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Twisted cohomological equations for translation flows

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Dedicated to Anatole Katok, who taught us how to think

Abstract. We prove by methods of harmonic analysis a result on the existence of solutions for twisted cohomological equations on translation surfaces with loss of derivatives at most $3+$ in Sobolev spaces. As a consequence we prove that product translation flows on (three-dimensional) translation manifolds which are products of a (higher-genus) translation surface with a (flat) circle are stable in the sense of A. Katok. In turn, our result on product flows implies a stability result of time-τ maps of translation flows on translation surfaces.

Key words: translation surfaces and flows, twisted cohomological equations, twisted invariant distributions, Sobolev estimates
2020 Mathematics Subject Classification: 37C10, 37E35, 37A20 (Primary); 30E20, 31A20 (Secondary)

1. *Introduction*

The first result on solutions of the cohomological equation for a parabolic non-homogeneous (but 'locally homogeneous') smooth flow was given by the author in [**F97**] in the case of translation flows (and their smooth time-changes) on higher-genus translation surfaces, by methods of harmonic analysis based on the theory of boundary behavior of holomorphic functions.

Since then refined versions of that result have been proved by (dynamical) renormalization methods based on 'spectral gap' (and hyperbolicity) properties of the Rauzy–Veech–Zorich cocycle [**MMY05**, **MY16**], of the Kontsevich–Zorich cocycle over the Teichmüller flow [**F07**] and, more recently, of the transfer operator of a pseudo-Anosov map on appropriate anisotropic Banach spaces of currents [**FGL**]. The renormalization approach has the immediate advantage of a refined control on the regularity loss and of more explicit conditions of Diophantine type on the dynamics, and in particular applies to self-similar translation flows or interval exchange transformations. It also gives a direct approach to results for almost all translation surfaces, while an extension to almost all directions *for any given translation surface* had to wait for the work of Chaika and Eskin [**CE**] based on the breakthroughs of Eskin and Mirzakhani [**EM**], Eskin, Mirzakhani and Mohammadi [**EMM**] and Filip [**Fi**].

In this paper we apply a twisted version of the arguments of [**F97**] to the solution of the twisted cohomological equation for translation flows and derive results on the cohomological equation for three-dimensional 'translation flows' on products of a higher-genus translation surface with a circle. For these problems no renormalization approach is available at the moment, although steps in that direction have been taken in the work of Bufetov and Solomyak [**BS14**, **BS18a**, **BS18b**, **BS18c**, **BS19**] and of the author [**F19**], who have introduced twisted versions of the Rauzy–Veech–Zorich and Kontsevich–Zorich cocycles respectively, and proved 'spectral gap' results for them.

The study of solutions of the twisted cohomological equation is therefore motivated in part by its connection to the question of asymptotics of twisted integrals, investigated in the above-mentioned papers. In fact, twisted integrals of twisted coboundaries with bounded transfer function are bounded for all times. However, there are other more direct motivations, related to the theory of cohomological equations. From results on the twisted cohomological equation (Theorem 1.1 below) we derive results on the (untwisted) cohomological equation for translation flows on the three-dimensional product of a higher-genus surface with a circle (Theorem 1.4) and results on the cohomological equation for time-τ maps of translation flows (Corollary 1.5). These are the first results of this kind for non-homogeneous 'parabolic flows' beyond the case of (translation) flows on higher-genus surfaces.

For any translation surface (M, h) (a pair of a Riemann surface M and an abelian differential h on M) let $H^s_h(M)$ denote the scale of (weighted) Sobolev spaces (introduced in [**F97**], and recalled below in §2), based on the Hilbert space $H^0_h(M) := L^2(M, \omega_h)$ of square-integrable functions with respect to the area form ω_h of the abelian differential, and defined by the Lie derivative operators given by translation vector fields on the translation surface (M, h).

For the horizontal translation flow $\phi^h_{\mathbb{R}}$ on M (of generator the horizontal vector field S) and for any $\sigma \in \mathbb{R}$, let $\mathcal{I}^s_{h,\sigma} \subset H^{-s}_h(M)$ denote the space of $(S + \imath\sigma)$-*invariant distributions*, that is, the subspace

$$\mathcal{I}^s_{h,\sigma} := \{D \in H^{-s}_h(M) \mid (S + \imath\sigma)D = 0 \in H^{-(s+1)}_h(M)\}.$$

For all $\theta \in \mathbb{T}$, let $h_\theta := e^{-\imath\theta}h$ denote the rotated abelian differential and let S_θ denote the generator of the horizontal translation flow on (M, h_θ).

We prove the following results.

THEOREM 1.1. *For any translation surface (M, h), for almost all $\theta \in \mathbb{T}$ and for almost all $\sigma \in \mathbb{R}$ (with respect to the Lebesgue measure), the following statement holds. For all $s > 0$ the space $\mathcal{I}^s_{h_\theta,\sigma} \subset H^{-s}_h(M)$ of $(S_\theta + \imath\sigma)$-invariant distributions has finite dimension (bounded above and below by linear functions of the regularity $s > 0$). For all $f \in H^s_h(M)$ with $s > 3$, satisfying the distributional conditions $D(f) = 0$ for all $(S_\theta + \imath\sigma)$-invariant distributions $D \in \mathcal{I}^s_{h_\theta,\sigma}$, the twisted cohomological equation $(S_\theta + \imath\sigma)u = f$ has a solution $u \in H^r_h(M)$ for all $r < s - 3$, and there exists a constant $C_{r,s}(\theta, \sigma) > 0$*

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such that

$$|u|_r \leq C_{r,s}(\theta, \sigma)|f|_s.$$

In other words, the theory of the twisted cohomological equation of translation flows is analogous, *for Lebesgue almost all twisting parameters*, to the untwisted theory of the cohomological equation for translation flows.

Remark 1.2. In the untwisted case the optimal loss of regularity of solutions of the cohomological equation is known to be $1+$ for L^2 Sobolev norms, for almost all translation flows with respect to any $SL(2, \mathbb{R})$ invariant measure under the hypothesis of hyperbolicity of the Kontsevich–Zorich cocycle [**F07**]. Marmi and Yoccoz [**MY16**] proved a similar, but slightly weaker, statement for Hölder norms. For self-similar translation flows, the loss of $1+$ derivatives for Hölder norms should follow from the recent work of Faure, Gouëzel and Lanneau [**FGL**], although spaces with fractional exponents are not explicitly considered in their paper.

It is natural to conjecture that the optimal loss of derivatives is $1+$ also in the twisted case, and it seems plausible that the whole argument of [**F07**] would carry over under the (equivalent?) hypotheses of hyperbolicity of the twisted cocycles introduced in [**BS18c**, **F19**]. At the moment the only known results on such twisted exponents are upper bounds (in particular, that the top exponent is less than 1) [**BS18c**, **BS19**, **F19**] but no lower bounds are known.

Remark 1.3. The problem of solution of the cohomological equation $(S + \lambda)u = f$ is interesting only in the case, considered in Theorem 1.1, of $\lambda \in \imath\mathbb{R}$. In fact, the Lie derivative operator with respect to a translation flow S on a higher-genus translation surface (M, h) is essentially skew-adjoint (see [**Ne59**, Lemma 3.10]), hence its spectrum is contained in $\imath\mathbb{R}$. For $\lambda \notin \imath\mathbb{R}$, that is, for λ in the resolvent set of the operator, we have the standard L^2 a priori estimate

$$\mathrm{Re}(\lambda)^2|u|_0^2 \leq \mathrm{Re}(\lambda)^2|u|_0^2 + |(S + \imath\mathrm{Im}(\lambda))u|_0^2 = |(S + \lambda)u|_0^2.$$

From the above a priori estimate, it is possible to derive, for all $\lambda \notin \imath\mathbb{R}$, the existence of (unique) solutions $u \in L^2(M, \omega_h)$ of the cohomological equation $(S + \lambda)u = f$, for all data $f \in H_h^0(M)$, and by regularization, the existence of regular solutions $u \in H_h^s(M)$ for all regular data $f \in H_h^s(M)$ (in the appropriate weighted Sobolev spaces), with no loss of derivatives.

A result on the existence of solutions of the cohomological equation for twisted horocycle flows was recently proved by Flaminio, Forni and Tanis [**FFT16**], who were motivated by applications to the cohomological equation for horocycle time-τ maps (see also [**Ta12**]) and to deviation of ergodic averages for twisted horocycle integrals and horocycle time-τ maps. Twisted nilflows are still nilflows, hence the theory of twisted cohomological equations in the nilpotent case is covered by the general results of Flaminio and Forni [**FlaFo07**]. As for results on deviation of ergodic averages for nilflows, they are related to bounds on Weyl sums for polynomials. The Heisenberg, and the general step-2, case are better understood by renormalization methods (see, for instance, [**FlaFo06**]),

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while the higher-step case is not renormalizable, hence harder (see, for instance, [GT12, FlaFo14]). Results on twisted ergodic integrals of translation flows and applications to effective weak mixing were recently proved by the author [F19].

For all $(s, \nu) \in \mathbb{N} \times \mathbb{N}$, let $H_h^{s,\nu}(M \times \mathbb{T})$ denote the L^2 Sobolev space on $M \times \mathbb{T}$ with respect to the invariant volume form $\omega_h \wedge d\phi$ and the vector fields S, T, and $\partial/\partial\phi$: for all $(s, \nu) \in \mathbb{N} \times \mathbb{N}$, we define

$$H_h^{s,\nu}(M \times \mathbb{T}) := \left\{ f \in L^2(M \times \mathbb{T}, d\mathrm{vol}) \,\Big|\, \sum_{i+j \le s} \sum_{\ell \le \nu} \left\| S^i T^j \frac{\partial^\ell f}{\partial \phi^\ell} \right\|_0 < +\infty \right\};$$

the space $H_h^{s,\nu}(M \times \mathbb{T})$ can be defined for all $(s, \nu) \in \mathbb{R}^+ \times \mathbb{R}^+$ by interpolation [LM] and the space $H_h^{-s,-\nu}(M \times \mathbb{T})$ is defined as the dual of the space $H_h^{s,\nu}(M \times \mathbb{T})$.

The space $L^2(M \times \mathbb{T}, d\mathrm{vol})$ of the product manifold with respect to the invariant volume form $\omega_h \wedge d\phi$ decomposes as a direct sum of the eigenspaces $\{H_n^0 \mid n \in \mathbb{Z}\}$ of the circle action:

$$L^2(M \times \mathbb{T}, d\mathrm{vol}) = \bigoplus_{n \in \mathbb{Z}} H_n^0.$$

Let now $X_{\theta,c} = S_\theta + c(\partial/\partial\phi)$ denote a translation vector field on the translation manifold $M \times \mathbb{T}$, and let $\mathcal{I}_{h_\theta,c}^{s,\nu} \subset H_h^{-s,-\nu}(M \times \mathbb{T})$ denote the space of $X_{\theta,c}$-invariant distributions. The subspace of $X_{\theta,c}$-invariant distributions in $\mathcal{I}_{h_\theta,c}^{s,\nu}$ supported on the Sobolev subspace of $H_n^s \subset H_n^0$ has finite and non-zero dimension, uniformly bounded with respect to $n \in \mathbb{N}$. It follows that the space $\mathcal{I}_{h_\theta,c}^{s,\nu}$ has countable dimension.

THEOREM 1.4. *For any translation surface (M, h), for almost all $\theta \in \mathbb{T}$ and for almost all $c \in \mathbb{R}$ (with respect to the Lebesgue measure), the following statement holds.*

For all $s > 3$ and $\nu > 2$, the space $\mathcal{I}_{h_\theta,c}^{s,\nu} \subset H_h^{-s,-\nu}(M \times \mathbb{T})$ of $X_{\theta,c}$-invariant distributions has infinite (countable) dimension, and for all $f \in H_h^{s,\nu}(M \times \mathbb{T})$ such that $D(f) = 0$ for all $D \in \mathcal{I}_{h_\theta,c}^{s,\nu}$, the cohomological equation $X_{\theta,c} u = f$ has a solution $u \in H_h^{r,\mu}(M \times \mathbb{T})$ for all $r < s - 3$ and $\mu < \nu - 2$. In addition, there exists a constant $C_{r,s}^{(\mu,\nu)}(\theta, c) > 0$ such that

$$\|u\|_{r,\mu} \le C_{r,s}^{(\mu,\nu)}(\theta, c) \|f\|_{s,\nu}.$$

Following Katok (see [Ka01, §3.1] or [Ka03, §11.1]) a smooth flow on a smooth compact, connected manifold $\mathcal{M}$ is called C^∞-*stable* if the range of the Lie derivative operator on $C^\infty(\mathcal{M})$ is closed in $C^\infty(\mathcal{M})$ (see also [F08]). In this terminology, Theorem 1.4 implies that for almost all $(\theta, c) \in \mathbb{T} \times \mathbb{R}$, the flow of the vector field $X_{\theta,c}$ on $M \times \mathbb{T}$ is C^∞-stable. Indeed, the closed-range property is an immediate consequence of the (Sobolev) estimates on the solutions of the cohomological equation. Our theorem implies more generally stability in weighted Sobolev spaces of finitely differentiable functions.

In fact, ours is the first example of a C^∞-stable non-homogeneous (although locally homogeneous), non-hyperbolic (partially hyperbolic) hyperbolic flow on a manifold of dimension at least 3. Indeed, we recall that the only known examples of stable (and renormalizable) three-dimensional parabolic flows are (up to smooth conjugacies and time-changes) homogeneous flows: horocycle flows of hyperbolic surfaces [FlaFo03]

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and Heisenberg nilflows [**FlaFo06**]. However, there has been recent progress (although conditional) on 'Ruelle asymptotics' and deviation of ergodic averages for horocycle flows for negatively curved metrics on surfaces [**AB**] (see also [**FG18**]), hence a proof of smooth stability (at least in low regularity) for such flows seems within reach of current methods for the analysis for the transfer operator of hyperbolic flows, along the lines of the work of Giulietti and Liverani [**GL**] for Anosov maps of tori.

We recall, again following A. Katok's terminology, that a stable flow on a smooth, compact, connected manifold $\mathcal{M}$ such that the range of the Lie derivative operator has codimension *one* in $C^\infty(\mathcal{M})$ is called *rigid* or *cohomology free*. The Katok–Hurder conjecture (known for flows only up to dimension 3; see [**F08**] and references therein) states that all smooth rigid flows are smoothly conjugate to linear Diophantine flows on tori. It is therefore expected that all non-toral uniquely ergodic stable flows have invariant distributions which are not signed measures, a property which is confirmed by our result, since the range of the Lie derivative has higher codimension. In fact, in all known examples of C^∞-stable flows, the space of invariant distributions is infinite-dimensional, with the only partial exception of typical translation flows on compact orientable surfaces, which have finite codimension in finite differentiability [**F97**]. Our result confirms once more the expectation that, in dimension greater than 2, the space of invariant distributions of a generic smooth non-toral stable flow has infinite dimension even in finite differentiability.

Finally, from Theorem 1.4 on the cohomological equation for the product of translation flows on higher-genus surfaces with translation flows on the circle we derive, by a general argument, a result on the cohomological equation for the time-τ maps of translation flows on higher-genus surfaces.

Let Φ_θ^τ denote the time-τ map of the horizontal translation flow of the abelian differential h_θ on M. For all $s \geq 0$, let $\mathcal{I}^s_{h_\theta,\tau} \subset H_h^{-s}(M)$ denote the space of Φ_θ^τ-invariant distributions, that is, the space of all distributions in $H_h^{-s}(M)$ which vanish on the subspace

$$\overline{\{u \circ \Phi_\theta^\tau - u \mid u \in H_h^\infty(M)\} \cap H_h^s(M)} \subset H_h^s(M).$$

We have the following result.

COROLLARY 1.5. *For any translation surface (M, h), for almost all $\theta \in \mathbb{T}$ and for almost all $T \in \mathbb{R}$ (with respect to the Lebesgue measure), the following statement holds. For all $f \in H_h^s(M)$ with $s > 3$, satisfying the distributional conditions $D(f) = 0$ for all Φ_θ^τ-invariant distributions $D \in \mathcal{I}^s_{h_\theta,\tau} \subset H_h^{-s}(M)$, the cohomological equation $u \circ \Phi_\theta^\tau - u = f$ has a solution $u \in H_h^r(M)$, for all $r < s - 3$, and there exists a constant $C_{r,s}(\theta, \tau) > 0$ such that*

$$|u|_r \leq C_{r,s}(\theta, \tau)|f|_s.$$

The paper is organized as follows. In §2 we recall basic facts of analysis on translation surfaces as developed by the author in [**F07**, **F97**]. In §3 we introduce a twisted version of the Beurling-type isometry of the L^2 space of a translation surface defined in [**F97**] (see also [**F02**]). Section 4 recalls results from the theory of boundary behavior of Cauchy integrals of finite measures on the circle and applications to the spectral theory of general

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unitary operators on Hilbert spaces, following [**F97**]. In §5 we prove the core result about the existence of solutions of the cohomological equation, by the following the presentation given in [**F07**] of the original argument of [**F97**], generalized to the twisted case. Section 5.1 is devoted to the core result about existence of distributional solutions, §5.2 to finiteness results for the spaces of twisted invariant distributions, and §5.3 to the proof of the main results on existence of smooth solutions for the twisted cohomological equations, product flows and time-τ maps.

