\cup A_{k+1}}|\phi-\phi_{A_k\cup A_{k+1}}|^2\,\mathrm{d}x\\
&\quad=\frac{1}{|A_k|+|A_{k+1}|}\left(\int_{A_k}|\phi-\phi_{A_k\cup A_{k+1}}|^2\,\mathrm{d}x+\int_{A_{k+1}}|\phi-\phi_{A_k\cup A_{k+1}}|^2\,\mathrm{d}x\right)\\
&\quad\geqslant\int_{A_k}|\phi-\phi_{A_k\cup A_{k+1}}|^2\,\mathrm{d}x\\
&\quad\geqslant\left|\int_{A_k}(\phi-\phi_{A_k\cup A_{k+1}})\,\mathrm{d}x\right|^2\\
&\quad=\left|\int_{A_k}\phi\,\mathrm{d}x-\frac{|A_k|}{|A_{k+1}|+|A_k|}\int_{A_k}\phi\,\mathrm{d}x-\frac{|A_k|}{|A_{k+1}|+|A_k|}\int_{A_{k+1}}\phi\,\mathrm{d}x\right|^2\\
&\quad=\frac{1}{|A_k|+|A_{k+1}|}\left||A_{k+1}|\int_{A_k}\phi\,\mathrm{d}x-|A_k|\int_{A_k}\phi\,\mathrm{d}x\right|^2\\
&\quad=\frac{|A_k||A_{k+1}|}{|A_k|+|A_{k+1}|}|\phi_{A_k}-\phi_{A_{k+1}}|^2
\end{aligned}
$$

and so

$$
\begin{aligned}
|\phi_{A_k}-\phi_{A_{k+1}}|^2&\leqslant c\frac{|A_k|+|A_{k+1}|}{|A_k||A_{k+1}|}\int_{A_k\cup A_{k+1}}|\phi-\phi_{A_k\cup A_{k+1}}|^2\,\mathrm{d}x\\
&\qquad\text{(taking into account that } |A_k|=c2^{kN}\left(2^N-1\right)\text{)}\\
&\leqslant c2^{-kN}\int_{A_k\cup A_{k+1}}|\phi-\phi_{A_k\cup A_{k+1}}|^2\,\mathrm{d}x
\end{aligned}
$$

$$\text{(we use (2.10) with } \lambda = 2^k\text{)}$$
$$\leqslant c2^{k(2-N)} \int_{A_k \cup A_{k+1}} |\nabla \phi|^2 \,\mathrm{d}x$$
$$\text{(we use (2.12))}$$
$$\leqslant c2^{k(2-N)} (1 + 2^{k\frac{p}{p-1}})^{\frac{N(p-2)}{p}} 2^{-k\frac{p-2}{p-1}} \int_{A_k \cup A_{k+1}} |\nabla U|^{p-2} |\nabla \phi|^2 \,\mathrm{d}x$$
$$\leqslant c \left(2^{k\frac{p-Np+N}{p-1}} + 2^{k\frac{p-N}{p-1}} \right) \int_{A_k \cup A_{k+1}} |\nabla U|^{p-2} |\nabla \phi|^2 \,\mathrm{d}x. \tag{2.16}$$

Now, we use the simple fact that for any $\eta > 0$ the following inequality holds

$$(a+b)^2 \leqslant (1+\eta) a^2 + \left(\frac{\eta+1}{\eta} \right) b^2 \text{ for any } a, b \in \mathbb{R}.$$

Then, if we choose $\eta > 0$ so that $1 + \eta = \eta_0 2^{-\frac{p}{p-1}+N}$ where $\eta_0 = \frac{2}{1+2^{-\frac{p}{p-1}+N}} < 1$, (this is possible because $N - \frac{p}{p-1} > 0$, since $p \geqslant 2$), we get

$$|\phi_{A_k}|^2 = |\phi_{A_k} - \phi_{A_{k+1}} + \phi_{A_{k+1}}|^2 \leqslant \eta_0 2^{-\frac{p}{p-1}+N} |\phi_{A_{k+1}}|^2 + \frac{\eta+1}{\eta} |\phi_{A_k} - \phi_{A_{k+1}}|^2.$$

