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A NEW EMBEDDING SCHEME FOR GROUPS AND SOME APPLICATIONS

VIATCHESLAV N. OBRAZTSOV

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Abstract

In this paper a scheme of an 'economical' embedding of an arbitrary set of groups without involutions in an infinite group with a proper simple normal subgroup is presented. This scheme is then applied to construction of groups with new properties.

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1. Main result and its corollaries

Many properties of a group are closely connected with the structure of its subgroups. In [7] was proved a theorem on embeddability of every at most countable group A without involutions in a simple 2-generator group in which every proper subgroup is either a cyclic group or contained in a subgroup conjugate to A, and an embedding scheme of an arbitrary set of groups without involutions in a simple group G with 'well-described' lattice of subgroups was established in [8]. But for the solution of some group-theoretical problems, we need a generalization of these embedding schemes giving a group G with a proper normal subgroup.

Let $\{G_i\}_{i\in I}$, $|I| > 1$, be an arbitrary set of non-trivial groups without involutions. We denote by Ω^1 the *free amalgam* of the groups G_i, $i \in I$, that is, the set $\bigcup_{i\in I} G_i$ with $G_i \cap G_j = 1$ whenever $i \neq j$. We say that the mapping $g : \Omega^1 \to G$ is an *embedding* of Ω^1 into G if it is injective and its restriction to every G_i is a homomorphism.

Let $\Omega = \Omega^1 \setminus \{1\} = \{a_j,\ j \in J\}$. Then as in [8], a mapping $f : 2^\Omega \setminus \{\emptyset\} \to 2^\Omega$ is called *generating* on the set Ω if the following conditions hold:

(1) if $C \subseteq G_i$ for some $i \in I$ then $f(C) = \mathrm{gp}\{C\} \setminus \{1\}$;

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(2) if C is a finite subset of Ω and $C \not\subseteq G_i$ for each $i \in I$, then $f(C) = B$, where B is an arbitrary finite or countable subset of Ω such that $C \subseteq B$ and if D is a finite subset of B, then $f(D) \subseteq B$;

(3) if C is an infinite subset of Ω and $C \not\subseteq G_i$ for each $i \in I$, then $f(C) = \bigcup_{A \in T} f(A)$, where T is the set of all finite subsets of C.

For example, a generating mapping f on Ω can be defined in the following way: if $C \in 2^{\Omega} \setminus \{\emptyset\}$ and $C = \bigcup_{i \in I} C_i$, where $C_i = C \cap G_i$, $i \in I$, then $f(C) = (\bigcup_{i \in I} \mathrm{gp}\{C_i\}) \setminus \{1\}$.

We denote by $G(1)$ the free product of groups G_i, $i \in I$. A group G having the presentation

$$G = \langle G(1) \mid R = 1;\, R \in D \rangle \tag{1.1}$$

is called (*diagrammatically*) *aspherical* ((*diagrammatically*) *atoroidal*) if every diagram on the sphere (torus) over (1.1) is either non-reduced or consists entirely of 0-cells. (All necessary information about diagrams can be found in [10].)

Let $G = \mathrm{gp}\{\Omega\}$, f an arbitrary generating mapping on Ω. We say that X is a *minimal* word of a group G if it is follows from $X = Y$ in G that $|X| \leq |Y|$, where $|Z|$ denotes the length of the word Z. Let W be the set of all non-empty words over the alphabet Ω written in the *normal form*, that is, every element X in W is written in the form $X_1 \ldots X_k$, where each X_l, $1 \leq l \leq k$, is a non-trivial element of $G_{\mu(l)}$, $\mu(l) \in I$, and $\mu(l) \neq \mu(l+1)$ for $l = 1, \ldots, k-1$. Then a mapping $F : 2^W \setminus \{\emptyset\} \to 2^{\Omega}$ is defined in the following way: if $C \subseteq W$ and $C \neq \emptyset$ then let V be the set of all letters occuring in the expressions of words of C. Then we set $F(C) = f(V)$.

The main result of this paper is the following embedding scheme:

THEOREM A. *Let m be a sufficiently large odd number or $m = \infty$, $g_i : G_i \to H$ a set of arbitrary homomorphisms of groups with kernels N_i, $i \in I$, such that a system of subgroups $\{g_i(G_i)\}_{i \in I}$ generates H, let $\{N_j\}_{j \in I_1}$, $I_1 \subseteq I$, be the set of nontrivial groups of the set $\{N_i\}_{i \in I}$, Ω_1^1 the free amalgam of the groups N_j, $j \in I_1$, and let f be an arbitrary generating mapping on Ω such that $f(C) \cap \Omega_1^1 \neq \emptyset$ if $C \not\subseteq G_i$ for each $i \in I$. If $|I_1| > 1$ then the free amalgam Ω^1 of the groups G_i can be embedded in an aspherical atoroidal group $G = \mathrm{gp}\{\Omega\}$ with the following properties:*

(1) *the free amalgam Ω_1^1 is embedded in a normal simple infinite subgroup L of G such that $G/L \cong H$;*

(2) *if $X \in L$ and X is not conjugate in G to an element of one of the groups G_i, $i \in I$, then either X is equal to a power of an element Y, where Y is of infinite order and whose homomorphic image in H has even order, or X is of order dividing m (of infinite order in the case $m = \infty$);*

(3) $\mathrm{Aut}\, L \cong G$ *(and so* $\mathrm{Out}\, L \cong H$*) and if $g \in G_i \setminus \Omega_1^1$, $i \in I$, then the mapping $g : L \to g^{-1}Lg$ is a regular automorphism of L (that is, $g(a) = a$ if and only if*

$a = 1$) if and only if there is no $c \in G_i \cap \Omega_1$, where $\Omega_1 = \Omega_1^1 \setminus \{1\}$, such that $[g, c] = 1$;

(4) *every subgroup M of G is either a cyclic group or $M \cap L = 1$ and the homomorphic image of M in $H \cong G/L$ has an element of infinite order, or M is conjugate in G to an extension $G_{C,H'}$ of a group H' by a normal subgroup L_C (that is, $G_{C,H'}/L_C \cong H'$), where $H' \le H$ and if every element of L_C is a minimal word of G, then $C = F(L_C \setminus \{1\})$ or $C = \emptyset$ in the case $L_C = \{1\}$;*

(5) *$L_C \le R_C \cap L$, where $R_C = \mathrm{gp}\{C\}$, $C \in 2^{\Omega} \setminus \{\emptyset\}$ or $R_C = \{1\}$ in the case $C = \emptyset$, and if $C \not\subseteq G_i$ for each $i \in I$, then $L_C = R_C \cap L$ and $G_{C,H'} \le R_C$;*

(6) *if $C \not\subseteq G_i$ for each $i \in I$, then for each $a \in f(C) \cap \Omega_1$, $L_C = \mathrm{gp}\{bab^{-1}, b \in f(C)\}$ (in particular, $L = \mathrm{gp}\{bab^{-1}, b \in \Omega\}$, where a is an arbitrary element of Ω_1);*

(7) *if X is a minimal non-trivial word of the group G, then $X \in R_C$ if and only if $F(\{X\}) \subseteq f(C)$;*

(8) *if $\{G_j\}_{j \in J}$, $J \subseteq I$, is a set of all groups having non-trivial intersections with a subgroup R_C of G and $X \in Z^{-1}R_C Z$, where $|Z|$ is the minimal among all words in $R_C Z$ and $G_j Z$ for each $j \in J$, then $F(\{Z\}) \subseteq F(\{X\})$;*

(9) *if $C \not\subseteq G_i$ for each $i \in I$, M is a subgroup of G in which every element is a minimal word of G, then $\mathrm{gp}\{L_C, M\} \cap L = L_{C_1}$, where $C_1 = F(C \cup (M \setminus \{1\}))$;*

(10) *if $H = G_s$ for some $s \in I$ and the homomorphism $g_j : G_j \to H$ is trivial for each $j \in I \setminus \{s\}$, then G is the semidirect product of H and L.*

The first corollary of Theorem A is devoted to the groups of outer automorphisms of simple infinite groups. Matumoto [5] proved that every group is isomorphic to the outer automorphism group of some group, and a scheme of an 'economical' embedding of an arbitrary set of groups without involutions in a simple *complete* group (that is, a group with trivial centre and no outer automorphisms) was established in [9]. Now we have

THEOREM B. *Let $\{G_i\}_{i \in I}$, $|I| > 1$, be an arbitrary set of non-trivial groups without involutions, H an arbitrary (in particular, trivial) group without involutions, Ω^1 the free amalgam of the groups H and G_i, $i \in I$, and let f be an arbitrary generating mapping on $\Omega = \Omega^1 \setminus \{1\}$, m a sufficiently large odd number or $m = \infty$. Then the free amalgam Ω^1 can be embedded in an aspherical atoroidal group $G = \mathrm{gp}\{\Omega\}$ with the following properties:*

(1) *the free amalgam of the groups G_i is embedded in a simple normal infinite subgroup L of G and $G/L \cong H$;*

(2) Out *$L \cong H$ and for each $g \in H \setminus \{1\}$, g is a regular automorphism of L;*

(3) *every non-trivial subgroup of L is a cyclic group of order dividing m (an infinite cyclic group in the case $m = \infty$) or contained in a subgroup conjugate in*

G to some G_i, or conjugate in G to a subgroup $L_C = R_C \cap L$, where $C \in 2^{\Omega} \setminus 2^H$, $R_C = \mathrm{gp}\{C\}$, and $L_C = \mathrm{gp}\{bab^{-1}, b \in f(C)\}$ for each $a \in f(C) \setminus H$.