Note. Although the paper is mostly self-contained, it is largely based on ideas and techniques from [**F97**], already revisited and streamlined in [**F07**, §§2 and 3]. A familiarity with these earlier works will therefore greatly facilitate the reading of the present work.

2. *Analysis on translation surfaces*

This section gathers basic results on the flat Laplacian of a translation surface, following [**F97**, §§2 and 3] and [**F07**, §2].

Let $\Sigma_h := \{p_1, \ldots, p_\sigma\} \subset M_h$ be the set of zeros of the holomorphic abelian differential h on a Riemann surface M, of orders $(k_1, \ldots, k_\sigma)$ respectively with $k_1 + \cdots + k_\sigma = 2g - 2$. Let $R_h := |h|$ be the flat metric with cone singularities at Σ_h induced by the abelian differential h on M and let ω_h denote its area form. With respect to a holomorphic local coordinate $z = x + \imath y$ at a regular point, the abelian differential h has the form $h = \phi(z)dz$, where ϕ is a locally defined holomorphic function, and, consequently,

$$R_h = |\phi(z)|(dx^2 + dy^2)^{1/2}, \quad \omega_h = |\phi(z)|^2\, dx \wedge dy. \tag{1}$$

The metric R_h is flat, degenerate at the finite set Σ_h of zeros of h and has trivial holonomy, hence h induces a *translation surface* structure on M.

The weighted L^2 space is the standard space $L^2_h(M) := L^2(M, \omega_h)$ with respect to the area element ω_h of the metric R_h. Hence the weighted L^2 norm $|\cdot|_0$ is induced by the hermitian product $\langle\cdot, \cdot\rangle_h$ defined as follows: for all functions $u, v \in L^2_h(M)$,

$$\langle u, v\rangle_h := \int_M u\, \bar{v}\, \omega_h. \tag{2}$$

Let $\mathcal{F}_{\mathrm{Im}(h)}$ be the *horizontal foliation*, $\mathcal{F}_{\mathrm{Re}(h)}$ be the *vertical foliation* for the holomorphic abelian differential h on M. The foliations $\mathcal{F}_{\mathrm{Im}(h)}$ and $\mathcal{F}_{\mathrm{Re}(h)}$ are measured foliations (in the sense of Thurston): $\mathcal{F}_{\mathrm{Im}(h)}$ is the foliation given by the equation $\mathrm{Im} h = 0$ endowed with the invariant transverse measure $|\mathrm{Im}\, h|$, $\mathcal{F}_{\mathrm{Re}(h)}$ the foliation given by the equation $\mathrm{Re} h = 0$ endowed with the invariant transverse measure $|\mathrm{Re}\, h|$. Since the metric R_h is flat with trivial holonomy, there exist commuting vector fields S_h and T_h on $M \setminus \Sigma_h$ such that:

(1) the frame $\{S_h, T_h\}$ is a parallel orthonormal frame with respect to the metric R_h for the restriction of the tangent bundle TM to the complement $M \setminus \Sigma_h$ of the set of cone points;
(2) the vector field S_h is tangent to the horizontal foliation $\mathcal{F}_{\mathrm{Im}(h)}$, and the vector field T_h is tangent to the vertical foliation $\mathcal{F}_{\mathrm{Re}(h)}$ on $M \setminus \Sigma_h$ [**F07**, **F97**].

In the following we will often drop the dependence of the vector fields S_h, T_h on the abelian differential in order to simplify the notation. The symbols $\mathcal{L}_S$, $\mathcal{L}_T$ denote the Lie

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derivatives, and $\imath_S$, $\imath_T$ the contraction operators with respect to the vector field S, T on $M \setminus \Sigma_h$. We have:

(1) $\mathcal{L}_S \omega_h = \mathcal{L}_T \omega_h = 0$ on $M \setminus \Sigma_h$, that is, the area form ω_h is invariant with respect to the flows generated by S and T;

(2) $\imath_S \omega_h = \mathrm{Re} h$ and $\imath_T \omega_h = \mathrm{Im} h$, hence the 1-forms $\eta_S := \imath_S \omega_h$, $\eta_T := -\imath_T \omega_h$ are smooth and closed on M and $\omega_h = \eta_T \wedge \eta_S$.

Let $C_0^\infty(M \setminus \Sigma_h)$ denote the space of complex-valued smooth functions with compact support in $M \setminus \Sigma_h$. It follows from the area-preserving property (1) that the vector fields S, T are anti-symmetric as densely defined operators on $L^2_h(M)$, that is, for all functions u, $v \in C_0^\infty(M \setminus \Sigma_h)$ (see [**F97**, formula (2.5)]),

$$\langle Su, v\rangle_h = -\langle u, Sv\rangle_h, \quad \langle Tu, v\rangle_h = -\langle u, Tv\rangle_h, \tag{3}$$

respectively. In fact, by Nelson's criterion [**Ne59**, Lemma 3.10], the anti-symmetric operators S, T are *essentially skew-adjoint* on the Hilbert space $L^2_h(M)$.

The *weighted Sobolev space* $H^k_h(M)$, with integer exponent $k \in \mathbb{Z}$, introduced in [**F97**], is defined for $k > 0$ as the common domain

$$H^k_h(M) := \bigcap_{i+j \le k} D(\bar{S}^i \bar{T}^j) \cap D(\bar{T}^i \bar{S}^j)$$

of the products of non-negative powers of the closures $\bar{S}$, $\bar{T}$ of the essentially skew-adjoint operators S, T on $L^2_h(M)$ up to order $k \in \mathbb{N}$. It is endowed with the *weighted Sobolev norms* $|\cdot|_k$, with integer exponent $k > 0$, which are the euclidean norms induced by the following hermitian product: for all functions u, $v \in H^k_h(M)$,

$$\langle u, v\rangle_k := \frac{1}{2} \sum_{i+j \le k} \langle S^i T^j u, S^i T^j v\rangle_h + \langle T^i S^j u, T^i S^j v\rangle_h. \tag{4}$$

The *weighted Sobolev spaces* with integer exponent $-k < 0$ are defined to be the dual Hilbert spaces, endowed with the dual norms.

The weighted Sobolev space $H^k_h(M)$, with integer exponent $k \in \mathbb{Z}$, coincides with the Hilbert space obtained as the completion with respect to the norm $|\cdot|_k$ of the maximal *common invariant domain*

$$H^\infty_h(M) := \bigcap_{i,j \in \mathbb{N}} D(\bar{S}^i \bar{T}^j) \cap D(\bar{T}^i \bar{S}^j) \tag{5}$$

of the closures $\bar{S}$, $\bar{T}$ of the essentially skew-adjoint operators S, T on $L^2_h(M)$.

Since the vector fields S, T commute as operators on $C_0^\infty(M \setminus \Sigma_h)$, the following weak commutation identity holds on M.

LEMMA 2.1. [**F97**, Lemma 3.1] *For all functions u, $v \in H^1_h(M)$,*

$$\langle Su, Tv\rangle_h = \langle Tu, Sv\rangle_h. \tag{6}$$

By the anti-symmetry property (3) and the commutativity property (6), the frame $\{S, T\}$ yields an essentially skew-adjoint action of the Lie algebra $\mathbb{R}^2$ on the Hilbert space $L^2_h(M)$ with common domain $H^1_h(M)$.

If $\Sigma_h \neq \emptyset$, the (flat) Riemannian manifold $(M \setminus \Sigma_h, R_h)$ is not complete, hence its Laplacian Δ_h is not essentially self-adjoint on $C_0^\infty(M \setminus \Sigma_h)$. By a theorem of Nelson [**Ne59**, §9], this is equivalent to the non-integrability of the action of $\mathbb{R}^2$ as a Lie algebra (to an action of $\mathbb{R}^2$ as a Lie group).

Following [**F97**], the Fourier analysis on the flat surface (M, h) will be based on a canonical self-adjoint extension Δ_h^F of the Laplacian Δ_h, called the *Friedrichs extension*, which is uniquely determined by the *Dirichlet hermitian form* $\mathcal{Q} : H_h^1(M) \times H_h^1(M) \to \mathbb{C}$. We recall that, for all $u, v \in H_h^1(M)$,

$$\mathcal{Q}(u, v) := \langle Su, Sv \rangle_h \; + \; \langle Tu, Tv \rangle_h. \tag{7}$$

THEOREM 2.2. [**F97**, Theorem 2.3] *The hermitian form $\mathcal{Q}$ on $L_h^2(M)$ has the following spectral properties.*

(1) *$\mathcal{Q}$ is positive semi-definite and the set $EV(\mathcal{Q})$ of its eigenvalues is a discrete subset of* $[0, +\infty)$.
(2) *Each eigenvalue has finite multiplicity. In particular,* $0 \in EV(\mathcal{Q})$ *is simple and the kernel of $\mathcal{Q}$ consists only of constant functions.*
(3) *The space $L_h^2(M)$ splits as the orthogonal sum of the eigenspaces. In addition, all eigenfunctions are C^∞ (real analytic) on M.*

The *Weyl asymptotics* holds for the eigenvalue spectrum of the Dirichlet form. For any $\Lambda > 0$, let $N_h(\Lambda) := \text{card}\{\lambda \in \text{EV}(\mathcal{Q}) \,/\, \lambda \leq \Lambda\}$, where each eigenvalue $\lambda \in \text{EV}(\mathcal{Q})$ is counted according to its multiplicity.

THEOREM 2.3. [**F97**, Theorem 2.5] *There exists a constant* $C > 0$ *such that*

$$\lim_{\Lambda \to +\infty} \frac{N_h(\Lambda)}{\Lambda} \;=\; \text{vol}(M, R_h). \tag{8}$$

Let $\partial_h^\pm := S_h \pm \imath\, T_h$ (with $\imath = \sqrt{-1}$) be the *Cauchy–Riemann operators*induced by the holomorphic abelian differential h on M, introduced in [**F97**, §3]. Let $\mathcal{M}_h^\pm \subset L_h^2(M)$ be the subspaces of meromorphic (respectively, anti-meromorphic) functions (with poles at Σ_h). By the Riemann–Roch theorem, the subspaces $\mathcal{M}_h^\pm$ have the same complex dimension equal to the genus $g \geq 1$ of the Riemann surface M. In addition, $\mathcal{M}_h^+ \cap \mathcal{M}_h^- = \mathbb{C}$, hence

$$H_h := (\mathcal{M}_h^+)^\perp \oplus (\mathcal{M}_h^-)^\perp = \left\{ u \in L_h^2(M) \;\middle|\; \int_M u\, \omega_h \;=\; 0 \right\}. \tag{9}$$

Let $H_h^1 := H_h \cap H_h^1(M)$. By Theorem 2.2, the restriction of the hermitian form to H_h^1 is positive definite, hence it induces a norm. By the Poincaré inequality (see [**F97**, Lemma 2.2] or [**F02**, Lemma 6.9]), the Hilbert space $(H_h^1, \mathcal{Q})$ is isomorphic to the Hilbert space $(H_h^1, \langle \cdot, \cdot \rangle_1)$.

PROPOSITION 2.4. [**F97**, Proposition 3.2] *The Cauchy–Riemann operators $\partial_h^\pm$ are closable operators on the common domain $C_0^\infty(M \setminus \Sigma_h) \subset L_h^2(M)$ and their closures (denoted by the same symbols) have the following properties:*

(1) *the domains $D(\partial_h^\pm) = H_h^1(M)$ and the kernels $N(\partial_h^\pm) = \mathbb{C}$;*

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(2) *the ranges* $R_h^{\pm} := Ran(\partial_h^{\pm}) = (\mathcal{M}_h^{\mp})^{\perp}$ *are closed in* $L_h^2(M)$*;*
(3) *the operators* $\partial_h^{\pm} : (H_h^1, \mathcal{Q}) \to (R^{\pm}, \langle \cdot, \cdot \rangle_h)$ *are isometric.*

Let $\mathcal{E} = \{e_n \mid n \in \mathbb{N}\} \subset H_h^1(M) \cap C^{\infty}(M)$ be an orthonormal basis of the Hilbert space $L_h^2(M)$ of eigenfunctions of the Dirichlet form (7) and let $\lambda : \mathbb{N} \to \mathbb{R}^+ \cup \{0\}$ be the corresponding sequence of eigenvalues:

$$\lambda_n := \mathcal{Q}(e_n, e_n) \quad \text{for each } \ n \in \mathbb{N}. \tag{10}$$

The sequence $\{\lambda_n \mid n \in \mathbb{N}\}$ can be equivalently defined as the sequence of eigenvalues of the non-negative Friedrichs extension $-\Delta_h^F$ of the flat Laplacian Δ_h on M.

We then recall the definition of the *Friedrichs (fractional) weighted Sobolev norms and spaces* introduced in [**F07**, §2.2].

Definition 2.5.

(i) The *Friedrichs (fractional) weighted Sobolev norm* $\| \cdot \|_s$ of order $s \geq 0$ is the norm induced by the hermitian product defined as follows: for all $u, v \in L_h^2(M)$,

$$(u, v)_s := \sum_{n \in \mathbb{N}} (1 + \lambda_n)^s \ \langle u, e_n \rangle_h \ \langle e_n, v \rangle_h. \tag{11}$$

(ii) The *Friedrichs weighted Sobolev space* $\bar{H}_h^s(M)$ of order $s \geq 0$ is the Hilbert space

$$\bar{H}_h^s(M) := \left\{ u \in L_h^2(M) \Bigg/ \sum_{n \in \mathbb{N}} (1 + \lambda_n)^s |\langle u, e_n \rangle_h|^2 < +\infty \right\} \tag{12}$$

endowed with the hermitian product given by (11).

(iii) The *Friedrichs weighted Sobolev space* $\bar{H}_h^{-s}(M)$ of order $-s < 0$ is the dual space of the Hilbert space $\bar{H}_h^s(M)$.

As stated in [**F07**, Lemma 2.6], the family of Friedrichs (fractional) weighted Sobolev spaces is a holomorphic interpolation family in the Lions–Magenes sense [**LM**, Ch. 1], endowed with the canonical interpolation norm.

The family $\{H_h^s(M)\}_{s \in \mathbb{R}}$ of *fractional weighted Sobolev spaces* will be defined as follows. Let $[s] \in \mathbb{N}$ denote the *integer part* and $\{s\} \in [0, 1)$ the *fractional part* of any real number $s \geq 0$.

Definition 2.6. [**F07**, Definition 2.7]

(i) The *fractional weighted Sobolev norm* $| \cdot |_s$ of order $s \geq 0$ is the euclidean norm induced by the hermitian product defined as follows: for all functions $u, v \in H_h^{\infty}(M)$,

$$\langle u, v \rangle_s := \frac{1}{2} \sum_{i+j \leq [s]} (S^i T^j u, S^i T^j v)_{\{s\}} + (T^i S^j u, T^i S^j v)_{\{s\}}. \tag{13}$$

(ii) The *fractional weighted Sobolev norm* $| \cdot |_{-s}$ of order $-s < 0$ is defined as the dual norm of the weighted Sobolev norm $| \cdot |_s$.

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(iii) The *fractional weighted Sobolev space* $H^s_h(M)$ of order $s \in \mathbb{R}$ is defined as the completion with respect to the norm $|\cdot|_s$ of the maximal common invariant domain $H^\infty_h(M)$.

It can be proved that the weighted Sobolev space $H_h^{-s}(M)$ is isomorphic to the dual space of the Hilbert space $H^s_h(M)$, for all $s \in \mathbb{R}$.

The definition of the fractional weighted Sobolev norms is motivated by the following basic result.

LEMMA 2.7. [**F07**, Lemma 2.9] *For all $s \geq 0$, the restrictions of the Cauchy–Riemann operators $\partial_h^\pm : H^1_h(M) \to L^2_h(M)$ to the subspaces $H^{s+1}_h(M) \subset H^1_h(M)$ yield bounded operators*

$$\partial_s^\pm : H^{s+1}_h(M) \to H^s_h(M)$$

(which do not extend to operators $\bar{H}^{s+1}_h(M) \to \bar{H}^s_h(M)$ unless M is the torus). On the other hand, the Laplace operator

$$\Delta_h = \partial_h^+ \partial_h^- = \partial_h^- \partial_h^+ : H^2_h(M) \to L^2_h(M) \tag{14}$$

yields a bounded operator $\bar{\Delta}_s : \bar{H}^{s+2}_h(M) \to \bar{H}^s_h(M)$, defined as the restriction of the Friedrichs extension $\Delta^F_h : \bar{H}^2_h(M) \to L^2_h(M)$.

We do not know whether the fractional weighted Sobolev spaces form a holomorphic interpolation family. However, the fractional weighted Sobolev norms do satisfy interpolation inequalities (see [**F07**, Lemma 2.10 and Corollary 2.26]).

A detailed comparison between Friedrichs weighted Sobolev norms and weighted Sobolev norms and the corresponding weighted Sobolev spaces is carried out in [**F07**, §2]. In particular, we have the following result.

Let $H^s(M)$, $s \in \mathbb{R}$, denote a family of standard Sobolev spaces on the compact manifold M (defined with respect to a Riemannian metric).

