VALUATION OF MORTGAGE INSURANCE CONTRACTS WITH COUNTERPARTY DEFAULT RISK: REDUCED-FORM APPROACH

BY

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ABSTRACT

In the recent subprime mortgage crisis, which has caused banks and insurance companies to go bankrupt or into acquisition, the lender and insurer have exhibited not only correlated defaults when exposed to common risk factors but also counterparty default risk, which is triggered by mortgage defaults. Given the correlated defaults and the counterparty default risk, we use the reduced-form approach to derive the closed-form formulas of mortgage insurance contracts with premium refunds, annual premiums and upfront premiums. Regardless of the nature of the premium structures, the numerical analysis with parameter calibration demonstrates that both the correlated defaults and the counterparty default risk significantly impact mortgage insurance premiums, particularly in long-term mortgage loans.

KEYWORDS

Correlated defaults, counterparty default risk, reduced-form model, mortgage insurance premium structures.

1. INTRODUCTION

Canner and Passmore (1994) indicate that mortgage insurance (MI) plays an important role in the functioning of the housing finance markets because it transfers the credit risk exposure from lenders to insurers and facilitates the creation of secondary mortgage markets. The U.S. mortgage guaranty industry is dominated by six insurance groups: MGIC Investment Corporation, Radian Group, Genworth Financial, PMI Group, American International Group and Old Republic International Corporation. Subsidiaries of them wrote 93% of the $4.4 billion of premiums in 2010. Owing to the subprime mortgage crisis, the increasing foreclosure rate of borrowers has resulted in capital scarcity for many mortgage insurers. These same companies also recorded 71% of the combined $2.4 billion losses in 2010. Two substantial groups, PMI and Old Republic, wrote

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24.6% of 2010 earned premiums but were forced to stop writing new policies due to insufficient capital at the end of the third quarter of 2011. Two PMI subsidiaries were placed into receivership by the state insurance regulator. One, PMI Mortgage Insurance Company, recorded 11.6% of the total mortgage premiums earned in 2010. Huge losses, $2.4 billion in 2010, threaten to destroy the MI business. The concern is that the losses will continue to grow and, with limited growth in real estate sales requiring MI, there will be additional withdrawals from the market and/or potential failures. As a consequence, default probabilities of mortgage insurers have become a critical factor in valuing MI contracts.

Previous studies of pricing MI contracts, such as those performed by Kau *et al.* (1992, 1993, 1995) and Kau and Keenan (1995, 1999), have used a structural approach with two static variables, interest rate and housing price, to model endogenously prepayments as an American call option and defaults as an American put option. Other studies, such as those performed by Schwartz and Torous (1992), Dennis *et al.* (1997) and Bardhan *et al.* (2006), exogenously model the unconditional probability of default. Dennis *et al.* (1997) propose an actuarial pricing method in which the actuarially fair premiums of different MI structures, including upfront premiums, annual premiums and premium refunds, are determined by the present value of the expected losses (plus a gross margin) equal to that of the expected premium revenue. However, the expected losses are simply a constant fraction of the balance of the loan if a borrower is in default during the life of a mortgage. Using a process that assumes risk-neutral agents, housing prices follow geometric Brownian motion and constant interest rates. Bardhan *et al.* (2006) derive the closed-form formulas of upfront MI premiums, where the expected losses for an insurer are represented as a portfolio of put options on the collateral of borrowers. Chang *et al.* (2012) extend Bardhan *et al.* (2006) to employ an option-pricing framework to price and hedge the fair premium of MI by using a linear regression on the comovement of macroeconomic factors and housing prices. Chang *et al.* (2012) further incorporate the default risk of the mortgage insurer by using structured models and then simulate vulnerable American puts (upfront MI premiums) by the Least-Squares Monte Carlo algorithm.

Some studies (e.g., Deng and Quigley, 2002; Lambrecht *et al.*, 2003; and Caselli *et al.*, 2008) indicate that the default and prepayment rates of mortgages are highly dependent on macroeconomic variables, such as interest rates, housing prices and employment rates. If the default and prepayment processes are modeled to incorporate the macroeconomic variables under a structural approach, it is hard to derive the closed-form formulas for MI premiums. However, this problem can be solved by an alternative method: the reduced-form approach. A relative merit of this approach is its flexibility and ability to derive an analytical solution with two correlated effects: correlated defaults and correlated prepayments. It is important to incorporate the counterparty default risk, which has been considered by few previous studies, into the MI pricing model, particularly in the case of a mortgage crisis. Therefore, to fill this gap in the existing literature, this article aims to extend the reduced-form model presented by Jarrow and Yu (2001) as its first goal. Specifically, we model the default and prepayment processes as functions of exogenous variables. These variables include interest rates, housing price indices and employment rates, all of which lead to changes in the probabilities of default and prepayment. This model can consider the possibility that lenders, insurers and borrowers default prior to the maturities of MI contracts. The model can also consider the possibility that borrowers' defaults increase the lenders' and insurers' default rates and default probabilities. Furthermore, most of the MI pricing articles (e.g., Kau and Keenan, 1995, 1999; Bardhan *et al.*, 2006) focus on upfront premiums instead of annual premiums or premium refunds. Hence, the second goal of this article is to propose a risk-neutral pricing model to derive the fair upfront premiums, annual premiums and premium refunds with correlated defaults, correlated prepayments and counterparty default risk, respectively. Through numerical analysis, we demonstrate that fair upfront premiums are the cheapest premium structure and that fair annual premiums are larger than premium refunds. Furthermore, a borrower's correlated default is positively related to the premium, whereas a borrower's correlated prepayment is negatively related to the premium. The correlated default is dominated by the correlated prepayment, which implies that if the housing price index (HPI) decreases and the unemployment rate increases, the borrower will decide to default rather than make a prepayment. An increase in the borrower's default probability will contribute to an increase in the probability that the insurer incurs a loss and will thus cause the fair MI premium to increase. In addition, the longer the loan maturity is, the higher the MI premium. If the default risk of a lender or a mortgage insurer (counterparty default risk) increases, the effectiveness of the mortgage insurer's protection against the borrower's default decreases, as does the borrower's willingness to pay an MI premium to the insurer. In summary, correlated effects, time to loan maturity and counterparty default risk are all critical factors in pricing an MI contract.

The next section describes the structure of MI contract. Section 3 models the hazard processes of the lender, insurer and borrower with correlated effects. The counterparty default risk is also considered with respect to the lender and insurer. Section 4 develops the framework for determining the MI premiums of different structures. Section 5 provides numerical examples of existing premium structures with parameter calibration to emphasize the impacts of correlated effects and counterparty default risk on MI premiums. The last section draws conclusions about our findings and discusses their implications.

2. The structure of mortgage insurance contract

In practice, if a borrower's loan-to-value ratio exceeds 80%, the borrower is required by the lender to enter a private MI contract. Figure 1 shows the structure of a private MI contract. According to the private MI contract, the borrower has to pay the MI premiums to the mortgage insurer, who, in turn, guarantees that if

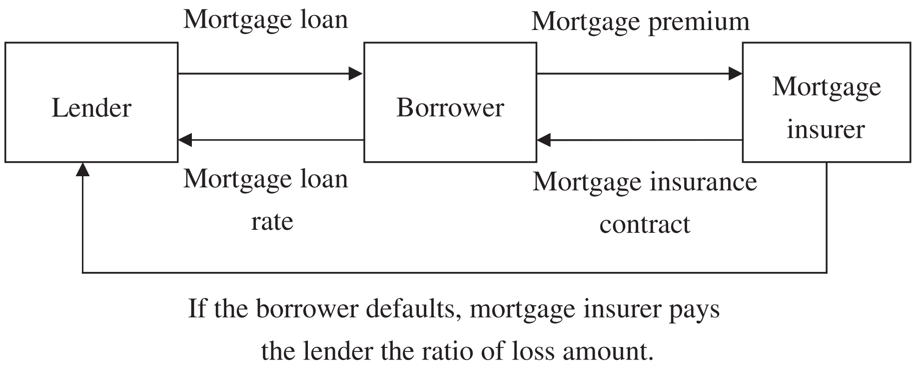

FIGURE 1: The structure of a private MI contract.

a borrower defaults on a loan, the mortgage insurer will pay up to 20% to 30% of the claim amount to the lender for any loss resulting from a property foreclosure. Hence, if the contract is immediately terminated after a borrower defaults, the payments (loss) that the mortgage insurer pays to the lender increases, which, in turn, increases the likelihood that the mortgage insurer will default, especially during the subprime mortgage crisis. In addition, if the borrower defaults, the lender's default rate may increase because only a part of the mortgage loss (i.e., approximately 20% to 30%) is covered by the insurer.

A feasible premium structure is one in which the present value of the expected losses (plus a gross margin) is equivalent to the present value of the expected revenue. The existing premium structures of private MI contracts can be classified as upfront premiums, monthly premiums, level annual premiums and standard annual premiums based on the frequency of the payments. Note that the monthly and annual MI premiums are refundable at prepayment for the fraction of the month or the year that has not passed. However, this type of refund differs from an upfront premium refund, which could occur more than one year after the premium payment. Generally speaking, private MI rates fall within the range of 0.5% to 1%. Federal Housing Administration (FHA) loans require a premium of 1.5% of the loan value at closing; monthly premiums are approximately 0.5% of the loan amount.

3. THE MODEL

According to the structure of a private MI contract, there are three types of agents in the economy: a lender, a mortgage insurer and a borrower. We first model the hazard processes of the borrower's default and prepayment. Then, considering the counterparty default risk, we design the hazard processes of the lender and the mortgage insurer. An abnormal jump in these processes is triggered if the borrower defaults during the life of the mortgage. Finally, we present the dynamics of the common risk factors that affect the hazard processes of the lender, the mortgage insurer and the borrower.

3.1. The hazard processes of the borrower's default and prepayment

Let uncertainty in the economy be described by the filtered probability space $(\Omega, F, Q, (F_t)_{t=0}^{T^*})$. Under the assumption that the market is complete and arbitrage free, there exists a probability measure[1] Q, the MI premium is determined if the present value of the expected losses is equal to that of the expected premium revenues. Recently, numerous empirical studies, such as those prepared by Schwartz and Torous (1989, 1993), Quigley and Van Order (1990, 1995), Deng *et al.* (2000), Deng and Quigley (2002), Lambrecht *et al.* (2003), and Caselli *et al.* (2008), have indicated that the patterns of default and prepayment are significantly explained by the following risk factors: interest rate, housing price return, loan-to-value ratio and unemployment rate. Prepayment probability is positively related to housing price return but negatively related to unemployment rate. Conversely, default probability is an increasing function of unemployment rate but a decreasing function of housing price return. Both probabilities are positively related to the interest rate. Therefore, the borrower may make a prepayment decision rather than a default decision if the housing price return is high and the unemployment rate is low.