PROOF. Let $g_i : G_i \to H$ be the trivial homomorphism for each $i \in I$, $g_H : H \to H$ the natural isomorphism. Then a system $\{N_i\}_{i \in I}$ of non-trivial kernels of the homomorphisms g_H and g_i, $i \in I$, is the same as the set of the groups G_i, $i \in I$, and hence Theorem A applies to Ω^1, f and m and yields the required G.

If the condition 'a group H has no involutions' is omitted, then the situation is more complicated.

THEOREM C. *Let $\{G_i\}_{i \in I}$, $|I| > 1$, be an arbitrary set of non-trivial groups without involutions, $H = \mathrm{gp}\{h_j\}_{j \in J}$ an arbitrary (in particular, trivial) group, n_j the order of h_j in H, $j \in J$, let $\{S_j = \mathrm{gp}\{s_j\}\}_{j \in J}$ be a set of infinite cyclic groups, Ω^1 the free amalgam of the groups $\{G_i\}_{i \in I}$ and $\{S_j\}_{j \in J}$, Ω_1^1 the free amalgam of the groups $\{G_i\}_{i \in I}$ and $\{\mathrm{gp}\{s_j^{n_j}\}\}_{j \in J}$, where $\mathrm{gp}\{s_j^{n_j}\} = \{1\}$ if $n_j = \infty$, and let f be an arbitrary generating mapping on $\Omega = \Omega^1 \setminus \{1\}$ such that $f(C) \cap \Omega_1^1 \neq \emptyset$ if $C \not\subseteq S_j$ for each $j \in J$. Then the free amalgam Ω^1 can be embedded in an aspherical atoroidal group $G = \mathrm{gp}\{\Omega\}$ with the following properties:*

(1) *the free amalgam Ω_1^1 is embedded in a simple normal infinite subgroup L of G and $G/L \cong H$;*
(2) Out $L \cong H$;
(3) *every non-trivial subgroup of L is an infinite cyclic or contained in a subgroup conjugate in G to some G_i, or conjugate in G to a subgroup $L_C = R_C \cap L$, where $C \in 2^{\Omega} \setminus \{\emptyset\}$, $R_C = \mathrm{gp}\{C\}$, and $L_C = \mathrm{gp}\{bab^{-1}, b \in f(C)\}$ for each $a \in f(C) \cap \Omega_1^1$.*

PROOF. Let $g_i : G_i \to H$ be the trivial homomorphism for each $i \in I$, and for each $j \in J$, we define a homomorphism $g_j : S_j \to H$ by setting $g_j(s_j^t) = h_j^t$, $t \geq 1$. Then Theorem A applies to Ω^1, f and $m = \infty$ and yields the required G.

For countable groups, we have the following important corollary:

THEOREM D. *Let $\{G_i\}_{i \in I}$, $|I| > 1$, be an at most countable set of non-trivial finite or countable groups without involutions, H an arbitrary at most countable group, m a sufficiently large odd number or $m = \infty$. Then the free amalgam of the groups G_i can be embedded in a simple infinite group L with the following properties:*

(1) Out $L \cong H$, *and if H has no involutions then for each $g \in H \setminus \{1\}$, g is a regular automorphism of L;*

(2) *every proper subgroup of L is either an infinite cyclic group (a cyclic group of order dividing m if H has no involutions and $m < \infty$) or contained in a subgroup $\psi(G_i)$ for some $\psi \in \operatorname{Aut} L$ and $i \in I$.*

PROOF. If H has no involutions, then let Ω^1 be the free amalgam of the groups H and G_i, $i \in I$. If $H = \mathrm{gp}\{h_j\}_{j \in J}$ has involutions, then let Ω^1 be the free amalgam of the groups G_i, $i \in I$, and of infinite cyclic groups $S_j = \mathrm{gp}\{s_j\}$, $j \in J$. In any case, we define a generating mapping f on $\Omega = \Omega^1 \setminus \{1\}$ in the following way: if $C \subseteq \Omega$, $C \not\subseteq G_i$ for each $i \in I$ and $C \not\subseteq H$ (and $C \not\subseteq S_j$ for each $j \in J$ in the second case), then $f(C) = \Omega$. Then Theorem B or Theorem C applies to Ω^1, m and this mapping f and yields the group G with the required normal subgroup L.

COROLLARY. *Let H be an arbitrary at most countable group. Then for any sufficiently large prime number p or $p = \infty$, there exists a simple infinite group L all of whose proper subgroups are infinite cyclic (cyclic groups of order p if H has no involutions and $p < \infty$) such that* Out $L \cong H$, *and if H has no involutions then for each $g \in H \setminus \{1\}$, g is a reqular automorphism of L.*

PROOF. It is sufficient to take G_1 and G_2 to be cyclic groups of order p and L as the group in Theorem D for the set $\{G_1, G_2\}$ and $m = p$.

A group G is called a *K-group* if its subgroup lattice is complemented, that is, for each $A \leq G$ there exists $B \leq G$ such that $A \cap B = 1$ and $\mathrm{gp}\{A, B\} = G$. The following obvious remark will be used for proving results about K-groups: if $A, B \leq G$, $A \cap B = 1$ and $\mathrm{gp}\{A, B\} = G$, then the groups $Z^{-1}AZ$, $Z^{-1}BZ$ satisfy these conditions for each $Z \in G$.

It is easy to see that a subgroup of a K-group is not, in general, a K-group, as the following example shows: S_4 is a K-group with cyclic subgroups of order 4 which are not K-groups. Further information on subgroups of K-groups is contained in

THEOREM E. *Let m be a sufficiently large odd number or $m = \infty$, $\{G_i\}_{i \in I}$, $|I| > 1$, an arbitrary set of non-trivial groups without involutions, G_0 a cyclic group of order m. Then the free amalgam Ω^1 of the groups G_0 and G_i, $i \in I$, can be embedded in a simple infinite K-group $G = \mathrm{gp}\{\Omega\}$, where $\Omega = \Omega^1 \setminus \{1\}$, such that every proper subgroup of G is either a cyclic group of order dividing m (an infinite cyclic group in the case $m = \infty$) or conjugate to a subgroup $R_C = \mathrm{gp}\{C\}$ for some $C \in 2^\Omega \setminus \{\emptyset\}$, where if $C \cap G_0 \neq 1$ and $C \not\subseteq G_0$, then $G_0 \subseteq C$, and $b \in R_C \cap G_i$, $i \in I \cup \{0\}$, if and only if $b \in C \cap G_i$.*

PROOF. We set $H = \{1\}$ and define a generating mapping f on Ω in the following way: if $C \subseteq \Omega$, $C \not\subseteq G_0$ and $C = \bigcup_{i \in I \cup \{0\}} C_i$, where $C_i = C \cap G_i$, $i \in I \cup \{0\}$, then

$$f(C) = (G_0' \cup \bigcup_{i \in I} \mathrm{gp}\{C_i\}) \setminus \{1\},$$

where $G_0' = G_0$ in the case $C_0 \neq \emptyset$, for otherwise $G_0' = \{1\}$. It remains to prove that the group G taken as the group in Theorem A for $\{G_i\}_{i \in I \cup \{0\}}$, m and the mapping f is a K-group.

Let M be a proper subgroup of G, $\Omega_1 = \Omega \setminus G_0$ and $G_0 = \mathrm{gp}\{a\}$. We consider the following cases:

(1) if $M = R_C$ and $\Omega_1 \subseteq C$, then $G_0 \cap C = \emptyset$ (since for otherwise $M = G$) and by Theorem A, $R_C \cap a^{-1} R_{\Omega_1} a = 1$ and $\mathrm{gp}\{R_C, a^{-1} R_{\Omega_1} a\} = G$;

(2) if $M = R_C$ and there is $b \in \Omega_1 \setminus C$, then it follows from Theorem A that $R_C \cap b^{-1} a^{-1} R_{\Omega_1} ab = 1$ and $\mathrm{gp}\{R_C, b^{-1} a^{-1} R_{\Omega_1} ab\} = G$;

(3) if $M = \mathrm{gp}\{X\}$ is a cyclic group, then it is obvious that there is $Y \in G$ such that $a \in F(\{Y^{-1} X Y\})$, and by Theorem A, $M \cap Y R_{\Omega_1} Y^{-1} = 1$ and $\mathrm{gp}\{M, Y R_{\Omega_1} Y^{-1}\} = G$.

