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COMPONENTS AND MINIMAL NORMAL SUBGROUPS OF FINITE AND PSEUDOFINITE GROUPS

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Abstract. It is proved that there is a formula $\pi(h, x)$ in the first-order language of group theory such that each component and each non-abelian minimal normal subgroup of a finite group G is definable by $\pi(h, x)$ for a suitable element h of G; in other words, each such subgroup has the form $\{x \mid x \models \pi(h, x)\}$ for some h. A number of consequences for infinite models of the theory of finite groups are described.

§1. Introduction. A group G is called *pseudofinite* if it is an infinite model for the first-order theory of finite groups, in other words, if G is infinite and G satisfies all first-order sentences in the language of group theory that hold in all finite groups. The study of pseudofinite groups was begun by Felgner [3] and further developed in [10], [8], and [9]. Some results about finite groups lead directly to results for pseudofinite groups because they can be formulated using first-order formulae. For example, a result of [11] asserts that there is a first-order formula $\rho(x)$ such that an element g of a finite group G satisfies $\rho(g)$ if and only if g lies in the largest soluble normal subgroup $\mathrm{R}(G)$ of G. It follows easily that every pseudofinite group G has a unique largest definable normal subgroup $\mathrm{R}(G)$ that satisfies all first-order sentences holding in all finite soluble groups; moreover $G/\mathrm{R}(G)$ is finite or pseudofinite and $\mathrm{R}(G/\mathrm{R}(G)) = 1$ (see for example [9, Propositions 2.16, 2.17]).

We recall that a finite group is called *quasisimple* if it is perfect and simple modulo its centre, and that a *component* of a finite group G is a quasisimple subnormal subgroup. Distinct components centralize each other, and the components of G can also be described as the quasisimple subgroups L that centralize all conjugates of L distinct from L; cf. Isaacs [5, Chapter 9].

A pseudofinite group S is called *definably simple* if it is has no definable normal subgroups apart from 1 and S. An argument of Felgner ([3]; cf. [10, Proposition 2,7]) shows that such groups are precisely the groups elementarily equivalent to ultraproducts of finite simple groups; in particular, an ultraproduct of alternating groups of unbounded rank is definably simple but not simple. We call a definable subgroup L of a pseudofinite group G a (*definable*) *component* of G if it has no non-trivial definable abelian quotients, is definably simple modulo its centre, and centralizes all conjugates of L distinct from L.

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For a first-order formula $\pi(h,x)$ and a group element h we write $\{x \mid \pi(h,g)\}$ for the set $\{x \mid x \models \pi(h,x)\}$. In Section 4 below we study definable components and minimal definable normal subgroups in pseudofinite groups. In particular, we prove the following result.

THEOREM 1.1. *Let G be a pseudofinite group. Then*

(a) *every non-trivial definable normal subgroup of G contains either a non-trivial abelian normal subgroup of G or a non-abelian minimal definable normal subgroup of G;*
(b) *each non-abelian minimal definable normal subgroup M is of the form $S \times \mathrm{C}_M(S)$ for a definably simple component S;*
(c) *distinct components centralize each other, and so the subgroup generated by finitely many components is their product and is definable;*
(d) *all non-abelian minimal definable normal subgroups and all of the products in (c) are sets of the form $\{x \mid \pi(h,x)\}$ for elements $h \in G$, where $\pi(h,x)$ is a first-order formula independent of G.*

In particular, from (a) and (b), if $\mathrm{R}(G) = 1$ then G has both minimal definable normal subgroups and definably simple components.

For a finite group G, the relation between the various products of components is particularly easy to describe: they constitute a Boolean lattice with respect to natural operations. For a pseudofinite group G there is also such a lattice which is interpretable in G and whose atoms correspond to the components of G (see Proposition 4.5 below). Theorem 1.2 describes the structure of G in the case when this lattice is finite.

THEOREM 1.2. *Let G be a pseudofinite group that has only finitely many definable components and let P be their product. Then G has a characteristic definable subgroup G_1 of finite index such that $G_1/P\mathrm{C}_G(P)$ is metabelian. In particular, if $\mathrm{R}(G) = 1$, then G/P is virtually metabelian.*

The key to these results is a first-order formulation of properties of components and perfect minimal normal subgroups of finite groups.

