Elementary Considerations relating to the Absolute.

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Non-Euclidean geometry in the narrowest sense is that system of geometry which is usually associated with the names of Lobachevskij[1] and Bolyai[2], and which arose from the substitution for Euclid's parallel-postulate of a postulate admitting an infinity of lines through a fixed point not intersecting a given line, the two limits between the intersectors and the non-intersectors being called the parallels to the given line through the fixed point. In a wider sense, any system of geometry which denies one or more of the fundamental assumptions upon which Euclid's system is based is a non-euclidean geometry. Of special interest are, however, those which touch only the question of parallel lines; and there exists, in addition to Lobachevskij's geometry, another, commonly associated with the name of Riemann[3], in which the parallels to any line through a fixed point are imaginary. The three geometries, Lobachevskij's, Euclid's, and Riemann's, thus form a trio characterised by the existence of real, coincident, or imaginary pairs of parallels through a given point to a given line. With reference to this criterion, a consistent nomenclature was introduced by Klein, who called these three geometries respectively Hyperbolic, Parabolic, and Elliptic.

Various concrete representations of these systems have been given within the domain of ordinary geometry. It is proposed to explain briefly that one which was devised by Cayley[5] and elaborated by Klein[6], and which deals primarily with the analytic expression for the distance between two elements, points or lines, and the representation of the infinitely distant elements.

In ordinary plane geometry[9] the metrical properties of figures are referred to a special line, the line infinity, u, and two special (imaginary) points on this line, the circular points at infinity, ω, ω'.

The line infinity appears in point-coordinates as an equation of the first degree, $u=0$, while every finite point satisfies the identity $u \equiv$ const. In trilinear coordinates, for example, if a, b, c are the sides of the triangle of reference, $\triangle$, $u \equiv ax+by+cz \equiv 2\triangle$.

The circular points appear in line-coordinates as an equation of the second degree, $\omega\omega' = 0$, while every ordinary line satisfies the identity $\omega\omega' \equiv$ const. In trilinear coordinates

$$\omega\omega' \equiv \xi^2 + \eta^2 + \zeta^2 - 2\eta\zeta\cos A - 2\zeta\xi\cos B - 2\xi\eta\cos C$$
$$\equiv (\xi - \eta e^{iC} - \zeta e^{-iB})(\xi - \eta e^{-iC} - \zeta e^{iB}) \equiv 4\Delta^2.$$

In rectangular cartesian coordinates, made homogeneous by the introduction of a third variable z, the equation of the line infinity is $z = 0$, while for finite points $z = 1$. The line-coordinates of the line $lx + my + nz = 0$ are l, m, n, and in general $l^2 + m^2 =$ const., $= 1$, when the equation is in the "perpendicular" form. But for the line infinity $l = 0$ and $m = 0$ so that $l^2 + m^2 = 0$, and this is true also for any line $y = \pm ix + b$, *i.e.* for any line passing through one or other of the points of intersection of the line $z = 0$ with the locus $x^2 + y^2 = 0$.

Now an equation of the second degree in point-coordinates or in line-coordinates represents a conic. But the equation $l^2 + m^2 = 0$ represents a degenerate conic consisting of two (imaginary) pencils of lines, since $l^2 + m^2$ decomposes into linear factors. Similarly $z = 0$ as a point-equation, when written $z^2 = 0$, represents a degenerate conic consisting of two coincident straight lines. These conics are just one conic considered from the two different points of view of a locus and of an envelop, for the reciprocal of the equation $l^2 + m^2 = cn^2$ is $c(x^2 + y^2) = z^2$. When $c = 0$ the point-equation represents a circle of infinite radius $z^2 = 0$, and the line-equation represents the two pencils of lines passing through the two points through which all circles pass. This degenerate conic is called the *Absolute*.