To incorporate the empirical results into our pricing model, we define the enlarged filtration F, under which default and prepayment are interrelated as follows:

$$F_t = F_t^r \vee F_t^M \vee H_t^d \vee H_t^p,$$

where $F_t^r = \sigma(r(s), s \leq t)$, $F_t^M = \sigma(M_i(s), s \leq t, i = 1, \ldots, n)$ and $H_t^i = \sigma(1_{\{\tau^i \leq s\}}, s \leq t)$ for $i = d$ or p; $1_{\{.\}}$ is an indicator function; $r(t)$ is the spot rate at time t; $M(t)$, an R^n-valued stochastic process denotes the time t common risk factors, such as housing price and unemployment rate. As a result, $F_{T^*}^r \vee F_{T^*}^M$ contains complete information on the interest rate and all of the common risk factors. Let τ^d and τ^p denote a borrower's default time and prepayment time, respectively, and satisfy

$$\tau^i = \inf\left\{t : \int_t^T \lambda^i(r(s), M(s))ds \geq E^i\right\}, \quad i = d \text{ or } p,$$

where the default time and prepayment time are considered to be the first jump time of a doubly stochastic Poisson process (also called a Cox process) combined with a hazard rate process $\lambda^i(r(t), M(t))$; E^i is an exponential random variable independent of state variables; and $\lambda^i \cdot (r(t), M(t))_{t=0}^{T^*}$, which represents the static variables underlying the evolution of the economy, is a process that is right continuous with left limits R^{n+1}-valued that is used to predict the likelihood of default or prepayment. Consequently, the conditional survival probability of τ^i,

$i = d$ or p, takes on the following form:

$$P\left(\tau^i > t \middle| F^r_{T^*} \vee F^M_{T^*}\right) = \exp\left(-\int_0^t \lambda^i[r(s), M(s)]ds\right), \quad t \in [0, T^*].$$

As a function of the interest rate and common risk factors, the hazard rates of default and prepayment can be rewritten as follows:

$$\lambda^i(r(u), M(u)) = f[r(u, M_1(u), \ldots, M_n(u))], \quad i = d \text{ or } p, u \in [0, T^*]. \tag{1}$$

By defining the hazard rates of default and prepayment as linear functions of the spot rate and the excess returns of influential variables, we have the following representation:

$$\lambda^i(u) = \lambda^i_0 + \lambda^i_r r(u) + \sum_{j=1}^{n} \lambda^i_x \log\left(\frac{M_j(u)}{B(u)}\right), \quad \text{for} \quad i = d \text{ or } p, \tag{2}$$

where $\lambda^d_0(\lambda^p_0)$ denotes the baseline hazard rate of default (prepayment) at time u; $\lambda^d_r(\lambda^p_r)$ measures the magnitude of default (prepayment) to the level of spot rate; and $\lambda^d_j(\lambda^p_j)$ represents the magnitude of default (prepayment) to the excess return of common risk factor $j = 1, 2, \ldots, n$. $B(u) = \exp(\int_0^u r(s)ds)$, the savings account, corresponds to the wealth accumulated by an initial \$1 investment at spot rate $r(u)$ in each subsequent period.

Most studies use the Cox proportional hazard model to specify a hazard function as the product of a baseline hazard rate and an exponential function of covariates. However, under the pricing framework of martingale measure, Miltersen *et al.* (1997) indicate that a double exponential expression causes an infinite expectation of cumulative factors if the influential variables are lognormally distributed. Therefore, Jarrow and Turnbull (1995), Duffee (1999), Jarrow and Turnbull (2000), Jarrow and Yu (2001), Calem and LaCour-Little (2004) and Liao *et al.* (2008) assume that the hazard rate function is a linear function of the spot rate and the influential variables. One problem with this assumption is that the hazard rate function may be negative. However, Jarrow and Turnbull (1997) indicate that this difficulty can be avoided by using non-linear transformations in lattice-based models. Furthermore, Duffee (1999) also argues that this problem can be ignored if the model accurately prices the relevant instruments.

3.2. Hazard processes of the lender and insurer

In addition to risk exposure to the interest rate and common risk factors, mortgage defaults will affect the default rates of the lender and mortgage insurer, particularly during a mortgage crisis. We define the default times of the lender and the mortgage insurer as follows:

$$\tau^j = \inf\left\{t : \int_t^T \lambda^j(r(s),\ M(s), \tau^d)ds \geq E^j\right\}, \quad j = I \text{ or } L, \tag{3}$$

where the superscripts I and L denote the mortgage insurer and the lender, respectively. By virtue of (3), their default times are connected not only to the interest rate and common risk factors but also to the default time of the borrower. From this definition, the conditional survival probability of τ^j is given by

$$P\left(\tau^j > t \mid F^r_{T^*} \vee F^M_{T^*} \vee H^d_{T^*}\right) = \exp\left(-\int_0^t \lambda^j[r(s),\ M(s),\ \tau^d]\, ds\right), \quad t \in [0, T^*],$$
$$j = I \text{ or } L.$$

Using the law of iterated expectation, we obtain

$$P(\tau^j > t) = E^Q\left[\exp\left(-\int_0^t \lambda^j[r(s),\ M(s),\ \tau^d]\, ds\right)\right], \quad t \in [0, T^*],$$
$$j = I \text{ or } L.$$

Extending the model of Jarrow and Yu (2001), we assume that the hazard rates of the lender and the mortgage insurer are linear and incorporate n common risk factors as well as the spillover effect of the borrower's default to derive the following expression:

$$\lambda^j(u) = \lambda_0^j + \lambda_r^j r(u) + \sum_{x=1}^n \lambda_x^j \log\left(\frac{M_x(u)}{B(u)}\right) + 1_{\{\tau^d \leq u\}}\alpha_0^j, \quad j = I \text{ or } L, \tag{4}$$

where $\lambda_o^I(\lambda_o^L)$ denotes the baseline default rate of the mortgage insurer (lender); $\lambda_r^I(\lambda_r^L)$ measures the magnitude of default to the level of spot rate of the mortgage insurer (lender); λ_x^j has the anagogic definition; $\alpha_0^I(\alpha_0^L)$ represents the jump size of the mortgage insurer's (lender's) default intensity if a borrower defaults. For example, if $\alpha_0^j > 0$, $j = I$ or L, which implies that the contract is immediately terminated in the case of the borrower's default, the MI premiums (revenue) paid by the borrower to the mortgage insurer decrease, but the payment (loss) from the mortgage insurer to the lender increases, which, in turn, leads to an increase in the likelihood of default by the mortgage insurer. Conversely, if the borrower defaults, the default rate of the lender may increase because part of the loss is compensated by the mortgage insurer. Therefore, a borrower default may lead to a shock in the hazard rates of the lender and the mortgage insurer. Otherwise, if $\alpha_0^j = 0$, $j = I$ or L, a borrower default may not influence the hazard rates of the lender and the mortgage insurer.

3.3. Interest rate and common risk *factor* processes

We use a one-factor model — the extended Vasicek model (see Hull and White 1990) — to describe the evolution of the term structure. That is,[2]

$$dr(t) = [\theta(t) - \alpha(t)r(t)]dt + \sigma^r dW_t^r, \tag{5}$$

where $\theta(t)/\alpha(t)$ denotes the long-term equilibrium value of the process; $\alpha(t)$ is a non-negative mean reversion speed; σ_r is the volatility of the spot rate; and W_t^r is a Brownian motion with respect to F_t.

We follow Kau *et al.* (1992, 1993, 1995) and Liao *et al.* (2008) in assuming that common risk factors, such as the HPI and the unemployment rate, follow a geometric Brownian motion.[3] That is,

$$\frac{dM_j(t)}{M_j(t)} = r(t)dt + \sigma_j dW_t^j, \quad j = 1, 2, \ldots, n, \tag{6}$$

where σ_j is the constant volatility of the common risk factor j. W_t^j, a standard Brownian motion under Q, is correlated with W_t^r and W_t^i and satisfies $E(dW_t^r dW_t^j) = \rho_{rj}dt$ and $E(dW_t^i dW_t^j) = \rho_{ij}dt$, where $\rho_{ij}(\rho_{rj})$ is the correlation coefficient between the common risk factor i (spot rate) and the common risk factor j and satisfies $\rho_{jj} = 1$.

4. Valuation of mortgage insurance contracts

In this section, we present a framework under which feasible MI premium structures can be constructed given the counterparty default risk. In an MI contract, a borrower pays insurance premiums to an insurer. A fair premium structure is one in which the present value of the expected losses is equivalent to the present value of the expected revenue. The premium structures can include upfront premiums, annual premiums or premium refunds.

Consider a situation in which a lender issues a T-year mortgage loan to a borrower. This loan originates at time 0 and is secured by the housing property. For an MI contract prior to maturity, the borrower determines whether to prepay, default or make the scheduled payment. Denoted by R_s, the remaining mortgage balance at the instant after the time-s payment is $s \in [t, T]$. g_s is the ratio of the refund to the remaining mortgage balance. If the borrower decides to prepay part of the mortgage loan, the mortgage insurer will refund a portion of the remaining mortgage balance $R_s g_s$ to the borrower. If the borrower decides to default on the mortgage loan, the mortgage insurer incurs losses proportional to the remaining mortgage balance $L_R R_s$, where L_R denotes the ratio of the losses to the remaining mortgage balance.

In what follows, we first present the discounted values of expected losses and revenue for the MI contracts subject to counterparty default risk from the

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insurer's viewpoint. Then we discuss alternative premium structures (e.g., refund, annual and upfront premiums).

4.1. Present value of the expected accumulated revenues for the insurer

During the life of a mortgage, we assume that a borrower pays the insurance premium $c_s ds$ to a mortgage insurer at each subinterval $[s, s + ds]$. The MI contract will be terminated early and the premium payments $c_s ds$ will cease if one of the following conditions is fulfilled: (1) the borrower defaults prior to maturity; (2) the borrower prepays prior to maturity; (3) the borrower does not default but the MI contract is terminated early because the lender has declared bankruptcy[4] or (4) the borrower does not default, but the mortgage insurer goes bankrupt.