THEOREM 1.3. *There exist formulae $\pi(h,y)$, $\pi'(h)$ in the first-order language of group theory such that, for every finite group G, the products of components of G are precisely the sets $\{x \mid \pi(h,x)\}$ for the elements h of G satisfying $\pi'(h)$.*

Moreover there exist formulae $\pi'_{\mathrm{c}}(h)$, $\pi'_{\mathrm{m}}(h)$ such that, for each finite G, the components and non-abelian minimal normal subgroups of G are the sets $\{x \mid \pi(h,x)\}$ with h satisfying, respectively, $\pi'_{\mathrm{c}}(h)$ and $\pi'_{\mathrm{m}}(h)$.

The formula $\pi(h,y)$ appearing in Theorem 1.3 is the same one as in Theorem 1.1.

To prove these results we use ideas concerning double centralizers which have proved fruitful in other contexts (cf. [12]). We also need some consequences of the classification of the finite simple groups, including the positive solutions to the Schreier and Ore conjectures. Somewhat weaker structural results can be proved using only the solubility of outer automorphism groups of finite simple groups, by working directly with double centralizers of components.

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§**2. Preliminary results.** We begin by listing the consequences of the classification of finite simple groups that we shall require.

(CFSG) (a) *Each finite quasisimple group can be generated by two elements.*

(b) *Let S be a finite non-abelian simple group. Then*

(i) *the outer automorphism group* $\mathrm{Out}(S) = \mathrm{Aut}(S)/\mathrm{Inn}(S)$ *of S has a metabelian subgroup of index at most 2, and has a series* $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_5 = \mathrm{Out}(S)$ *with G_i/G_{i-1} cyclic for $i = 1, \ldots, 5$.*

(ii) *every element of S is a commutator in S.*

(c) *All elements of finite quasisimple groups are products of two commutators.*

Assertion (a) is an immediate consequence of the fact, proved by Aschbacher and Guralnick [1], that finite non-abelian simple groups can be generated by two elements. For (b)(i), see for example Gorenstein [4], Theorem 4.237 and the discussion preceding it, and for (b)(ii) and (c) see [6] and [7]. Here is an easy and well-known consequence of (CFSG) (b)(i).

Lemma 2.1. *Let M be a perfect finite group with all components normal and with no non-central nilpotent normal subgroups of class at most 2. Then M is the product of its components.*

Proof. Let $L_1, \ldots, L_n$ be the components of M and M_0 their product. Write $S_i = L_i/\mathrm{Z}(L_i)$ for each i (where $\mathrm{Z}(H)$ denotes the centre of a group H). Conjugation in M induces a homomorphism from M to $A = \mathrm{Aut}(S_1) \times \cdots \times \mathrm{Aut}(S_n)$, with kernel D, say, and M_0 maps to $I = \mathrm{Inn}(S_1) \times \cdots \times \mathrm{Inn}(S_n)$. Because each group $\mathrm{Aut}(S_i)/\mathrm{Inn}(S_i)$ is soluble so is A/I, and thus the perfect group M must also map to I. Hence, $M = M_0 D$. Suppose that $D \nleqslant M_0$ and choose $N \lhd M$ minimal with respect to $N \leqslant D$, $N \nleqslant M_0$. Thus, $N/(N \cap M_0)$ is either abelian or a direct product of non-abelian simple groups. Since D centralizes $M_0/\mathrm{Z}(M_0)$ we have $N \cap M_0 \leqslant [D, M_0] \leqslant \mathrm{Z}(M_0) \leqslant \mathrm{Z}(M)$. Hence, either N is nilpotent of class at most 2, or N is perfect and a product of simple groups modulo its centre. In the latter case N is a product of components, and in both cases we have a contradiction. ⊣

Our notation for conjugates and commutators is as follows: $x^y = x^{-1}yx$ and $[x, y] = x^{-1}y^{-1}xy$ for elements x, y of a group G. We write $\mathrm{C}_G^2(X)$ as shorthand for $\mathrm{C}_G(\mathrm{C}_G(X))$ for each subset X of G. Evidently $\langle X \rangle \leqslant \mathrm{C}_G^2(X)$ for each X.