If we now replace the degenerate conic by a proper conic, we get a more general form of geometry which includes ordinary Euclidean geometry as a special case. It also includes as special cases the geometries of Lobachevskij and Riemann, the former when the conic is real, the latter when it is imaginary. There are obviously other cases—for example, when the conic degenerates to two distinct lines—and there will be corresponding systems of geometry. Most of these geometries are very bizarre. In one, for example, the perimeter of any triangle is constant. The only ones which at all resemble the geometry of experience are the three just mentioned.

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We have now to obtain the expressions for the distance between two points and the angle between two straight lines. As the absolute in ordinary geometry is less degenerate as an envelop than as a locus (the equation in line-coordinates being of the second degree) it will be simpler to take first the angle between two lines.

The expression must be such as to admit of extension to the case of a proper conic. Now Laguerre[4] has shown that the angle between two straight lines can be expressed in terms of a cross-ratio. Consider two lines $y = x\tan\theta$, $y = x\tan\theta'$, passing through 0. We have also through 0 the two (isotropic) lines, $y = ix$, $y = -ix$, which pass through the circular points. The cross-ratio of the pencil formed by these four lines is

$$(uu',\ \omega\omega') = \frac{\tan\theta - i}{\tan\theta' - i} \div \frac{\tan\theta + i}{\tan\theta' + i}$$

$$= \frac{e^{i\theta}}{e^{i\theta'}} \div \frac{e^{-i\theta}}{e^{-i\theta'}} = e^{2i(\theta - \theta')}.$$

Hence $$\theta' - \theta = \tfrac{1}{2}i\log(uu',\ \omega\omega').$$

We can now extend this to the general case. Through the point of intersection L of two straight lines p, q there are two lines belonging to the absolute considered as an envelop, viz., the two tangents from L. Call these x, y. The angle (pq) is then defined to be

$$k\log(pq,\ xy)$$

where k is a constant depending upon the angular unit employed. It is usual to take $k = \frac{1}{2}i$ so that the angle between two rays which form one straight line is $\frac{1}{2}i\log 1 = \frac{1}{2}i \,.\, 2in\pi = m\pi$. This corresponds to the circular system of angular measurement, and we see that the angle between two rays is periodic with period 2π. The angle between two lines with undefined sense has, however, the period π.

An analogous definition is given for the distance between two points. On the line l joining two points P, Q there are two points belonging to the absolute considered as a locus, viz., the two points of intersection with l. Call these X, Y. The distance (PQ) is then defined to be

$$\mathrm{K}\log(\mathrm{PQ},\ \mathrm{XY})$$

where K is a constant depending upon the linear unit employed.

When the absolute is imaginary X, Y are conjugate imaginary points, and log(PQ, XY) is a pure imaginary. In order that the

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distance may be real, K must then be a pure imaginary, and, as in the case of angles, we see that distance is a periodic function with period $2\pi Ki$. By taking $K = \frac{1}{2}i$ the period becomes π, and we make linear measurement correspond with angular. This case will be seen to correspond to *spherical* geometry, but the period (the radius of the sphere being unity) is not π but 2π. This is exactly analogous to the case of two rays, or lines with defined sense. On the sphere two antipodal points define the same pencil of great circles, but with opposite sense of rotation. If we leave the sense of rotation undefined, then they determine exactly the same pencil, and must be considered identical, or together as forming a single point; just as two rays, which make an angle π, together form a single line. On the sphere two lines (great circles) determine two antipodal points or pencils of opposite rotations; two points determine two rays of opposite directions. It is convenient thus to consider antipodal points as identical, or we may conceive a geometry in which this is actually the case. This is the geometry to which the name *elliptic* is generally confined, the term *spherical* being retained for the case in which antipodal points are distinct.* In the Cayley-Klein representation spherical geometry is conveniently excluded since two lines only intersect once.