Let ER denote the present value of the mortgage insurer's expected revenues from time t to T. Given the information on the interest rate, the common risk factors and the default (prepayment) of a borrower before t, which is denoted by F_t, we have

$$\begin{aligned} ER &\equiv E^Q \left\{ \int_t^T \frac{B(t)}{B(s)} \left[c_s R_s 1_{\{\tau^L > s, \tau^I > s, \tau^d > s, \tau^p > s\}} - 1_{\{\tau^d > s, t < \tau^p \leq s\}} g_s R_t \right] ds \,\middle|\, F_t \right\} \\ &= G_1(t, T) - G_2(t, T), \quad \text{if} \quad \tau^d > t, \tau^p > t, \tau^I > t, \tau^L > t, \end{aligned} \tag{7}$$

where $c_s R_s 1_{\{\tau^L > s, \tau^I > s, \tau^d > s, \tau^p > s\}}$ indicates that if the borrower does not prepay and the mortgage lender, insurer and borrower do not default at time S, the revenue of the mortgage insurer is $c_s R_s$. $1_{\{\tau^d > s, t < \tau^p \leq s\}} g_s R_t$ indicates that if the borrower prepays but does not default at time s, then the mortgage insurer will refund a portion of the remaining mortgage balance to the borrower. The closed-form of the first term $G_1(t, T)$ in (7) becomes

$$G_1(t, T) = \int_t^T c_s R_s \exp\left(-\lambda_0^K (s - t) + \Gamma' \mu_X^{t,s} + \frac{\Gamma' \Sigma_t^{s,s} \Gamma}{2} \right) ds,$$

where

$$\begin{aligned} \lambda_0^K &= \lambda_0^L + \lambda_0^I + \lambda_0^d + \lambda_0^p, \quad \lambda_r^K = 1 + \lambda_r^L + \lambda_r^I + \lambda_r^d + \lambda_r^p, \\ \lambda_i^K &= \lambda_i^L + \lambda_i^I + \lambda_i^d + \lambda_i^p, \quad \Gamma = \left[-\lambda_r^K, -\lambda_1^K, \ldots, -\lambda_n^K \right]', \\ \mu_X^{t,s} &= [\mu_{X_0}(t, s), \mu_{X_1}(t, s), \ldots, \mu_{X_n}(t, s)]', \end{aligned}$$

$\mu_{X_0}(t, s) = \int_t^s f(t, u) du + \frac{1}{2} \sigma_{X_0^{t,s} X_0^{t,s}}$, $\mu_{X_i}(t, s) = \frac{(s-t)^2}{4} \sigma_i^2$; $f(t, T), 0 \leq t \leq T \leq T^*$, denotes the forward rate at time t for instantaneous borrowing and lending at time T. $\Sigma_t^{s,y} = (\sigma_{X_i^{t,s} X_j^{t,y}})$ is a $(n+1)-$ by $-(n+1)$ covariance matrix; $\sigma_{X_i^{t,s} X_j^{t,y}}$

is the covariance between $X_i^{t,s}$ and $X_j^{t,y}$ for $s > y$, where

$$X^{t,s} = \left[X_0^{t,s}, X_1^{t,s}, \ldots, X_n^{t,s}\right]$$
$$= \left[\int_t^s r(u)du, \int_t^s \log\left[\frac{M_1(u)}{B(u)}\right]du, \ldots, \int_t^s \log\left[\frac{M_n(u)}{B(u)}\right]du\right],$$

$\sigma_{X_0^{t,s} X_0^{t,y}} = \int_t^y b(u,s)b(u,y)du$, is the covariance of $\int_t^s r(u)du$ and $\int_t^y r(u)du$, $\sigma_{X_0^{t,s} X_i^{t,s}} = -\sigma_i \rho_{ri} \int_t^s (s-u)b(u,s)\, du$ is the covariance of $\int_t^s r(u)du$ and $\int_t^s \log[\frac{M_i(u)}{B(u)}]du$, $\sigma_{X_i^{t,s} X_j^{t,s}} = \frac{(s-t)^3}{3}\sigma_i\sigma_j\rho_{ij}$ is the covariance of $\int_t^s \log[\frac{M_i(u)}{B(u)}]du$ and $\int_t^s \log[\frac{M_j(u)}{B(u)}]du$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$,

$$b(u,s) = -\sigma_r D(u,s), \quad D(u,s) = \int_u^s \exp\{-[\Phi(y) - \Phi(u)]\}dy,$$
$$\Phi(u) = \int_0^u \alpha(y)dy.$$

$G_1(t, T)$ is the discount value of MI premiums adjusted by the borrower's default and prepayment risks and by the lender's and insurer's default risks. Furthermore, the closed-form of the second term $G_2(t, T)$ in (7) becomes

$$G_2(t,T) = \int_t^T g_s R_t \exp\left(-\lambda_0^d(s-t) + \phi'\mu_X^{t,s} + \frac{\phi'\Sigma_t^{s,s}\phi}{2}\right) ds$$
$$- \int_t^T g_s R_t \exp\left(-\lambda_0^p(s-t) + \Upsilon'\mu_X^{t,s} + \frac{\Upsilon'\Sigma_t^{s,s}\Upsilon}{2}\right) ds,$$
$$\text{if} \quad \tau^d > t, \tau^p > t, \tau^I > t, \tau^L > t,$$

where

$$\phi = \left[-\left(1+\lambda_r^d\right), -\lambda_1^d, \ldots, -\lambda_n^d\right]',$$
$$\Upsilon = \left[-\left(1+\lambda_r^d+\lambda_r^p\right), -\left(\lambda_1^d+\lambda_1^p\right), \ldots, -\left(\lambda_n^d+\lambda_n^p\right)\right]'.$$

The detailed proof is shown in Appendix A. In (7) the present value of the insurer's expected revenue is composed of two parts — $G_1(t, T)$, the discounted value of MI payments adjusted by default and prepayment risks, and $G_2(t, T)$, the discounted value of refund premiums — conditional on no prepayments and bankruptcies between the lender, mortgage insurer and borrower prior to time t.

4.2. Present value of the expected accumulated losses for the insurer

During the life of a mortgage, if a borrower defaults without prepaying the mortgage before the lender and mortgage insurer default (the indicator functions $1_{\{t<\tau^d\leq s\}}1_{\{\tau^p>\tau^d\}}1_{\{\tau^I>\tau^d\}}1_{\{\tau^L>\tau^d\}}$ all hold), the mortgage insurer will compensate the lender for the losses on the mortgage $L_R R_{\tau^d}$ at the default time of borrower τ^d. Hence, the present value of the expected losses from t to T is denoted by EL and is given by

$$\begin{aligned} EL &\equiv E^Q\left[\int_t^T L_R R_{\tau^d}\frac{B(t)}{B(\tau^d)}1_{\{t<\tau^d\leq s\}}1_{\{\tau^p>\tau^d\}}1_{\{\tau^I>\tau^d\}}1_{\{\tau^L>\tau^d\}}ds|F_t\right] \\ &= G_3(t,T), \quad \text{if} \quad \tau^d>t, \tau^p>t, \tau^I>t, \tau^L>t, \end{aligned} \tag{8}$$

where

$$\begin{aligned} &G_3(t,T) \\ &= \int_t^T L_R\left\{\int_t^s R_v \exp\left(-\left(\lambda_0^K+\alpha_0^I+\alpha_0^L\right)(v-t)\right)\left[\left(\lambda_0^d+\Psi'\mu_W^{t,v}+\Psi'\bar{\Sigma}_t^{v,v}\Gamma\right)\right.\right. \\ &\quad \times \exp\left(\Gamma'\mu_X^{t,v}+\frac{1}{2}\left(\Gamma'\Sigma_t^{v,v}\Gamma\right)\right) \\ &\quad +\left(\alpha_0^L+\alpha_0^I\right)\int_t^v \exp\left(-[\lambda_0^d-(\alpha_0^I+\alpha_0^L)](x-t)\right) \\ &\quad \times \exp\left(\Gamma'\mu_X^{t,v}+\Psi'\mu_X^{t,x}+\frac{1}{2}\left(\Gamma'\Sigma_t^{v,v}\Gamma\right)+\left(\Psi'\Sigma_t^{v,x}\Gamma\right)+\frac{1}{2}\left(\Psi'\Sigma_t^{x,x}\Psi\right)\right) \\ &\quad \left.\left.\times\left(\lambda_0^d+\Psi'\mu_W^{t,v}+\Psi'\bar{\Sigma}_t^{v,v}\Psi+\Psi'\bar{\Sigma}_t^{x,v}\Psi\right)dx\right]dv\right\}ds, \end{aligned}$$

where $\Psi=[\lambda_r^d,\lambda_1^d,\ldots,\lambda_n^d]'$, $\mu_W=[\mu_r(t,v),\mu_{M_1}(t,v),\ldots,\mu_{M_n}(t,v)]'$, and satisfies

$$\mu_r(t,v)=f(t,v)+\frac{b(t,v)^2}{2} \text{ and } \mu_{M_i}(t,v)=-\frac{1}{2}\sigma_i^2(v-t) \text{ for } i=1,2,\ldots,n;$$

$\bar{\Sigma}_t^{y,y}=(\sigma_{X_i^{t,y}W_j^{t,y}})$ is a $(n+1)$–by–$(n+1)$ covariance matrix; $W^{t,y}$ is given by

$$W^{t,y}=\left[W_0^{t,y},W_1^{t,y},\ldots,W_n^{t,y}\right]'=\left[r(y),\log\left(\frac{M_1(y)}{B(y)}\right),\ldots,\log\left(\frac{M_n(y)}{B(y)}\right)\right]',$$

$$\sigma_{X_0^{t,v}W_0^{t,x}}=\int_t^x b(u,v)b(u,x)du, \quad i=0,1,2,\ldots,n;$$

$\sigma_{X_i^{t,y} W_j^{t,y}}$, which represents the covariance between $X_i^{t,y}$ and $W_j^{t,y}$, satisfies

$$\sigma_{r^{t,y} X_0^{t,y}} = \int_t^y -\Phi(u, y)b(u, y)du,$$

$$\sigma_{X_i^{t,y} r^{t,y}} = \sigma_i \rho_{ri} \int_t^y \Phi(u, y)(y - u)\, du,$$

$$\sigma_{X_0^{t,y} W_i^{t,y}} = -\sigma_i \rho_{ri} \int_t^y b(u, y)(y - u)\, du,$$

$$\sigma_{X_j^{t,y} W_i^{t,y}} = -\frac{1}{2}\sigma_i \sigma_j \rho_{ij} (y - t)^2 \text{ for } i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n, \; y = x \text{ or } v.$$

$G_3(t, T)$ is the present value of expected losses (adjusted by the borrower's default and prepayment risks as well as by the lender's and insurer's default risks). Appendix B provides the detailed proof of $G_3(t, T)$.

In our pricing model, the fair premium is determined by when the present value of the expected losses is equal to that of the expected premium revenues. However, the mortgage insurers are expected to earn a profit margin beyond the fair premium. Thus, following Equation (5) of Dennis *et al.* (1997) and Equation (6) of Bardhan *et al.* (2006), we determine the mortgage premium charged by the insurers by adding the profit margin and the fair premium as follows:

$$ER = (1 + q)EL. \tag{9}$$

In the *Refund case*, the MI premium collected at time t is proportional to the remaining mortgage balance at time t (R_t), and if the borrower decides to prepay part of the mortgage loan, the mortgage insurer will refund a portion of the remaining mortgage balance to the borrower. Hence, this rate is called the level annual premium rate with refund. Thus, according to (7)–(9), the condition of no arbitrage is $ER = (1 + q)EL$, subject to $c_s = c^{RF}$, which implies that the closed-form formula of *a* level annual premium rate with refund c^{RF} is as follows:

$$c^{RF} = \frac{(1 + q)G_3(t, T) + G_2(t, T)}{G_1^*(t, T)}, \quad \text{if} \quad \tau^d > t, \tau^p > t, \tau^I > t, \tau^L > t, \tag{10}$$

where

$$G_1^*(t, T) = \int_t^T R_s \exp\left(-\lambda_0^K (s - t) + \Gamma' \mu_X^{t,s} + \frac{\Gamma' \Sigma_t^{s,s} \Gamma}{2}\right) ds.$$

In light of (10), the annual premium rate is equal to the ratio of the sum of the discount values of the expected losses (plus a gross margin) and refund premiums to that of the remaining mortgage balance at each time if there is no default or prepayment prior to evaluation time t.