Lemma 2.2. *Let G be a finite group. If L is a component of G then $L \lhd \mathrm{C}_G^2(L)$.*

Proof. Let T be the product of all components $N \neq L$ of G; then $T \leqslant \mathrm{C}_G(L)$ and so $\mathrm{C}_G^2(L) \leqslant \mathrm{C}_G(T)$. If $c \in \mathrm{C}_G(T)$ and $L^c \neq L$ then $L^c \leqslant T$ and $L \leqslant T^{c^{-1}} = T$, a contradiction. ⊣

For each element h of a group G define

$$X_h = \{[h^{-1}, h^g] \mid g \in G\} \quad \text{and} \quad W_h = \bigcup (X_h^f \mid f \in G, [X_h, X_h^f] \neq 1).$$

Lemma 2.3. *Let M be the product of some components $L_1, \ldots, L_r$ of a finite group G.*

(a) *If X is a subset of M whose projection in each $L_i/\mathrm{Z}(L_i)$ is non-trivial then $M = \langle X^g \mid g \in M, [X, X^g] \neq 1 \rangle$.*

(b) *Suppose that M centralizes all conjugates $M^g \neq M$ and let h be an element of M that projects non-trivially to each $L_i/\mathrm{Z}(L_i)$. Then $M = \langle W_h \rangle$.*

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PROOF. (a) Write $H = \langle X \rangle$. Thus, $[X, X^g] \neq 1$ if and only if $[H, H^g] \neq 1$. The following elegant proof based on an idea of Chris Parker replaces an earlier longer argument.

Since $\langle H^g \mid g \in M \rangle$ is normal in M and has non-trivial projection in each $L_i/\mathrm{Z}(L_i)$ we have $\langle H^g \mid g \in M \rangle = M$. Let $K = \langle H^g \mid g \in G, [H, H^g] \neq 1 \rangle$. Then $\mathrm{N}_M(H)$ contains the conjugates of H that centralize H and permutes the remaining conjugates, and so $\mathrm{N}_M(H)$ normalizes K. Thus, $\langle H^g \mid g \in M \rangle \leqslant \langle K, \mathrm{N}_M(H) \rangle = \mathrm{N}_M(H)K$ and $M = \mathrm{N}_M(H)K$. Clearly $H^{g_0} \leqslant K$ for some $g_0 \in M$. Let $g \in M$ and write $g_0 = n_0 k_0$, $g = nk$ with $n_0, n \in \mathrm{N}_M(H)$, $k_0, k \in K$. Then $H^g = H^{nn_0^{-1}g_0k_0^{-1}k} = H^{g_0k_0^{-1}k} \leqslant K^{k_0^{-1}k} = K$, and the result follows.

(b) For each $g \in G$, either g normalizes M or $[M, M^g] = 1$; hence, $[h^{-1}, h^g] \in M$. Therefore $X_h \subseteq M$, and for $f \in G$ we have $X_{h^f} \subseteq M^f$; thus, if X_{h^f} and X_h do not commute then $X_{h^f} \subseteq M$. It follows that $W_h \subseteq M$.

Write $h = h_1 \dots h_r$ with $h_i \in L_i$ for each i. For each i there is an element $s \in L_i$ such that $[h_i^{-1}, h_i^s] \notin \mathrm{Z}(L_i)$, and clearly $[h^{-1}, h^s] = [h_i^{-1}, h_i^s] \in L_i$. Thus the subset $\{[h^{-1}, h^f] \mid f \in M\}$ of M satisfies the hypothesis on X in (a), and hence $M \subseteq \langle W_h \rangle$. ⊣

We define the words δ_r for $r \geqslant 1$ recursively by $\delta_1(x_1, x_2) = [x_1, x_2]$ and

$$\delta_r(x_1, \dots, x_{2^r}) = [\delta_{r-1}(x_1, \dots, x_{2^{r-1}}), \delta_{r-1}(x_{2^{r-1}+1}, \dots, x_{2^r})] \quad \text{for } r > 1.$$

LEMMA 2.4. *Let M be the product of some components $L_1, \dots, L_r$ of a finite group G and let $M \leqslant K \leqslant \mathrm{C}_G^2(M) \cap \bigcap_{i \leqslant r} \mathrm{N}_G(L_i)$. Then M is the set of products of two δ_4-values in K.*

PROOF. By (CFSG) (b)(ii) and induction, every element of a non-abelian finite simple group is a δ_n-value for all n. Therefore, for each i, every element of L_i is congruent modulo $\mathrm{Z}(L_i)$ to a δ_3-value in L_i, and every commutator of two elements of L_i is a δ_4-value in L_i. Thus, from (CFSG) (c) each element of L_i is a product of two δ_4-values in L_i. The corresponding statement now follows for the (external) direct product P of the groups L_i, and since M is a homomorphic image of P, every element of M is a product of two δ_4-values in M.