and using (2.16) we deduce

$$\begin{aligned}
&2^{k\left(-\frac{p}{p-1}+N\right)} |\phi_{A_k}|^2 \\
&\quad \leqslant 2^{k\left(-\frac{p}{p-1}+N\right)} 2^{\left(-\frac{p}{p-1}+N\right)} \eta_0 |\phi_{A_{k+1}}|^2 \\
&\qquad + c 2^{k\left(-\frac{p}{p-1}+N\right)} \left(2^{k\frac{p-Np+N}{p-1}} + 2^{k\frac{p-N}{p-1}} \right) \int_{A_k \cup A_{k+1}} |\nabla U|^{p-2} |\nabla \phi|^2 \,\mathrm{d}x \\
&\quad = 2^{(k+1)\left(-\frac{p}{p-1}+N\right)} \eta_0 |\phi_{A_{k+1}}|^2 + c \left(1 + 2^{kN\frac{p-2}{p-1}} \right) \int_{A_k \cup A_{k+1}} |\nabla U|^{p-2} |\nabla \phi|^2 \,\mathrm{d}x.
\end{aligned}$$

We sum upon k and we get

$$\begin{aligned}
&\sum_{k=-\infty}^{m-1} 2^{k\left(-\frac{p}{p-1}+N\right)} |\phi_{A_k}|^2 \\
&\quad \leqslant \eta_0 \sum_{k=-\infty}^{m-1} 2^{(k+1)\left(-\frac{p}{p-1}+N\right)} |\phi_{A_{k+1}}|^2 \\
&\qquad + \sum_{k=-\infty}^{m-1} \left(1 + 2^{kN\frac{p-2}{p-1}} \right) \int_{A_k \cup A_{k+1}} |\nabla U|^{p-2} |\nabla \phi|^2 \,\mathrm{d}x \\
&\quad \leqslant \eta_0 \sum_{k=-\infty}^{m} 2^{k\left(-\frac{p}{p-1}+N\right)} |\phi_{A_k}|^2 + \left(1 + \underbrace{\sum_{k=-\infty}^{m-1} 2^{kN\frac{p-2}{p-1}}}_{<+\infty} \right) \int_{\mathbb{R}^N} |\nabla U|^{p-2} |\nabla \phi|^2 \,\mathrm{d}x,
\end{aligned}$$

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which implies

$$(1-\eta_0)\sum_{k=-\infty}^{m-1} 2^{k\left(-\frac{p}{p-1}+N\right)}|\phi_{A_k}|^2 \leqslant c|\phi_{A_m}|^2 + c\int_{\mathbb{R}^N}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x. \tag{2.17}$$

On the other hand, we have

$$\begin{aligned}
&|\phi_{A_m}|^2 \quad \text{(we use Hölder's inequality)}\\
&\quad\leqslant c\int_{A_m}|x|^{-\frac{(N-1)(p-2)}{p-1}}\phi^2\,\mathrm{d}x \quad (\text{the annulus } A_m\subset\mathbb{R}^N\setminus B_{2^m(0)})\\
&\quad c\int_{\mathbb{R}^N\setminus B_{2^m(0)}}|x|^{-\frac{(N-1)(p-2)}{p-1}}\phi^2\,\mathrm{d}x \quad (\text{we use (2.8)})\\
&\quad\leqslant c\int_{\mathbb{R}^N}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x.
\end{aligned}\tag{2.18}$$

Finally, combining (2.15) with (2.17) (remember that $\eta_0<1$) and (2.18) we get

$$\int_{B_{2^m(0)}} U^{p*-2}\phi^2\,\mathrm{d}x \leqslant c\int_{\mathbb{R}^N}|\nabla U|^{p-2}|\nabla\phi|^2\,\mathrm{d}x. \tag{2.19}$$

□

3. Proof of Theorem 1.1

3.1. A wave decomposition

First of all, let us rewrite the linear equation (1.7) as

$$|x|^2\Delta\phi + (p-2)\sum_{i,j=1}^{N}\partial^2_{ij}\phi x_i x_j + \frac{(p-2)N}{1+|x|^{\frac{p}{p-1}}}(\nabla\phi,x) + \gamma_{N,p}\frac{|x|^{\frac{p}{p-1}}}{(1+|x|^{\frac{p}{p-1}})^2}\phi = 0 \tag{3.1}$$

where $\gamma_{N,p} := N\frac{Np-N+p}{p-1}$. Indeed a straightforward computation shows that