The proof of Theorem E is complete.

The following result is devoted to construction of K-groups having proper normal subgroups.

THEOREM F. *If in the statement of Theorem A the map $g_i : G_i \to H$ is an isomorphism for some $i \in I$, the homomorphism $g_j : G_j \to H$ is trivial for each $j \in I \setminus \{i\}$, H is a K-group and a generating mapping f on Ω is defined in such a way that $F(H \cup \{a\}) = \Omega$ for each $a \in \Omega_1$, then G is a K-group.*

PROOF. It follows from the statement of Theorem F that G is the semidirect product of H and L. Let M be a proper subgroup of G. Then the following cases are possible.

(1) If $M \cap L = 1$ and M_1 is the homomorphic image of M in H, then there is a subgroup M_2 of H such that $M_1 \cap M_2 = 1$ and $\mathrm{gp}\{M_1, M_2\} = H$. Hence by Theorem A, $M \cap M_2 L = 1$ and $\mathrm{gp}\{M, M_2 L\} = G$.

(2) If $M \cap L \neq 1$ and $M \cap H = M_1$, then there is a subgroup M_2 of H such that $M_1 \cap M_2 = 1$ and $\mathrm{gp}\{M_1, M_2\} = H$. Then it follows from Theorem A that $M \cap M_2 = 1$ and $\mathrm{gp}\{M, M_2\} \supseteq \mathrm{gp}\{H, M \cap L\} = G$, as required.

By Theorem E, every group without involutions is a subgroup of some simple K-group. The situation with normal subgroups of K-groups is less clear. Emaldi asked in [4, problem 11.128] whether normal subgroups of K-groups are K-groups.

COROLLARY. *There exists a K-group G containing a normal simple infinite subgroup L such that if $A, B \leq L$ and $\mathrm{gp}\{A, B\} = L$, then either $A = L$ or $B = L$.*

PROOF. Let m be a sufficiently large odd number or $m = \infty$, $\{G_i = \mathrm{gp}\{a_i\}\}_{i \geq 1}$ a set of cyclic groups of order m (of infinite cyclic groups in the case $m = \infty$), Ω_1^1

the free amalgam of the groups $\{G_j\}_{j\geq 3}$. Then Theorem E applies to the set $\{G_j\}_{j\geq 3}$ and m and yields the K-group $H = \text{gp}\{\Omega_1\}$, where $\Omega_1 = \Omega_1^1 \setminus \{1\}$, in which every proper subgroup is either a cyclic group of order dividing m or conjugate to a subgroup $R_C = \text{gp}\{C\}$ for some $C \in 2^{\Omega_1} \setminus \{\emptyset\}$, where if $C \cap G_3 \neq 1$ and $C \not\subseteq G_3$, then $G_3 \subseteq C$, and $a \in R_C \cap G_j$, $j \geq 3$, if and only if $a \in C \cap G_j$.

Let Ω^1 be the free amalgam of the groups H, G_1 and G_2. A generating mapping f on $\Omega = \Omega^1 \setminus \{1\}$ is defined in the following way: if C is a finite subset of Ω, $C \not\subseteq H$ and $C \not\subseteq G_i$, $i = 1, 2$, then k is the maximal index of letters of $\Omega' = \{a_i\}_{i\geq 1}$ occurring in the expressions of words of C (over the alphabet Ω'), then $f(C) = (\bigcup_{s\leq k} G_s) \setminus \{1\}$. Finally, if C is an infinite subset of Ω, $C \not\subseteq H$ and $C \not\subseteq G_i$, $i = 1, 2$, then $f(C) = \bigcup_{A\in T} f(A)$, where T is the set of all finite subsets of C. Then Theorem F applies to the set $\{H, G_1, G_2\}$ (with trivial homomorphisms $g_i : G_i \to H$, $i = 1, 2$) and the mapping f and yields the K-group G with the simple infinite normal subgroup L.

Let for each $k \geq 2$, Ω_k^1 be the free amalgam of the groups $\{G_i\}_{1\leq i\leq k}$. Then by Theorem A, every proper subgroup of L is either a cyclic group of order dividing m or conjugate to a subgroup S_k consisting of all minimal words T of L with $F(\{T\}) \subseteq \Omega_k^1$, $k \geq 2$.

Let A and B be proper subgroups of L. For each minimal word D of L, we denote by $M(D)$ the maximal index of letters occurring in the expression of D (over the alphabet Ω'). Assume first that $A = \text{gp}\{X\}$ and $B = \text{gp}\{Y\}$, where X and Y are minimal words in L. Then it follows from Theorem A that $\text{gp}\{A, B\} \leq S_k$, where $k = \max(M(X), M(Y), 2)$. We now consider the second case when $A = Z^{-1}S_kZ$, $B = \text{gp}\{X\}$, where Z, X are minimal words in L. Then it follows from Theorem A that $\text{gp}\{A, B\} \leq S_t$, where $t = \max(k, M(Z), M(X))$. The case when $A = \text{gp}\{X\}$, $B = Z^{-1}S_kZ$ can be considered in a similar way. Finally if $A = Z_1^{-1}S_kZ_1$, $B = Z_2^{-1}S_lZ_2$ and Z_1, Z_2 are minimal words in L, then by Theorem A, $\text{gp}\{A, B\} \leq S_t$, where $t = \max(k, l, M(Z_1), M(Z_2))$. This completes the proof of the corollary.

A group G is called *normally factorized* if for each normal subgroup A of G there is $B \leq G$ such that $A \cap B = 1$ and $AB = G$. It is obvious that every K-group is normally factorized. Moreover, these conditions coincide in some classes of groups, in particular, in the class of all soluble groups (Napolitani [6]), and in [3] it was noted that there were no examples to show that these conditions were distinct.

COROLLARY 2. *The group L in Corollary 1 provides an example of a simple* (*and hence normally factorized*) *group which is not a K-group.*

The following result is connected with a question about Frattini subgroups. The *Frattini subgroup* $\Phi(G)$ of a group G is the intersection of all the maximal subgroups of G ($\Phi(G) = G$ when G has no maximal subgroups). In [2] and [7] were constructed countable simple groups without maximal subgroups. Of course, for each such group G, $\Phi(G)$ is a simple group. In his report at the Conference on Group Theory (Trento, Italy, 1993) J. Wiegold asked about the existence of a finitely generated group G with non-trivial simple Frattini subgroup $\Phi(G)$.

THEOREM G. *Let H be an arbitrary periodic or abelian group with $d(H) = k$, $k \geq 2$, where $d(H)$ is the minimal number of generators of H, and let s be a sufficiently large odd number or $s = \infty$. Then there exists a k-generator group G such that*

(1) *G has a normal simple infinite subgroup L such that all proper subgroups of L are infinite cyclic (cyclic groups of order dividing s if H has no involutions and $s < \infty$) and $G/L \cong H$;*
(2) *every non-cyclic subgroup of G contains L;*
(3) *$\Phi(G)$ is isomorphic to an extension of the group $\Phi(H)$ by L (that is, $\Phi(G)/L \cong \Phi(H)$); in particular, if $\Phi(H) = 1$ then $\Phi(G) = L$.*

PROOF. Let $\{b_i\}_{1 \leq i \leq k}$ be an arbitrary set of generators of H, $G_i = \mathrm{gp}\{a_i\}$, $1 \leq i \leq k$, an infinite cyclic group (a cyclic group of order sn_i if H has no involutions and $s < \infty$), where n_i is the order of b_i in H, Ω^1 the free amalgam of the groups G_i. Then for each i, $1 \leq i \leq k$, we define a homomorphism $g_i : G_i \to H$ by setting $g_i(a_i^t) = b_i^t$, $t \geq 1$. A generating mapping f on $\Omega = \Omega^1 \setminus \{1\}$ is defined in the following way: if $C \subseteq \Omega$ and $C \nsubseteq G_i$ for each i, $1 \leq i \leq k$, then $f(C) = \Omega$. Hence Theorem A applies to Ω^1, $m = \infty$ (or $m = s$ if H has no involutions) and the mapping f and yields the k-generator group G satisfying assertion (1) of the theorem.

By the statement of the theorem, H is a periodic or abelian group. Then it follows from Theorem A and [10, Theorem 33.7] that every non-cyclic subgroup of G has a non-trivial intersection with L.

Let M be a non-cyclic subgroup of G. Then $M \cap L \neq 1$ and it follows from Theorem A and the definition of the mapping f that $L \leq M$.