In the *Annual case*, the annual premium collected at time t is proportional to the remaining mortgage balance R_t and is not refundable if the borrower prepays. Hence, the *Refund case* can be reduced to the *Annual case* if $g_s = 0(G_2(t, T) = 0)$. Thus, by virtue of (11), as long as there is no default or prepayment prior to time t, an annual premium rate c^{AN} is equivalent to the ratio of the present value of the expected losses (plus a gross margin) to the sum of the present values of the remaining mortgage balance at each time.

$$c^{AN} = \frac{(1+q)G_3(t, T)}{G_1^*(t, T)}, \quad \text{if} \quad \tau^d > t, \tau^p > t, \tau^I > t, \tau^L > t. \tag{11}$$

Note that in the *Upfront case*, neither annual premium nor premium refund is collected; only an upfront fee is paid. The upfront premium is determined by the present value of the expected losses (plus a gross margin). Hence, the upfront premium c^{UP} in equilibrium is given by

$$c^{UP} = (1+q)EL = (1+q)G_3(t, T). \tag{12}$$

5. Implementing the model

To emphasize the impacts of correlated effects and counterparty default risk on the pricing of MI contracts, we first describe the calibration procedures for implementing the model. Then, through numerical analysis, we show that the MI premiums would be overpriced if we did not consider the correlated effects and counterparty default risk, which are two major sources of risks during a subprime mortgage crisis.

5.1. Estimation of parameters

To illustrate the impacts of both correlated effects and counterparty default risk, we demonstrate the calibration procedures, including the estimation of the Vasicek interest rate model, the variance-covariance matrix of common risk factors and the coefficients for the linear hazard rate functions of the lender, the insurer and the borrower. Without loss of generality, we adopt two static variables, the HPI and unemployment rate, as the common risk factors.

By virtue of (10), the MI premium is determined jointly by the following parameters: (1) the initial yield curve $f(t, s)$ and the parameters α, θ and σ_r in the Vasicek model under the risk-neutral measure; (2) the volatilities of common risk factors σ_i and the correlation parameters ρ_{ri} and ρ_{ij}, $i, j = 1, 2, \ldots, n$; and (3) the parameters of the hazard rates λ_0^d, λ_r^d, λ_1^d, λ_2^d, λ_0^p, λ_r^p, λ_1^p and λ_2^p for the borrower and λ_0^L, λ_r^L, λ_1^L, λ_2^L, α_0^L, λ_0^I, λ_r^I, λ_1^I, λ_2^I and α_0^I for the lender and the insurer under the risk-neutral measure.

To estimate the parameters α, θ and σ_r in Vasicek model under the risk-neutral measure, we use the swap curve with maturities of one, two, three, four, five, seven, ten and thirty years on April 30, 2008, available from U.S. Federal

Reserve. Given a number of payment dates T_i, $i = \tilde{\alpha}+1, \ldots, \tilde{\beta}$ (called the tenor structure) and $\delta_i = T_i - T_{i-1}$, we calculate theoretical price of swap in Vasicek model under the risk-neutral measure, using formula

$$S_{\tilde{\alpha},\tilde{\beta}}(t) = \frac{B(t, T_{\tilde{\alpha}}) - B(t, T_{\tilde{\beta}})}{\sum_{i=\tilde{\alpha}+1}^{\tilde{\beta}} \delta_i B(t, T_{\tilde{\alpha}})},$$

where $B(t, T_i)$ denotes the price of a zero-coupon bond with maturity T_i at time t, given by

$$B(t, T_i) = A(t, T_i)e^{-r(t)C(t,T_i)}, \quad C(t, T_i) = \frac{1 - e^{-\alpha(T_i - t)}}{\alpha},$$

$$A(t, T_i) = \exp\left\{\left(\frac{\theta}{\alpha} - \frac{\sigma_r^2}{2\alpha^2}\right)(C(t, T_i) - T_i + t) - \frac{\sigma_r^2}{4\alpha}C^2(t, T_i)\right\}.$$

Our calibrations are performed by minimizing the sum of the squared errors between theoretical curve and market curve. Hence, α, θ and σ_r are estimated as 0.192, 0.003 and 0.021, respectively.[5] Furthermore, the instantaneous forward interest rate with maturity T_i is given by

$$f(t, T_i) = \left\{\left(\theta - \frac{\sigma_r^2}{2}C(t, T_i)\right)C(t, T_i) + r(t)e^{-\alpha(T_i - t)}\right\}.$$

The excess returns, the sample variances and the correlation coefficients of two common risk factors — housing price and unemployment rate — are estimated from the historical market data as follows:

$$\hat{\sigma}_1^2 = \mathrm{Var}\left(\frac{M_i(t+\Delta t) - M_i(t)}{M_i(t)}\right)\frac{1}{\Delta t}, \quad \hat{\sigma}_2^2 = Var\,(M_i(t+\Delta t) - M_i(t))\,\frac{1}{\Delta t},$$

$$\hat{\rho}_{ri} = \mathrm{Corr}\left(\frac{M_i(t+\Delta t) - M_i(t)}{M_i(t)}, r(t+\Delta t) - r(t)\right), \quad i = 1, 2,$$

$$\hat{\rho}_{12} = \mathrm{Corr}\left(\frac{M_1(t+\Delta t) - M_1(t)}{M_1(t)}, M_2(t+\Delta t) - M_2(t)\right),$$

where M_1 denotes the U.S. HPI and M_2 denotes the unemployment rate. Based on monthly data for the U.S. unemployment rate and the Office of Federal Housing Enterprise Oversight's HPI from January 2004 to April 2008, the corresponding parameters are $\hat{\sigma}_1 = 0.006$, $\hat{\sigma}_2 = 0.001$, $\hat{\rho}_{r1} = 0.558$, $\hat{\rho}_{r2} = -0.079$ and $\hat{\rho}_{12} = -0.258$.

The default and prepayment rates are measured by the monthly delinquency rates and the monthly annualized prepayment rates, respectively, from January 2004 to April 2008 (provided by Freddie Mac). Using a linear regression model, we obtain the parameters for the borrower's hazard rates as follows:

$$\lambda^{d}(u) = 0.019 + 0.054 \times r(u) - 0.003 \times \log\left(\frac{M_1(u)}{B(u)}\right) + 0.0034 \times \log\left(\frac{M_2(u)}{B(u)}\right), \tag{13}$$

$$\lambda^{p}(u) = 0.381 + 0.58 \times r(u) + 0.104 \times \log\left(\frac{M_1(u)}{B(u)}\right) - 0.114 \times \log\left(\frac{M_2(u)}{B(u)}\right). \tag{14}$$

Consistent with past empirical studies, the default rate is significantly and negatively related to the unemployment rate but positively related to housing price. Conversely, the prepayment rate is positively related to housing price but negatively related to the unemployment rate. Both the default rate and the prepayment rate are positively related to the interest rate.

The parameter values for the lender's and insurer's default rates can be obtained with virtually identical procedures. Without loss of generality, we assume that $\lambda_0^L = \lambda_0^I = 0.01$, $\lambda_r^L = \lambda_r^I = 0.05$, $\lambda_1^L = \lambda_1^I = 0.01$, $\lambda_2^L = \lambda_2^I = 0.01$, $\alpha_0^L = \alpha_0^I = 0.1$ and $\beta_0^L = \beta_0^I = 0.1$. Using the above parameters, we can price the MI premiums in different MI structures, either while considering or without considering the correlated effects (i.e., correlated default and correlated prepayment) and counterparty default risk. Note that these parameters of U.S. HPI, unemployment rate and default/prepayment risk are fitted under the real measure P. We assume that these parameters are identical under real measure P and risk-neutral measure Q.

5.2. Sensitivity analysis

To the best of our knowledge, no previous study has incorporated the possibility of both the lender and the insurer defaulting during the life of the mortgage into the MI pricing model. This omission misprices the MI premiums. Consequently, this section investigates the impacts of both correlated effects and counterparty default risk on the MI premiums with different MI premium structures.

To adapt our model to the criterion concerning the maximum loan-to-value ratio (LTV) in the United States, we set $LTV = 95\%$. In addition, we assume that the loss ratio L_R is 30%, that the refund ratio g is 0.05%, that the gross profit margin q is 10% and that the average size of the insured property is $250,000.

Some studies, such as those by Schwartz and Torous (1992), Dennis *et al.* (1997) and Bardhan *et al.* (2006), simply consider the borrower's default/prepayment probability and model the unconditional default/prepayment probability exogenously. To examine whether the correlated effects and the

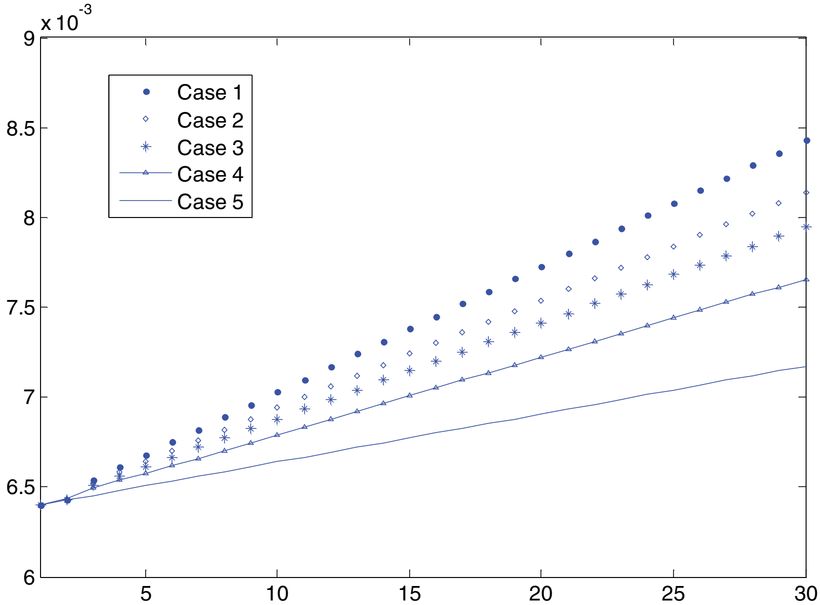

FIGURE 2: *Refund case*: the relationship between the premium rate and the loan maturity under different cases. Case 1: the borrower without correlated effects (dotted line). Case 2: the borrower with correlated effects (diamond line). Case 3: the lender and the borrower with correlated effects (asterisk line). Case 4: all the participants with correlated effects (triangle line). Case 5: general model considering not only correlated effects but also counterparty default risk (solid line). (Color online)

counterparty default risk play a crucial role in pricing MI, we considered the following five cases:

Case 1: The lender and insurer are default-free; because the hazard rate of a borrower is not influenced by the correlated effects, neither the default rate nor the prepayment rate is relevant to the interest rate, HPI and unemployment rate (the case assumed in previous studies on pricing MI).

Case 2: The lender and the insurer are default-free. The correlated effect is only considered for the borrower.

Case 3: The insurer is default-free. The hazard rates of the lender and borrower are impacted by the correlated effects.

Case 4: Neither agent is default-free. All of the hazard rates are impacted by the correlated default.

Case 5: Neither agent is default-free. All of the hazard rates are impacted by not only the correlated effects but also the counterparty default risk.