$$\begin{aligned}
&\operatorname{div}\left(|\nabla U|^{p-2}\nabla\phi\right) + (p-2)\operatorname{div}\left(|\nabla U|^{p-4}(\nabla U,\nabla\phi)\nabla U\right)\\
&\quad= |\nabla U|^{p-2}\Delta\phi + \left(\nabla|\nabla U|^{p-2},\nabla\phi\right)\\
&\qquad+(p-2)|\nabla U|^{p-4}(\nabla U,\nabla\phi)\,\Delta U\\
&\qquad+(p-2)(\nabla U,\nabla\phi)\left(\nabla|\nabla U|^{p-4},\nabla U\right)\\
&\qquad+(p-2)|\nabla U|^{p-4}\left(\nabla(\nabla U,\nabla\phi),\nabla U\right).
\end{aligned}$$

and

$$\nabla U = -c_{N,p}\frac{|x|^{\frac{2-p}{p-1}}x}{(1+|x|^{\frac{p}{p-1}})^{\frac{N}{p}}}$$

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$$|\nabla U|^{p-4} = c_{N,p}^{p-4} \frac{|x|^{\frac{p-4}{p-1}}}{(1+|x|^{\frac{p}{p-1}})^{\frac{N(p-4)}{p}}}$$

$$\big(\nabla|\nabla U|^{p-4}, \nabla U\big) = -c_{N,p}^{p-3} \frac{p-4}{p-1} \frac{|x|^{-\frac{2}{p-1}}}{(1+|x|^{\frac{p}{p-1}})^{\frac{N(p-4)}{p}+\frac{N}{p}}}$$

$$+ c_{N,p}^{p-3} N \frac{p-4}{p-1} \frac{|x|^{\frac{p-2}{p-1}}}{(1+|x|^{\frac{p}{p-1}})^{\frac{N(p-4)}{p}+\frac{N}{p}+1}}$$

$$(\nabla U, \nabla\phi) = -c_{N,p}^{p-3} \frac{|x|^{\frac{2-p}{p-1}}}{(1+|x|^{\frac{p}{p-1}})^{\frac{N}{p}}} (\nabla\phi, x)$$

$$(\nabla(\nabla U, \nabla\phi), \nabla U) = \frac{c_{N,p}^2}{(1+|x|^{\frac{p}{p-1}})^{\frac{2N}{p}}} (\nabla\phi, x) \left[\frac{1}{p-1}|x|^{2\frac{2-p}{p-1}} - \frac{N}{p-1} \frac{|x|^{\frac{4-p}{p-1}}}{1+|x|^{\frac{p}{p-1}}}\right]$$

$$+ c_{N,p}^2 \frac{|x|^{2\frac{2-p}{p-1}}}{(1+|x|^{\frac{p}{p-1}})^{\frac{2N}{p}}} \sum_{i,j} \partial_{ij}^2 \phi x_i x_j$$

$$|\nabla U|^{p-4}(\nabla U, \nabla\phi)\Delta U = \frac{c_{N,p}^{p-2}}{(1+|x|^{\frac{p}{p-1}})^{\frac{N(p-4)}{p}+\frac{2N}{p}}} (\nabla\phi, x)$$

$$\times \left[\left(\frac{2-p}{p-1} + N\right)|x|^{-\frac{p}{p-1}} - \frac{N}{p-1} \frac{1}{1+|x|^{\frac{p}{p-1}}}\right]$$

$$(\nabla|\nabla U|^{p-2}, \nabla\phi) = \frac{c_{N,p}^{p-2}}{(1+|x|^{\frac{p}{p-1}})^{\frac{N(p-2)}{p}}} (\nabla\phi, x)$$

$$\times \left[\frac{p-2}{p-1}|x|^{-\frac{p}{p-1}} - \frac{N(p-2)}{p-1} \frac{1}{1+|x|^{\frac{p}{p-1}}}\right]$$

where $c_{N,p} := \alpha_{N,p}^{\frac{N-p}{p}} \frac{N-p}{p-1}$.