It remains to prove that the Frattini subgroup of the group G is isomorphic to an extension of the group $\Phi(H)$ by L. It is sufficient to show that every maximal subgroup M of G is an extension of a maximal subgroup of H by the group L. But M is not cyclic, for otherwise, G is an extension of a cyclic group by L, which contradicts the hypothesis of the theorem. Then by assertion (2) of the theorem, $L \leq M$. The homomorphic image M_1 of M in H is a maximal subgroup of H, since M is a maximal subgroup of G; hence M is an extension of M_1 by L. This completes the proof of the theorem.

Another application of Theorem G was noted by H. Smith and J. Wiegold. It is devoted to the solution of the following problem of J. C. Lennox. Let π be an arbitrary set of prime numbers, G a finitely generated group such that if $M \leq G$ and G/M^G is a finite π-group, where M^G is the normal closure of M in G, then $|G : M|$ is a finite π-number. Lennox asked in [4, problem 8.32] whether the group G is nilpotent and noted that it is true for finitely generated soluble groups. A negative answer to this question follows immediately from

COROLLARY. *There is a 2-generator group G having a normal simple infinite subgroup L such that all proper subgroups of L are infinite cyclic, G/L is isomorphic to the free abelian group of rank 2 and if G/M^G is a finite group for some subgroup M of G, then M is a normal subgroup of G.*

PROOF. It is sufficient to take H to be the free abelian group of rank 2 and G as the group in Theorem G for H and $s = \infty$. Then if $M \leq G$ is such that G/M^G is a finite group, it is easy to see that M is not cyclic, and by assertion (2) of Theorem G, $L \leq M$. Now it is follows from the commutativity of $G/L \cong H$ that $M = M^G$.

A subgroup L of a group G is said to be *dual-standard* if for any subgroups X, Y of G, $\text{gp}\{X, Y\} \cap L = \text{gp}\{X \cap L, Y \cap L\}$. Dual-standard subgroups of finite groups were studied by Zappa [12], those of torsion-free locally soluble groups by Ivanov [1], and Stonehewer and Zacher [11] gave a characterization of dual-standard subgroups of non-periodic locally soluble groups. One more type of dual-standard subgroups is given by the following theorem.

THEOREM H. *Let H be an arbitrary non-trivial, at most countable, periodic group, s a sufficiently large odd number or $s = \infty$. Then there exists a group G having a normal dual-standard infinite subgroup L such that $H \cong G/L$ and all proper subgroups of L are infinite cyclic (cyclic groups of order dividing s if H has no involutions and $s < \infty$).*

PROOF. Let $\{b_i\}_{i \in I}$ be an arbitrary set of generators of H. We define groups G_i, homomorphisms g_i, $i \in I$, a set Ω and a generating mapping f on Ω as in the proof of Theorem G (if we consider the set I instead of $\{1, \dots, k\}$). Then Theorem A applies to $\{G_i\}_{i \in I}$, $m = \infty$ (or $m = s$ if H has no involutions) and the mapping f and yields the group G with the normal infinite subgroup L such that $H \cong G/L$ and all proper subgroups of L are infinite cyclic (cyclic groups of order dividing s if H has no involutions).

By the assumption of the theorem, H is a periodic group; it then follows from Theorem A that every proper subgroup of G has a non-trivial intersection with L. Let A, B be arbitrary proper subgroups of G. We consider the following cases.

(1) If $\text{gp}\{A, B\}$ is cyclic then it is not hard to show that $\text{gp}\{A, B\} \cap L = \text{gp}\{A \cap L, B \cap L\}$.

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(2) If $\mathrm{gp}\{A, B\}$ is not cyclic then $\mathrm{gp}\{A, B\} \cap L \neq 1$ and it follows from Theorem A and the definition of the mapping f that $L \leq \mathrm{gp}\{A, B\}$. On the other hand, it follows from Theorem A that $\mathrm{gp}\{A \cap L, B \cap L\}$ is not cyclic, and hence $L = \mathrm{gp}\{A \cap L, B \cap L\}$, as required.

In this paper we use the results from [9] and the geometric method of graded diagrams developed by Ol'shanskii (see [10]). Unless otherwise stated, all definitions and notation may be found in [10].

2. Construction of the group G

As in [10], we introduce the positive parameters $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \iota$, where all the parameters are arranged according to 'height': that is, the small positive value β is chosen after α, γ after β, and so on. Our proofs are based on a system of inequalities involving these parameters. The value of the parameters can be chosen in such a way that all the inequalities hold. We then use the following notation:

$$\alpha' = 1/2 + \alpha, \quad \beta' = 1 - \beta, \quad \gamma' = 1 - \gamma, \quad h = \delta^{-1}, \quad d = \eta^{-1}, n = \iota^{-1}.$$

We also use the notation introduced at the beginning of Section 1 and fix a sufficiently large odd integer n_0 such that $n = [(h+1)^{-1} n_0]$, where $[k]$ denotes the integer part of k. We set $m = n_0$ in the case $m < \infty$.

On the set W we introduce a total order such that $|X| \leq |Y|$ implies $X \leq Y$.

We may assume that I_1 is a well-ordered set, t_1 and t_2 are the minimal and the maximal elements of I_1, respectively (if such a t_2 exists), and $\Omega_1 = \Omega_2 \cup \Omega_2^{-1}$ is the union of two subsets Ω_2 and Ω_2^{-1} such that $\Omega_2 \cap \Omega_2^{-1} = \emptyset$ and $\Omega_2^{-1} = \{a^{-1}, a \in \Omega_2\}$. We also may assume that Ω_2 is a well-ordered set such that if $a \in N_i$ and $b \in N_j$, where $i < j$, then $a < b$.

By the statement of Theorem A, there is a homomorphism of the free product $G(1)$ of groups $G_i,\ i \in I$, onto H such that its restriction to every group G_i is equal to g_i. Suppose that the kernel of this homomorphism is N.

Let $D_1 = \emptyset$, and suppose, by induction, that we have defined the set of relators $D_{i-1} \subseteq N,\ i \geq 2$, and set $G(i-1) = \langle G(1) \mid R = 1; R \in D_{i-1} \rangle$.

A word X is called *free* in rank $i-1$ if X is not conjugate in rank $i-1$ to an element of Ω^1, that is, to an image in $G(i-1)$ of an element of one of the free factors G_j. A non-empty word Y is said to be *simple* in rank $i-1$ if it is free in rank $i-1$, not conjugate in rank $i-1$ (that is, in $G(i-1)$) to a power of a shorter word and not conjugate in rank $i-1$ to a power of a period of rank $k < i$.

Now let P_i denote a set of words of length i which are simple in rank $i-1$ with the property that $A, B \in P_i$ and $A \not\equiv B$ implies that A is not conjugate in rank $i-1$

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to B or B^{-1}. The words in P_i are called *periods* of rank i. We may assume (see [9, Lemma 3.1]) that if $a, b \in N_l \cap \Omega_2$, $c \in N_j \cap \Omega_2$, $d \in N_s \cap \Omega_2$, where $l < j$ (if such a j exists), $s \neq l$ and $a < b$, then the words ac, $A_1 = adbd$, $[a, c]$, $A_2 = a[a, c]^t$, $A_3 = a[a, c]^{-t}$, $A_4 = c[a, c]^{-t}$, $A_5 = c^{-1}[a, c]^{-t}$, $A_6 = [(ac)^k, c]$, $A_7 = (ac)^k A_6^t$, $A_8 = (ac)^k A_6^{-t}$ are periods of some ranks for each k, t, where $100\zeta^{-1} < k < 10^5\zeta^{-2}$, $\zeta n_0/300 \leq t \leq n_0/2$.

For each period $A \in P_i \cap N$, we fix a maximal subset Y_A such that:

(1) if $T \in Y_A$, then $1 \leq |T| < d|A|$;
(2) each double coset of the pair $\mathrm{gp}\{A\}$, $\mathrm{gp}\{A\}$ of subgroups of $G(i)$ contains at most one word in Y_A and this word is of minimal length among the words representing this double coset;
(3) if $T \in Y_A$, then $T \in N$ and $F(\{T\}) \subseteq F(\{A\})$.

We may assume (see [9, Lemma 3.1]) that if a period A of some rank is conjugate to a word $(BC^t)^\varepsilon$, where C is a period not equal to $[a, c]$ or A_6, $|\varepsilon| = 1$, $\zeta n_0/300 \leq t \leq n_0/2$ and $B \in Y_C$, then $\varepsilon = 1$.