LEMMA 2.8. [**F07**, Lemma 2.11] *The following continuous embedding and isomorphisms of Banach spaces hold:*

(1) $H^s(M) \subset H^s_h(M) \equiv \bar{H}^s_h(M)$ *for* $0 \leq s < 1$*;*
(2) $H^s(M) \equiv H^s_h(M) \equiv \bar{H}^s_h(M)$ *for* $s = 1$*;*
(3) $H^s_h(M) \subset \bar{H}^s_h(M) \subset H^s(M)$ *for* $s > 1$.

For $s \in [0, 1]$, the space $H^s(M)$ is dense in $H^s_h(M)$ and, for $s > 1$, the closure of $H^s_h(M)$ in $\bar{H}^s_h(M)$ or $H^s(M)$ has finite codimension.

We also have the following sharp version of [**F97**].

THEOREM 2.9. [**F07**, Lemma 2.5 and Corollary 2.25] *For each $k \in \mathbb{Z}^+$ there exists a constant $C_k > 1$ such that, for any holomorphic abelian differential h on M and for all $u \in H^k_h(M)$,*

$$C_k^{-1}\, |u|_k \leq \|u\|_k \leq C_k\, |u|_k. \tag{15}$$

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For any $0 < r < s$ *there exists constants* $C_r > 0$ *and* $C_{r,s} > 0$ *such that, for all* $u \in H^s_h(M)$*, the following inequalities hold:*

$$C_r^{-1}\|u\|_r \le |u|_r \le C_{r,s}\,\|u\|_s. \tag{16}$$

3. *The twisted Beurling transform*

For every $\sigma \in \mathbb{R}$ and for every abelian differential h, we introduce a family of partial isometries $U_{h,\sigma}$ (of Beurling transform type), defined on a finite-codimensional subspace of $L^2_h(M) := L^2(M, \omega_h)$, which generalized the partial isometry $U_q = U_{h,0}$ (for $q = h^2$) first introduced in [**F97**, §3], in the study of the cohomological equation for translation flows.

The partial isometry $U_{h,\sigma}$ is extended in an arbitrary way to a unitary operator $U_{J,\sigma}$ on the whole space $L^2_h(M)$. Resolvent estimates for $U_{J,\sigma}$ will appear to be related with a priori estimates for the twisted cohomological equations for translation flows on (M, h). Consequently, we derive our results on twisted cohomological equations from basic estimates on the limiting behavior as $z \to \partial D$ of the resolvent $\mathcal{R}_U(z) := (U - zI)^{-1}$, defined on the unit disk $D \subset \mathbb{C}$, of a unitary operator U on a general Hilbert space. Such estimates, established in [**F97**], are based on fundamental facts of classical harmonic analysis, in particular on Fatou's theory on the boundary behavior of holomorphic functions. The results obtained are then specialized to the case of the unitary operator $U := U_{J,\sigma}$.

Let h be a holomorphic abelian differential on a Riemann surface M of genus $g \ge 2$. Let $\{S, T\}$ be the orthonormal frame for TM on $M \setminus \Sigma$ introduced in §2. We recall that the 1-forms $\eta_S = \imath_S\omega_h$ and $\eta_T = -\imath_T\omega_h$ are closed and describe the horizontal (respectively, vertical) foliation of ω on M. It is possible to associate to h a one-parameter family of measured foliations parametrized by $\theta \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ in the following way. Let $h_\theta := e^{-\imath\theta}\, h$ and let $\mathcal{F}_\theta$ be the horizontal foliation of the abelian differential h_θ, that is, the foliation defined by the closed 1-form

$$\operatorname{Im} h_\theta = \{e^{-\imath\theta}(\eta_T + \imath\eta_S) - e^{\imath\theta}(\eta_T - \imath\eta_S)\}/2\imath.$$

The foliation $\mathcal{F}_\theta$ can also be obtained by integrating the dual vector field

$$S_\theta := (\cos\theta)S + (\sin\theta)T = \{e^{-\imath\theta}(S + \imath\, T) + e^{\imath\theta}(S - \imath\, T)\}/2, \tag{17}$$

which corresponds to the rotation of the vector field S by an angle $\theta \in \mathbb{T}$ in the positive direction.

In the following we will denote by $\partial_h^\pm$ the *Cauchy–Riemann*operators $S \pm \imath\, T$, respectively. The *twisted Cauchy–Riemann*operators

$$\partial^\pm_{h,\sigma} := (S + \imath\sigma) \pm \imath T = \partial_h^\pm + \imath\sigma \tag{18}$$

will play a crucial role. For all $\theta \in \mathbb{T}$, let $\sigma_\theta := \sigma\,\cos\theta$. We remark that, for every $\theta \in \mathbb{T}$, we have

$$S_\theta + \imath\sigma_\theta := \{e^{-\imath\theta}\partial^+_{h,\sigma} + e^{\imath\theta}\partial^-_{h,\sigma}\}/2,$$

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hence we have the formal factorization

$$\begin{aligned} S_\theta + \imath\sigma_\theta &= \frac{e^{-\imath\theta}}{2}\,(((\partial^+_{h,\sigma})(\partial^-_{h,\sigma})^{-1} + e^{2\imath\theta}))\,\partial^-_{h,\sigma} \\ &= \frac{e^{\imath\theta}}{2}\,(((\partial^-_{h,\sigma})(\partial^+_{h,\sigma})^{-1} + e^{-2\imath\theta}))\,\partial^+_{h,\sigma}. \end{aligned} \tag{19}$$

Let $Q_{h,\sigma}$ denote the bilinear form defined, for all $u, v \in H^1_h(M)$, as follows:

$$Q_{h,\sigma}(u, v) := \langle (S + \imath\sigma)u, (S + \imath\sigma)v\rangle_h + \langle Tu, Tv\rangle_h.$$

Let $K_{h,\sigma} \subset H^1_h(M) \cap C^\infty(M \setminus \Sigma)$ denote the finite-dimensional subspace

$$K_{h,\sigma} := \{u \in H^1_h(M) \cap C^\infty(M \setminus \Sigma) | (S + \imath\sigma)u = Tu = 0\}. \tag{20}$$

LEMMA 3.1. *The twisted bilinear form $Q_{h,\sigma}$ induces a norm on $K^\perp_{h,\sigma} \cap H^1_h(M)$. In fact, for all $\sigma \in \mathbb{R}$, there exists a constant $C_h > 1$ such that, for all $u \in K^\perp_{h,\sigma} \cap H^1_h(M)$,*

$$C_h^{-1} Q_{h,0}(u, u) \le Q_{h,\sigma}(u, u) \le C_h(1 + \sigma^2)\bigg(Q_{h,0}(u, u) + \bigg|\int_M u\omega_h\bigg|\bigg).$$

Proof. Since translation flows are area-preserving, hence symmetric on their common domain, we have, for all $u \in H^1_h(M)$,

$$\begin{aligned} Q_{h,\sigma}(u, u) :&= \langle (S + \imath\sigma)u, (S + \imath\sigma)u\rangle_h + \|Tu\|^2_{L^2_h(M)} \\ &= \|Su\|^2_{L^2_h(M)} + 2\imath\sigma\langle u, Su\rangle_h + \sigma^2\|u\|^2_{L^2_h(M)} + \|Tu\|^2_{L^2_h(M)}. \end{aligned} \tag{21}$$

By the Cauchy–Schwarz inequality, we have

$$|\langle u, Su\rangle_h| \le \|u\|_{L^2_h(M)}\|Su\|_{L^2_h(M)} \le \frac{\|Su\|^2_{L^2_h(M)} + \|u\|^2_{L^2_h(M)}}{2}.$$

It follows that

$$\|Su\|^2_{L^2_h(M)} + 2\imath\sigma\langle u, Su\rangle_h \le (1 + |\sigma|)\|Su\|^2_{L^2_h(M)} + |\sigma|\|u\|^2_{L^2_h(M)},$$

hence we derive that

$$Q_{h,\sigma}(u, u) \le (1 + |\sigma|)Q_{h,0}(u, u) + (\sigma^2 + |\sigma|)\|u\|^2_{L^2_h(M)}.$$

By the Poincaré inequality there exists a constant $C_h > 0$ such that, for all $u \in H^1_h(M)$, we have

$$\begin{aligned} Q_{h,\sigma}(u, u) &\le (1 + |\sigma|)Q_{h,0}(u, u) + (\sigma^2 + |\sigma|)\|u\|^2_{L^2_h(M)} \\ &\le [(1 + |\sigma|) + C(\sigma^2 + |\sigma|)]Q_{h,0}(u, u) + (\sigma^2 + |\sigma|)\bigg|\int_M u\omega_h\bigg|. \end{aligned}$$

The upper bound in the statement is therefore proved.

To prove the lower bound, we proceed as follows. By the definition of $Q_{h,\sigma}$, for the splitting $u = v + \bar{u} \in \mathbb{C}^\perp \oplus^\perp \mathbb{C} \subset H^1_h(M)$ we have

$$Q_{h,\sigma}(u, u) = Q_{h,\sigma}(v, v) + \sigma^2\bar{u}^2; \tag{22}$$

hence without loss of generality we can reduce the argument to functions $u \in H^1_h(M)$ of zero average. By the compact embedding $H^1_h(M) \to L^2(M)$ we derive that there exists a constant $c_h > 0$ such that, for all $u \in H^1_h(M)$, we have

$$\|u\|^2_{L^2_h(M)} \geq c_h^2 Q_{h,0}(u,u) \geq c_h^2 \|Su\|^2_{L^2_h(M)}.$$

It follows then by formula (21) that, for $|\sigma| \geq 2c_h^{-1}$, we have

$$Q_{h,\sigma}(u,u) \geq Q_{h,0}(u,u) + |\sigma| \|u\|_{L^2_h(M)} (|\sigma| \|u\|_{L^2_h(M)} - 2\|Su\|_{L^2_h(M)}) \geq Q_{h,0}(u,u).$$

It remains to prove the bound for $|\sigma| \leq 2c_h^{-1}$. Let us then assume by contradiction that for all $n \in \mathbb{N}$ there exist a bounded sequence (σ_n) and a sequence $u_n \in K^\perp_{h,\sigma_n} \subset H^1_h(M)$ of zero average such that

$$Q_{h,0}(u_n, u_n) \geq n Q_{h,\sigma_n}(u_n, u_n).$$

After normalizing, it is not restrictive to assume that $Q_{h,0}(u_n, u_n) = 1$, for all $n \in \mathbb{N}$, hence $Q_{h,\sigma_n}(u_n, u_n) \to 0$. By the Poincaré inequality, it follows that after passing to a subsequence we can assume that $u_n \to u$ in $L^2(M)$ and u has zero average, as well as that $\sigma_n \to \sigma \in \mathbb{R}$. Let $\Phi^{\mathbb{R}}_S$ and $\Phi^{\mathbb{R}}_T$ denote the horizontal and the vertical flow, respectively. By assumption, since

$$\begin{aligned}
\|e^{\imath \sigma_n t} u_n \circ \Phi^t_S - u_n\|_{L^2_h(M)} &= \left\| \int_0^t (S + \imath \sigma_n) u_n \circ \Phi^s_S \, ds \right\|_{L^2_h(M)} \\
&\leq \int_0^t \|(S + \imath \sigma_n) u_n \circ \Phi^s_S\|_{L^2_h(M)} \, ds \leq t Q^{1/2}_{h,\sigma_n}(u_n, u_n) \to 0, \\
\|u_n \circ \Phi^t_T - u_n\|_{L^2_h(M)} &= \left\| \int_0^t T u_n \circ \Phi^s_T \, ds \right\|_{L^2_h(M)} \\
&\leq \int_0^t \|T u_n \circ \Phi^s_T\|_{L^2_h(M)} \, ds \leq t Q^{1/2}_{h,\sigma_n}(u_n, u_n) \to 0,
\end{aligned}$$

it follows that the limit function $u \in K^\perp_{h,\sigma} \subset L^2_h(M)$ is a zero-average eigenfunction of eigenvalue $-\imath\sigma$ for the flow $\Phi^{\mathbb{R}}_S$ and it is invariant for the flow $\Phi^{\mathbb{R}}_T$. It follows, in particular, that $u \in C^\infty(M \setminus \Sigma) \cap H^1_h(M)$, which implies that u is constant on all minimal components of the flow $\Phi^{\mathbb{R}}_T$ and $\Phi^{\mathbb{R}}_T$-invariant on the cylindrical component. In particular, $u \in K_{h,\sigma}$, hence $u = 0$. However, from $u_n \to 0$ in $L^2_h(M)$ and $\sigma_n \to \sigma$, from the identity in formula (21) we then derive

$$0 = \lim_{n\to\infty} Q_{h,\sigma_n}(u_n, u_n) = \lim_{n\to\infty} Q_{h,0}(u_n, u_n) = 1,$$

a contradiction. We have thus proved that there exists $C_h > 1$ such that, for all $u \in K^\perp_{h,\sigma} \cap H^1_h(M)$,

$$C_h^{-1} Q_{h,0}(u,u) \leq Q_{h,\sigma}(u,u). \qquad \square$$

The twisted Cauchy–Riemann operators appear in the Hodge decomposition of the twisted exterior differential, which we now introduce (see [**V02**] or [**De96**] for a general introduction to Hodge theory). For all $\sigma \in \mathbb{R}$, the *twisted exterior differential* $d_{h,\sigma}$ is

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defined on the space $\Omega^*(M)$ of complex-valued smooth forms on the surface M by the formula

$$d_{h,\sigma}\alpha := d\alpha + \imath\sigma \mathrm{Re}(h) \wedge \alpha \quad \text{for all } \alpha \in \Omega^*(M).$$

Since $d^2_{h,\sigma} = 0$ (as is readily verified), the cohomology $H^*_{h,\sigma}(M, \mathbb{C})$ of the complex $(\Omega^*(M), d_{h,\sigma})$ is well defined and will be called the *twisted cohomology* of the translation surface (M, h). We have the following more general definition.

Definition 3.2. Let $\Omega^*(M)$ denote the space of all smooth differential forms on M. Let η be a real closed smooth 1-form on D and let d_η denote the twisted exterior derivative defined as

$$d_\eta\alpha = d\alpha + \imath\eta \wedge \alpha \quad \text{for all } \alpha \in \Omega^*(D).$$

The twisted cohomology (with complex coefficients) $H^*_\eta(M, \mathbb{C})$ is the cohomology of the differential complex $(\Omega^*(M), d_\eta)$.

Let $\Omega^*(M, \Sigma_h)$ denote the space of all smooth differential forms on M vanishing at Σ_h. The relative twisted cohomology (with complex coefficients) $H^*_\eta(M, \Sigma_h, \mathbb{C})$ is the cohomology of the differential complex $(\Omega^*(M, \Sigma_h), d_\eta)$.

By definition we have $H^*_{h,\sigma}(M, \mathbb{C}) := H^*_{\sigma\mathrm{Re}(h)}(M, \mathbb{C})$. A twisted cohomology cocycle on a twisted cohomology bundle over the Teichmüller flow is introduced in [**F19**] in the study of twisted ergodic integral and effective weak mixing.

The twisted cohomology has a *Hodge decomposition* with respect to the Hodge operator $*_h$ associated to the flat metric on the translation surface (M, h), which by definition satisfies the identities

$$*_h(\mathrm{Re}(h)) = \mathrm{Im}(h) \quad \text{and} \quad *_h(\mathrm{Im}(h)) = -\mathrm{Re}(h).$$

By the Hodge decomposition of the space of 1-forms into a holomorphic and an anti-holomorphic part (respectively, the eigenspaces of eigenvalues $\mp\imath$ of the Hodge operator), there exist twisted Cauchy–Riemann differentials $d^\pm_\eta$ on forms such that

$$d_\eta = \frac{d^+_\eta + d^-_\eta}{2}$$

(here d^+_η denotes the projection onto the anti-holomorphic part, and d^-_η the projection onto the holomorphic part). As a consequence, the twisted cohomology $H^1_\eta(M, \mathbb{C})$ splits a direct (Hodge orthogonal) sum of the holomorphic part $H^{1,+}_\eta(M, \mathbb{C})$ and the anti-holomorphic part $H^{1,-}_\eta(M, \mathbb{C})$:

$$H^1_\eta(M, \mathbb{C}) = H^{1,+}_\eta(M, \mathbb{C}) \oplus H^{1,-}_\eta(M, \mathbb{C}).$$

The twisted Cauchy–Riemann differentials $d^\pm_\eta$ on functions can be written in terms of twisted Cauchy–Riemann operators $\partial^\pm_\eta$ as follows: for all $f \in H^1_h(M)$,

$$d^+_\eta f = (\partial^+_\eta f)\bar{h} \quad \text{and} \quad d^-_\eta f = (\partial^-_\eta f)h.$$

The twisted Cauchy–Riemann operators $\partial^{\pm}_{h,\sigma}$ introduced in formula (18) correspond to the special case of $\eta = \sigma \mathrm{Re}(h)$:

$$\partial^{\pm}_{h,\sigma} = \partial^{\pm}_{h} + \imath\sigma = \partial^{\pm}_{\sigma \mathrm{Re}(h)}.$$

The dimension of the twisted cohomology can be exactly computed. In fact we have the following results.