Now, since U is radial we can make a partial wave decomposition of (3.1), namely we can write

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k(r) Y_k(\theta), \quad \text{where} \quad \psi_k(r) = \int_{S^{N-1}} \phi(r,\theta) Y_k(\theta)\, \mathrm{d}\theta, \tag{3.2}$$

where $r = |x|$, $\theta = \frac{x}{|x|} \in S^{N-1}$ and $Y_k(\theta)$ denotes the k-th spherical harmonic satisfying ($\Delta_{S^{N-1}}$ stands for the Laplace–Beltrami operator)

$$-\Delta_{S^{N-1}} Y_k = \lambda_k Y_k. \tag{3.3}$$

It is known that this equation has a sequence of eigenvalues

$$\lambda_k = k(N+k-2), \quad k = 0, 1, 2, \ldots, \tag{3.4}$$

whose multiplicity is finite. In particular $\lambda_0 = 0$ has multiplicity 1 and $\lambda_1 = N - 1$ has multiplicity N.

Let us write the equations satisfied by the radial functions ψ_k. It is known that (hereafter $'$ stands for $\frac{d}{dr}$)

$$\Delta(\psi_k(r)Y_k(\theta)) = Y_k(\theta)\left(\psi_k'' + \frac{N-1}{r}\psi_k'\right) + \frac{1}{r^2}\psi_k(r)\Delta_{S^{N-1}}Y_k(\theta). \tag{3.5}$$

Now, we have to compute the other terms in (3.1). It is easy to see that

$$\partial_{x_i}\phi = \psi_k'(r)\frac{x_i}{r}Y_k(\theta) + \psi_k(r)\frac{\partial Y_k}{\partial\theta_h}\frac{\partial\theta_h}{\partial x_i}$$

and

$$\begin{aligned}\partial^2_{x_ix_j}\phi &= \psi_k''(r)\frac{x_ix_j}{r^2}Y_k(\theta) + \psi_k'(r)\left(\frac{\delta_{ij}}{r} - \frac{x_ix_j}{r^3}\right)Y_k(\theta) + \psi_k'(r)\frac{x_i}{r}\frac{\partial Y_k}{\partial\theta_h}\frac{\partial\theta_h}{\partial x_j}\\ &\quad + \psi_k'(r)\frac{x_j}{r}\frac{\partial Y_k}{\partial\theta_h}\frac{\partial\theta_h}{\partial x_i} + \psi_k(r)\frac{\partial^2 Y_k}{\partial\theta_h\partial\theta_\ell}\frac{\partial\theta_\ell}{\partial x_j}\frac{\partial\theta_h}{\partial x_i} + \psi_k(r)\frac{\partial Y_k}{\partial\theta_h}\frac{\partial^2\theta_h}{\partial x_i\partial x_j}.\end{aligned}$$

Hence

$$(\nabla\phi, x) = \sum_{i=1}^N x_i\partial_{x_i}\phi = \psi_k'(r)rY_k(\theta) + \psi_k(r)\frac{\partial Y_k}{\partial\theta_h}\sum_{i=1}^N\frac{\partial\theta_h}{\partial x_i}x_i = \psi_k'(r)rY_k(\theta) \tag{3.6}$$

and

$$\begin{aligned}\sum_{i,j=1}^N \partial^2_{x_ix_j}\phi x_ix_j &= \psi_k''(r)r^2Y_k(\theta) + 2\psi_k'(r)r\sum_{i=1}^N\frac{\partial Y_k}{\partial\theta_h}\frac{\partial\theta_h}{\partial x_i}x_i\\ &\quad + \psi_k(r)\sum_{i,j=1}^N\frac{\partial^2 Y_k}{\partial\theta_h\partial\theta_\ell}\frac{\partial\theta_\ell}{\partial x_j}x_j\frac{\partial\theta_h}{\partial x_i}x_i\\ &\quad + \psi_k(r)\sum_{i,j=1}^N\frac{\partial Y_k}{\partial\theta_h}\frac{\partial^2\theta_h}{\partial x_i\partial x_j}x_ix_j = \psi_k''(r)r^2Y_k(\theta).\end{aligned} \tag{3.7}$$

because it holds true that

$$\sum_{i=1}^N\frac{\partial\theta_h}{\partial x_i}x_i = 0 \quad\text{and}\quad \sum_{i,j=1}^N\frac{\partial^2\theta_h}{\partial x_i\partial x_j}x_ix_j = 0, \quad h = 1,\dots,N-1.$$