For each period $A \in P_i \cap N$, we introduce the ordering of the set of natural numbers (or a finite segment of it) on the set Y_A such that the first element of the set Y_A belongs to Ω_1 (it follows from the statement of Theorem A that $Y_A \cap \Omega_1 \neq \emptyset$) and if $A = A_k$, $1 \leq k \leq 8$, or $A = [a, c]$ for some $a, b \in N_l \cap \Omega_2$, $c \in N_j \cap \Omega_2$, $d \in N_s \cap \Omega_2$, where $a < b$, $s \neq l$ and $l < j$, then a is the first element of the set Y_A. We denote this order by $\leq_A$.

The set of relators S_i of rank i is constructed as follows. Firstly, if $A \in P_i$, $m < \infty$ and there is a minimal positive integer k such that $A^k \in N$, then in the case that k is an odd number, we include in S_i a word of the form A^{kn_0} (a *relator of the first type*) and call a relation

$$A^{kn_0} = 1 \tag{2.1}$$

a *defining relation of the first type of rank* i.

For each period $A \in P_i \cap N$, $i \geq 3$, we now construct some relations of the second type. Let a be the minimal element of the set $F(\{A\})$. If $A = A_j$, $j \in \{2, 3\}$, for some $a \in N_l \cap \Omega_2$, $c \in N_s \cap \Omega_2$, where $l < s$, then for each k, $5 \leq k \leq 15$, we introduce the following relations:

$$c^{-1} A^n c A^{n+k} c A^{n+30+k} \cdots c A^{n+30(h-2)+k} = 1, \tag{2.2}$$

and

$$a^{-1} A^n a A^{n+k} a A^{n+30+k} \cdots a A^{n+30(h-2)+k} = 1. \tag{2.3}$$

If $A = A_j$, $j \in \{7, 8\}$, for some $a \in N_l \cap \Omega_2$, $c \in N_s \cap \Omega_2$, where $l < s$, then for each k, t, where $16 \leq k \leq 25$, $100\zeta^{-1} < t < 10^5\zeta^{-2}$, we consider the relation

$$ac A^n (ac)^t A^{n+k} (ac)^t A^{n+30+k} \cdots (ac)^t A^{n+30(h-2)+k} = 1. \tag{2.4}$$

Let $T \in Y_A$ and $T \neq a, c$ in the case $A = A_j,\ j \in \{2, 3\}$. If a is not contained in $\mathrm{gp}\{A\} \subset G(i-1)$, T is outside $\mathrm{gp}\{A\}a\ \mathrm{gp}\{A\}$, then we introduce the relation

$$aA^nTA^{n+10}TA^{n+40}\cdots TA^{n+30(h-2)+10} = 1, \tag{2.5}$$

and if T belongs to $\mathrm{gp}\{A\}a\ \mathrm{gp}\{A\}$, then it follows from [10, Lemma 25.18] that T is not contained in $\mathrm{gp}\{A\}a^{-1}\,\mathrm{gp}\{A\}$ in $G(i-1)$, and we set

$$a^{-1}A^nTA^{n+10}TA^{n+40}\cdots TA^{n+30(h-2)+10} = 1. \tag{2.6}$$

If $T \in Y_A$ and $T \neq (ac)^t$, $100\zeta^{-1} < t < 10^5\zeta^{-2}$, in the case $A = A_j,\ j \in \{7, 8\}$, then we introduce the relation

$$a^{-1}A^nTA^{n+20}TA^{n+50}\cdots TA^{n+30(h-2)+20} = 1. \tag{2.7}$$

And if $T \in Y_A$ then let T_1 be the minimal element of the set Y_A such that T_1 is not contained in neither $\mathrm{gp}\{A\} \subset G(i-1)$ nor in $\mathrm{gp}\{A\}a^{\pm1}\,\mathrm{gp}\{A\}$ and $T <_A T_1$ (if such an element T_1 exists). Then we consider the relation

$$T_1A^nTA^{n+30}TA^{n+60}\cdots TA^{n+30(h-1)} = 1. \tag{2.8}$$

Relations (2.2)–(2.8) are called *defining relations of the second type of rank* i, and their left-hand sides are called *relators of the second type of rank* i, and are included in S_i. For each $i \geq 2$, we set $D_i = D_{i-1} \cup S_i$, and the group $G(i)$ is defined by its presentation:

$$G(i) = \langle G(1) \mid R = 1;\ R \in D_i\rangle. \tag{2.9}$$

Finally, we define $G = \langle G(1) \mid R = 1;\ R \in D = \bigcup_{i\geq1} D_i\rangle$.

By a *diagram of rank* i, where $i \geq 2$, we mean a diagram over the presentation (2.9). Relators of the first type (in the case $m < \infty$) correspond, in the diagrams under considerations, to *cells of the first type* whose *contour* (that is, boundary path) is taken as one long cyclic section. But if a cell Π corresponds to a word of the form (2.2)–(2.8), then it is called a *cell of the second type*. Its contour splits into sections according to (2.2)–(2.8). Those sections of Π with labels $(A^{n+s})^{\pm1}$ are called *long sections* while the others (with labels $T^{\pm1}$, $a^{\pm1}$, $(ac)^{\pm1}$ and $T_1^{\pm1}$) are called *short sections* of the contour.

3. Auxiliary lemmas

Immediate verification shows that the above presentations of the groups $G(i)$ satisfy condition R (see [10, §§25, 34]). So we can apply to diagrams over the presentation (2.9) all the results in [10, Chapter 11].

LEMMA 1. *Let $X = Y$ in G, where Y is a minimal word of the group G. Then $F(\{Y\}) \subseteq F(\{X\})$.*

PROOF. Let Δ be a reduced circular diagram of some rank with contour $p_1 p_2$, where $\phi(p_1) \equiv X^{-1}$, $\phi(p_2) \equiv Y$. If $r(\Delta) = 0$, then we derive the conclusion of the lemma from the definition of the mapping F.

If $r(\Delta) > 0$, then by [10, Theorem 22.1], there is a γ-cell π in Δ. We proceed by induction on $|\Delta(2)|$. It follows from [10, Lemma 21.7] that there is a contiguity submap Γ of π to p_1 such that $(\pi, \Gamma, p_1) > \varepsilon$, since $\gamma' - \alpha' > \varepsilon$. Repeating the proof of [10, Theorem 22.2], we obtain that there is a long section p of a D-cell Π in Δ and a contiguity submap Γ_1 of p to p_1 such that $r(\Gamma_1) = 0$ and $(p, \Gamma_1, p_1) \geq \varepsilon$. By the definition of the relations of G, if Π is a cell of the second type and t_1, t_2 are its short and long sections, respectively, then $F(\{\phi(t_1)\}) \subseteq F(\{\phi(t_2)\})$. Therefore, excising Π from Δ together with Γ_1, we obtain a diagram Δ_1 of an equation $X_1^{-1}Y = 1$ with $|\Delta_1(2)| < |\Delta(2)|$, and $F(\{X_1\}) \subseteq F(\{X\})$. By the induction hypothesis we can assume the lemma is true for this equation. Hence $F(\{Y\}) \subseteq F(\{X_1\}) \subseteq F(\{X\})$, as required.

LEMMA 2. *Let Γ be a contiguity submap of q_1' to q_2' in a B-diagram Δ and $\phi(q_1'), \phi(q_2')$ minimal words in G, where q_1' and q_2' are sections of cells or of contours of Δ. If $\partial(q_1', \Gamma, q_2') = p_1 q_1 p_2 q_2$, then the following conditions hold:*

(1) *$F(\{\phi(p_i)\}) \subseteq F(\{\phi(q_j)\})$ for each $i, j \in \{1, 2\}$;*
(2) *$F(\{\phi(q_1)\}) = F(\{\phi(q_2)\})$.*

PROOF. We denote by E_1 and E_2 the bonds defining Γ. If E_1 and E_2 are 0-bonds, then $|p_1| = |p_2| = 0$, and we derive the conclusion of the lemma from Lemma 1.

Let π be the principal cell of E_1 and $r(\pi) = k > 0$. By definition of the bond, there are contiguity submaps Γ_1, Γ_2 of long sections t_1 and t_2 of π to q_1' and q_2' such that $(t_i, \Gamma_i, q_i') \geq \varepsilon$, $i = 1, 2$. We denote by $p_1^i q_1^i p_2^i q_2^i$ the standard decomposition of the contour $\partial\Gamma_i$, where $\Gamma_i \wedge q_i' = q_2^i, \Gamma_i \wedge t_i = q_1^i$, $i = 1, 2$. Since Γ_1 and Γ_2 have fewer D-cells than Δ, then by the induction hypothesis,

$$F(\{\phi(q_1^i)\}) = F(\{\phi(q_2^i)\}), \qquad i = 1, 2, \tag{3.1}$$

and

$$F(\{\phi(p_j^i)\}) \subseteq F(\{\phi(q_2^i)\}) \tag{3.2}$$

for each $i, j \in \{1, 2\}$. It follows from the definition of the mapping F and the relations of G that

$$F(\{\phi(\partial\pi)\}) = F(\{\phi(q_1^i)\}) \tag{3.3}$$

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for each $i \in \{1, 2\}$.