Write $S_i = L_i/\mathrm{Z}(L_i)$ for each i. Since each L_i is normal in K, conjugation in K yields a homomorphism from K to $A = \mathrm{Aut}(S_1) \times \cdots \times \mathrm{Aut}(S_r)$, with kernel D, say. The image I of M is equal to $\mathrm{Inn}(S_1) \times \cdots \times \mathrm{Inn}(S_r)$. From (CFSG) (b) each group $\mathrm{Aut}(S_i)/\mathrm{Inn}(S_i)$ is soluble of derived length at most 3, and hence so is A/I. Therefore all δ_3-values in A lie in I, and so all δ_3-values in K lie in MD. Since D acts trivially on each S_i we have $[M, D] \leqslant \mathrm{Z}(M)$ and so $[[M, D], M] = 1$. Therefore from the 3-lemma we have $[M, D] = [[M, M], D] = 1$. Thus, $D \leqslant \mathrm{C}_G(\mathrm{C}_G(M)) \cap \mathrm{C}_G(M)$ and so D and MD/M are abelian. It follows that every δ_4-value in K lies in M and the lemma is proved. ⊣

We shall need the following elementary lemma.

LEMMA 2.5. *Let H be a finite non-abelian group with no non-central nilpotent normal subgroups of class at most 2. Then, H has a component.*

PROOF. Let L be a minimal subnormal subgroup subject to $L \nleqslant \mathrm{Z}(H)$. Thus, $L/(L \cap \mathrm{Z}(H))$ is simple and either L is perfect, and hence quasisimple and a component of H, or L is nilpotent. Suppose that L is nilpotent. Then the subgroup N generated by its conjugates is normal and nilpotent. Write N_i for the i-th term

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of the lower central series of N for each i, and let d be the smallest integer with $N_{2d} = 1$. Since $[N_d, N_d] \leqslant N_{2d}$ the subgroup N_d is abelian and normal in H; hence $N_d \leqslant \mathrm{Z}(H)$ and $N_{d+1} = 1$. Thus $d \leqslant 2$ and $N_4 = 1$. But then N_2 is abelian and lies in $\mathrm{Z}(H)$, so that $N_3 = 1$. Therefore $L \leqslant N \leqslant \mathrm{Z}(H)$, and a contradiction ensues. ⊣

Although far better results than the following are known we include an *ad hoc* proof for the reader's convenience.

Lemma 2.6. *Let G be a group having a series $1 = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_n = G$ with cyclic factors and let $K = \langle g^2 \mid g \in G\rangle$. Then* (a) $|G/K| \leqslant 2^n$ *and* (b) *every element of K is a product of $3n$ squares in G.*

Proof. Since G/K has exponent at most 2 and has a series of length at most n with cyclic factors assertion (a) is clear. Assertion (b) is clear for $n = 1$. Let $n \geqslant 2$ and write $A = H_{n-1}$. Then $B = \langle a^2 \mid a \in A\rangle$ is normal in G and by induction we may assume that B consists of products of $3(n-1)$ squares. It suffices now to show that each product of squares in G is congruent modulo B to a product of three squares. Thus, we may pass to G/B and assume that A has exponent at most 2. Write $G = \langle t\rangle A$ and $D = \{a^t a \mid a \in A\}$. Then $D \lhd G$ and G/D is abelian, and so every product of squares in G is congruent to a square modulo D. Since $a^t a = t^{-2}(ta)^2$ for each $a \in A$ the result follows. ⊣

§3. Finite groups: Proof of Theorem 1.3. The following first-order characterization of quasisimple groups is similar to a characterization of simple groups given by Felgner [3].