LEMMA 3.3. *For all closed real smooth 1-form $\eta \in \Omega^1(M)$, the following dimension identity holds:*

$$\dim_{\mathbb{C}} H^0_\eta(M, \mathbb{C}) = \begin{cases} 1 & \text{if } [\eta] \in H^1(M, \mathbb{Z}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We repeat here for the convenience of the reader the argument given in [**F19**, Lemma 4.1]. Let $Z^0_\eta(M, \mathbb{C})$ denote the space of functions $f \in C^\infty(M)$ such that

$$d_\eta f = df + 2\pi\imath\eta f = 0.$$

It follows from the above equation that the function f is constant along each leaf of the measured foliation $\mathcal{F}_\eta = \{\eta = 0\}$, hence all the leaves of $\mathcal{F}_\eta$ are compact. In addition, we have

$$d(f\bar{f}) = (df)\bar{f} + (\overline{df})f = -2\pi\imath\eta f\bar{f} + 2\pi\imath\eta f\bar{f} = 0,$$

hence either $f \equiv 0$ or there exists $c_f \in \mathbb{C} \setminus \{0\}$ such that $f/c_f : M \to U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ and there exists a real-valued function $\theta : M \to \mathbb{R}/\mathbb{Z}$ such that

$$f(x) = \exp(-2\pi\imath\theta(x)) \quad \text{for all } x \in M.$$

By definition we have $df = -2\pi\imath f\, d\theta$, and since by assumption $f \in Z^0_\eta(M, \mathbb{C})$, the space of d_η-closed 0-forms, that is, complex-valued functions, and $f(x) \neq 0$ for all $x \in M$, it follows that $d\theta = \eta$. Since $\theta : M \to \mathbb{R}/\mathbb{Z}$, we conclude that $\eta \in H^1(M, \mathbb{Z})$.

Conversely, let us assume that $[\eta] \in H^1(M, \mathbb{Z})$. Given any point $p \in M$, the function

$$f_p(x) = \exp\left(-2\pi\imath \int_p^x \eta\right) \quad \text{for all } x \in M,$$

is a well-defined, non-zero element of $Z^0_\eta(M, \mathbb{C})$ since

$$df_p = -2\pi\imath f_p \eta.$$

In addition, given any $g \in Z^0_\eta(M, \mathbb{C})$ we have

$$d(\bar{f}_p g) = (\overline{df_p})g + \bar{f}_p(dg) = 2\pi\imath\bar{f}_p g\eta - 2\pi\imath\bar{f}_p g\eta = 0,$$

hence $\bar{f}_p g$ is a constant, which implies that $H^0_\eta(M, \mathbb{C})$ has dimension equal to 1. □

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LEMMA 3.4. *For all closed real smooth 1-forms $\eta \in \Omega^1(M)$, the following dimension identities hold:*

$$\dim_{\mathbb{C}} H^1_\eta(M, \mathbb{C}) = 2 \dim_{\mathbb{C}} H^0_\eta(M, \mathbb{C}) + 2g - 2,$$
$$\dim_{\mathbb{C}} H^1_\eta(M, \Sigma_h, \mathbb{C}) = \#(\Sigma_h) + \dim_{\mathbb{C}} H^0_\eta(M, \mathbb{C}) + 2g - 2.$$

Proof. The proof of the above dimension relation is given in greater generality by Goldman in [**Go84**, §1.5], and also explained in our context in [**F19**, Lemma 4.3]. We outline the argument for the convenience of the reader.

Let $\mathcal{L}_\eta$ denote the local system on M defined as the subbundle of the space $\Omega^*(\hat{M})$ of complex-valued forms $\hat{\alpha}$ on the universal cover $\hat{M}$ such that

$$\gamma^*(\hat{\alpha}) = \exp\left(\imath \int_\gamma \eta\right)\hat{\alpha} \quad \text{for all } \gamma \in \pi_1(M, *).$$

The twisted cohomology $H^*_\eta(M, \mathbb{C})$, defined as the cohomology of the complex of the twisted differential d_η on complex-valued forms $\Omega^*(M)$, is isomorphic to the cohomology $H^*(M, \mathcal{L}_\eta)$, defined as the cohomology of the complex of the exterior differential d on $\mathcal{L}_\eta$-valued forms $\Omega^*(M, \mathcal{L}_\eta)$.

The cohomology $H^1(M, \mathcal{L}_\eta)$, defined as the de Rham cohomology of the corresponding local system $\mathcal{L}_\eta$, can be identified with other cohomologies such as the singular, Čech, or simplicial cohomologies with local coefficients in $\mathcal{L}_\eta$. In simplicial cohomology, we have that the (finite-dimensional) cochain complex is independent of the flat connection which defines the local system, so its Euler characteristic equals $2 - 2g$, since the local system $\mathcal{L}_\eta$ has rank equal to 1. Now the Euler characteristic is invariant under taking the cohomology of the complex, so the Euler characteristic of the graded cohomology space also equals $2 - 2g$. Finally, since M is a closed orientable surface, by the Poincaré duality $H^2(M, \mathcal{L}_\eta) \equiv H^0(M, \mathcal{L}_\eta)$. By definition of Euler characteristic of a complex, we have

$$\dim_{\mathbb{C}} H^0(M, \mathcal{L}_\eta) - \dim_{\mathbb{C}} H^1(M, \mathcal{L}_\eta) + \dim_{\mathbb{C}} H^2(M, \mathcal{L}_\eta) = 2 - 2g,$$

so that $\dim_{\mathbb{C}} H^1(M, \mathcal{L}_\eta) = 2 \dim_{\mathbb{C}} H^0(M, \mathcal{L}_\eta) + 2g - 2$, as stated.

It remains to prove the dimension formula for the relative cohomology. Since $\Omega^1(M, \Sigma_h) \subset \Omega^1(M)$ and any twisted cohomology class in $H^1(M, \mathbb{C})$ can be represented by a twisted closed 1-form with compact support in $M \setminus \Sigma_h$ there exists a natural surjective map $H^1(M, \Sigma_h, \mathbb{C}) \to H^1(M, \mathbb{C})$. It remains to compute the kernel of such a map. An element of its kernel can be represented by a 1-form α with compact support in $M \setminus \Sigma_h$ such that there exists a function $f \in C^\infty(M)$ with $\alpha = d_\eta f$. The function f is uniquely determined up to kernel of the twisted differential, that is, up to cohomology classes in $H^0(M, \mathbb{C})$. By a local calculation, since the form $\alpha = d_\eta f$ vanishes near Σ_h, it has vanishing cohomology class in $H^1(M, \Sigma_h, \mathbb{C})$ if and only if the function f vanishes at Σ_h. It follows that the kernel of the map has dimension $\#(\Sigma_h) - \dim_{\mathbb{C}} H^0(M, \mathcal{L}_\eta)$, and the dimension formula for the relative homology follows from that for the absolute homology. □

The twisted Cauchy–Riemann operators $\partial^\pm_{h,\sigma}$ on the Hilbert space $L^2_h(M)$ are described in the following proposition.

PROPOSITION 3.5. *The Cauchy–Riemann operators $\partial^{\pm}_{h,\sigma}$ are closable operators on $C_0^\infty(M \setminus \Sigma) \subset L^2_h(M)$ and their closures (denoted by the same symbols) have the following properties.*

(i) $D(\partial^{\pm}_{h,\sigma}) = H^1_h(M)$ *and* $N(\partial^{\pm}_{h,\sigma}) = K_{h,\sigma} \subset H^1_h(M)$.

(ii) *The kernels* $\mathcal{M}^{\pm}_{\Sigma}(\sigma) \subset L^2_h(M)$ *of the adjoint operators* $(\partial^{\mp}_{h,\sigma})^*$ *have finite dimensions* $d^{\pm}(\sigma)$ *such that*

$$d^+(\sigma) = d^-(-\sigma) = g - 1 + \dim_{\mathbb{C}}(K_{h,\sigma}) \quad \textit{for all } \sigma \in \mathbb{R}.$$

(iii) *The adjoints* $(\partial^{\pm}_{h,\sigma})^*$ *of* $\partial^{\pm}_{h,\sigma}$ *are extensions of* $-\partial^{\mp}_{h,\sigma}$, *and we have closed ranges*

$$R^{\pm}_{h,\sigma} := Ran\ (\partial^{\pm}_{h,\sigma}) = [\mathcal{M}^{\mp}_{\Sigma}(\sigma)]^{\perp}.$$

(iv) *The operators* $\partial^{\pm}_{h,\sigma} : (K^{\perp}_{h,\sigma} \cap H^1_h(M), Q_{h,\sigma}) \to (R^{\pm}_{h,\sigma}, \langle \cdot, \cdot \rangle_h)$ *are isometric.*

Proof. If $u, v \in H^1_h(M)$, Lemma 2.1 implies the following identity:

$$\begin{aligned}\langle \partial^{\pm}_{h,\sigma} u, \partial^{\pm}_{h,\sigma} v\rangle_h &= \langle (S + \imath\sigma)u, (S + \imath\sigma)v\rangle_h + \langle Tu, Tv\rangle_h \\ &\quad \pm \imath(\langle Tu, (S + \imath\sigma)v\rangle_h - \langle (S + \imath\sigma)u, Tv\rangle_h) = Q_{h,\sigma}(u, v).\end{aligned} \tag{23}$$

It follows immediately that the operators $\partial^{\pm}_{h,\sigma}$ are closed with domain $D(\partial^{\pm}_{h,\sigma}) = H^1_h(M)$ and that their kernels are both equal to $K_{h,\sigma} \subset H^1_h(M)$.

Since, for all $u, v \in H^1_h(M)$, we have

$$\begin{aligned}\langle \partial^{\pm}_{h,\sigma} u, v\rangle_h &= \langle [(S + \imath\sigma) \pm \imath T]u, v\rangle_h \\ &= -\langle u, [(S + \imath\sigma) \mp \imath T]v\rangle_h = -\langle u, \partial^{\mp}_{h,\sigma} v\rangle_h,\end{aligned}$$

and the adjoint $(\partial^{\pm}_{h,\sigma})^*$ of $\partial^{\pm}_{h,\sigma}$ is an extension of $-\partial^{\mp}_{h,\sigma}$.

The dimension of the kernels $\mathcal{M}^{\pm}_{\Sigma}(\sigma)$ of the Cauchy–Riemann operators $\partial^{\mp}_{h,\sigma}$ in $L^2_h(M)$ follows from the fact that the quotients $\mathcal{M}^{\pm}_{\Sigma}(\sigma)/K_{h,\sigma}$ are, respectively, isomorphic to the anti-holomorphic and holomorphic twisted cohomology spaces $H^{1,\pm}_{h,\sigma}(M, \mathbb{C})$, which have half the dimension of the twisted cohomology $H^1_{h,\sigma}(M, \mathbb{C})$. We describe the isomorphism below. The maps $j^{\pm}_h : \mathcal{M}^{\pm}_{\Sigma}(\sigma) \to H^{1,\pm}_{h,\sigma}(M, \mathbb{C})$ defined as

$$j^+_h(m^+) = [m^+ h] \quad \text{and} \quad j^-_h(m^-) = [m^- \bar{h}] \quad \text{for all } m^{\pm} \in \mathcal{M}^{\pm}_{\Sigma}(\sigma),$$

are isomorphisms. In, fact the maps $j^{\pm}_h$ are onto since by definition any cohomology class $c^{\pm} \in H^{1,\pm}_{h,\sigma}(M, \mathbb{C})$ is represented by a twisted holomorphic section $m^+ h$ (respectively, by a twisted anti-holomorphic section $m^- \bar{h}$), hence we have, in the weak sense,

$$0 = d^+_{h,\sigma}(m^+ h) = (\partial^+_{h,\sigma} m^+)(\bar{h} \wedge h) \quad \text{and} \quad 0 = d^-_{h,\sigma}(m^- \bar{h}) = (\partial^-_{h,\sigma} m^-)(h \wedge \bar{h})$$

which implies that $\partial^{\pm}_{h,\sigma} m^{\pm} = 0$, hence $m^{\pm} \in \mathcal{M}^{\pm}_{\Sigma}(\sigma)$. We the prove that the maps $j^{\pm}$ are injective. The kernels of the maps $j^{\pm}_h$ can be described as follows. By definition, we have that $j^{\pm}(m^{\pm}) = 0$ if and only if there exists $f^{\pm} \in H^1_h(M)$ such that $d_{h,\sigma} f^+ = m^+ h$ and

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$d_{h,\sigma} f^- = m^- \bar{h}$, respectively. The latter identities imply that

$$\partial^{\pm}_{h,\sigma} f^{\pm} = 0 \quad \text{and} \quad \frac{\partial^{\mp}_{h,\sigma} f^{\pm}}{2} = m^{\pm}.$$

Since the condition $\partial^{\pm}_{h,\sigma} f^{\pm} = 0$ for functions $f^{\pm} \in H^1_h(M)$ implies that $f^{\pm} \in K_{h,\sigma}$, by definition it follows that $(S + \imath\sigma) f^{\pm} = T f^{\pm} = 0$, hence $m^{\pm} = \partial^{\mp}_{h,\sigma} f^{\pm}/2 = 0$. We have thus proved that the spaces $\mathcal{M}^{\pm}_{\Sigma}(\sigma)$ are isomorphic to the spaces $H^{1,\pm}_{h,\sigma}(M, \mathbb{C})$, respectively, hence they have the same dimensions. Finally, by definition, the cohomology $H^0_{h,\sigma}(M, \mathbb{C})$ coincides with the subspace $K_{h,\sigma}$, hence by Lemma 3.4 we derive

$$\begin{aligned}\dim_{\mathbb{C}}(\mathcal{M}^{\pm}_{\Sigma}(\sigma)) &= \dim_{\mathbb{C}}(H^{1,\pm}_{h,\sigma}(M, \mathbb{C})) = \tfrac{1}{2}\dim_{\mathbb{C}}(H^1_{h,\sigma}(M, \mathbb{C})) \\ &= g - 1 + \dim_{\mathbb{C}}(H^0_{h,\sigma}(M, \mathbb{C})) = g - 1 + \dim_{\mathbb{C}}(K_{h,\sigma}).\end{aligned}$$

The formula for the range $R^{\pm}_{h,\sigma}$ follows from a general fact of Hilbert space theory, as soon as we have proved that the range is closed. It follows from Lemma 3.1 that $R^{\pm}_{h,\sigma}$ are closed. In fact, the subspaces $R^{\pm}_{h,\sigma}$ coincide with the range of restrictions of the operators $\partial^{\pm}_{h,\sigma}$ to the subspace $K^{\perp}_{h,\sigma} \cap H^1_h(M)$. By Lemma 3.1 these restrictions have closed range.

Finally, (iv) is a direct consequence of the identity in formula (23). □

The results just proved in Proposition 3.5, in particular (iv), allow us to give a precise meaning to the formal factorization (19), by introducing a family of unitary operators on $L^2_h(M)$, which, as will be seen, contains a great deal of information about the properties of the differential operator $S_\theta + \imath\sigma$ ($\theta \in \mathbb{T}$) defined in formula (17). Let $U_{h,\sigma} : R^-_{h,\sigma} \to R^+_{h,\sigma}$ be defined as

$$U_{h,\sigma} := (\partial^+_{h,\sigma})(\partial^-_{h,\sigma})^{-1}. \tag{24}$$

It is an immediate consequence of assertion (iv) of Proposition 3.5 that $U_{h,\sigma}$ is a partial isometry. Thus, we extend in the natural way the domain of definition of $U_{h,\sigma}$ as follows. Let

$$J : \mathcal{M}^+_{\Sigma}(\sigma) \to \mathcal{M}^-_{\Sigma}(\sigma) \tag{25}$$

be an isometric operator, with respect to the euclidean structures induced on $\mathcal{M}^+_{\Sigma}(\sigma)$ and $\mathcal{M}^-_{\Sigma}(\sigma)$ by the Hilbert space $L^2_h(M)$. The existence of J is a consequence of the fact that the *deficiency subspaces* $\mathcal{M}^+_{\Sigma}(\sigma)$ and $\mathcal{M}^-_{\Sigma}(\sigma)$ are isomorphic finite-dimensional vector spaces of the same complex dimension (equal to the genus g of the surface M). In fact, there exists a whole family of operators J as required, parametrized by the Lie group $U(g, \mathbb{C})$. Let $\pi^{\pm}_{h,\sigma} : L^2_h(M) \to R^{\pm}_{h,\sigma}$ be the orthogonal projections. We recall that $R^{\pm}_{h,\sigma}$ are the orthogonal complements of $\mathcal{M}^{\mp}_{\Sigma}(\sigma)$, respectively (Proposition 3.5). Once an isometric operator J as in formula (25) is fixed, the partial isometry $U_{h,\sigma}$, associated with the holomorphic abelian differential h on M and $\sigma \in \mathbb{R}$ as in formula (24), will be extended to a unitary operator $U_{J,\sigma}$ on the whole $L^2_h(M)$ by the formula

$$U_{J,\sigma}(u) := U_{h,\sigma}\pi^-_{h,\sigma}(u) + J(I - \pi^-_{h,\sigma})(u) \quad \text{for all } u \in L^2_h(M)$$

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(the dependence of the unitary operator $U_{J,\sigma}$ on the abelian differential is omitted in the notation for convenience).