Given a suitable set-up for the parameters, the MI premiums are examined in three MI premium structures[6] (Figure 2, *Refund case*; Figure 3, *Annual case*; and Figure 4, *Upfront case*) in light of (10)–(12). Each figure provides five premium curves for the five above-mentioned cases. Figures 2–4 indicate that the MI premium is an increasing function of loan maturity. That is, the premium is lower

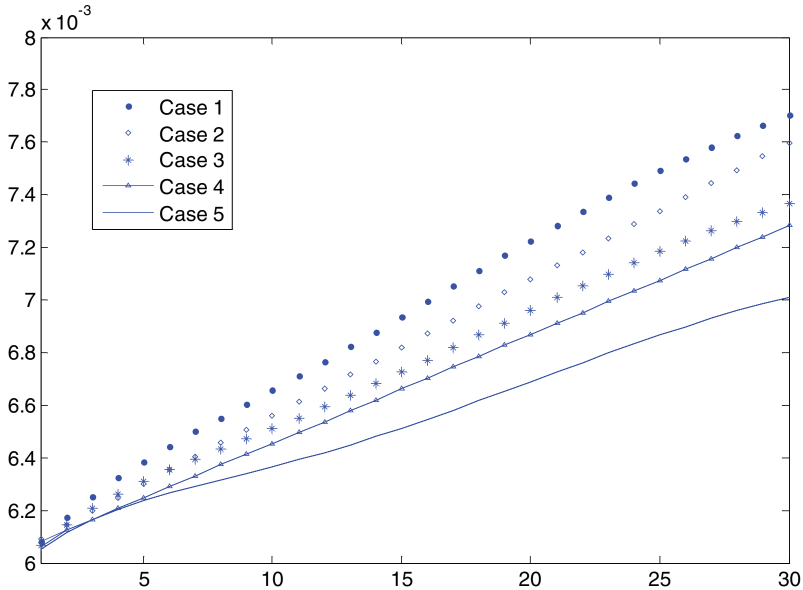

FIGURE 3: *Annual case*: the relationship between the premium rate and the loan maturity under different cases. Case 1: the borrower without correlated effects (dotted line). Case 2: the borrower with correlated effects (diamond line). Case 3: the lender and the borrower with correlated effects (asterisk line). Case 4: all the participants with correlated effects (triangle line). Case 5: general model considering not only correlated effects but also counterparty default risk (solid line). (Color online)

if the life of the mortgage is shorter. Corresponding to a given loan maturity, we see an interesting pattern with respect to the correlated effects and counterparty default risk: if more correlated effects and counterparty default risk are considered when pricing MI, the MI premiums are lower. In the following paragraphs, we discuss some interesting insights in more detail.

First, the premium in Case 1 is higher than that in Case 2. The likelihood that a borrower's correlated default increases the probability that an insurer incurs a loss is positively related to the premium, whereas the likelihood that a borrower's correlated prepayment increases the probabilities of freeing an insurer from the obligation is negatively related to the premium. Because the estimated parameters of a borrower's default rate λ_r^d, λ_1^d and λ_2^d in (13) and (14) are virtually identical to zero, the correlated default dominated by the correlated prepayment will, in turn, lead to a decline in the premium. Overall, if the borrower's correlated effects are not considered, the premium in Case 1 should be higher than that in Case 2.

Second, comparing Case 2 with Cases 3 and 4, we can see that the extra effects induced by the lender's and insurer's correlated defaults are negatively related to the premiums. Considering the correlated defaults in both the lender and the insurer will increase the default probabilities as well as the probabilities that the MI contracts will be terminated early. Therefore, similar to the MI

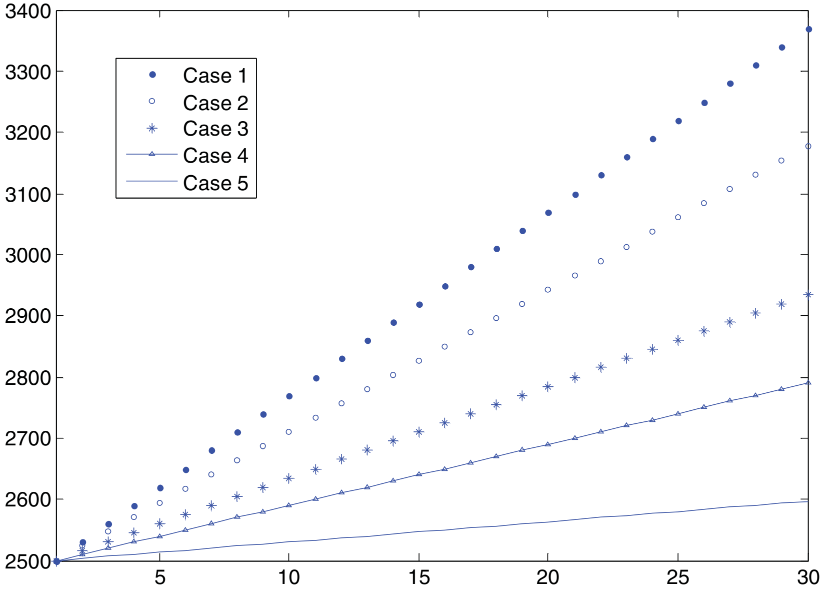

FIGURE 4: *Upfront case*: the relationship between the premium and the loan maturity under different cases. Case 1: the borrower without correlated effects (dotted line). Case 2: the borrower with correlated effects (diamond line). Case 3: the lender and the borrower with correlated effects (asterisk line). Case 4: all the participants with correlated effects (triangle line). Case 5: general model considering not only correlated effects but also counterparty default risk (solid line). (Color online)

contract with a shorter loan maturity, the correlated default is negatively related to the premiums.

Finally, the counterparty default risk, under which the borrower's default can trigger a jump in the insurer's default rate, increases the probability of joint defaults for the insurer and the borrower while decreasing the effectiveness of the insurer's protection against the borrower's default. Therefore, the premium in Case 5 — with the counterpart default risk — is lower than that in Case 4. Furthermore, if the counterparty default risk is considered, a longer time to maturity leads to a bigger decline in premium.

Other features that can be observed in Figure 2 are also presented in Figures 3 and 4. In brief, the correlated effects, time to loan maturity and counterparty default risk, all play highly important roles in the pricing of MI contracts. Without considering either the correlated effects or the counterparty default risk, the premiums would be overpriced regardless of what the premium structures are, especially for long-term mortgages.

6. CONCLUSIONS

Mortgage defaults have risen sharply during the subprime mortgage crisis, with major adverse consequences for both lenders and insurers. Therefore, the lender's and insurer's default risks triggered by the mortgage defaults should be seriously considered to prevent MI premiums from being overpriced. Further, according to previous empirical studies, the default and prepayment rates depend closely on macroeconomic variables, such as the interest rates, housing price and employment rates. Using the reduced-form approach, we consider the correlated effects along with the counterparty default risk to price suitable premiums for three MI structures: premium refunds, annual premiums and upfront premiums.

Sensitivity analysis shows that the MI premium is an increasing function of loan maturity. Corresponding to a given maturity, the correlated defaults of the lender and the insurer are negatively related to the MI premiums. Triggered by the mortgage defaults, the counterparty default risk increases the probability of joint defaults for both the insurer and the borrower and decreases the effectiveness of the insurer's protection against mortgage defaults. The premium is lower and the decline in premium for a mortgage with longer maturity is larger if the counterparty default risk is considered.

In conclusion, correlated effects, time to loan maturity and counterparty default risk play crucial roles in the pricing of MI contracts. Without considering either the correlated effects or the counterparty default risk, the premiums would be overpriced regardless of the premium structures, especially for long-term mortgages.

There are several potential improvements to, and possible extensions of, this model. First, we suggest modeling the house price index as a jump diffusion model (Chen *et al.*, 2010) or an ARMA-GARCH model (e.g., Brown *et al.*, 1997; Crawford and Fratantoni, 2003) for further research. Next, this paper fails to consider that an insurer default may impact a lender default and that a lender default does not imply that the lender's claims disappear. Therefore, it is also an important issue in pricing MI contracts.

NOTES

1. Some risk factors (e.g., the unemployment rate used in numerical applications) in our model are not necessarily freely traded, the market considered is well incomplete and the risk neutral measure is not unique. For simplification, we follow Chang *et al.* (2012) to assume that the market is complete and arbitrage-free.

2. This model is widely used in the mortgage pricing literature. See, for example, Liao, Tsai and Chiang (2008).

3. (6) is true only if the market is complete. In certain circumstances, the growth rate can set to another value if the risk is not traded and if the market is risk averse. However, we follow some pricing studies with non-traded mortality/longevity risk or macroeconomic factors risk by using the arbitrage-free or risk-neutral pricing framework (e.g., Milevsky and Promislow, 2001; Dahl, 2004; Blake, Cairns and Dowd, 2006; Chang *et al.*, 2012).

4. If the mortgage insurance is payable to the holder of the mortgage regardless of the survival or failure of the mortgage lender (i.e., mortgage insurance contract will not be early terminated if the lender goes bankrupt), we can assume $\lambda_0^L = \lambda_1^L = \ldots = \lambda_n^L = \alpha_0^L = 0$ in the closed-form formula to compute the mortgage premiums for this special case.

5. The quasi-Newton algorithm is used to estimate the parameters. The initial value of α, θ and σ_r is set as 0.2, 0.006 and 0.02, respectively, by using the moment method proposed

by Stanton (1995) and the daily data of three-month Treasury rates from January 2004 to April 2008.

6. The parameters of Case 1 are given by $\lambda_0^d = 0.019$, $\lambda_0^p = 0.381$, $\lambda_0^I = \lambda_0^L = \alpha_0^L = \alpha_0^I = 0$, $\lambda_r^d = \lambda_r^p = \lambda_1^d = \lambda_1^p = \lambda_2^d = \lambda_2^p = \lambda_r^L = \lambda_r^I = \lambda_1^L = \lambda_1^I = \lambda_2^L = \lambda_2^I = 0$; the parameters of Case 2 are expressed as $\lambda_0^d = 0.019$, $\lambda_0^p = 0.381$, $\lambda_0^I = \lambda_0^L = \alpha_0^L = \alpha_0^I = 0$, $\lambda_r^d = 0.054$, $\lambda_r^p = 0.586$, $\lambda_1^d = -0.003$, $\lambda_1^p = 0.104$, $\lambda_2^d = 0.0034$, $\lambda_2^p = -0.114$, $\lambda_r^L = \lambda_r^I = \lambda_1^L = \lambda_1^I = \lambda_2^L = \lambda_2^I = 0$; the parameters of Case 3 are set as $\lambda_0^d = 0.019$, $\lambda_0^p = 0.381$, $\lambda_0^L = 0.01$, $\lambda_0^I = \alpha_0^L = \alpha_0^I = 0$, $\lambda_r^d = 0.054$, $\lambda_r^p = 0.586$, $\lambda_1^d = -0.003$, $\lambda_1^p = 0.104$, $\lambda_2^d = 0.0034$, $\lambda_2^p = -0.114$, $\lambda_r^L = 0.05$, $\lambda_1^L = \lambda_2^L = 0.01$, $\lambda_r^I = \lambda_1^I = \lambda_2^I = 0$; the parameters of Case 4 satisfy $\lambda_0^d = 0.019$, $\lambda_0^p = 0.381$, $\lambda_0^L = \lambda_0^I = 0.01$, $\alpha_0^L = \alpha_0^I = 0$, $\lambda_r^d = 0.054$, $\lambda_r^p = 0.586$, $\lambda_1^d = -0.003$, $\lambda_1^p = 0.104$, $\lambda_r^L = \lambda_r^I = 0.05$, $\lambda_1^L = \lambda_1^I = \lambda_2^L = \lambda_2^I = 0.01$; and the parameters of Case 5 are defined as $\lambda_0^d = 0.019$, $\lambda_0^p = 0.381$, $\lambda_0^L = \lambda_0^I = 0.01$, $\alpha_0^L = \alpha_0^I = 0.1$, $\lambda_r^d = 0.054$, $\lambda_r^d = 0.054$, $\lambda_r^p = 0.586$, $\lambda_1^d = -0.003$, $\lambda_1^p = 0.104$, $\lambda_2^d = 0.0034$, $\lambda_2^p = -0.114$, $\lambda_r^L = \lambda_r^I = 0.05$, $\lambda_1^L = \lambda_1^I = \lambda_2^L = \lambda_2^I = 0.01$.