Putting together (3.3), (3.5), (3.6) and (3.7) into (3.1) we get the following equations for any ψ_k, $k = 0,1,2,\dots$,

$$\psi_k'' + \frac{\psi_k'}{r}\left(\frac{N-1}{p-1} + \frac{(p-2)N}{p-1}\frac{1}{1+r^{\frac{p}{p-1}}}\right) - \frac{\lambda_k}{r^2}\psi_k + \gamma_{N,p}\frac{r^{\frac{p}{p-1}-2}}{(1+r^{\frac{p}{p-1}})^2}\psi_k = 0, \tag{3.8}$$

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which can be rewritten in a weak form as

$$\mathcal{L}_k(\psi_k) = 0, \quad k = 0, 1, 2, \ldots, \tag{3.9}$$

where the operator $\mathcal{L}_k$ is defined by

$$\mathcal{L}_k(\psi) := \left(r^{N-1}|U'(r)|^{p-2}\psi'\right)' + (p^*-1)r^{N-1}\left(U(r)\right)^{p^*-2}\psi - \lambda_k r^{N-3}|U'(r)|^{p-2}\psi.$$

Since we are concerned with solutions $\psi \in \mathcal{D}^{1,2}_*(\mathbb{R}^N)$ to the linear equation (1.7), we will look for solutions ψ_k to (3.8) or (3.9) in the space $\mathcal{D}_k$ which is the completion of $C^1_c([0,+\infty))$ with respect to the norm

$$\|\psi\|_k := \left(\int_0^{+\infty} r^{N-1}|U'(r)|^{p-2}|\psi'(r)|^2\,\mathrm{d}r + \lambda_k \int_0^{+\infty} r^{N-3}|U'(r)|^{p-2}|\psi(r)|^2\,\mathrm{d}r\right)^{\frac{1}{2}}.$$

3.2. Solving the equations $\mathcal{L}_k(\psi) = 0$

- *The case* $k = 0$.
  We know that the function Z_0 defined in (1.5) as

$$Z_0(x) = \frac{N-p}{p(p-1)}\alpha_{N,p}^{\frac{N-p}{p}} \underbrace{\frac{p-1-|x|^{\frac{p}{p-1}}}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{N}{p}}}}_{:=\psi_0(|x|)}$$

solves the equation (3.1). We claim that all the solutions in $\mathcal{D}_0$ to $\mathcal{L}_0(\psi) = 0$ are given by $\psi = c\psi_0$, $c \in \mathbb{R}$. Indeed, for $k = 0$ we have that $\lambda_0 = 0$ and a straightforward computation shows that $\psi_0 \in \mathcal{D}_0$ and $\mathcal{L}_0(\psi_0) = 0$.

We look for a second linearly independent solution of the form

$$w(r) = c(r)\psi_0(r).$$

Then we get

$$c''(r)\psi_0(r) + c'(r)\left[2\psi_0'(r) + \frac{\psi_0(r)}{r}\left(\frac{N-1}{p-1} + \frac{N(p-2)}{p-1}\frac{1}{1+r^{\frac{p}{p-1}}}\right)\right] = 0$$

and hence

$$\frac{c''(r)}{c'(r)} = -2\frac{\psi_0'(r)}{\psi_0(r)} - \frac{1}{r}\left(\frac{N-1}{p-1} + \frac{N(p-2)}{p-1}\frac{1}{1+r^{\frac{p}{p-1}}}\right).$$

A direct computation shows that

$$c'(r) = A\frac{\left(1+r^{\frac{p}{p-1}}\right)^{\frac{N(p-2)}{p}}}{(\psi_0(r))^2 r^{\frac{N-1+N(p-2)}{p-1}}} \text{ for some } A \in \mathbb{R}\setminus\{0\}.$$

Therefore

$$c(r) \sim Br^{\frac{N-p}{p-1}} \quad \text{and} \quad w(r) = c(r)\psi_0(r) \sim B \text{ as } r \to +\infty \text{ with } B \neq 0.$$

However $w \notin \mathcal{D}_0$ because of Lemma 2.1.

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- *The case* $k = 1$.