The following version of the formal identities (19) holds on $H^1_h(M)$:

$$S_\theta + \imath\sigma_\theta = \frac{e^{-\imath\theta}}{2}\,((U_{J,\sigma} + e^{2\imath\theta}))\,\partial^-_{h,\sigma} = \frac{e^{\imath\theta}}{2}\,((U^{-1}_{J,\sigma} + e^{-2\imath\theta}))\,\partial^+_{h,\sigma}. \tag{26}$$

A priori estimates for S_θ are related by formula (26) to estimates for the resolvent $\mathcal{R}_{J,\sigma}(z) := (U_{J,\sigma} - zI)^{-1}$ of any of the operators $U_{J,\sigma}$, as $z \to \mathbb{T}$ non-tangentially. Since these are unitary operators, their spectrum is contained in the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. As a consequence, the resolvent $\mathcal{R}_{J,\sigma}(z)$ is a well-defined operator-valued *holomorphic*function on the unit disk $D := \{z \in \mathbb{C} \mid |z| < 1\}$. In addition, by the *spectral theorem*for unitary operators, it is given by a *Cauchy integral*on ∂D of the spectral measure associated with $U_{J,\sigma}$.

4. *Spectral theory of unitary operators*

Let $U : \mathcal{H} \to \mathcal{H}$ be *any*unitary operator on a (separable) Hilbert space $\mathcal{H}$. By the spectral theorem [**Yo**, XI.4], its resolvent $\mathcal{R}_U(z) := (U - zI)^{-1}$ can be represented as a Cauchy integral of the spectral family as follows. For any $u, v \in \mathcal{H}$,

$$(\mathcal{R}_U(z)u, v)_{\mathcal{H}} = \int_0^{2\pi} (z - e^{it})^{-1}\, d(E_U(t)u, v)_{\mathcal{H}} \quad \text{for all } z \in D,$$

where $(\,\cdot,\,\cdot\,)_{\mathcal{H}}$ denotes the inner product in $\mathcal{H}$ and $\{E_U(t)\}_{0\le t\le 2\pi}$ denotes the spectral family associated with the unitary operator U on $\mathcal{H}$. Our approach from [**F97**] is based on the fundamental property of holomorphic functions on D, which can be represented as Cauchy integrals on ∂D of complex measures, of having non-tangential boundary values *almost everywhere*.

We recall below general results on the boundary behavior of Cauchy integrals of complex measures. The modern theory of these singular integrals, based on the Lebesgue integral, was initiated by Fatou in his thesis [**Ft**]. The results gathered below were originally obtained by Riesz [**Rz**], Smirnov [**Sm**], Hardy and Littlewood [**HL**]. The arguments, given in [**F97**, §3], follow the approach of Zygmund [**Zy**, VII.9], based on real variables methods, and are taken from the books by Rudin [**Rd**] and Stein and Weiss [**SW**].

Let μ be a complex Borel measure (of finite total mass) on ∂D. The Cauchy integral of μ is the holomorphic function I_μ on D defined as

$$I_\mu(z) := \int_0^{2\pi} (z - e^{it})^{-1}\, d\mu(t) \quad \text{for all } z \in D. \tag{27}$$

LEMMA 4.1. [**F97**, Lemma 3.3A] *The non-tangential limit*

$$I_\mu(z) \to I^*_\mu(\theta) \quad \textit{as } z \to e^{\imath\theta},$$

exists almost everywhere with respect to the (normalized) Lebesgue measure $\mathcal{L}$ on the circle $\mathbb{T}$. In addition, there exists a constant $C > 0$ such that the following weak type

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estimate holds:

$$\mathcal{L}\{\theta \in \mathbb{T} \mid |I^*_\mu(\theta)| > t\} \le \frac{C}{t}\,|\mu| \quad \textit{for all } t > 0,$$

where $|\mu|$ *denotes the total mass of the measure* μ.

Lemma 4.1 is not enough for our purposes, since it gives no information concerning the behavior of Cauchy integrals (27) as the convergence to the limiting boundary values takes place. The necessary estimates are given below, following [**F97**], in terms of non-tangential maximal functions, the definition of which we recall below following [**Rd**, §11.18].

For $0 < \alpha < 1$, we define the non-tangential approach region Ω_α to be the cone over $D(0, \alpha)$ of vertex $z = 1$, that is, the union of the disk $D(0, \alpha)$ and the line segments from the point $z = 1$ to the points of $D(0, \alpha)$. Rotated copies of Ω_α, having vertex at $e^{\imath\theta}$, will be denoted by $\Omega_\alpha(\theta)$.

For any complex function Φ on the unit disk D and $0 < \alpha < 1$, its *non-tangential maximal function* $N_\alpha(\Phi)$ is defined on $\mathbb{T}$ as

$$N_\alpha(\Phi)(\theta) := \sup\{|\Phi(z)| \mid z \in \Omega_\alpha(\theta)\}. \tag{28}$$

We would like to complete Lemma 4.1 with estimates on the non-tangential maximal function $N_\alpha(I_\mu)$ of the Cauchy integral in formula (27). This can be accomplished by a standard argument of basic Hardy space theory.

For the convenience of the reader, we will recall the definition of Hardy spaces $H^p(D)$ on the unit disk D [**Rd**, §§7 and 17.6]. Let Φ be a complex function on D. For $0 < r < 1$, we define the functions Φ_r on $\mathbb{T}$ by the formula

$$\Phi_r(\theta) := \Phi(re^{\imath\theta}) \quad \text{for all } \theta \in \mathbb{T},$$

and, for $0 < p \le \infty$, we define

$$|\Phi|_p = \sup\{|\Phi_r|_p \mid 0 \le r < 1\},$$

where $|\cdot|_p$ denotes the L^p norm on $\mathbb{T}$ with respect to the Lebesgue measure. The *Hardy space* $H^p(D)$ is defined to be the space of holomorphic functions Φ on the unit disk D such that $|\Phi|_p < \infty$.

LEMMA 4.2. [**F97**, Lemma 3.3B] *The holomorphic function* I_μ*, given as a Cauchy integral (see formula (27)) of a Borel complex measure* μ *on* ∂D*, belongs to the Hardy spaces* $H^p(D)$*, for any* $0 < p < 1$*. (Consequently, it admits a non-tangential limit almost everywhere on* ∂D*.) In addition, its non-tangential maximal function* $N_\alpha(I_\mu)$ *belongs to* $L^p(\mathbb{T}, \mathcal{L})$*, for any* $0 < p < 1$ *and for all* $\alpha < 1$*, and there exist constants* $A_\alpha, A_{\alpha,p} > 0$*, with* $A_{\alpha,p} \to \infty$ *as* $p \to 1$*, such that the following estimates hold:*

$$|N_\alpha(I_\mu)|_p \le A_\alpha |I_\mu|_p \le A_{\alpha,p}|\mu|,$$

where $|\mu|$ *denotes the total mass of the measure* μ.

The general harmonic analysis lemmas (Lemmas 4.1 and 4.2) are then applied via the spectral theorem to the resolvent of an arbitrary unitary operator on a Hilbert space. The

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abstract Hilbert space result which we obtain in this way will then be applied to the unitary operators $U_{J,\sigma}$, $U_{J,\sigma}^{-1}$ introduced in §3.

COROLLARY 4.3. [**F97**, Corollary 3.4] *Let $\mathcal{R}_U(z) : \mathcal{H} \to \mathcal{H}$, $z \in D$, denote the resolvent of a unitary operator $U : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$. Then, for any $u, v \in \mathcal{H}$, the holomorphic functions $\Phi(u, v)(\cdot) := (\mathcal{R}(\cdot)u, v)_{\mathcal{H}}$ belong to the Hardy spaces $H^p(D)$, for any $0 < p < 1$. Consequently, they admit a non-tangential limit almost everywhere on ∂D. Furthermore, their non-tangential maximal functions $N_\alpha(u, v)$ belong to $L^p(\mathbb{T}, \mathcal{L})$, for any $0 < p < 1$ and for all $\alpha < 1$, and there exist constants A_α, $A_{\alpha,p} > 0$, with $A_{\alpha,p} \to \infty$ as $p \to 1$, such that the following estimates hold:*

$$|N_\alpha(u, v)|_p \le A_\alpha |\Phi(u, v)|_p \le A_{\alpha,p} |u|_{\mathcal{H}}\, |v|_{\mathcal{H}},$$

where $|\cdot|_{\mathcal{H}}$ denotes the Hilbert space norm.

5. *Solutions of the twisted cohomological equation*

In this section we adapt to the twisted cohomological equation the streamlined version [**F07**] of the main argument of [**F97**, Theorem 4.1] given with the goal of establishing the sharpest bound on the loss of Sobolev regularity within the reach of the methods of [**F97**].

We prove in §5.1 a priori Sobolev bounds for distributional solutions of the cohomological equation with respect to the Friedrichs weighted Sobolev norms, then in §5.3 for smooth solutions with respect to fractional weighted Sobolev norms.

In §5.2 we prove results on the structure of the space of twisted invariant distributions, which give obstructions to the existence of solutions of the cohomological equations.

5.1. *Distributional solutions.*

We derive results on distributional solutions of the twisted cohomological equation from the harmonic analysis results of §4 about the boundary behavior of the resolvent of a unitary operator.

We recall the definition of the Friedrichs weighted Sobolev spaces and norms, given in Definition 2.5. The Friedrichs weighted Sobolev space $\bar{H}_h^s(M)$ can be defined, for all $s \in \mathbb{R}$, as the maximal domain of the (fractional) power $(-\Delta_h^F)^{s/2}$ of the Friedrichs extension $-\Delta_h^F$ of the non-negative flat Laplacian, endowed with the following norms. Let $\{\lambda_k \mid k \in \mathbb{N}\}$ denote the sequence of eigenvalues (repeated according to their multiplicities) of the non-negative Friedrichs Laplacian $-\Delta_h^F$ relative to a orthonormal basis $\mathcal{E} = \{e_k \mid k \in \mathbb{N}\}$ of eigenfunctions. The Friedrichs weighted Sobolev norms are given, for all $s \in \mathbb{R}$, as

$$\|u\|_s = \bigg(\sum_{k\in\mathbb{N}} (1+\lambda_k)^s |\langle u, e_k\rangle|^2 \bigg)^{1/2} \quad \text{for all } u \in \bar{H}_h^s(M).$$

Definition 5.1. Let h be an abelian differential and let $\sigma \in \mathbb{R}$. A distribution $u \in \bar{H}_h^{-r}(M)$ will be called a *(distributional) solution* of the cohomological equation $(S + \imath\sigma)u = f$ for a given function $f \in \bar{H}_h^{-s}(M)$ if

$$\langle u, (S + \imath\sigma)v\rangle = -\langle f, v\rangle \quad \text{for all } v \in H_h^{r+1}(M) \cap \bar{H}_h^s(M).$$

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Let $h_\theta = e^{-\imath\theta} h$ be its rotation and let $\sigma_\theta := \sigma \cos\theta$. Let $\{S_\theta\}$ denote the one-parameter family of rotated vector fields introduced in formula (17):

$$S_\theta := \{e^{-\imath\theta}(S + \imath\, T) + e^{\imath\theta}(S - \imath\, T)\}/2.$$

For all $s \in \mathbb{R}$, let $\mathcal{H}^s_h(M) \subset H^s_h(M)$ and $\bar{\mathcal{H}}^s_h(M) \subset \bar{H}^s_h(M)$ the subspaces of distributions vanishing on constant functions. By Lemma 2.8, for all $s \in [-1, 1]$, we have $\bar{H}^s_h(M) \equiv H^s_h(M)$, hence $\bar{\mathcal{H}}^s_h(M) \equiv \mathcal{H}^s_h(M)$.

THEOREM 5.2. *Let h be an abelian differential on M with minimal vertical foliation. Let $r > 2$ and $p \in (0, 1)$ be such that $rp > 2$. For any $\sigma \in \mathbb{R}$, there exists a bounded linear operator*

$$\mathcal{U}_\sigma : \mathcal{H}^{-1}_h(M) \to L^p(\mathbb{T}, \bar{H}^{-r}_h(M))$$

such that the following statement holds. For any $\sigma \in \mathbb{R}$ and any $f \in H^{-1}(M)$ there exists a full measure subset $\mathcal{F}_r(\sigma, f) \subset \mathbb{T}$ such that $u := \mathcal{U}_\sigma(f)(\theta) \in \bar{H}^{-r}_h(M)$ is a distributional solution of the cohomological equation $(S_\theta + \imath\sigma_\theta)u = f$, for all $\theta \in \mathcal{F}_r(f, \sigma)$. In addition, there exists a constant $B_h := B_h(p, r) > 0$ such that, for all $f \in \mathcal{H}^{-1}_h(M)$, vanishing on constant functions,

$$|\mathcal{U}_\sigma(f)|_p := \left(\int_{\mathbb{T}} \|\mathcal{U}_\sigma(f)(\theta)\|^p_{-r}\, d\theta\right)^{1/p} \leq B_h\, \|f\|_{-1}.$$

The above theorem is a consequence of the following estimate.

LEMMA 5.3. *Let h be an abelian differential on M with minimal vertical foliation. Let $r > 2$ and $p \in (0, 1)$ be such that $rp > 2$. For any $\sigma \in \mathbb{R}$ and any $f \in \mathcal{H}^{-1}_h(M)$, vanishing on constant functions, there exists a measurable function $A_{h,\sigma}(f) := A_{h,\sigma}(f, p, r) \in L^p(\mathbb{T}, \mathcal{L})$ such that, for all $v \in H^{r+1}_h(M)$, we have*

$$|\langle f, v\rangle| \leq A_{h,\sigma}(f, \theta)\, \|(S_\theta + \imath\sigma_\theta)v\|_r. \tag{29}$$

In addition, the following bound for the L^p norm of the function $A_{h,\sigma}(f)$ holds. There exists a constant $B_h := B_h(p, r) > 0$ such that, for every $\sigma \in \mathbb{R}$ and for every $f \in H^{-1}_h(M)$, vanishing on constant functions, we have

$$|A_{h,\sigma}(f)|_p \leq B_h\, \|f\|_{-1}. \tag{30}$$

Proof. We recall formulas (26):

$$S_\theta + \imath\sigma_\theta = \frac{e^{-\imath\theta}}{2}\,((U_{J,\sigma} + e^{2\imath\theta}))\,\partial^-_{h,\sigma} = \frac{e^{\imath\theta}}{2}\,((U^{-1}_{J,\sigma} + e^{-2\imath\theta}))\,\partial^+_{h,\sigma}. \tag{31}$$

It appears from the above formulas that, since the twisted Cauchy–Riemann operators, which are elliptic partial differential operators, have finite-dimensional kernel and cokernel, a formal solution of the cohomological equation $(S_\theta + \imath\sigma_\theta)u = f$ (up to a finite-dimensional cokernel) can be written in terms of the inverses of the operator $U_{J,\sigma} + e^{2\imath\theta} I$ or $U^{-1}_{J,\sigma} + e^{-2\imath\theta} I$. However, since U_J and U^{-1}_J are unitary operators, their spectrum in contained in the unit circle, hence the inverses of the operators $U_{J,\sigma} + e^{2\imath\theta} I$ or $U^{-1}_{J,\sigma} + e^{-2\imath\theta} I$ are a priori unbounded on $L^2_h(M)$. (It can in fact be proved that the

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spectra of U_J and U_J^{-1} coincide with the full unit circle.) The key idea of the argument is to construct inverses in the distributional (weak) sense by taking non-tangential boundary limits of the resolvents $(U_{J,\sigma} + zI)^{-1}$ or $(U_{J,\sigma}^{-1} + zI)^{-1}$, which are well-defined bounded operators on the Hilbert space $L^2_h(M)$ as long as the spectral parameter $z \in \mathbb{C}$ belongs to the interior of the unit disk.

Thus, our proof of estimate (29) is based on properties of the resolvent of the operators $U_{J,\sigma}, U_{J,\sigma}^{-1}$. In fact, it is based on the *general* results, summarized below in §4, concerning the non-tangential boundary behavior of the resolvent of a unitary operator on a Hilbert space, applied to the operators $U_{J,\sigma}$, $U_{J,\sigma}^{-1}$ on $L^2_h(M)$. The Fourier analysis of [**F97**, §2], also plays a relevant role through Lemma 2.9 and the Weyl's asymptotic formula (Theorem 2.3).

Since the vertical foliation of h is minimal, it follows that all T-invariant functions in the space $H^1_h(M)$ are constant, hence the common kernel of the twisted Cauchy–Riemann operators $K_{h,\sigma} \subset H^1_h(M)$ coincides with the subspace of constant functions.