Consider next the case where the absolute is a real proper conic. This divides the plane into two distinct regions which we may call the interior and the exterior, and it is of no moment whether the conic be an ellipse, a parabola, or a hyperbola. It is convenient to picture it as an ellipse. If the points P, Q are in different regions, then (PQ, XY) is negative and log(PQ, XY) is a complex number of the form $a + (2n+1)i\pi$, or simply $a + i\pi$, to take its principal value. a is zero only when (PQ, XY) = − 1. K log(PQ, XY) also will in general be complex whatever be the value of K. Of course it is possible to choose $K = a - i\pi$, which would make the distance real, but for points in the vicinity of Q the distance (PQ) would still be complex. On the other hand, if P, Q are in the same region, (PQ, XY) is either real, when X, Y are real, or purely

* Some writers have distinguished these two geometries as single or polar elliptic and double or antipodal elliptic. The idea of elliptic geometry is due to Klein, but it was worked out independently by Newcomb[7] and by Frankland[8]. Spherical geometry, as an independent geometry not subsumed in Euclidean space of three dimensions, owes its origin to Riemann[3].

imaginary, when X, Y are conjugate imaginary points. Then by taking K either real or a pure imaginary we can make the distance between two points in the same region real when measured along a certain class of lines, purely imaginary when measured along another class: these are the lines which do or do not cut the absolute. Hence we are led to consider certain points and lines as *ideal.*

Suppose we consider points within the absolute as *actual* points. The line joining two actual points always cuts the absolute, and we must take K real. Then all points outside the absolute are ideal points, for the distance between an exterior point and an interior point is complex (or purely imaginary in the case of harmonic conjugates). If Q lies on the absolute, while P does not, (PQ, XY) is either zero or infinite and log(PQ, XY) is infinite. Hence the absolute is the assemblage of points at infinity. Two lines cutting in an actual point O make a real angle if k is a pure imaginary, since the tangents from O are conjugate imaginaries.

This then completes the representation of Hyperbolic Geometry. Actual points are represented by the points within a real proper conic. The conic itself consists of all the points at infinity, while points outside it are ideal.

If now we consider points outside the absolute as actual points there are two cases according as K is taken to be real or imaginary. In the first case the distance between two points will be imaginary if the line joining them does not cut the absolute. Such a line must therefore be considered ideal, and we get in any pencil of lines with an actual point as vertex a class of ideal lines and a class of actual lines, and these are separated by the two tangents to the absolute. As these tangents are real, k must now be taken to be real, and we get a system of angular measurement of an entirely different nature from that with which we are familiar. The period of the angle is now $2i\pi k$ which is imaginary, and complete rotation about a point becomes impossible. If the line q is a tangent to the absolute $\log(pq, xy)$ is infinite. The angle between two lines thus tends to infinity as one line is rotated. Further, if the line PQ touches the absolute $\log(\mathrm{PQ}, \mathrm{XY})=0$, *i.e.* $(\mathrm{PQ})=0$, or the distance between any two points on an absolute line is zero. This curious result can be found to hold even in ordinary geometry if we consider imaginary points. If the line PQ passes through one of the circular points, so that $y_1-y_2=i(x_1-x_2)$, then

$$\mathrm{PQ}^2=(x_1-x_2)^2+(y_1-y_2)^2=0.$$

We have now to examine if the logarithmic expression for the distance between two points holds in ordinary geometry. In this case the two absolute points X, Y on any line PQ coincide, and (PQ, XY) = 1. The distance between any two points would thus be zero if K is finite. As the distance between any two points must, however, in general be finite, it follows that we must make K infinite.

Let PY = PX + ϵ where ϵ is small.

Then

$$(PQ,\, XY) = \frac{PX}{QX} \cdot \frac{QX+\epsilon}{PX+\epsilon} = \left(1+\frac{\epsilon}{QX}\right) \div \left(1+\frac{\epsilon}{PX}\right) = 1 + \epsilon\left(\frac{1}{QX} - \frac{1}{PX}\right)$$

and $\quad (PQ) = K \log(PQ,\, XY) = K\epsilon\left(\frac{1}{QX} - \frac{1}{PX}\right).$

Choose K infinite in such a way that $K\epsilon$ is finite and $= \lambda$.