The path p_1 has the form $p_1^2 u p_2^1$, where u^{-1} is a subpath in $\partial\pi$. Then by the definition of the mapping F,

$$F(\{\phi(u)\}) \subseteq F(\{\phi(\partial\pi)\}). \tag{3.4}$$

It follows from (3.1)–(3.4) that

$$F(\{\phi(p_1)\}) \subseteq F(\{\phi(p_1^2), \phi(u), \phi(p_2^1)\}) \subseteq F(\{\phi(q_2^i)\})$$

for each $i \in \{1, 2\}$. Hence $F(\{\phi(p_1)\}) \subseteq F(\{\phi(q_i)\})$, $i = 1, 2$.

Similarly we obtain the required assertion for $F(\{\phi(p_2)\})$. Now it follows from Lemma 1 that

$$F(\{\phi(q_i)\}) \subseteq F(\{\phi(q_{3-i}), \phi(p_1), \phi(p_2)\}) \subseteq F(\{\phi(q_{3-i})\})$$

for each $i \in \{1, 2\}$. This completes the proof of the lemma.

LEMMA 3. *Let V be a minimal word in G and $V = Z^{-1}A^tZ$, where A is a period of some rank, Z is a minimal word in G, or $V = Z^{-1}a_jZ$, where $a_j \in G_i$ for some $i \in I$ and Z is of minimal length among the words representing a coset G_iZ. Then $F(\{V\}) = F(\{Z, A\})$ $(F(\{V\}) = F(\{Z, a_j\}))$.*

PROOF. Consider, for example, the first case (the other case of the lemma can be considered in the same manner).

By Lemma 1, $F(\{V\}) \subseteq F(\{Z, A\})$: hence it is necessary to show the reverse inclusion. We note that for this purpose it is sufficient to find $X \in G$ such that $V = X^{-1}A^tX$ and $F(\{V\}) \supseteq F(\{X, A\})$, since by [10, Lemma 34.9] $ZX^{-1} \in \mathrm{gp}\{A\}$, and it follows from Lemma 1 that $F(\{Z\}) \subseteq F(\{X, A\})$, hence $F(\{Z, A\}) \subseteq F(\{X, A\}) \subseteq F(\{V\})$.

Let $V = Y^{-1}V_1Y$ in the group $G(1)$, where V_1 is cyclically minimal in $G(1)$. Then there is a reduced annular diagram Δ of some rank with contours p and q, where $\phi(p) \equiv V_1$ and $\phi(q) \equiv A^{-t}$.

Repeating the proof of Lemma 1, we obtain that for each cell π in Δ, $F(\{\phi(\partial\pi)\}) \subseteq F(\{V_1\})$ and

$$F(\{A\}) \subseteq F(\{V_1\}). \tag{3.5}$$

Therefore, there exists a word L such that $V_1 = L^{-1}A^tL$ and

$$F(\{L\}) \subseteq F(\{V_1, A\}) \subseteq F(\{V_1\}). \tag{3.6}$$

We have that $V = (LY)^{-1}A'(LY) = X^{-1}A'X$, and by Lemma 1 and (3.5), (3.6),

$$F(\{X, A\}) \subseteq F(\{V_1, L, Y\}) \subseteq F(\{V\}).$$

The proof of the lemma is complete.

A reduced diagram Δ of rank i on a sphere with three holes with contours q_1^0, q_2^0, q_3^0 is called *I-diagram* if the following conditions hold:

I1. sections q_1^0 and q_2^0 have labels A^k and A^{-k}, respectively, where A is either a simple word in rank i or a period of rank $j \le i$, $100\zeta^{-1} < k$ (and $k \le n_0/2$ if A is a period of the first type);

I2. the section q_3^0 is cyclically reduced;

I3. if Γ is a contiguity submap of $q_{i_1}^0$ to $q_{i_2}^0$, where $i_1, i_2 \in \{1, 2\}$ and $i_1 \ne i_2$, then $(q_{i_1}^0, \Gamma, q_{i_2}^0) < 1/100$ and $(q_{i_2}^0, \Gamma, q_{i_1}^0) < 1/100$;

I4. if Γ is a contiguity submap of a long section p of a cell Π to q_3^0, then $(p, \Gamma, q_3^0) < \varepsilon$.

LEMMA 4. *In any I-diagram Δ, there are contiguity submaps Γ_1 and Γ_2 of q_1^0 to q_3^0 and q_2^0 to q_3^0, respectively, such that $r(\Gamma_i) = 0$ and $(q_i^0, \Gamma_i, q_3^0) > 1/10$, $i = 1, 2$.*

PROOF. We consider the following cases.

(1) Let s be a section of a cell π or a subpath of a section q_i^0, $i = 1, 2$, and Γ a contiguity submap of s to q_3^0. Then by condition I4, Γ is the 0-contiguity submap with contour $p_1 s_1 p_2 s_2$, where $|p_1| = |p_2| = 0$ and s_1, s_2 are subpaths of sections s and q_3^0, respectively. If $r(\Gamma) > 0$ then by [10, Theorem 22.1], there is a γ-cell Π in Γ. It follows from [10, Lemma 21.7] that for any contiguity submap Γ_1 of Π to s_1, the Γ_1-contiguity degree of Π to s_1 is less than α'; hence there exists a contiguity submap Γ_2 of Π to q_3^0 such that $(\Pi, \Gamma_2, q_3^0) > \varepsilon$, and we arrive at a contradiction to condition I4. Thus $r(\Gamma) = 0$.

(2) Let Γ be a contiguity submap of q_3^0 to q_3^0. Then by condition I4 and [10, Theorem 22.1], we obtain, as in case 1, that $r(\Gamma) = 0$, since $2\varepsilon < \gamma'$.

(3) We define the distinguished contiguity submaps in an I-diagram in the same way as for E-maps. The Ω-edges of the contiguity arcs of q_i^0 to $q_{i'}^0$, where $i \in \{1, 2\}$, $i' \in \{1, 2, 3\}$, for the distinguished submaps are called *outer* edges in Δ while all the other edges are called *inner*. The construction of the estimating graphs and the weight function is left unchanged. We obtain estimates for the sums H', C', D' and G' in the same way as in [10, Lemma 24.6].

Let K' be defined for an I-diagram in the same way as in [10, Lemma 23.8] for a C-map. If $q_2' = q_3^0$ then by case 1 and condition I4, $|q_2| = |q_1| < \varepsilon |q_1'|$ (notation from [10, Lemma 23.8]). Then, as in [10, Lemma 23.8], we obtain $K' \le 10\varepsilon^{2/3} M$.

Now L' can be defined as the sum L in [10, Lemma 23.12] (sections of the contour of the first kind are now replaced by q_1^0, q_2^0 and q_3^0). If $q = q_3^0$ then by case 1, $|q_2| = |q_1| < dk$ (notation from [10, Lemma 23.12]). As in that lemma, we have $L' \leq \alpha M$. Then as in [10, Lemma 24.6], immediate verification shows that

$$M < \alpha\nu(\Delta). \tag{3.7}$$

(4) Let Γ be a contiguity submap of q_3^0 to q_3^0 and $\partial(q_3^0, \Gamma, q_3^0) = p_1 s_1 p_2 s_2$. Then by case 2, $r(\Gamma) = 0$, and it follows from condition I2 that Δ consists of two annular subdiagrams Δ_1 and Δ_2 with contours $t_1 q_1$ and $t_2 q_2$, respectively, where t_1, t_2 are subpaths of q_3^0, such that Δ_1 and Δ_2 are joined in Δ by subpaths s_1, s_2 of q_3^0. Applying condition I4, [10, Theorem 22.1 and Lemma 21.7] to Δ_i, $i = 1, 2$, we obtain, as in case 1, that $r(\Delta_i) = 0$, which completes the proof of the lemma in this case.

(5) It remains to consider the case when Δ has no contiguity submaps of q_3^0 to q_3^0. It follows from [10, Lemma 25.8] that there is no contiguity submap Γ_i of q_i^0 to q_i^0, $i = 1, 2$, such that $(q_i^0, \Gamma_i, q_i^0) > 1/100$. Then by (3.7) and condition I3, there are distinguished contiguity submaps Γ_1, Γ_2 of q_1^0, q_2^0 to q_3^0, respectively, such that the sum of the weights of the contiguity arcs $s_1 = \Gamma_1 \wedge q_1^0$ and $s_2 = \Gamma_2 \wedge q_2^0$ is greater than

$$(1 - \alpha - 4/100)\nu(\Delta) > 9\nu(\Delta)/10. \tag{3.8}$$

But by condition I1 and the definition of the weight function,

$$\nu(q_1^0) = \nu(q_2^0) = \nu(\Delta)/2. \tag{3.9}$$

It follows from (3.8) and (3.9) that Γ_i exists for each $i \in \{1, 2\}$, and in the light of case 1, we have the conclusion of the lemma.