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APPENDIX A

In the appendix, the derivation of (7) is provided. In view of (7), it can be rewritten as

$$\begin{aligned} ER &\equiv E^Q\left\{\int_t^T \frac{B(t)}{B(s)}\left[c_s R_s 1_{\{\tau^d>s,\tau^p>s,\tau^I>s,\tau^L>s\}} - 1_{\{\tau^d>s,t<\tau^P\le s\}} g_s R_t\right]ds \mid F_t\right\} \\ &= E^Q\left[\int_t^T \frac{B(t)}{B(s)} c_s R_s 1_{\{\tau^d>s,\tau^p>s,\tau^I>s,\tau^L>s\}} ds \mid F_t\right] \\ &\quad - E^Q\left[\int_t^T \frac{B(t)}{B(s)} 1_{\{\tau^d>s,t<\tau^P\le s\}} g_s R_t ds \mid F_t\right] \equiv A - B. \end{aligned}$$

Applying the law of iterated expectation, A satisfies

$$\begin{aligned} A &= E^Q\left\{\int_t^T \frac{B(t)}{B(s)} c_s R_s 1_{\{\tau^d>s\}} 1_{\{\tau^p>s\}} \right. \\ &\quad \left. \times E^Q\left[1_{\{\tau^L>s\}} 1_{\{\tau^I>s\}} \mid F^r_{T^*} \vee F^M_{T^*} \vee H^d_{T^*} \vee H^p_{T^*} \vee H^I_t \vee H^L_t\right] ds \mid F_t\right\} \\ &= E^Q\left\{\int_t^T \frac{B(t)}{B(s)} c_s R_s 1_{\{\tau^d>s\}} 1_{\{\tau^p>s\}} 1_{\{\tau^L>t\}} \exp\left(\int_0^t \lambda^L(u)du\right) \right. \\ &\quad \left. \times E^Q\left[1_{\{\tau^L>s\}} 1_{\{\tau^I>s\}} \mid F^r_{T^*} \vee F^M_{T^*} \vee H^d_{T^*} \vee H^p_{T^*} \vee H^I_t\right] ds \mid F_t\right\} \\ &= E^Q\left\{\int_t^T \frac{B(t)}{B(s)} c_s R_s 1_{\{\tau^d>s\}} 1_{\{\tau^p>s\}} 1_{\{\tau^L>t\}} 1_{\{\tau^I>t\}} \exp\left(\int_0^t \lambda^L(u)+\lambda^I(u)du\right) \right. \\ &\quad \left. \times E^Q\left[1_{\{\tau^L>s\}} 1_{\{\tau^I>s\}} \mid F^r_{T^*} \vee F^M_{T^*} \vee H^d_{T^*} \vee H^p_{T^*}\right] ds \mid F_t\right\}. \end{aligned} \tag{A1}$$

Under the complete information of $F^r_{T^*} \vee F^M_{T^*} \vee H^d_{T^*} \vee H^p_{T^*}$, the default time and prepayment time of the borrower are known, and the default time of the lender τ^L and the default time of the insurer τ^I are conditional independent. Hence, (A1) can be rewritten as follows:

$$
\begin{aligned}
A = {} & E^Q\left\{\int_t^T \frac{B(t)}{B(s)} c_s R_s 1_{\{\tau^d>s\}} 1_{\{\tau^p>s\}} 1_{\{\tau^L>t\}} 1_{\{\tau^I>t\}}\right.\\
& \times \exp\left(\int_0^t [\lambda^L(u)+\lambda^I(u)]du\right) P\left[\tau^L > s \,\middle|\, F^r_{T^*} \vee F^M_{T^*} \vee H^d_{T^*} \vee H^p_{T^*}\right]\\
& \left.\times P\left[\tau^I > s \,\middle|\, F^r_{T^*} \vee F^M_{T^*} \vee H^d_{T^*} \vee H^p_{T^*}\right] ds \,|F_t\right\}\\
= {} & E^Q\left\{\int_t^T \frac{B(t)}{B(s)} c_s R_s 1_{\{\tau^d>s\}} 1_{\{\tau^p>s\}} 1_{\{\tau^L>t\}} 1_{\{\tau^I>t\}} \exp\left(\int_0^t [\lambda^L(u)+\lambda^I(u)]du\right)\right.\\
& \left.\times \exp\left(-\int_0^s \lambda^L(u)du\right) \exp\left(-\int_0^s \lambda^I(u)du\right) ds|F_t\right\}\\
= {} & 1_{\{\tau^d>t\}} 1_{\{\tau^p>t\}} E^Q\left[\int_t^T c_s R_s 1_{\{\tau^d>s\}} 1_{\{\tau^p>s\}}\right.\\
& \left.\times \exp\left(-\int_t^s [r(u)+\lambda^L(u)+\lambda^I(u)]du\right) ds \,|F_t\right].
\end{aligned} \tag{A2}
$$

Substituting the linear hazard rate functions of the lender and the insurer into the following equation,

$$
1_{\{\tau^d>s\}} 1_{\{\tau^p>s\}} \exp\left(-\int_t^s [r(u)+\lambda^L(u)+\lambda^I(u)]du\right),
$$

we have

$$
\begin{aligned}
& 1_{\{\tau^d>s\}} 1_{\{\tau^p>s\}} \exp\left\{-\int_t^s \left[\left(\lambda_0^L+\lambda_0^I\right) + \left(1+\lambda_r^L+\lambda_r^I\right) r(u)\right.\right.\\
& \quad + \left(\lambda_1^L+\lambda_1^I\right)\log\left(\frac{M_1(u)}{B(u)}\right) + \dots\\
& \quad \left.\left. + (\lambda_n^L+\lambda_n^I)\log\left(\frac{M_n(u)}{B(u)}\right)\right]du - 1_{\{t<\tau^d\le s\}}\left(\alpha_0^L+\alpha_0^I\right))(s-\tau^d)\right\}.
\end{aligned} \tag{A3}
$$

Due to the fact that $1_{\{t<\tau^d \le s\}}(\alpha_0^L + \alpha_0^I)(s-\tau^d)$ is clearly 0 conditional on the set $\{\tau^d > s\}$, (A3) is expressed by

$$1_{\{\ \tau^d > s\}} 1_{\{\tau^p > s\}} \exp\left\{ - \int_t^s \left[\left(\lambda_0^L + \lambda_0^I\right) + \left(1 + \lambda_r^L + \lambda_r^I\right) r(u) \right.\right.$$
$$+ \left(\lambda_1^L + \lambda_1^I\right) \log\left(\frac{M_1(u)}{B(u)}\right) + \ldots$$
$$\left.\left. + \left(\lambda_n^L + \lambda_n^I\right) \log\left(\frac{M_n(u)}{B(u)}\right)\right] du \right\}. \tag{A4}$$

Substituting (A4) into (A2) and using the law of iterated expectation, we have

$$A = 1_{\{\ \tau^d > t\}} 1_{\{\tau^p > t\}} E^Q \left\{ \int_t^T c_s R_s \exp\left(- \int_t^s \left[\left(\lambda_0^L + \lambda_0^I\right) \right.\right.\right.$$
$$+ \left(1 + \lambda_r^L + \lambda_r^I\right) r(u) + \left(\lambda_1^L + \lambda_1^I\right) \log\left(\frac{M_1(u)}{B(u)}\right)$$
$$\left.\left. + \ldots + \left(\lambda_n^L + \lambda_n^I\right) \log\left(\frac{M_n(u)}{B(u)}\right)\right] du \right)$$
$$\left. \times\ E^Q \left[1_{\{\ \tau^d > s\}} 1_{\{\tau^p > s\}} \,\middle|\, F_{T^*}^r \vee F_{T^*}^M \vee H_t^d \vee H_t^p \vee H_t^L \vee H_t^I \right] ds | F_t \right\}$$
$$= 1_{\{\tau^L > t\}} 1_{\{\tau^I > t\}} 1_{\{\ \tau^d > t\}} 1_{\{\tau^p > t\}} \int_t^T c_s R_s \exp\left[- \left(\lambda_0^L + \lambda_0^I + \lambda_0^d + \lambda_0^p\right)(s-t)\right]$$
$$\times E^Q \left\{ \exp\left(- \int_t^s \left[(1 + \lambda_r^L + \lambda_r^I + \lambda_r^d + \lambda_r^p) r(u) \right.\right.\right.$$
$$+ \left(\lambda_1^L + \lambda_1^I + \lambda_1^d + \lambda_1^p\right) \log\left(\frac{M_1(u)}{B(u)}\right)$$
$$\left.\left.\left. + \ldots + \left(\lambda_n^L + \lambda_n^I + \lambda_n^d + \lambda_n^p\right) \log\left(\frac{M_n(u)}{B(u)}\right)\right] du \right) | F_t \right\} ds. \tag{A5}$$

Assuming $X_0^{t,s} \equiv \int_t^s r(u) du$ and $X_i^{t,s} \equiv \int_t^s \log[\frac{M_i(u)}{B(u)}] du$, $i = 1, 2, \ldots, n$, we obtain

$$X_i^{t,s} = \int_t^s \log\left[\frac{M_i(u)}{B(u)}\right] du = \int_t^s \left(\log\left[\frac{M_i(t)}{B(t)}\right] - \frac{1}{2}\sigma_i^2 (u-t) + \sigma_i \int_t^u dW_v^i \right) du$$
$$= \log\left[\frac{M_i(t)}{B(t)}\right](s-t) + \frac{1}{2}\sigma_i^2 \int_t^s (u-t) du - \sigma_i \int_t^s (s-v) dW_v^i.$$

Without loss of generality, we assume that $M_i(t)/B(t) = 1$; therefore, we have

$$\begin{aligned}
X_i^{t,s} &= \frac{(s-t)^2}{4}\sigma_i^2 - \sigma_i \int_t^s (s-v)\, dW_v^i, \quad i = 1, 2, \ldots, n, \\
\mu_{X_i}^{t,s} &= E^Q\left(X_i^{t,s}\right) = \frac{(s-t)^2}{4}\sigma_i^2, \\
\sigma_{X_i^{t,s} X_i^{t,s}} &= \operatorname{Var}\left(X_i^{t,s}\right) = \frac{(s-t)^3}{3}\sigma_i^2, \quad i = 1, 2, \ldots, n,
\end{aligned}$$

where $E^Q(\cdot)$ and $\operatorname{Var}(\cdot)$ are correspondingly the conditional expectation and variance with respect to F_t.