Then $\quad (PQ) = \lambda \cdot \frac{PQ}{PX \,.\, QX}.$

Now to fix λ we must choose some point E so that (PE) = 1, the unit of length. Then $\quad 1 = \lambda \cdot \frac{PE}{PX \,.\, EX}$

and $$(PQ) = \frac{PX \,.\, EX}{PE} \cdot \frac{PQ}{PX \,.\, QX} = \frac{XE}{PE} \div \frac{XQ}{PQ} = (XP,\, EQ).$$

If we take P as origin $= 0$,

$$(0Q) = (X0,\, EQ) = (0\infty,\, Q1) = \frac{0Q}{01} \div \frac{\infty Q}{\infty 1},$$

which agrees with the ordinary expression since $\frac{\infty Q}{\infty 1} = 1.$

It will be noticed that this case differs in one marked respect from the case of elliptic geometry. In that system there is a natural unit of length, which may be taken as the length of the complete straight line—the period, in fact, of linear measurement; just as in ordinary angular measurement there is a natural unit of angle, the complete revolution. In Euclidean geometry, however, the unit of length has to be chosen conventionally, the natural unit having become infinite. The same thing appears to occur in the hyperbolic case, since the period is there imaginary, but, K being imaginary, iK is real, and this forms a natural linear standard.

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It still remains for us to consider the cases in which the absolute degenerates as an envelop to two coincident points and as a locus to two straight lines which may be real, coincident or imaginary. In these cases k is seen to be infinite, and it appears as in the analogous case just considered that there is now no natural unit of angle available, as the period is infinite. A unit must be chosen conventionally.

The geometries in the case in which k is infinite or real present a somewhat bizarre appearance, and are generally on that account excluded from discussion, the objection being that complete rotation about a point is impossible, and the right angle has no real existence. Yet, if we go outside the bounds of plane geometry, such geometries will present themselves when we consider the metrical relations subsisting on certain planes, ideal or at infinity.

Let us consider the case of hyperbolic geometry of three dimensions. Here the absolute is a real, not ruled, quadric surface, say an ellipsoid, and actual points are within. Actual lines and planes are those which cut the absolute, and the geometry upon an actual plane is hyperbolic. But an ideal plane cuts the absolute in an imaginary conic, and the geometry upon such a plane is elliptic. A tangent plane to the absolute cuts the surface in two coincident points and a pair of imaginary lines. The geometry on such a plane is the reciprocal of Euclidean geometry, *i.e.* the measurement of distances is elliptic while angular measurement is parabolic. In this geometry the perimeter of a triangle is constant and equal to π, just as in Euclidean geometry the sum of the angles is constant and equal to π. Now if we make use of the theorem that the angle between two planes is equal to the distance between their poles with respect to the absolute,* we see that the geometry of a bundle of planes passing through a point on the absolute is Euclidean. The sum of the three dihedral angles of three planes whose lines of intersection are parallel is therefore always equal to π—a result which was obtained by Lobachevskij and Bolyai.

In this brief sketch an attempt only has been made to show how the generalised conception of measurement can be evolved from the ordinary conception. The results are all well known. The following list, selected from the great literature of the subject, will form a guide to anyone seeking to probe deeper into this branch of

* See the author's paper, "Classification of Geometries with Projective Metric," §4.

geometry. 1–6 are fundamental memoirs, 7 and 8 are independent accounts of elliptic geometry, 9–13 are some of the most accessible general expositions, while 14–16 are historical and bibliographical.

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[*At the request of members of the Society interested in the papers of this Session on Non-Euclidean geometry, Dr Sommerville kindly consented to write this paper primarily to serve as an introduction to his other paper on geometries with projective metric.*—EDITOR.]