Proposition 3.1. *Let G be a finite group. Then G is quasisimple if and only if G satisfies the sentence* $\mathrm{QS}_1 \wedge \mathrm{QS}_2 \wedge \mathrm{QS}_3$, *where* QS_1, QS_2, *and* QS_3 *are respectively defined as follows*:

$$(\exists u)(u \neq 1) \wedge (\forall x)(\exists y_1 \exists y_2 \exists y_3 \exists y_4)\ (x = [y_1, y_2][y_3, y_4]);$$
$$(\forall x)((\forall u)[x, x^u] \in \mathrm{Z}(G)) \to x \in \mathrm{Z}(G);$$
$$(\forall x \forall y)(x \notin \mathrm{Z}(G) \wedge \mathrm{C}_G(x, y) > \mathrm{Z}(G) \to \textstyle\bigcap_{g \in G}(\mathrm{C}_G(x, y)\mathrm{C}_G^2(x, y))^g = \mathrm{Z}(G)).$$

These sentences are clearly equivalent to (considerably longer) sentences written in the primitive first-order language of group theory.

Proof. First suppose that G is quasisimple. Then QS_1 holds by (CFSG) (c), while an element x that commutes modulo $\mathrm{Z}(G)$ with all of its conjugates generates an abelian normal subgroup modulo $\mathrm{Z}(G)$, and therefore lies in $\mathrm{Z}(G)$. If $\mathrm{Z}(G) < H < G$ then $H\mathrm{C}_G(H) < G$ and $\bigcap_g (H\mathrm{C}_G(H))^g = \mathrm{Z}(G)$; therefore QS_3 certainly holds.

Now let G be a group satisfying $\mathrm{QS}_1 \wedge \mathrm{QS}_2 \wedge \mathrm{QS}_3$. Then G is non-trivial and perfect and so it has a minimal normal subgroup K subject to $K \nleqslant \mathrm{Z}(G)$. Moreover $K\mathrm{Z}(G)/\mathrm{Z}(G)$ cannot be abelian by QS_2. Therefore G has a component T. By (CFSG) (a) we can find x, y with $T = \langle x, y\rangle$; evidently $x \notin \mathrm{Z}(G)$. Write $H = \mathrm{C}_G(x, y) = \mathrm{C}_G(T)$. Then the product of all components of G is normal and lies in HT, so that $\bigcap_{g \in G}(\mathrm{C}_G(x, y)\mathrm{C}_G^2(x, y))^g \neq \mathrm{Z}(G)$. Therefore by QS_3 we have $H = \mathrm{Z}(G)$; in particular T is the only component of G, and $T \lhd G$. Write $D = \mathrm{C}_G(T/\mathrm{Z}(T))$. Since G/D is perfect and embeds in $\mathrm{Aut}(T/\mathrm{Z}(T))$ we have $G = TD$

by (CFSG) (b)(i). Since T is perfect and $[[T, D], T] = 1$ we have $[T, D] = 1$ from the 3-lemma. Therefore $D = H = Z(G)$ and $G = T$. ⊣

PROOF OF THEOREM 1.3. We begin with the following formulae (cf. [12, Section 4]):

$$\begin{aligned}
\varphi(h, x):&\quad (\exists y)(x = [h^{-1}, h^y]);\\
\psi(h, x):&\quad (\exists t \exists y_1 \exists y_2)(\varphi(h, y_1) \wedge \varphi(h^t, y_2) \wedge \varphi(h^t, x) \wedge\ [y_1, y_2] \neq 1);\\
\gamma^1(h, x):&\quad (\forall y)(\psi(h, y) \rightarrow [x, y] = 1);\\
\gamma(h, x):&\quad (\forall y)(\gamma^1(h, y) \rightarrow [x, y] = 1);\\
\alpha^1(h, x):&\quad (\exists y_1 \cdots \exists y_{16})\big(\textstyle\bigwedge_{n=1}^{16} \gamma(h, y_n)\big) \wedge (x = \delta_4(y_1, \ldots, y_{16})),\\
\alpha(h, x):&\quad (\exists y_1 \exists y_2)(\alpha^1(h, y_1) \wedge \alpha^1(h, y_2) \wedge x = y_1 y_2).
\end{aligned}$$

For a group G and element h these formulae express, respectively, that $x \in X_h$, $x \in W_h$, $x \in C_G(W_h)$, $x \in C_G^2(W_h)$, x is a δ_4-value in $C_G^2(W_h)$ and that x is a product of two δ_4-values in $C_G^2(W_h)$. Let $\alpha'(h)$ be a first-order formula asserting that $\{x \mid \alpha(h, x)\}$ is a subgroup satisfying $QS_1 \wedge QS_2 \wedge QS_3$ that centralizes all of its other conjugates.