Following [**F97**, Proposition 4.6A] or [**F02**, Lemma 7.3], we prove that there exists a constant $C_h > 0$ such that the following statement holds. For any $\sigma \in \mathbb{R}$ and for any distribution $f \in H_h^{-1}(M)$ there exist (weak) solutions $F^\pm_\sigma \in L^2_h(M)$ of the equations $\partial^\pm_{h,\sigma} F^\pm_\sigma = f$ such that

$$|F^\pm_\sigma|_0 \leq C_h \, \|f\|_{-1}. \tag{32}$$

In fact, the maps given by

$$\partial^\pm_{h,\sigma} v \to -\langle f, v\rangle \quad \text{for all } v \in H^1_h(M), \tag{33}$$

are bounded linear functionals on the (closed) ranges $R^\pm_{h,\sigma} \subset L^2_h(M)$ of the twisted Cauchy–Riemann operator $\partial^\pm_{h,\sigma} : H^1_h(M) \to L^2_h(M)$. In fact, the functionals are well defined since by assumption $K_{h,\sigma} = \mathbb{C}$ and f vanishes on constant functions, and they are bounded since, by Lemma 3.1 and Proposition 3.5, there exists a constant $C_h > 0$ such that, for any $\sigma \in \mathbb{R}$ and for any $v \in H^1_h(M)$ of zero average,

$$\begin{aligned} |\langle f, v\rangle| &\leq \|f\|_{-1} |v|_1 \leq C'_h \|f\|_{-1} Q_{h,0}(v) \\ &\leq C_h \|f\|_{-1} Q_{h,\sigma}(v) = C_h \, \|f\|_{-1} \, |\partial^\pm_{h,\sigma} v|_0. \end{aligned} \tag{34}$$

Let $\Phi^\pm_\sigma$ be the unique linear extension of the linear map (33) to $L^2_h(M)$ which vanishes on the orthogonal complement of $R^\pm_{h,\sigma}$ in $L^2_h(M)$. By (34), the functionals $\Phi^\pm_\sigma$ are bounded on $L^2_h(M)$ with norm

$$\|\Phi^\pm_\sigma\| \leq C_h \, \|f\|_{-1}.$$

By the Riesz representation theorem, there exist two (unique) functions $F^\pm_\sigma \in L^2_h(M)$ such that

$$\langle v, F^\pm_\sigma\rangle_h = \Phi^\pm_\sigma(v) \quad \text{for all } v \in L^2_h(M).$$

The functions $F^\pm_\sigma$ are by construction (weak) solutions of the twisted Cauchy–Riemann equations $\partial^\pm_{h,\sigma} F^\pm_\sigma = f$ satisfying the required bound (32).

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The identities (31) immediately imply that

$$\langle \partial^{\pm}_{h,\sigma} v, F^{\pm}_{\sigma} \rangle_h = 2e^{\mp i\theta}\, \langle \mathcal{R}^{\pm}_{J,\sigma}(z)(S_\theta + \imath\sigma_\theta)v, F^{\pm}_{\sigma}\rangle_h \\ - (z + e^{\mp 2\imath\theta})\langle \mathcal{R}^{\pm}_{J,\sigma}(z)\partial^{\pm}_h v, F^{\pm}_{\sigma}\rangle_h, \tag{35}$$

where $\mathcal{R}^{+}_{J,\sigma}(z)$ and $\mathcal{R}^{-}_{J,\sigma}(z)$ denote the resolvents of the unitary operators $U_{J,\sigma}$ and $U^{-1}_{J,\sigma}$ respectively, which yield holomorphic families of bounded operators on the unit disk $D \subset \mathbb{C}$.

Let $r > 2$ and let $p \in (0, 1)$ be such that $pr > 2$. Let $\mathcal{E} = \{e_k\}_{k\in\mathbb{N}}$ denote the orthonormal Fourier basis of the Hilbert space $L^2_h(M)$ of eigenfunctions of the Friedrichs Laplacian, described in §2. By Corollary 4.3 all holomorphic functions

$$\mathcal{R}^{\pm}_{h,\sigma,k}(z) := \langle \mathcal{R}^{\pm}_{J,\sigma}(z)e_k, F^{\pm}_{\sigma}\rangle_h, \quad k \in \mathbb{N}, \tag{36}$$

belong to the Hardy space $H^p(D)$, for any $0 < p < 1$. The corresponding non-tangential maximal functions $N^{\pm}_k$ (over cones of arbitrary fixed aperture $0 < \alpha < 1$) belong to the space $L^p(\mathbb{T}, \mathcal{L})$ and for all $0 < p < 1$ there exists a constant $A_{\alpha,p} > 0$ such that, for any abelian differential h on M, for every $\sigma \in \mathbb{R}$ and $k \in \mathbb{N}$, the following inequalities hold:

$$|N^{\pm}_{h,\sigma,k}|_p \le A_{\alpha,p}|e_k|_0\, |F^{\pm}_{\sigma}|_0 = A_{\alpha,p}\, |F^{\pm}_{\sigma}|_0 \le A_{\alpha,p}\, C_h\, |f|_{-1}. \tag{37}$$

Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the sequence of the eigenvalues of the Dirichlet form $\mathcal{Q} := \mathcal{Q}_{h,0}$ introduced in §2. Let $w \in \bar{H}^r_h(M)$. We have

$$\langle \mathcal{R}^{\pm}_{J,\sigma}(z)w, F^{\pm}_{\sigma}\rangle_h = \sum_{k=0}^{\infty} \langle w, e_k\rangle_h\ \mathcal{R}^{\pm}_{h,\sigma,k}(z), \tag{38}$$

hence, by the Cauchy–Schwarz inequality,

$$\begin{aligned} |\langle \mathcal{R}^{\pm}_{J,\sigma}(z)w, F^{\pm}_{\sigma}\rangle_h| &\le \sum_{k=0}^{\infty} \frac{|\mathcal{R}^{\pm}_{h,\sigma,k}(z)|}{(1+\lambda_k)^r}(1+\lambda_k)^r|\langle w, e_k\rangle_h| \\ &\le \bigg(\sum_{k=0}^{\infty} \frac{|\mathcal{R}^{\pm}_{h,\sigma,k}(z)|^2}{(1+\lambda_k)^r}\bigg)^{1/2}\bigg(\sum_{k=0}^{\infty}(1+\lambda_k)^r|\langle w, e_k\rangle_h|^2\bigg)^{1/2} \\ &= \bigg(\bigg(\sum_{k=0}^{\infty} \frac{|\mathcal{R}^{\pm}_{h,\sigma,k}(z)|^2}{(1+\lambda_k)^r}\bigg)\bigg)^{1/2}\|w\|_r. \end{aligned} \tag{39}$$

Let $N^{\pm}_{h,\sigma}(\theta)$ be the functions defined as

$$N^{\pm}_{h,\sigma}(\theta) := \bigg(\bigg(\sum_{k=0}^{\infty} \frac{|N^{\pm}_{h,\sigma,k}(\theta)|^2}{(1+\lambda_k)^r}\bigg)\bigg)^{1/2}. \tag{40}$$

Let $N^{\pm}_{h,\sigma}(w)$ denote the non-tangential maximal function for the holomorphic function $\langle \mathcal{R}^{\pm}_{J,\sigma}(z)w, F^{\pm}_{\sigma}\rangle_h$. By formulas (39) and (40), it follows that, for all $\theta \in \mathbb{T}$ and all functions $w \in \bar{H}^r_h(M)$, we have

$$N^{\pm}_{h,\sigma}(w)(\theta) \le N^{\pm}_{h,\sigma}(\theta)\, \|w\|_r. \tag{41}$$

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The functions $N^{\pm}_{h,\sigma} \in L^p(\mathbb{T}, \mathcal{L})$ for any $0 < p < 1$. In fact, by formula (37) and (following a suggestion of Stephen Semmes) by the 'triangular inequality' for the space $L^{p/2}$ with $0 < p < 1$, we have

$$|N^{\pm}_{h,\sigma}|_p^p \leq (A_{\alpha,p}\, C_h)^p \bigg(\sum_{k=0}^{\infty} \frac{1}{(1+\lambda_k)^{pr/2}} \bigg) \, \|f\|_{-1}^p \; < \; +\infty. \tag{42}$$

The series in formula (42) is convergent by the Weyl asymptotics (Theorem 2.3) since $pr/2 > 1$. Let then

$$B_h(p,r) := (A_{\alpha,p}\, C_h) \bigg(\sum_{k=0}^{\infty} \frac{1}{(1+\lambda_k)^{pr/2}} \bigg)^{1/p}.$$

By taking the non-tangential limit as $z \to -e^{\mp 2\imath\theta}$ in identity (35), formula (41) implies that, for all $\theta \in \mathbb{T}$ such that $N^{\pm}_{h,\sigma}(\pi \mp 2\theta) < +\infty$,

$$|\langle \partial^{\pm}_{h,\sigma} v, F^{\pm}_{\sigma} \rangle_h| \leq N^{\pm}_{h,\sigma}(\pi \mp 2\theta)\, \|(S_\theta + \imath\sigma_\theta) v\|_r,$$

hence the required estimates (29) and (30) are proved with the choice of the function $A_{h,\sigma}(f,\theta) := N^{+}_{h,\sigma}(\pi - 2\theta)$ or $A_{h,\sigma}(f,\theta) := N^{-}_{h,\sigma}(\pi + 2\theta)$, for all $\theta \in \mathbb{T}$. □

Proof of Theorem 5.2. By the estimate (29) of Lemma 5.3, the linear map given by

$$(S_\theta + \imath\sigma_\theta) v \to -\langle f, v\rangle \quad \text{for all } v \in H^{r+1}_h(M), \tag{43}$$

is well defined and extends by continuity to the closure of the range $\bar{R}^r_\sigma(\theta)$ of the linear operator $S_\theta + \imath\sigma_\theta$ in $\bar{H}^r_h(M)$. Let $\mathcal{U}_\sigma(f)(\theta)$ be the extension uniquely defined by the condition that $\mathcal{U}_\sigma(f)(\theta)$ vanishes on the orthogonal complement of $\bar{R}^r_\sigma(\theta)$ in $\bar{H}^r_h(M)$. By construction, for almost all $\theta \in \mathbb{T}$ the linear functional $u := \mathcal{U}_\sigma(f)(\theta) \in \bar{H}^{-r}_h(M)$ yields a distributional solution of the cohomological equation $(S_\theta + \imath\sigma_\theta)u = f$ whose norm satisfies the bound

$$\|\mathcal{U}_\sigma(f)(\theta)\|_{-r} \leq A_{h,\sigma}(f,\theta).$$

By (30) the L^p norm of the measurable function $\mathcal{U}_\sigma(f) : \mathbb{T} \to \bar{H}^{-r}_h(M)$ satisfies the required estimate

$$|\mathcal{U}_\sigma(f)|_p := \bigg(\int_{\mathbb{T}} \|\mathcal{U}_\sigma(f)(\theta)\|^p_{-r}\, d\theta \bigg)^{1/p} \quad \leq \quad B_h\, \|f\|_{-1}. \qquad \square$$

THEOREM 5.4. *Let h be an abelian differential with minimal vertical foliation. For any $r > 2$ and $p \in (0,1)$ such that $pr > 2$, there exists a constant $C_{h,p,r} > 0$ such that, for all zero-average functions $f \in \bar{H}^{r-1}_h(M)$, for all $\sigma \in \mathbb{R}$ and for Lebesgue almost all $\theta \in \mathbb{T}$, the twisted cohomological equation $(S_\theta + \imath\sigma_\theta)u = f$ has a distributional solution $u_\theta \in \bar{H}^{-r}_h(M)$ satisfying the following estimate:*

$$\bigg(\int_{\mathbb{T}} \|u_\theta\|^p_{-r}\, d\theta \bigg)^{1/p} \leq C_{h,p,r}\, \|f\|_{r-1}. \tag{44}$$

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Proof. Let $\{\lambda_k | k \in \mathbb{N}\}$ denote the sequence of eigenvalues of the Friedrichs Laplacian relative to the orthonormal Fourier basis $\mathcal{E} = \{e_k\}_{k\in\mathbb{N}}$ of eigenfunctions of the Hilbert space $L^2_h(M)$, described in §2. Let $r > 2$ and $p \in (0, 1)$ be such that $pr > 2$.

By Theorem 5.2, for any $k \in \mathbb{N} \setminus \{0\}$ there exists a function with distributional values $u_k := \mathcal{U}_\sigma(e_k) \in L^p(\mathbb{T}, \bar{H}^{-r}_h(M))$ such that the following statement holds. There exists a constant $C_{h,r} := C_h(p, r) > 0$ such that

$$\left(\int_{\mathbb{T}} \|u_k(\theta)\|^p_{-r} \, d\theta \right)^{1/p} \leq C_{h,r} \, \|e_k\|_{-1} \leq C_{h,r} \, (1 + \lambda_k)^{-1/2}. \tag{45}$$

In addition, for any $k \in \mathbb{N} \setminus \{0\}$, there exists a full measure set $\mathcal{F}_k(\sigma) \subset \mathbb{T}$ such that, for all $\theta \in \mathcal{F}_k(\sigma)$, the distribution $u := u_k(\theta) \in \bar{H}^{-r}_h(M)$ is a (distributional) solution of the cohomological equation $(S_\theta + \imath\sigma_\theta)u = e_k$.

Any function $f \in \bar{H}^{r-1}_h(M)$ of zero average has a Fourier decomposition in $L^2_h(M)$:

$$f = \sum_{k\in\mathbb{N}\setminus\{0\}} \langle f, e_k\rangle_h \, e_k.$$

A (formal) solution of the cohomological equation $(S_\theta + \imath\sigma_\theta)u = f$ is therefore given by the series

$$u_\theta := \sum_{k\in\mathbb{N}\setminus\{0\}} \langle f, e_k\rangle_h \, u_k(\theta). \tag{46}$$

By the triangular inequality in $\bar{H}^{-r}_h(M)$ and by the Hölder inequality, we have

$$\|u_\theta\|_{-r} \leq \left(\sum_{k\in\mathbb{N}\setminus\{0\}} \frac{\|u_k(\theta)\|^2_{-r}}{(1 + \lambda_k)^{r-1}} \right)^{1/2} \|f\|_{r-1},$$

hence by the 'triangular inequality' for L^p spaces (with $0 < p < 1$) and by estimate (45),

$$\int_{\mathbb{T}} \|u_\theta\|^p_{-r} \, d\theta \leq C^p_{h,r} \left(\sum_{k\in\mathbb{N}\setminus\{0\}} \frac{1}{(1 + \lambda_k)^{pr/2}} \right) \|f\|^p_{r-1}. \tag{47}$$

Since $pr/2 > 1$ the series in (47) is convergent, hence $u_\theta \in \bar{H}^{-r}_h(M)$ is a solution of the equation $(S_\theta + \imath\sigma_\theta)u = f$ which satisfies the required bound (44). □

5.2. *Twisted invariant distributions and basic currents.* In this section we describe the structure of the space of obstructions to the existence of solutions of the twisted cohomological equation $(S + \imath\sigma)u = f$.

Definition 5.5. For all $r > 0$, let $\mathcal{I}^r_{h,\sigma} \subset H^{-r}_h(M)$ denote the space of distributions invariant for the twisted Lie derivative operator $S + \imath\sigma$, that is, the space

$$\mathcal{I}^r_{h,\sigma} := \{D \in H^{-r}_h(M) | (S + \imath\sigma)D = 0\}.$$

Twisted invariant distributions are in one-to-one correspondence with twisted basic currents. We introduce Sobolev space of 1-forms and of one-dimensional currents.

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Definition 5.6. Let $\mathcal{S}(M, T^*M)$ denote the space of all Borel measurable sections of the cotangent bundle T^*M. For all $r > 0$, the weighted Sobolev space of 1-forms $W^r_h(M)$ is defined as follows:

$$W^r_h(M) := \{\alpha \in \mathcal{S}(M, T^*M) \mid (\imath_S\alpha, \imath_T\alpha) \in H^r_h(M) \times H^r_h(M)\}. \tag{48}$$

The weighted Sobolev space of 1-currents $W^{-r}_h(M)$ is defined as the dual space of the weighted Sobolev space of 1-forms $W^r_h(M)$.

Twisted basic currents are defined as follows.

Definition 5.7. For all $r > 0$, let $\mathcal{B}^r_{h,\sigma} \subset H^{-r}_h(M)$ denote the space of twisted basic currents, that is, the space

$$\mathcal{B}^r_{h,\sigma} := \{C \in W^{-r}_h(M) \mid (\mathcal{L}_S + \imath\sigma)C = \imath_S C = 0\}.$$

(Here $\mathcal{L}_S$ denotes the Lie derivative operator on currents in the direction of the vector field S on $M \setminus \Sigma_h$.)

The notions of twisted invariant distributions and twisted basic currents are related (for the untwisted case see [**F02**, Lemmas 6.5 and 6.6] or [**F07**, Lemma 3.14]):

LEMMA 5.8. *A one-dimensional current $C \in \mathcal{B}^r_{h,\sigma}$ if and only if the distribution $C \wedge$ $\mathrm{Re}(h) \in \mathcal{I}^r_{h,\sigma}$. In addition, the map*

$$\mathcal{D}_h : C \to -C \wedge \mathrm{Re}(h) \tag{49}$$

is a bijection from the space $\mathcal{B}^r_{h,\sigma}$ onto the space $\mathcal{I}^r_{\sigma,h}$.