LEMMA 5. *Let A and C be periods of the group G, $V \equiv C^k$, where $100\zeta^{-1} < k$ (and $k \leq n_0/2$ if C is a period of the first type), W a word which does not commute with V in G and whose length is minimal among all words in the double coset $\mathrm{gp}\{C^k\} W \mathrm{gp}\{C^k\}$, and also let $C^k W C^{-k} W^{-1} = Z^{-1} A^l Z$, where Z is a minimal word in G (and $|l| \leq n_0/2$ if A is a period of the first type). Then $|l| \leq 100\zeta^{-1}$ and, by a simultaneous conjugation in G, we can bring $([C^k, W], C^k)$ to the form (A^l, B), where B is a minimal word in G, $|B| < d|A|$ and*

$$F(\{A\}) = F(\{C, W\}), \qquad F(\{B\}) \subseteq F(\{A\}). \tag{3.10}$$

PROOF. By [10, Lemma 25.21], it remains to prove only (3.10). It follows from Lemmas 1 and 3 that $F(\{A, Z\}) \subseteq F(\{C, W\})$; hence

$$F(\{A\}) \subseteq F(\{C, W\}). \tag{3.11}$$

Now let Δ be a reduced annular diagram (of some rank) with contours p and q such that $\phi(q) \equiv A^{-l}$, $p = p_1 p_2 p_3 p_4$, $\phi(p_1) \equiv \phi(p_3^{-1}) \equiv C^k$, $\phi(p_2) \equiv \phi(p_4^{-1}) \equiv W$. Pasting together paths p_2 and p_4^{-1}, we obtain a diagram Δ' on a sphere with three holes whose reduced form (that is, with j-pairs removed) is denoted by Δ_0. The cyclic sections p_1, p_3 and q can be assumed smooth in Δ_0 if we modify their labels in accordance with [10, Lemma 13.3].

It is obvious that Δ_0 satisfies conditions I1 and I2. Suppose that there is a contiguity submap Γ of p_{i_1} to p_{i_2}, where $i_1, i_2 \in \{1, 3\}$ and $i_1 \neq i_2$, such that $(p_{i_1}, \Gamma, p_{i_2}) \geq 1/100$. We have that $|C| = |C^{-1}|$; then by [10, Lemma 25.10], p_1 and p_3 are C-compatible in Δ_0, and using [10, Lemma 24.9], we arrive at a contradiction to the choice of the word W. Thus Δ_0 satisfies condition I3.

Now we assume that there is a long section t of a D-cell π in Δ_0 and a contiguity submap Γ of t to q such that $(t, \Gamma, q) \geq \varepsilon$. Then repeating the proof of [10, Theorem 22.2], we obtain that there is a cell π_1 and a contiguity submap Γ_1 of a long section t_1 of π_1 to q such that $r(\Gamma_1) = 0$ and $(t_1, \Gamma_1, q) \geq \varepsilon$. Excising the cell π_1 together with Γ_1 from Δ_0, we obtain a diagram Δ_1 on a sphere with three holes with cyclic sections p_1, p_3 and q_1 such that $|\Delta_1(2)| < |\Delta(2)|$. We can assume that the section q_1 is cyclically reduced, and by the definition of the relations of G, $F(\{\phi(q_1)\}) \subseteq F(\{\phi(q)\})$. Then, by repeating the same trick several times, we obtain an I-diagram Δ_r with cyclic sections p_1, p_3 and q_r such that

$$F(\{C\}) \subseteq F(\{\phi(q_r)\}) \subseteq F(\{\phi(q)\}) = F(\{A\}). \tag{3.12}$$

Moreover, the initial points of p_1 and p_3 can be joined in Δ_r by a path s of the form $s_1 s' s_3$, where s', s_1 and s_3 are subpaths of q_r, p_1 and p_3, respectively. Then by [10, Lemma 24.9], a word $\phi(s)$ is contained in $\mathrm{gp}\{C^k\} W \,\mathrm{gp}\{C^k\}$, and it follows from the choice of the word W, Lemma 1 and (3.12) that

$$F(\{W\}) \subseteq F(\{A\}). \tag{3.13}$$

It follows from (3.11)–(3.13) that $F(\{C, W\}) = F(\{A\})$, and by Lemmas 1 and 3 that $F(\{Z, A\}) \subseteq F(\{C, W\}) = F(\{A\})$. Hence

$$F(\{Z\}) \subseteq F(\{A\}). \tag{3.14}$$

But the word B is minimal in G and equal in G to the word ZC^kZ^{-1}. Then by Lemma 3, (3.12) and (3.14), $F(\{B\}) = F(\{Z, C\}) \subseteq F(\{A\})$, which completes the proof of the lemma.

LEMMA 6. *Let $R = \mathrm{gp}\{C^k, W\}$, where C is a period, $C^k \in N \setminus \{1\}$ and W is a minimal word in G such that W is not contained in* $\mathrm{gp}\{C\}$. *Then R contains a period $C_1 \in N$ such that $F(\{C_1\}) = F(\{C, W\})$ and $n|C| < |C_1|$.*

PROOF. We can assume that $C^t \in R \cap N$, where $n_0/5 < t$ (and $t \leq n_0/2$ if C is a period of the first type). By [10, Lemma 34.9], $[C^t, W] \neq 1$. It follows from Lemma 1 that we can assume W has minimal length among all words in the double coset $\mathrm{gp}\{C^t\}W\,\mathrm{gp}\{C^t\}$, and by Lemma 5 and [10, Lemma 34.7], $[C^t, W] = Z^{-1}A^f Z$, where A is a period, Z is a minimal word in G, $|f| \leq 100\zeta^{-1}$, $|B| < d|A|$ for a word B which is minimal in G and equal in G to the word ZC^tZ^{-1}, and condition (3.10) holds. Moreover, it follows from the proof of [10, Lemma 25.21] that

$$|A| > 10^{-2}\zeta^2|C^t| > \zeta^2 n_0|C|/600 \tag{3.15}$$

and

$$|Z| < 400\zeta^{-2}|A|. \tag{3.16}$$

Raising A^f to a suitable power we consider the subgroup $\mathrm{gp}\{B, A^p\}$ of the group $R_1 = ZRZ^{-1}$, where $n_0/3 \leq p \leq 2n_0/3$. Repeating the proof of [10, Lemma 27.3], we obtain that $BA^p = Z_1^{-1}C_1^{\varepsilon}Z_1$, where $|\varepsilon| = 1$, C_1 is a period of some rank such that $C_1 \in N$, Z_1 is a minimal word in G and

$$|Z_1| < 2|C_1|, \qquad n_0|A|/100 < |C_1|. \tag{3.17}$$

Now let Δ denote a reduced annular diagram for this conjugacy. Let zl and q be the contours of Δ, where $\phi(z) \equiv B$, $\phi(l) \equiv A^p$, $\phi(q^{-1}) \equiv C_1^{\varepsilon}$. Then, as in the proof of [10, Lemma 27.3], there is a contiguity submap Γ of l to q in Δ such that $(l, \Gamma, q) > \beta'$. Hence by Lemma 2, $F(\{A\}) \subseteq F(\{C_1\})$. But it follows from Lemma 3 and (3.10) that

$$F(\{Z_1, C_1\}) \subseteq F(\{B, A\}) \subseteq F(\{A\}).$$

Thus

$$F(\{C_1\}) = F(\{A\}),\ F(\{Z_1\}) \subseteq F(\{A\}). \tag{3.18}$$

We consider the subgroup $\mathrm{gp}\{C_1, Z_2\}$ of the group $R_2 = Z_1R_1Z_1^{-1} = (Z_1Z)R(Z_1Z)^{-1}$, where Z_2 is a minimal word in G which is equal in G to the word $Z_1BZ_1^{-1}$. It follows from the proof of [10, Lemma 27.3] that $|Z_2| < 3|C_1|$, and by Lemma 1 and (3.10), (3.18),

$$F(\{Z_2\}) \subseteq F(\{Z_1, B\}) \subseteq F(\{C_1\}). \tag{3.19}$$

It follows from Lemma 1, (3.10) and (3.16)–(3.19) that there are $Z_i' \in Y_{C_1}$, $i \in \{1, 2\}$, and $Z' \in Y_{C_1}$ such that $Z_i \in \mathrm{gp}\{C_1\}Z_i'\,\mathrm{gp}\{C_1\}$, $i \in \{1, 2\}$, and $Z \in \mathrm{gp}\{C_1\}Z'$ $\mathrm{gp}\{C_1\}$. By the definition of the relation (2.5) for C_1 and Z_2', the minimal element a

of the set Y_{C_1} is contained in R_2. Now using the defining relation (2.8) for C_1 and a, we obtain that $a_1 \in R_2$, where a_1 is the minimal element of the set $Y_{C_1} \setminus \{a\}$, and so on. Thus we have that Z' and Z'_1 are contained in R_2; hence $Z, Z_1 \in R_2$ and $R = R_2$.