Let $B(t, T)$ be the time t price of a zero coupon bond paying \$1 at time T. As proved by Heath *et al.* (1992) and Chiarella and Kwon (2001), the dynamic of the price of a zero coupon bond is

$$\frac{dB(t, T)}{B(t, T)} = r(t)dt + b(t, T)dW_t^r, \tag{A6}$$

where $b(u, s) = -\sigma_r D(u, s)$, $D(u, s) = \int_u^s \exp\{-[\Phi(y) - \Phi(u)]\}dy$ and $\Phi(u) = \int_0^u \alpha(y)dy$. By virtue of (A6), using Ito's lemma, we have

$$\ln B(s, s) = \ln B(t, s) + \int_t^s r(u)du - \frac{1}{2}\int_t^s b(u, s)^2 du + \int_t^s b(u, s)dW_u^r.$$

Since $B(s, s) = 1$ and $B(t, s) = \exp(-\int_t^s f(t, u)du)$ by the definition of forward rate, we obtain

$$\begin{aligned}
X_0^{t,s} &\equiv \int_t^s r(u)du = \int_t^s f(t, u)du + \frac{1}{2}\int_t^s b(u, s)^2 du - \int_t^s b(u, s)dW_u^r, \\
\mu_{X_0}^{t,s} &\equiv E^Q\left[X_0^{t,s}\right] = \int_t^s f(t, u)du + \frac{1}{2}\sigma_{X_0^{t,s} X_0^{t,s}}, \\
\sigma_{X_0^{t,s} X_0^{t,s}} &\equiv \operatorname{Var}[X_0^{t,s}] = \int_t^s b(u, s)^2 du, \\
\sigma_{X_0^{t,s} X_i^{t,s}} &\equiv \operatorname{Cov}\left[-\int_t^s b(u, s)\, dW_u^r,\ \sigma_i \int_t^s (s-v)\, dW_v^i\right] \\
&= -\sigma_i \rho_{ri} \int_t^s (s-u) b(u, s)\, du, \quad i = 1, 2, \ldots, n,
\end{aligned}$$

where Cov$(\cdot, \cdot)$ denotes the covariance function conditional on F_t. In sum, (A5) can be rewritten as

$$A = 1_{\{\tau^d > t\}} 1_{\{\tau^p > t\}} 1_{\{\tau^I > t\}} 1_{\{\tau^L > t\}} \int_t^T c_s R_s \exp\left(- \left(\lambda_0^K \right) (s - t) + \Gamma' \mu_X^{t,s} + \frac{\Gamma' \Sigma_t^{s,s} \Gamma}{2} \right) ds,$$

where $\lambda_0^K = \lambda_0^L + \lambda_0^I + \lambda_0^d + \lambda_0^p$, $\lambda_r^K = 1 + \lambda_r^L + \lambda_r^I + \lambda_r^d + \lambda_r^p$, $\lambda_i^K = \lambda_i^L + \lambda_i^I + \lambda_i^d + \lambda_i^p$, $\Gamma = [-\lambda_r^K, -\lambda_1^K, \ldots, -\lambda_n^K]'$ and $\mu_X^{t,s} = [\mu_{X_0}(t,s), \mu_{X_1}(t,s), \ldots, \mu_{X_n}(t,s)]'$; $\Sigma_t^{s,s} = (\sigma_{X_i^{t,s} X_j^{t,s}})$ is a $(n+1)$−by−$(n+1)$ covariance matrix; $\sigma_{X_i^{t,s} X_j^{t,s}}$ is the covariance between $X_i^{t,s}$ and $X_j^{t,s}$, $i = 0, 1, 2, \ldots, n$, $j = 0, 1, 2, \ldots, n$, and satisfies

$$\left[X_0^{t,s}, X_1^{t,s}, \ldots, X_n^{t,s} \right] = \left[\int_t^s r(u) du, \int_t^s \log \left[\frac{M_1(u)}{B(u)} \right] du, \ldots, \int_t^s \log \left[\frac{M_n(u)}{B(u)} \right] du \right].$$

Similarly, we can derive the solution of B as follows:

$$\begin{aligned} B &= E^Q \left[\int_t^T \frac{B(t)}{B(s)} 1_{\{\tau^d > s, t < \tau^P \le s\}} g_s R_t ds \,\middle|\, F_t \right] \\ &= 1_{\{\tau^P > t\}} E^Q \left[\int_t^T \frac{B(t)}{B(s)} 1_{\{\tau^d > s\}} g_s R_t ds \,\middle|\, F_t \right] \\ &\quad - E^Q \left[\int_t^T \frac{B(t)}{B(s)} 1_{\{\tau^d > s, \tau^P > s\}} g_s R_t ds \,\middle|\, F_t \right] \equiv C - D. \end{aligned}$$

Using the law of iterated expectation, we have

$$\begin{aligned} C &= 1_{\{\tau^P > t\}} 1_{\{\tau^d > t\}} \int_t^T g_s R_t E^Q \left[1_{\{\tau^d > s\}} \exp \left(- \int_t^s r(u) du \right) \right] ds \\ &= 1_{\{\tau^P > t\}} 1_{\{\tau^d > t\}} \int_t^T g_s R_t \exp \left(- \lambda_0^d (s - t) + \phi' \mu_X^{t,s} + \frac{\phi' \Sigma_t^{s,s} \phi}{2} \right) ds, \end{aligned}$$

where $\phi = [-(1+\lambda_r^d), -\lambda_1^d, \ldots, -\lambda_n^d]'$. Furthermore, D can be obtained in the same way as follows:

$$
\begin{aligned}
D &= E^Q\left[\int_t^T \frac{B(t)}{B(s)} 1_{\{\tau^d>s,\tau^p>s\}} g_s R_t ds \,\middle|\, F_t\right] \\
&= \int_t^T g_s R_t E^Q\left[\exp\left(-\int_t^s [r(u)+\lambda^d(u)+\lambda^p(u)]du\right) ds \,\middle|\, F_t\right] \\
&= 1_{\{\tau^d>t\}} 1_{\{\tau^p>t\}} \int_t^T g_s R_t \exp\left[-\left(\lambda_0^d+\lambda_0^p\right)(s-t)\right] \\
&\quad \times E^Q\left\{\exp\left(-\int_t^s \left[(1+\lambda_r^d+\lambda_r^p) r(u)\right.\right.\right. \\
&\quad \left.\left.\left. + \sum_{i=1}^n (\lambda_i^d+\lambda_i^p)\log\left(\frac{M_i(u)}{B(u)}\right)\right] du\right) \,\middle|\, F_t\right\} ds \\
&= 1_{\{\tau^p>t\}} \int_t^T g_s R_t \exp\left(-\lambda_0^p(s-t) + \Upsilon'\mu_X^{t,s} + \frac{\Upsilon'\Sigma_t^{s,s}\Upsilon}{2}\right) ds,
\end{aligned}
$$

where $\Upsilon = [-(1+\lambda_r^d+\lambda_r^p), -(\lambda_1^d+\lambda_1^p), \ldots, -(\lambda_n^d+\lambda_n^p)]'$.

APPENDIX B

In the appendix, the derivation of (8) is provided. Using the law of iterated expectation, we have

$$
\begin{aligned}
EL &= E^Q\left\{E^Q\left[\int_t^T \frac{B(t)}{B(\tau^d)} L_R R_{\tau^d} 1_{\{t<\tau^d\le s\}} 1_{\{\tau^p>\tau^d\}}\right.\right. \\
&\quad \left.\left. \times 1_{\{\tau^L>\tau^d\}} 1_{\{\tau^I>\tau^d\}} ds \,\middle|\, F_{T*}^r \vee F_{T*}^M \vee H_t^d \vee H_{T*}^p \vee H_{T*}^L \vee H_{T*}^I\right] \,\middle|\, F_t\right\} \\
&= E^Q\left\{\int_t^T L_R 1_{\{\tau^d>t\}} 1_{\{\tau^p>t\}} \exp\left(\int_0^t [\lambda^d(u)+\lambda^p(u)]\,du\right)\right. \\
&\quad \times E^Q\left[\frac{B(t)}{B(\tau^d)} R_{\tau^d} 1_{\{t<\tau^d\le s\}} 1_{\{\tau^p>\tau^d\}}\right. \\
&\quad \left.\left. \times 1_{\{\tau^I>\tau^d\}} ds \,\middle|\, F_{T*}^r \vee F_{T*}^M \vee H_{T*}^L \vee H_{T*}^I\right] \,\middle|\, F_t\right\}
\end{aligned}
$$

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$$= 1_{\{\tau^d>t\}} 1_{\{\tau^p>t\}} E^Q \left\{ \int_t^T L_R \exp\left(\int_0^t [\lambda^d(u) + \lambda^p(u)]\, du \right) \right.$$
$$\times \int_t^{>s} 1_{\{\tau^p>s\}} 1_{\{\tau^I>v\}} 1_{\{\tau^L>v\}} \frac{B(t)}{B(v)} R_v \lambda^d(v)$$
$$\left. \times \exp\left(-\int_0^v [\lambda^d(u)]\, du \right) dv\, ds \,\middle|\, F_t \right\}$$
$$= 1_{\{\tau^d>t\}} 1_{\{\tau^p>t\}} E^Q \left\{ \int_t^T L_R \int_t^s \frac{B(t)}{B(v)} 1_{\{\tau^p>v\}} 1_{\{\tau^L>v\}} 1_{\{\tau^I>v\}} \right.$$
$$\left. \times R_v \lambda^d(v) \exp\left(-\int_t^v [\lambda^d(u)]\, du \right) dv\, ds \,\middle|\, F_t \right\}.$$