Proof. The map $\mathcal{D}_h$ is well defined since for any $C \in \mathcal{B}^r_{h,\sigma}$ we have

$$(\mathcal{L}_S + \imath\sigma)[C \wedge \mathrm{Re}(h)] = [(\mathcal{L}_S + \imath\sigma)C] \wedge \mathrm{Re}(h) = 0.$$

The inverse map is the map $\mathcal{B}_h : \mathcal{I}^r_{\sigma,h} \to \mathcal{B}^r_{h,\sigma}$ defined as

$$\mathcal{B}_h(D) = \imath_S D \quad \text{for all } D \in \mathcal{I}^r_{\sigma,h}.$$

(Here $\imath_S$ denotes the contraction operator with respect the vector field S on $M \setminus \Sigma_h$, which maps distributions, as currents of degree 2, to currents of degree 1.)

The map $\mathcal{B}_h$ is well defined since $\imath_S \circ \imath_S = 0$, and

$$(\mathcal{L}_S + \imath\sigma) \circ \imath_S = \imath_S \circ (\mathcal{L}_S + \imath\sigma).$$

It follows that if $D \in \mathcal{I}^r_{h,\sigma}$ then $C = \imath_S D \in \mathcal{B}^r_{h,\sigma}$ since

$$(\mathcal{L}_S + \imath\sigma)C = \imath_S \circ (\mathcal{L}_S + \imath\sigma)D = 0 \quad \text{and} \quad \imath_S C = \imath_S(\imath_S D) = 0.$$

Finally, the map $\mathcal{B}_h$ is the inverse of the map $\mathcal{D}_h$. In fact, since $\imath_S C = 0$ (as C is basic) and $D \wedge \mathrm{Re}(h) = 0$ (as a current of degree 3), and $\imath_S \mathrm{Re}(h) = 1$, we have

$$(\mathcal{B}_h \circ \mathcal{D}_h)(C) = -\imath_S(C \wedge \mathrm{Re}(h)) = -\imath_S C \wedge \mathrm{Re}(h) + C \wedge \imath_S \mathrm{Re} h = C,$$
$$(\mathcal{D}_h \circ \mathcal{B}_h)(D) = -(\imath_S D \wedge \mathrm{Re}(h)) = -\imath_S(D \wedge \mathrm{Re}(h)) + (D \wedge \imath_S \mathrm{Re}(h)) = D.$$

The argument is complete. □

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We recall the definition of the twisted exterior differential $d_{h,\sigma}$:

$$d_{h,\sigma}\alpha := d\alpha + \imath\sigma \mathrm{Re}(h) \wedge \alpha \quad \text{for all } \alpha \in W_h^s(M).$$

The twisted exterior differential extends to currents by duality.

Definition 5.9. A current $C \in W_h^{-s}(M)$ is $d_{h,\sigma}$-closed if

$$d_{h,\sigma} C = dC + \imath\sigma(\mathrm{Re}(h) \wedge C) = 0.$$

LEMMA 5.10. *A current $C \in \mathcal{B}_{h,\sigma}^r$ if and only if $\imath_S C = 0$ and C is $d_{h,\sigma}$-closed.*

Proof. If $C \in \mathcal{B}_{h,\sigma}^r$, then $\imath_S C = 0$ by definition, and

$$\imath_S d_{h,\sigma} C = \imath_S[dC + \imath\sigma(\mathrm{Re}(h) \wedge C)] = \mathcal{L}_S C + \imath\sigma C = 0,$$

so that $\imath_S d_{h,\sigma} C = 0$, which implies $d_{h,\sigma} C = 0$, as $d_{h,\sigma} C$ is a current of degree 2 (and dimension 0) and the contraction operator $\imath_S$ is surjective onto 2-forms.

Conversely, if $\imath_S C = 0$ and $d_{h,\sigma} C = 0$, then by the above formula $\mathcal{L}_S C + \imath\sigma C = 0$, hence $C \in \mathcal{B}_{h,\sigma}^r$, thereby completing the argument. □

By the de Rham theorem for twisted cohomology, it is possible to attach a twisted cohomology class to any $d_{h,\sigma}$-closed current.

For any real closed 1-form η on M, let $H_\eta^1(M, \Sigma_h, \mathbb{C})$ denote the relative twisted cohomology introduced in Definition 3.2. For every abelian differential h on M and $\sigma \in \mathbb{R}$, let us adopt the notation

$$H_{h,\sigma}^1(M, \Sigma_h, \mathbb{C}) := H_{\sigma \mathrm{Re}(h)}^1(M, \Sigma_h, \mathbb{C}).$$

LEMMA 5.11. *For every $r \geq 0$, the cohomology map $j_r : \mathcal{B}_{h,\sigma}^r \to H_{h,\sigma}^1(M, \Sigma_h, \mathbb{C})$ such that $j_r(C)$ is the twisted cohomology class of the twisted basic current $C \in \mathcal{B}_{h,\sigma}^r$ is a well-defined linear map.*

Proof. A current $C \in W_h^{-r}(M)$ does not in general extend to a linear functional on $C^\infty(M)$, hence is not a current on the compact surface M. However, since the space $\Omega_c^1(M \setminus \Sigma_h)$ of smooth 1-forms with compact support in $M \setminus \Sigma_h$ is a subspace of the Sobolev space of 1-forms $W_h^r(M)$, for all $r > 0$, it follows from the de Rham theorem for the twisted cohomology that the current $C \in W_h^{-r}(M)$ such that $d_{h,\sigma} C = 0$ in $W_h^{-(r+1)}(M)$ has a well-defined twisted cohomology class $[C] \in H_{h,\sigma}^1(M, \Sigma_h, \mathbb{C})$. In fact, C defines a linear functional on the twisted cohomology with compact support $H_{h,\sigma,c}^1(M \setminus \Sigma_h, \mathbb{C})$, which is dual to the relative cohomology $H_{h,\sigma}^1(M, \Sigma_h, \mathbb{C})$ by the intersection pairing on 1-forms given by integration. □

The structure of the space of basic currents with vanishing cohomology class, with respect to the filtration induced by weighted Sobolev spaces with integer exponent, was described in [**F02**, §7] (see also [**F07**, §3.3], with respect to the filtration induced by weighted Sobolev spaces with general real exponent). We extend these results below to the space of twisted basic currents.

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Let $\delta_r : \mathcal{B}^{r-1}_{h,\sigma} \to \mathcal{B}^r_{h,\sigma}$ be the linear maps defined as follows (see [**F02**, formula (7.18′)] and [**F07**, formulas (3.61) and (3.62)] for the untwisted case):

$$\delta_r(C) := (d_{h,\sigma} \circ \imath_T)(C) = -\, d_{h,\sigma}\left(\frac{C \wedge \mathrm{Re}(h)}{\omega_h}\right) \quad \text{for } C \in \mathcal{B}^r_{h,\sigma}. \tag{50}$$

Indeed, it can be proved by Lemma 5.8 and by the definition of the weighted Sobolev spaces $H^r_h(M)$ and $W^r_h(M)$ that formula (50) defines, for all $r > 0$, bounded linear maps $\delta_r : \mathcal{B}^{r-1}_{h,\sigma} \to \mathcal{B}^r_{h,\sigma}$.

Let $\mathcal{K}^r_{h,\sigma} \subset \mathcal{I}^r_{h,\sigma} \subset H^{-r}_h(M)$ denote the subspace of distributions which are twisted S-invariant and T-invariant, that is,

$$\mathcal{K}^r_{h,\sigma} := \{D \in H^{-r}_h(M) \mid (S + \imath\sigma)D = TD = 0\}.$$

Let $i_r : \mathcal{K}^r_{h,\sigma} \to \mathcal{B}^r_{h,\sigma}$ denote the restriction to $\mathcal{K}^r_{h,\sigma}$ of the inverse of the map $\mathcal{D}^h : \mathcal{I}^r_{h,\sigma} \to \mathcal{B}^r_{h,\sigma}$ (see Lemma 5.8), that is, the map defined as

$$i_r(D) := \imath_S D \quad \text{for all } D \in \mathcal{K}^r_{h,\sigma}.$$

THEOREM 5.12. *For all $r > 0$ there exist exact sequences*

$$0 \to \mathcal{K}^{r-1}_{h,\sigma} \xrightarrow{i_{r-1}} \mathcal{B}^{r-1}_{h,\sigma} \xrightarrow{\delta_r} \mathcal{B}^r_{h,\sigma} \xrightarrow{j_r} H^1_{h,\sigma}(M, \Sigma_h, \mathbb{C}). \tag{51}$$

Proof. The map $i_{r-1} : \mathcal{K}^{r-1}_{h,\sigma} \to \mathcal{B}^{r-1}_{h,\sigma}$ is by definition injective, since the contraction operator is surjective onto the space of functions (0-forms).

The identity $\mathrm{Im}(i_{r-1}) = \ker(\delta_r)$ holds since by Lemma 5.8 a current $C \in \mathcal{B}^{r-1}_{h,\sigma}$ if and only if $C = \imath_S D$ with $D \in \mathcal{I}^{r-1}_{h,\sigma}$ and in addition

$$\delta_r(\imath_S D) = d_{h,\sigma}(\imath_T \imath_S D) = \imath_T(S + \imath\sigma)D - \imath_S(TD) = -\imath_S(TD),$$

hence $\delta_r(\imath_S D) = 0$ if and only if $TD = 0$ (since TD has degree 2 and the contraction is surjective onto the space of functions (0-forms).

The identity $\mathrm{Im}(\delta_r) = \ker(j_r)$ holds by the following argument. Let $C' \in \mathcal{B}^r_{h,\sigma}$ be a current such that $[C'] = 0 \in H^1_{h,\sigma}(M, \Sigma_h, \mathbb{C})$, hence there exists a current U of degree 0 (and dimension 2) such that $C' = d_{h,\sigma}U$. Let $C = U \wedge \mathrm{Im}(h)$. By definition we have $C' = \delta_r(C)$. We claim that $C = U \wedge \mathrm{Im}(h) \in \mathcal{B}^{r-1}_{h,\sigma}$. In fact, by definition $\imath_S(U \wedge \mathrm{Im}(h)) = 0$, and since $\imath_S C' = 0$, we have

$$\begin{aligned}(\mathcal{L}_S + \imath\sigma)(U \wedge \mathrm{Im}(h)) &= (\mathcal{L}_S + \imath\sigma)(U) \wedge \mathrm{Im}(h) \\ &= \imath_S d_{h,\sigma} U \wedge \mathrm{Im}(h) = \imath_S C' \wedge \mathrm{Im}(h) = 0.\end{aligned}$$

The argument is thus complete. □

The above theorem and Lemma 5.8 imply the following finiteness result.

COROLLARY 5.13. *For any abelian differential h on M, for all $\sigma \in \mathbb{R}$ and for all $s \geq 0$, the spaces $\mathcal{I}^s_{h,\sigma}$ of twisted invariant distributions for the operator $S + \imath\sigma$ and the corresponding space of $\mathcal{B}^s_{h,\sigma}$ of twisted basic currents have finite dimension.*

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We conclude this subsection by proving a lower bound on the dimensions of the spaces of twisted invariant distributions.

COROLLARY 5.14. *Let h be an abelian differential with minimal vertical foliation. For all $\theta \in \mathbb{T}$, let $h_\theta := e^{-\imath\theta} h$ be the rotated abelian differential and let $\sigma_\theta := \sigma \cos\theta$. For any $r > 2$ and for almost all $\theta \in \mathbb{T}$, the subspace $j_r(\mathcal{B}^r_{h_\theta,\sigma_\theta}) \cap H^1_{h_\theta,\sigma_\theta}(M, \mathbb{C})$ has codimension at most equal to 1 in $H^1_{h_\theta,\sigma_\theta}(M, \mathbb{C})$.*

Proof. Let α be any twisted closed 1-form, that is, a 1-form such that

$$d_{h_\theta,\sigma_\theta}\alpha := d\alpha + \imath\sigma_\theta \mathrm{Re}(h_\theta) \wedge \alpha = 0.$$

Let $\alpha := f\mathrm{Re}(h_\theta) + g\mathrm{Im}(h_\theta)$ and assume that $f \in \bar{H}^{r-1}_h(M)$ with $r > 2$ and that f is orthogonal to constant functions. Then by Theorem 5.4 it follows that the cohomological equation

$$(S_\theta + \imath\sigma_\theta)u = f$$

has a distributional solution $u \in \bar{H}^{-r}_h(M)$ for almost all $\theta \in \mathbb{T}$. Let C denote the current of degree 1 (and dimension 1) uniquely determined by the formula

$$d_{h_\theta,\sigma_\theta} u := (d + \imath\sigma_\theta \mathrm{Re}(h_\theta))u = \alpha + C.$$

It is clear from the definition that C is closed with respect to the twisted differential $d_{h_\theta,\sigma_\theta}$, that is, $d_{h_\theta,\sigma_\theta} C = 0$, and in addition $\imath_{S_\theta} C = 0$, hence, by Lemma 5.10, the current C is a twisted basic current $d_{h_\theta,\sigma_\theta}$-cohomologous to the 1-form α.

Finally, it can be proved that for all $\sigma \in \mathbb{R}$ all cohomology classes in $H^1_{h,\sigma}(M, \mathbb{C})$ can be represented by $d_{h,\sigma}$-closed 1-forms $\alpha \in W^r_h(M)$ with $r > 1$. □

5.3. *Smooth solutions.* In this section we prove our main result on the existence of smooth solutions of the twisted cohomological equation for translation flows, which holds for *any* abelian differential in almost all directions, and derive as a corollary our result on cohomological equations for product translation flows.

LEMMA 5.15. *Let h be an abelian differential with minimal vertical foliation. For every $s > r \geq 0$ such that $s - r > 3$ there exists $p \in (0, 1)$ such that for every $\sigma \in \mathbb{R}$ there exists a function $A_{h,\sigma} := A_{h,\sigma}(p, r, s) \in L^p(\mathbb{T}, \mathcal{L})$ such that the following statement holds. For almost all $\theta \in \mathbb{T}$ and for all zero-average functions $v \in H^{s+1}_h(M)$, we have*

$$|v|_r \leq A_{h,\sigma}(\theta)\, |(S_\theta + \imath\sigma_\theta)v|_s, \tag{52}$$

and there exists a constant $B_h := B_h(p, r, s) > 0$ such that, for all $\sigma \in \mathbb{R}$, we have

$$|A_{h,\sigma}|_p \leq B_h.$$

Proof. Let $\{\lambda_k \mid k \in \mathbb{N}\}$ denote the sequence of eigenvalues of the Friedrichs extension $-\Delta^F_h$ of the non-negative flat Laplacian, relative to the orthonormal Fourier basis $\mathcal{E} = \{e_k \mid k \in \mathbb{N}\}$ of eigenfunctions of the Hilbert space $L^2_h(M)$, described in §2. We recall that the

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Friedrichs weighted Sobolev norms are given, for all $s > 0$, as follows (see Definition 2.5):

$$\|u\|_s = \bigg(\sum_{k\in\mathbb{N}}(1+\lambda_k)^s|\langle u, e_k\rangle|^2\bigg)^{1/2} \quad \text{for all } u \in \bar{H}^s_h(M).$$

Let $\alpha > 2$ and let $p \in (0, 1)$ be such that $\alpha p > 2$. By Lemma 5.3, for all $k \in \mathbb{N}\setminus\{0\}$ there exists a function $A^{(k)}_{h,\sigma} := A^{(k)}_{h,\sigma}(p, \alpha) \in L^p(\mathbb{T}, \mathcal{L})$ such that, for all $v \in H^{\alpha+1}_h(M)$ of zero average, we have

$$|\langle v, e_k\rangle| \le A^{(k)}_{h,\sigma}(\theta)\|(S_\theta + \imath\sigma_\theta)v\|_\alpha.$$

In addition, there exists a constant $B_h := B_h(p, \alpha)$ such that

$$|A^{(k)}_{h,\sigma}|_p \le B_h\|e_k\|_{-1} = B_h(1+\lambda_k)^{-1/2}.$$

Let $\beta > 1$ such that $(\beta+1)p > 2$. It follows that, for any $v \in H^{\alpha+1}_h(M)$ of zero average, we have

$$\|v\|_{-\beta} \le \bigg(\bigg(\sum_{k\in\mathbb{N}\setminus\{0\}}(1+\lambda_k)^{-\beta}A^{(k)}_{h,\sigma}(\theta)^2\bigg)\bigg)^{1/2}\|(S_\theta + \imath\sigma_\theta)v\|_\alpha.$$

Let then $A_{h,\sigma} := A_{h,\sigma}(p, \alpha, \beta)$ denote the function defined, for $\theta \in \mathbb{T}$, as follows:

$$A_{h,\sigma}(\theta) := \bigg(\bigg(\sum_{k\in\mathbb{N}\setminus\{0\}}(1+\lambda_k)^{-\beta}A^{(k)}_{h,\sigma}(\theta)^2\bigg)\bigg)^{1/2}.$$