Now let G be a finite group and L a component of G. From Lemma 2.3, for each $h \in L \setminus Z(L)$ we have $L = \langle W_h \rangle$ and hence $L \leqslant C_G^2(W_h)$. Lemma 2.4 now shows that L is the set of products of two δ_4-values in $C_G^2(W_h)$ and that $L = \{x \mid \alpha(h, x)\}$. Therefore $\alpha'(h)$ holds by Proposition 3.1. Conversely, if h satisfies $\alpha'(h)$, then $\{x \mid \alpha(h, x)\}$ is a component by Proposition 3.1.

We shall use this characterization of components to find formulae for arbitrary products of components.

Let $K = \bigcap_L (C_G(L) C_G^2(L))$, the intersection being over all components L of G. If L_1, L_2 are components then either $L_1 = L_2$ and hence $L_1 \leqslant C_G^2(L_2)$, or $L_1 \neq L_2$ and hence $L_1 \leqslant C_G(L_2)$. Thus, for each component L we have $L \leqslant K$, and indeed $L \lhd K$ by Lemma 2.2. From above, we have $K = \bigcap (C_G(W_h) C_G^2(W_h))$, the intersection being over all h satisfying $\alpha'(h)$; thus, K is defined by the formula

$$\kappa(x):\ (\forall h)(\alpha'(h) \rightarrow (\exists u \exists v)(x = uv \wedge \gamma^1(h, u) \wedge \gamma(h, v))).$$

Now we introduce formulae like those at the start of the proof but relative to the subgroup K:

$$\begin{aligned}
\varphi_K(h, x):&\quad (\exists y)(\kappa(y) \wedge x = [h^{-1}, h^y]);\\
\psi_K(h, x):&\quad (\exists t \exists y_1 \exists y_2)(\kappa(t) \wedge \varphi_K(h, y_1) \wedge \varphi_K(h^t, y_2) \wedge \varphi_K(h^t, x) \wedge\ [y_1, y_2] \neq 1);\\
\gamma_K^1(h, x):&\quad (\forall y)((\kappa(y) \wedge \psi_K(h, y)) \rightarrow [x, y] = 1);\\
\gamma_K(h, x):&\quad (\forall y)((\kappa(y) \wedge \gamma_K^1(h, y)) \rightarrow [x, y] = 1);\\
\pi^1(h, x):&\quad (\exists y_1\ \cdots \exists y_{16})\big(\big(\textstyle\bigwedge_{n=1}^{16} \gamma_K(h, y_n)\big) \wedge x = \delta_4(y_1, \ldots, y_{16})\big);\\
\pi(h, x):&\quad (\exists y_1 \exists y_2)(\pi^1(h, y_1) \wedge \pi^1(h, y_2) \wedge x = y_1 y_2).
\end{aligned}$$

Finally, let $\pi'(h)$ be the conjunction of $\kappa(h)$ and a first-order formula asserting that $\{x \mid \pi(h, x)\}$ is a normal subgroup of K satisfying QS_1 and QS_2 (that is, every element is a product of two commutators and K has no non-central nilpotent normal subgroups of class at most 2).

If $\pi'(h)$ holds then $\{x \mid \pi(h, x)\}$ is a product of components from Lemma 2.1. We claim that every product of components has this form. Let $L_1, \ldots, L_r$ (with $r \geqslant 0$) be components, write $M = L_1 \ldots L_r$ and choose $h \in M$ with non-trivial projections in all groups $L_i/Z(L_i)$. (Thus, if $r = 0$ we have $M = 1$ and $h = 1$.)

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So $M = \langle\{x \mid \psi_K(h,x)\}\rangle$, from Lemma 2.3 applied to the group K. By Lemma 2.4, M is the set of products of two δ_4-values in $\mathrm{C}_K^2(\{x \mid \psi_K(h,x)\})$ and so $M = \{x \mid \pi(h,x)\}$. Since M is a product of components, the formula $\pi'(h)$ holds and our claim follows.