Since the default rates of the lender and the insurer are a function of the interest rate, common risk factors and the default time of the borrower, using the law of iterated expectation, we obtain

$$EL = 1_{\{\tau^d>t\}} 1_{\{\tau^p>t\}} E^Q \left\{ \int_t^T L_R \int_t^s E^Q \left[\frac{B(t)}{B(v)} 1_{\{\tau^p>v\}} 1_{\{\tau^L>v\}} 1_{\{\tau^I>v\}} R_v \lambda^d(v) \right. \right.$$
$$\left. \left. \times \exp\left(-\int_t^v [\lambda^d(u)]\, du \right) dv \,\middle|\, F^r_{T*} \vee F^M_{T*} \vee H^d_{T^*} \vee H^p_{T^*} \vee H^L_t \vee H^I_t \right] ds \,\middle|\, F_t \right\}$$
$$= 1_{\{\tau^d>t\}} 1_{\{\tau^p>t\}} 1_{\{\tau^I>t\}} 1_{\{\tau^L>t\}}$$
$$\times E^Q \left\{ \int_t^T L_R \exp\left(\int_0^t [\lambda^p(u) + \lambda^L(u) + \lambda^I(u)]\, du \right) \right.$$
$$\times E^Q \left[\int_t^s 1_{\{\tau^p>v\}} 1_{\{\tau^I>v\}} 1_{\{\tau^L>v\}} \frac{B(t)}{B(v)} R_v \lambda^d(v) \right.$$
$$\left. \left. \times \exp\left(-\int_t^v [\lambda^d(u)]\, du \right) dv \,\middle|\, F^r_{T^*} \vee F^M_{T^*} \vee H^d_{T^*} \right] ds \,\middle|\, F_t \right\}$$
$$= 1_{\{\tau^d>t\}} 1_{\{\tau^p>t\}} 1_{\{\tau^I>t\}} 1_{\{\tau^L>t\}} E^Q \left\{ \int_t^T L_R\, E^Q \left[\int_t^s \frac{B(t)}{B(v)} R_v \lambda^d(v) \right. \right.$$
$$\left. \left. \times \exp\left(-\int_t^v [\lambda^d(u) + \lambda^p(u) + \lambda^I(u) + \lambda^L(u)]\, du \right) dv \right] ds \,\middle|\, F_t \right\}$$
$$= 1_{\{\tau^d>t\}} 1_{\{\tau^p>t\}} 1_{\{\tau^I>t\}} 1_{\{\tau^L>t\}} E^Q \left\{ \int_t^T L_R E^Q \left[\int_t^s \frac{B(t)}{B(v)} R_v \lambda^d(v) \right. \right.$$

$$\times \exp\left(-\int_t^v [\lambda^d(u)+\lambda^p(u)+\lambda^I(u)+\lambda^L(u)]\,du\right)dv$$
$$\times \left|F_t \vee F^r_{T^*} \vee F^M_{T^*}\right] ds \,|F_t\Big\}. \tag{B1}$$

Using Fubini's theorem and substituting (2) and (4) into (B1), we have

$$EL = 1_{\{\tau^d>t\}}1_{\{\tau^p>t\}}1_{\{\tau^I>t\}}1_{\{\tau^L>t\}}E^Q\left\{\int_t^T L_R\int_t^s R_v\lambda^d(v)\right.$$
$$\times \exp\left(-\int_t^v\left[\lambda_0^K+\lambda_r^K r(u)+\lambda_1^K\log\left(\frac{M_1(u)}{B(u)}\right)\right.\right.$$
$$\left.\left.+\ldots+\lambda_n^K\log\left(\frac{M_n(u)}{B(u)}\right)\right]du\right)$$
$$\times E^Q\left[\exp\left((\alpha_0^I+\alpha_0^L)(v-\tau^d)1_{\{t<\tau^d\le v\}}\right)\left|F_t\vee F^r_{T^*}\vee F^M_{T^*}\right]dv\,ds\,|F_t\right\}. \tag{B2}$$

Assuming that Z is the conditional expectation under the information $F_t \vee F^r_{T^*} \vee F^M_{T^*}$ in (B2), we have

$$Z = \left(\int_t^v+\int_v^\infty\right)\exp\left(-\left(\alpha_0^I+\alpha_0^L\right)(v-x)1_{\{t<x\le v\}}\right)\lambda^d(x)$$
$$\times\exp\left(-\lambda_0^d(x-t)-\int_t^x\left[\lambda_r^d\, r(y)+\lambda_1^d\log\left(\frac{M_1(y)}{B(y)}\right)\right.\right.$$
$$\left.\left.\times +\ldots+\lambda_n^d\log\left(\frac{M_n(y)}{B(y)}\right)\right]dy\right)dx$$
$$=\exp\left(-\left(\alpha_0^I+\alpha_0^L\right)(v-t)\right)\Big\{1+\left(\alpha_0^I+\alpha_0^L\right)$$
$$\times\int_t^v\exp\left(-[\lambda_0^d-\left(\alpha_0^I+\alpha_0^L\right)](x-t)\right.$$
$$\left.-\int_t^x\left[\lambda_r^d\, r(y)+\sum_{i=1}^n\lambda_i^d\log\left(\frac{M_i(y)}{B(y)}\right)\right]dy\right)dx\Big\}. \tag{B3}$$

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Substituting (B3) into (B2), we have

$$
\begin{aligned}
&1_{\{\tau^d>t\}}1_{\{\tau^p>t\}}1_{\{\tau^I>t\}}1_{\{\tau^L>t\}}E^Q\Bigg\{\int_t^T L_R\int_t^s R_v\lambda^d(v)\\
&\times\exp\left(-\int_t^v\left[\lambda_0^K+\lambda_r^K r(u)+\sum_{i=1}^n\lambda_i^K\log\left(\frac{M_i(u)}{B(u)}\right)\right]du\right)\\
&\times\exp\left(-(\alpha_0^I+\alpha_0^L)(v-t)\right)\Bigg[1+\left(\alpha_0^I+\alpha_0^L\right)\\
&\times\int_t^v\exp\Bigg(-[\lambda_0^d-(\alpha_0^I+\alpha_0^L)](x-t)-\int_t^x\Bigg[\lambda_r^d\,r(y)\\
&+\sum_{i=1}^n\lambda_i^d\log\left(\frac{M_i(y)}{B(y)}\right)\Bigg]dy\Bigg)\,dx\Bigg]dvds|F_t\Bigg\}.
\end{aligned}
\tag{B4}
$$

(B4) can be divided into two parts, J_1 and J_2, as follows:

$$
\begin{aligned}
J_1 &= 1_{\{\tau^d>t\}}1_{\{\tau^p>t\}}1_{\{\tau^I>t\}}1_{\{\tau^L>t\}}\\
&\times E^Q\left\{\int_t^T L_R\int_t^s\exp\left(-(\alpha_0^I+\alpha_0^L)(v-t)\right)R_v\lambda^d(v)\right.\\
&\left.\times\exp\left(-\int_t^v\left[\lambda_0^K+\lambda_r^K r(u)\;+\sum_{i=1}^n\lambda_i^K\log\left(\frac{M_i(u)}{B(u)}\right)\right]du\right)dvds|\,F_t\right\},\\
J_2 &= 1_{\{\tau^d>t\}}1_{\{\tau^p>t\}}1_{\{\tau^I>t\}}1_{\{\tau^L>t\}}\\
&\times E^Q\Bigg\{\int_t^T L_R\left(\alpha_0^I+\alpha_0^L\right)\int_t^s\exp\left(-\left(\alpha_0^I+\alpha_0^L\right)(v-t)\right)R_v\\
&\times\lambda^d(v)\exp\left(-\int_t^v\left[\lambda_0^K+\lambda_r^K r(u)\;+\;\sum_{i=1}^n\lambda_i^K\log\left(\frac{M_i(u)}{B(u)}\right)\right]du\right)\\
&\times\Bigg[\int_t^v\exp\Bigg(-\left[\lambda_0^d-(\alpha_0^L+\alpha_0^I)\right](x-t)-\int_t^x\Bigg[\lambda_r^d\,r(y)\\
&+\sum_{i=1}^n\lambda_i^d\log\left(\frac{M_i(y)}{B(y)}\right)\Bigg]dy\Bigg)dx\Bigg]dv\;ds|F_t\Bigg\}.
\end{aligned}
\tag{B5}
$$

To compute J_1, using the fact that

$$\frac{\partial E^Q(\exp(\Gamma' X^{t,v} + \Psi' W^{t,v}))}{\partial \Psi'} = E^Q\left(W^{t,v} \exp(\Gamma' X^{t,v} + \Psi' W^{t,v})\right)$$
$$= \exp\left(\Gamma' \mu_X^{t,v} + \Psi' \mu_W^{t,v} + \frac{1}{2}(\Gamma' \Sigma_t^{v,v} \Gamma + 2\Psi' \bar{\Sigma}_t^{v,v} \Gamma + \Psi' \Sigma_t^{v,v} \Psi)\right)$$
$$\times \left(\mu_W^{t,v} + \Sigma_t^{v,v} \Psi + \bar{\Sigma}_t^{v,v} \Gamma\right),$$

we obtain

$$J_1 = 1_{\{\tau^d > t\}} 1_{\{\tau^p > t\}} 1_{\{\tau^I > t\}} 1_{\{\tau^L > t\}} \int_t^T L_R \int_t^s \exp\left(-\left(\lambda_0^K + \alpha_0^I + \alpha_0^L\right)(v - t)\right) R_v$$
$$\times E^Q\left\{\left[\lambda_0^d + \lambda_r^d r(v) + \lambda_1^d \log\left(\frac{M_1(v)}{B(v)}\right) + \ldots + \lambda_n^d \log\left(\frac{M_n(v)}{B(v)}\right)\right]\right.$$
$$\left.\times \exp(\Gamma' X^{t,v}) \mid F_t\right\} dv\, ds.$$

Assuming that $\Psi = [\lambda_r^d, \lambda_1^d, \ldots, \lambda_n^d]'$ and $W^{t,v} = [r(v), \log(\frac{M_1(v)}{B(v)}), \ldots, \log(\frac{M_n(v)}{B(v)})]'$, we have

$$J_1 = 1_{\{\tau^d > t\}} 1_{\{\tau^p > t\}} 1_{\{\tau^I > t\}} 1_{\{\tau^L > t\}} \int_t^T L_R \int_t^s \exp\left(-\left(\lambda_0^K + \alpha_0^I + \alpha_0^L\right)(v - t)\right) R_v$$
$$\times E^Q\left\{\left(\lambda_0^d + \Psi' W^{t,v}\right)\left(\exp\left(\Gamma' X^{t,v}\right) \Big| F_t\right)\right\} dv\, ds$$
$$= 1_{\{\tau^d > t\}} 1_{\{\tau^p > t\}} 1_{\{\tau^I > t\}} 1_{\{\tau^L > t\}} \int_t^T L_R \int_t^s \exp\left(-(\lambda_0^K + \alpha_0^I + \alpha_0^L)(v - t)\right) R_v$$
$$\times \left\{\lambda_0^d E^Q(\exp(\Gamma' X^{t,v}) | F_t) + \Psi' E^Q(W^{t,v} \exp(\Gamma' X^{t,v}) | F_t)\right\} dv\, ds. \tag{B6}$$

Based on

$$\frac{\partial E^Q[\exp(\Gamma' X^{t,v} + \Psi' W^{t,v})]}{\partial \Psi'}\Big|_{\Psi=0} = E^Q(W^{t,v} \exp(\Gamma' X^{t,v}))$$
$$= \exp\left(\Gamma' \mu_X^{t,v} + \frac{1}{2}(\Gamma' \Sigma_t^{v,v} \Gamma)\right)\left(\mu_W^{t,v} + \Sigma_t^{v,v} \Psi\right),$$

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(B6) can be rewritten as

$$J_1 = 1_{\{\tau^d>t\}}1_{\{\tau^p>t\}}1_{\{\tau^I>t\}}1_{\{\tau^L>t\}}\int_t^T L_R \int_t^s \exp\left(-\left(\lambda_0^K+\alpha_0^I+\alpha_0^L\right)(v-t)\right) R_v$$
$$\times\left(\lambda_0^d+\Psi'\mu_W^{t,v}+\Psi'\bar{\Sigma}_t^{v,v}\Gamma\right)\exp\left(\Gamma'\mu_X^{t,v}+\frac{1}{2}(\Gamma'\Sigma_t^{v,v}\Gamma)\right)dv\,ds.$$

Following the similar procedure, we can derive J_2.

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