By the triangular inequality for the space $L^{p/2}$ (with $p/2 < 1$) we have

$$\begin{aligned}|A_{h,\sigma}|^p_p &= \bigg|\sum_{k\in\mathbb{N}\setminus\{0\}}(1+\lambda_k)^{-\beta}(A^{(k)}_{h,\sigma})^2\bigg|^{p/2}_{p/2}\\ &\le \sum_{k\in\mathbb{N}\setminus\{0\}}(1+\lambda_k)^{-\beta p/2}|A^{(k)}_{h,\sigma}|^p_p \le B^p_h \sum_{k\in\mathbb{N}\setminus\{0\}}(1+\lambda_k)^{-(\beta+1)p/2}.\end{aligned}$$

By the Weyl asymptotics, the series on the right-hand side of the above formula is convergent as soon as $(\beta+1)p > 2$. Let then $v \in H^{\alpha+3}_h(M)$ so that $\Delta_h v \in H^\alpha_h(M)$ and we have

$$\begin{aligned}\|v\|_{-\beta+2} &= \|(I - \Delta^F_h)v\|_{-\beta} \le A_{h,\sigma}(\theta)\|(S_\theta + \imath\sigma_\theta)(I - \Delta^F_h)v\|_\alpha\\ &= A_{h,\sigma}(\theta)\|(I - \Delta_h)(S_\theta + \imath\sigma_\theta)v\|_\alpha = A_{h,\sigma}(\theta)\|(S_\theta + \imath\sigma_\theta)v\|_{\alpha+2}.\end{aligned}$$

By the interpolation inequality for the Friedrichs norms and by Lemma 2.8, for every $\rho \in [0, 1)$, whenever $\alpha p > 2$, $(\beta+1)p > 2$ and $\rho + \beta \le 2$, we have

$$|v|_\rho = \|v\|_\rho \le A_{h,\sigma}(\theta)\|(S_\theta + \imath\sigma_\theta)v\|_{\alpha+\beta+\rho} \le A_{h,\sigma}(\theta)|(S_\theta + \imath\sigma_\theta)v|_{\alpha+\beta+\rho}.$$

Finally, for all $r \ge 0$, by applying the above bound to all functions S^iT^jv and T^iS^j, for all $i, j \le [r]$, we finally have that there exists a constant $C_r > 0$ such that

$$|v|_r \le C_r A_{h,\sigma}(\theta)|(S_\theta + \imath\sigma_\theta)v|_{\alpha+\beta+r}. \tag{53}$$

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Since, given $s > r \geq 0$ with $s - r > 3$, it is always possible to find $\alpha > 2$, $\beta > 1$ and $p \in (0, 1)$ such that

$$s = \alpha + \beta, \quad \alpha p > 2, \quad (\beta + 1)p > 1 \quad \text{and} \quad \{r\} + \beta \leq 2,$$

the bound in formula (52) follows immediately from that in the above formula (53), hence the argument is complete. □

THEOREM 5.16. *Let h be an abelian differential with minimal vertical foliation. For any $s > r \geq 0$ such that $s - r > 3$, there exist $p \in (0, 1)$ and a constant $C_{r,s} > 0$ such that the following statement holds. For any $\sigma \in \mathbb{R}$ and for almost all $\theta \in \mathbb{T}$, for any $f \in H^s_h(M)$ of zero average such that $\mathcal{D}(f) = 0$ for all twisted invariant distributions $\mathcal{D} \in \mathcal{I}^s_{h_\theta,\sigma_\theta}$, the cohomological equation $(S_\theta + \imath\sigma_\theta)u = f$ has a zero-average solution $\mathcal{U}_\theta(f) \in H^r_h(M)$ satisfying the following estimate:*

$$\left(\int_{\mathbb{T}} |\mathcal{U}_\theta(f)|^p_r \, d\theta\right)^{1/p} \leq C_{r,s} \, |f|_s. \tag{54}$$

Proof. It follows from the a priori bound of Lemma 5.15 that, for all $\sigma \in \mathbb{R}$ and for almost all $\theta \in \mathbb{T}$, the subspace

$$\{f \in \mathcal{H}^s_h(M) \mid f \in (S_\theta + \imath\sigma_\theta)[\mathcal{H}^r_h(M)]\}$$

is closed in $\mathcal{H}^s_h(M)$, hence it coincides with the kernel of the subspace $\mathcal{I}^{-s}_{h_\theta,\sigma_\theta} \cap \mathcal{H}^{-s}(M)$ of all twisted invariant distributions vanishing on constant functions. In addition, it follows by continuity that there exist $p \in (0, 1)$ and a function $A_{h,\sigma} \in L^p(\mathbb{T}, \mathcal{L})$ such that, for all $f \in \mathcal{H}^s_h(M) \cap \text{Ker}(\mathcal{I}^{-s}_{h_\theta,\sigma_\theta})$, the unique zero-average solution $\mathcal{U}_\theta(f) \in H^r_h(M)$ of the cohomological equation $(S_\theta + \imath\sigma_\theta)u = f$ satisfies the bound

$$|\mathcal{U}_\theta(f)|_r \leq A_{h,\sigma}(\theta)|f|_s.$$

From the above inequality and the bounds on the L^p norm of the function $A_{h,\sigma}$ established in Lemma 5.15, it follows immediately that

$$\left(\int_{\mathbb{T}} |\mathcal{U}_\theta(f)|^p_r \, d\theta\right)^{1/p} \leq |A_{h,\sigma}|_p \, |f|_s \leq B_h \, |f|_s.$$

The proof of the theorem is therefore complete. □

Proof of Theorem 1.1. The condition that h has a minimal vertical foliation it is not restrictive since the statement is rotation-invariant, and any abelian differential has a minimal direction [**Ma**, **AG**].

The finiteness of the dimension of the space $\mathcal{I}^s_{h,\sigma} \subset H^{-s}_h(M)$ of $(S + \imath\sigma)$-invariant distributions has been proved in Corollary 5.13 (for all $\sigma \in \mathbb{R}$ and all $\theta \in \mathbb{T}$). A linear upper bound on the dimension of the space $\mathcal{I}^s_{h,\sigma}$ follows from Theorem 5.12, and a linear lower bound on the dimension of the space $\mathcal{I}^s_{h_\theta,\sigma_\theta}$, for all $\sigma \in \mathbb{R}$ and for almost all $\theta \in \mathbb{T}$, follows from Theorem 5.12 and Corollary 5.14.

If the function $f \in H^s_h(M)$ is constant, then for $\sigma \neq 0$ the constant function $u = -\imath f/\sigma$ is a solution (which is unique in $L^2_h(M)$) for almost all $\theta \in \mathbb{T}$. For $\sigma = 0$, there is no

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solution unless $f = 0$, in which case the solution is the zero constant. The argument is therefore reduced to the case of functions of zero average.

By Theorem 5.16, for any abelian differential h with minimal vertical foliation, the twisted cohomological equation $(S_\theta + \imath\sigma \cos\theta)u = f$ can be solved with Sobolev bounds for all $f \in H^s_h(M)$ of zero average in the kernel of all twisted invariant distributions, *for all* $\sigma \in \mathbb{R}$ *and for almost all* $\theta \in \mathbb{T}$. Let then $\mathcal{F} \subset \mathbb{T} \times \mathbb{R}$ be the set of $(\theta, \sigma) \in \mathbb{T} \times \mathbb{R}$ such that the twisted cohomological equation $(S_\theta + \imath\sigma)u = f$ can be solved with Sobolev bounds for all $f \in H^s_h(M)$ of zero average in the kernel of all twisted invariant distributions. Since the map $(\theta, \sigma) \to (\theta, \sigma \cos\theta)$ from $\mathbb{T} \times \mathbb{R}$ into itself is absolutely continuous, it follows from Theorem 5.16 that the set $\mathcal{F}$ has full Lebesgue measure. Finally, the statement of the theorem follows by Fubini's theorem. □

For all $s, \nu \in \mathbb{N}$, let $H^{s,\nu}_h(M \times \mathbb{T})$ denote the L^2 Sobolev space on $M \times \mathbb{T}$ with respect to the invariant volume form $\omega_h \wedge d\phi$ and the vector fields S, T, and $\partial/\partial\phi$:

$$H^{s,\nu}_h(M \times \mathbb{T}) := \left\{ f \in L^2(M \times \mathbb{T}, d\text{vol}) \;\middle|\; \sum_{i+j\le s} \sum_{\ell \le \nu} \left\| S^i T^j \frac{\partial^\ell f}{\partial \phi^\ell} \right\|_0 < +\infty \right\};$$

the space $H^{s,\nu}_h(M \times \mathbb{T})$ can be defined for all $(s, \nu) \in \mathbb{R}^+ \times \mathbb{R}^+$ by interpolation [**LM**] and the space $H^{-s,-\nu}_h(M \times \mathbb{T})$ is defined as the dual of the space $H^{s,\nu}_h(M \times \mathbb{T})$.

Now let $V_{\theta,c} = S_\theta + c \cos\theta(\partial/\partial\phi)$ denote a translation vector field on $M \times \mathbb{T}$, and let $\mathcal{I}^{s,\nu}_{h_\theta,c}$ denote the space of $V_{\theta,c}$ invariant distributions.

The space $L^2(M \times \mathbb{T}, d\text{vol})$ of the product manifold with respect to the invariant volume form $\omega_h \wedge d\phi$ decomposes as a direct sum of the eigenspaces $\{H^0_n | n \in \mathbb{Z}\}$ of the circle action:

$$L^2(M \times \mathbb{T}, d\text{vol}) = \bigoplus_{n\in\mathbb{Z}} H^0_n.$$

COROLLARY 5.17. *Let h be an abelian differential with minimal vertical foliation. Let $s > r \ge 0$ be such that $s - r > 3$ and let $\nu > 2$ and $\mu < \nu - 2$. For all $c \in \mathbb{R}$ and for almost all $\theta \in \mathbb{T}$ there exists a constant $C^{(\mu,\nu)}_{r,s}(\theta, c) > 0$ such that the following statement holds. For any $f \in H^{s,\nu}_h(M \times \mathbb{T})$ such that $D(f) = 0$ for all $V_{\theta,c}$-invariant distributions $D \in \mathcal{I}^{s,\nu}_{h_\theta,c} \subset H^{-s,-\nu}_h(M \times \mathbb{T})$, the cohomological equation $V_{\theta,c}u = f$ has a solution $u := \mathcal{U}_\theta(f) \in H^{r,\mu}_h(M)$ satisfying the following estimate:*

$$|\mathcal{U}_\theta(f)|_{r,\mu} \le C^{(\mu,\nu)}_{r,s}(\theta, c)\, |f|_{s,\nu}. \tag{55}$$

Proof. By the Fourier decomposition with respect to the circle action, the argument is reduced to proving the existence of solutions for the cohomological equations

$$(S_\theta + 2\pi\imath cn \cos\theta)u_n = f_n. \tag{56}$$

For $n = 0$ the above equation reduces to the cohomological equation for the translation flow on M, so that the result already follows from [**F07**]. For every $n \in \mathbb{N} \setminus \{0\}$, let $\sigma^{c,n} := 2\pi cn \in \mathbb{R}$. The (finite-dimensional) space $\mathcal{I}^s_{h_\theta,\sigma^{c,n}_\theta} \subset H^{-s}_h(M)$ of twisted $(S_\theta + \imath\sigma^{c,n}_\theta)$-invariant distributions embeds as a subspace of the space $\mathcal{I}^{s,\nu}_{h_\theta,c} \subset H^{-s,-\nu}_h(M \times \mathbb{T})$

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of $V_{\theta,c}$-invariant distributions, for all $\nu \in \mathbb{N}$, by the formula

$$D\bigg(\sum_{n\in\mathbb{Z}} f_n e^{2\pi\imath\phi}\bigg) = D(f_n).$$

By Theorem 5.16 there exist constants $C_{r,s} > 0$ and $p \in (0, 1)$ and, for every $\epsilon > 0$, there exists a full measure set $\mathcal{F}_{c,n}(\epsilon) \subset \mathbb{T}$ of measure at least $1 - \epsilon/n^2$, such that for all $\theta \in \mathcal{F}_{c,n}(\epsilon)$, for every $f_n \in \mathcal{I}^s_{h_\theta,\sigma_\theta^{c,n}}$ there exists a solution $u_n \in H^r_h(M) \cap H^0_n$ of the cohomological equation (56) which satisfies the Sobolev estimate

$$|u_n|_r \le C_{r,s}\epsilon^{-1/p} n^{2/p} |f_n|_s.$$

In fact, the above claim follows immediately from Theorem 5.16.

From the claim it follows that for all functions $f \in H^{s,\nu}_h(M \times \mathbb{T})$ with $\nu > 2/p$, such that $f_n \in \mathcal{I}^{s,\nu}_{h_\theta,\sigma_\theta^{c,n}}$ for all $n \neq 0$, for all $\theta \in \mathcal{F}_c(\epsilon) := \bigcap_{n\neq 0} \mathcal{F}_{c,n}(\epsilon)$ the function $u = \sum_{n\neq 0} u_n \in H^{r,\nu-2/p}(M)$ is a solution of the cohomological equation $V_{\theta,c}u = f$. Since, for any $\epsilon > 0$, the set $\mathcal{F}_c(\epsilon)$ has Lebesgue measure at least $1 - C\epsilon$ with $C = \sum_{n\neq 0} 1/n^2$, the argument is complete. □

Proof of Theorem 1.4. The space $\mathcal{I}^{s,\nu}_{h_\theta,c}$ of $V_{\theta,c}$-invariant distributions is generated by the union of subspaces $\mathcal{I}^s_{h_\theta,\sigma_\theta^{c,n}}$ over all $n \in \mathbb{Z}$. The statement of the theorem then follows from Corollary 5.17 by Fubini's theorem. □

Proof of Corollary 1.5. For any $\phi_0 \in \mathbb{T}$, let $M_{\phi_0} = M \times \{\phi_0\} \subset M \times \mathbb{T}$. The return map of the flow of the vector field $X_{\theta,c}$ to the transverse surface $M_{\phi_0} \equiv M$ is smoothly conjugate to the time-$1/c$ map $\Phi^{1/c}_{\theta,c}$ of the translation flow generated by the horizontal vector field S_θ on M. Since the return time function is constant (equal to 1), it is possible to derive results on the cohomological equation for the return (Poincaré) map (the time-$1/c$ map) from results on the cohomological equation for the flow. In fact, the procedure is as follows. Let $\Phi^{\mathbb{R}}_{\theta,c}$ denote the flow of the vector field $X_{\theta,c}$ on $M \times \mathbb{T}$. Let $\chi \in C^\infty(\mathbb{T})$ be a smooth function with integral equal to 1 supported on a closed interval $I \subset \mathbb{T} \setminus \{\phi_0\}$. Let $F(f) \in H^{s,\infty}_h(M \times \mathbb{T})$ be the function defined as follows:

$$F(f) \circ \Phi^t_{\theta,c}(x, \phi_0) = \begin{cases} f(x)\chi(t) & \text{for } t \in I, \\ 0 & \text{for } t \notin I. \end{cases}$$

Let $f \in H^s_h(M)$ and let us assume that the cohomological equation $X_{\theta,c}U = F(f)$ has a solution $U \in H^{r,\mu}_h(M \times \mathbb{T})$. Then the restriction $u = U|M_{\phi_0}$ is a solution of the cohomological equation $u \circ \Phi^{1/c}_{\theta,c} - u = f$ for the time-$1/c$ map. In fact, for all $x \in M$, we have

$$u \circ \Phi^{1/c}_{\theta,c} - u = \int_0^{1/c} X_{\theta,c}U \circ \Phi^t_{\theta,c}\, dt = \int_0^{1/c} f\chi(\phi_0 + ct)\, dt = f(x)$$

By the Sobolev trace theorem, for any $\mu > 1/2$, the restriction $U|M_{\phi_0}$ of a function $U \in H^{r,\mu}_h(M \times \mathbb{T})$ is a function $u \in H^r_h(M)$, and there exists $C_\mu > 0$ such that

$$|u|_r \le C_\mu \|U\|_{H^{r,\mu}_h(M\times\mathbb{T})}.$$

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The result then follows from Theorem 1.4. In fact, for every $X_{\theta,c}$-invariant distribution $D \in \mathcal{I}^{s,\nu}_{h_\theta,c} \subset H_h^{-s,-\nu}(M \times \mathbb{T})$, we define the distribution $D_M \in H_h^{-s}(M)$ as

$$D_M(f) := D(F(f)).$$

By Theorem 1.4 it follows that, for $D_M(f) = 0$ for all $D \in \mathcal{I}^{s,\nu}_{h_\theta,c}$, there exists a solution $U \in H_h^{r,\mu}(M \times \mathbb{T})$ of the cohomological equation $X_{\theta,c} U = F(f)$, hence a solution $u = U|M \in H_h^r(M)$ of the equation $u \circ \Phi_\theta^{1/c} - u = f$. Finally, for all $u \in H_h^\infty(M)$ such that $u \circ \Phi_\theta^{1/c} - u \in H_h^s(M)$, we have

$$D_M(u \circ \Phi_{\theta,c}^{1/c} - u) = D(F(u \circ \Phi_{\theta,c}^{1/c} - u)) = D(F(u) \circ \Phi_{\theta,c}^{1/c} - F(u)) = 0,$$

since $D \in \mathcal{I}^{s,\nu}_{h_\theta,c}$ is by assumption $X_{\theta,c}$-invariant. □

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