Finally, it follows from (3.15) and (3.17) that $n|C| < |C_1|$, which completes the proof of the lemma.

4. Proof of Theorem A

Let L be the homomorphic image of the group N in G. Then L is a normal subgroup of G, and it follows from the definition of the relations of G made in Section 2 that $G/L \cong F/N \cong H$. By [10, Lemma 34.13], a group $\mathrm{gp}\{\Omega_1\}$ is infinite; hence L is an infinite subgroup of G. It follows from [10, Lemma 25.1] that the group G is aspherical and atoroidal.

If $X \in L$ and X is not conjugate in G to an element of any G_i, $i \in I$, then, by [10, Lemma 34.7], X is conjugate to a power of a period Y, and it follows from the definition of the relation (2.1) that either X is of order dividing m (of infinite order in the case $m = \infty$) or the homomorphic image of Y in H has even order and Y is of infinite order.

Repeating the proof of [9, Theorem A], we obtain that every automorphism of L is induced by an inner automorphism of G; hence $\mathrm{Aut}\, L \cong G$ and $\mathrm{Out}\, L \cong H$. The claim about regular automorphisms of L follows from [10, Lemmas 34.9 and 34.11].

Let M be an arbitrary non-cyclic subgroup of G. If M has no free elements, then by the proof of [10, Theorem 35.1], M is conjugate to a subgroup M_1 of a group G_i, $i \in I$, and so M is conjugate to a subgroup G_{C,M'_1}, where $C = (M_1 \cap L) \setminus \{1\}$ and M'_1 is the homomorphic image of M_1 in H.

Let M contain a free element X of G. By [10, Lemma 34.7], X is conjugate to a power of a period A. If $M \cap L = 1$, then it follows from the definition of the relation (2.1) that the image A in H has infinite order. In the opposite case, the group M is conjugate to a subgroup M_1 containing A^k and W, where $100\zeta^{-1} < k$ (and $k \le n_0/2$ if A is a period of the first type), W is a word which does not commute with A^k in G and whose length is minimal among all words in the double coset $\mathrm{gp}\{A^k\}W\,\mathrm{gp}\{A^k\}$, and moreover, $[A^k, W]$ is contained in L. It follows from Lemma 5 that M_1 is conjugate in G to a subgroup $M_2 = \mathrm{gp}\{C^l, \{W_j\}_{j\in J}\}$, where C is a period, $C^l \in L$ and for each $j \in J$, W_j is a minimal word in G such that W_j is not contained in $\mathrm{gp}\{C\}$.

Of course, M_2 is an extension of a group H' by a normal subgroup $L' = M_2 \cap L$, where H' is the homomorphic image of M_2 in H. Let $K = F(\{C\} \cup \{W_j\}_{j\in J})$. By Lemma 1, $M_2 \le R_K$ and $L' \le L_K = R_K \cap L$.

Now we prove that $L_K \le L'$. Let X be an arbitrary element of L_K. Then by the definition of a generating mapping on Ω, there are $W_{i_1}, \dots, W_{i_t}$, $t \ge 1$, such that $F(\{X\}) \subseteq F(\{C, W_{i_1}, \dots, W_{i_t}\})$. Applying Lemma 6 to the group $\mathrm{gp}\{C^l, W_{i_1}\}$, we obtain that the group L' contains a period C_1 such that $F(\{C_1\}) = F(\{C, W_{i_1}\})$. Similarly, $\mathrm{gp}\{C_1, W_{i_2}\}$ contains a period $C_2 \in L'$ such that $F(\{C_2\} = F(\{C, W_{i_1}, W_{i_2}\})$, and so on. As a result, we have a period $C_t \in L'$ such that $F(\{X\}) \subseteq F(\{C_t\})$. If $|X| > d|C_t|$, then by Lemma 6, the subgroup $\mathrm{gp}\{C_t, C^l\}$ contains a period C_{t+1} such that $F(\{X\}) \subseteq F(\{C_{t+1}\})$ and $n|C_t| < |C_{t+1}|$. Repeating the same trick several times, we have that L' contains a period B such that $F(\{X\}) \subseteq F(\{B\})$ and $|X| < d|B|$. We may assume that $C^l \in Y_B$, and it follows from the definition of the relation (2.5) for B and C^l that $a \in L'$, where a is the minimal element of the set Y_B. Now using the defining relation (2.8) for B and a, we obtain that $a_1 \in L'$, where a_1 is the minimal element of the set $Y_B \setminus \{a\}$, and so on. As a result, we have that $X_1 \in L'$, where $X_1 \in Y_B$ such that X is contained in $\mathrm{gp}\{B\}X_1\,\mathrm{gp}\{B\}$. Then $X \in L'$ and $L_K \leq L'$.

If $C \not\subseteq G_i$ for each $i \in I$, then by the statement of Theorem A, $f(C) \cap \Omega_1 \neq \emptyset$. Let $a \in f(C) \cap \Omega_1$ and $L'_C = \mathrm{gp}\{bab^{-1}, b \in f(C)\}$. It is obvious that $L'_C \leq L_C$. Now we prove that $L_C \leq L'_C$. We have that $C \not\subseteq G_i$ for each $i \in I$; then, by [10, Lemma 34.11] and the definition of the relations of G, there is $b \in f(C)$ and ε, $|\varepsilon| = 1$, such that $[a, b]^\varepsilon$ is a period. Let X be an arbitrary element of L_C. Then by the definition of a generating mapping on Ω, there are $b_1, \dots, b_t$, $t \geq 1$, such that $F(\{X\}) \subseteq F(\{[a, b], b_1ab_1^{-1}, \dots, b_tab_t^{-1}\})$. Repeating the previous considerations for X and the set $\{[a, b], b_1ab_1^{-1}, \dots, b_tab_t^{-1}\}$, we obtain that $X \in L'_C$. Then $L_C \leq L'_C$, as required.

Assertion 7 of Theorem A follows from Lemma 1.

Let $C \not\subseteq G_i$ for each $i \in I$, M be a subgroup of G in which every element is a minimal word of G, $L'_{C_1} = \mathrm{gp}\{L_C, M\} \cap L$ and $C_1 = F(C \cup (M \setminus \{1\}))$. It follows from Lemma 3 that $L'_{C_1} \leq L_{C_1}$. Now we prove that $L_{C_1} \leq L'_{C_1}$. We have that $C \not\subseteq G_i$ for each $i \in I$; then L'_{C_1} contains a power A^l of a perod A. Let $X \in L_{C_1}$. Then by the definition of a generating mapping on Ω, there are $W_{i_1}, \dots, W_{i_t} \in L'_{C_1}$, $t \geq 1$, such that $F(\{X\}) \subseteq F(\{A^l, W_{i_1}, \dots, W_{i_t}\})$ and for each s, $1 \leq s \leq t$, W_{i_s} is a minimal word in G not belonging to $\mathrm{gp}\{A\}$. Repeating the proof of assertion 5 of Theorem A, we obtain that $X \in L'_{C_1}$ and $L_{C_1} \leq L'_{C_1}$.

Assertions 8 and 10 of Theorem A follow from Lemma 3.

It remains to prove that L is simple. Let M be an arbitrary normal subgroup of L. If M is a proper subgroup, then we can assume that either M is a subgroup of some group G_i, $i \in I$, or $M = \mathrm{gp}\{A^t\}$, where A is a period, or $M = R_C$, where $C \not\subseteq G_i$ for each $i \in I$. We consider these cases.

(1) If M is a subgroup of some group G_i, $i \in I$, then there is $Z \in L \setminus G_i$, with $ZMZ^{-1} = M$, contradicting [10, Lemma 34.11].
(2) If $M = \mathrm{gp}\{A^t\}$, then there is $Z \in L \setminus \mathrm{gp}\{A\}$ such that $ZMZ^{-1} = M$ contradicting [10, Lemma 34.9].
(3) If $M = R_C$, where $C \not\subseteq G_i$ for each $i \in I$, then there is $Z \in L$ such that

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$F(\{Z\}) \not\subseteq f(C)$. The group M contains an element A^t, where A is a period; hence by Lemmas 3 and 1, ZA^tZ^{-1} is not contained in M, and we arrive at a contradiction to the choice of the group M.

Thus L is simple, and the proof of Theorem A is complete.

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Faculty of Mathematics and Physics
Kostroma Pedagogical University
First of May 14, Kostroma 156601
Russia

Current address:
Department of Mathematics
University of Melbourne
Parkville VIC 3052
Australia
e-mail: vobrazts@maths.unimelb.edu.au

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