We know that the function Z_i defined in (1.6) as

$$Z_i(x) = \frac{N-p}{p-1}\alpha_{N,p}^{\frac{N-p}{p}} \frac{x_i}{|x|} \underbrace{\frac{|x|^{\frac{1}{p-1}}}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{N}{p}}}}_{:=\psi_1(|x|)}, \; i = 1,\dots,N \tag{3.10}$$

solve the equation (3.1). We claim that all the solutions in $\mathcal{D}_1$ to $\mathcal{L}_1(\psi) = 0$ are given by $\psi = c\psi_1$, $c \in \mathbb{R}$. Indeed, for $k = 1$ we have that $\lambda_1 = N - 1$ and a straightforward computation shows that $\psi_1 \in \mathcal{D}_1$ and $\mathcal{L}_1(\psi_1) = 0$.

As above, we look for a second linearly independent solution of the form

$$w(r) = c(r)\psi_1(r).$$

Then we get

$$c''(r)\psi_1(r) + c'(r)\left[2\psi_1'(r) + \frac{\psi_1(r)}{r}\left(\frac{N-1}{p-1} + \frac{N(p-2)}{p-1}\frac{1}{1+r^{\frac{p}{p-1}}}\right)\right] = 0$$

and a direct computation shows that

$$c'(r) = A\frac{(1+r^{\frac{p}{p-1}})^{\frac{N(p-2)}{p}}}{(\psi_1(r))^2 r^{\frac{N-1+N(p-2)}{p-1}}} \text{ for some } A \in \mathbb{R}\setminus\{0\}.$$

Therefore

$$c(r) \sim Br^{\frac{N-1}{p-1}+1} \quad \text{and} \quad w(r) = c(r)\psi_1(r) \sim Br \text{ as } r \to +\infty \text{ with } B \neq 0.$$

However $w \notin \mathcal{D}_1$ because of Lemma 2.1.

- *The case* $k \geqslant 2$.

We claim that all the solutions in $\mathcal{D}_k$ of $\mathcal{L}_k(\psi) = 0$ are identically zero if $k \geqslant 2$. Assume there exists a function ψ_k such that $\mathcal{L}_k(\psi_k) = 0$, i.e. for any $r \geqslant 0$

$$\left(r^{N-1}|U'(r)|^{p-2}\psi_k'\right)' + r^{N-1}(U(r))^{\frac{Np-2N+2p}{N-p}}\psi_k - \lambda_k r^{N-3}|U'(r)|^{p-2}\psi_k = 0. \tag{3.11}$$

We claim that $\psi_k \equiv 0$ if $k \geqslant 2$. We argue by contradiction. Without the loss of generality, we can suppose that there exists $r_k > 0$ (possibly $+\infty$) such that $\psi_k(r) > 0$ for any $r \in (0, r_k)$ and $\psi_k(r_k) = 0$. In particular, $\psi_k'(r_k) \leqslant 0$.

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Now, let $\psi_1(r) = U'(r)$ (see (3.10)) be the solution of $\mathcal{L}_1(\psi_1) = 0$, i.e. for any $r \geqslant 0$

$$\left(r^{N-1}|U'(r)|^{p-2}\psi_1'\right)' + r^{N-1}\left(U(r)\right)^{\frac{Np-2N+2p}{N-p}}\psi_1 - \lambda_1 r^{N-3}|U'(r)|^{p-2}\psi_1 = 0. \tag{3.12}$$

We multiply (3.11) by ψ_1, (3.12) by ψ_k, we integrate between 0 and r_k, we subtract the two expressions and we get

$$\begin{aligned}(\lambda_k - \lambda_1)&\int_0^{r_k} r^{N-3}|U'(r)|^{p-2}\psi_k\psi_1\,\mathrm{d}r \\ &= \int_0^{r_k}\left(r^{N-1}|U'(r)|^{p-2}\psi_k'\right)'\psi_1\,dr - \int_0^{r_k}\left(r^{N-1}|U'(r)|^{p-2}\psi_1'\right)'\psi_k\,\mathrm{d}r\end{aligned}$$

(we integrate by part and we use that $\psi_k(r_k) = 0$)

$$= r_k^{N-1}|U'(r_k)|^{p-2}\psi_k'(r_k)\psi_1(r_k) \tag{3.13}$$

and a contradiction arises when $\lambda_k > \lambda_1$, (that is $k \geqslant 2$), since $\psi_k'(r_k) \leqslant 0$, $\psi_1(r) < 0$ for any $r > 0$ and $\int_0^{r_k} r^{N-3}|U'(r)|^{p-2}\psi_k\psi_1\,dr < 0$.

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