It is now easy to identify the elements h for which $\{x \mid \pi(h,x)\}$ is a non-abelian minimal normal subgroup or a component. We take $\pi'_{\mathrm{m}}(h) = \pi'(h) \wedge \mu'_1(h) \wedge \mu'_2(h)$ where $\mu'_1(h)$ is the formula

$$(\exists u \neq 1)\pi(h,u) \wedge (\forall x \forall y)(\pi(h,x) \rightarrow \pi(h,x^y)) \wedge (\forall v \neq 1)(\exists t)(\pi(h,v) \rightarrow [v,v^t] \neq 1)$$

asserting that the subgroup $\{x \mid \pi(h,x)\}$ is non-trivial, normal and contains no non-trivial abelian normal subgroup of G, and $\mu'_2(h)$ is the formula

$$(\forall k)(\pi'(k) \wedge \mu'_1(k) \wedge ((\forall x)\pi(k,x) \rightarrow \pi(h,x))) \rightarrow ((\forall y)\pi(h,y) \rightarrow \pi(k,y))$$

asserting minimality of this normal subgroup. For $\pi'_{\mathrm{c}}(h)$ we can take the conjunction of $\pi'(h)$ and a formula asserting that $\{x \mid \pi(h,x)\}$ is minimal among sets $\{x \mid \pi(k,x)\} \neq \{1\}$ for which $\pi'(k)$ holds. ⊣

§4. Pseudofinite groups.

Lemma 4.1. *Let G be a pseudofinite group. The following assertions hold.*

(a) *for all $h_1, h_2 \in G$,*
$$\pi'_{\mathrm{m}}(h_1) \wedge \pi'_{\mathrm{m}}(h_2) \wedge ((\forall x)\pi(h_1,x) \rightarrow \pi(h_2,x)) \rightarrow ((\forall y)\pi(h_1,y) \leftrightarrow \pi(h_2,y))$$
and
$$\pi'_{\mathrm{c}}(h_1) \wedge \pi'_{\mathrm{c}}(h_2) \wedge ((\forall x)\pi(h_1,x) \rightarrow \pi(h_2,x)) \rightarrow ((\forall y)\pi(h_1,y) \leftrightarrow \pi(h_2,y)).$$

(b) *Every non-trivial definable normal subgroup of G contains either a non-trivial abelian normal subgroup of G or a subgroup $\{x \mid \pi(h,x)\}$ with h satisfying $\pi'_{\mathrm{m}}(h)$.*

(c) *Let $h \in G$ satisfy $\pi'_{\mathrm{c}}(h)$. Then $\{x \mid \pi(h,x)\}$ is a (definable) component of G.*

Proof. (a) Since in a finite group the sets $\{x \mid \pi(h,x)\}$ for which $\pi'_{\mathrm{m}}(h)$ (resp. $\pi'_{\mathrm{c}}(h)$) holds are non-abelian minimal normal subgroups (resp. components), the two assertions hold for finite groups. Therefore they hold for G.

(b) Let $\theta(x)$ be a formula that in G defines a non-trivial normal subgroup and $\theta^+(x)$ the conjunction of $\theta(x)$ and a sentence asserting that $\{x \mid \theta(x)\}$ is closed for products, inverses, and conjugates. In each finite group the sentence

$$\begin{aligned}&(\exists a \neq 1)\theta^+(a)\\ &\quad \rightarrow (\exists b \neq 1)(\theta^+(b) \wedge (\forall x)[b,b^x] = 1) \vee (\exists h)(\pi'_{\mathrm{m}}(h) \wedge ((\forall y)\pi(h,y) \rightarrow \theta^+(y)))\end{aligned}$$

holds, since every non-trivial normal subgroup contains a non-trivial abelian normal subgroup or a non-abelian minimal normal subgroup. Therefore this sentence holds in G.

(c) Because of Theorem 1.3 and the properties of components of finite groups, $\pi(h,x)$ defines a subgroup H of G that centralizes all conjugates $H^g \neq H$. It remains to prove that H has no proper non-central definable normal subgroups. Let $\psi(x)$ be a formula that in G defines a non-central normal subgroup of H and $\psi^+(x)$ the conjunction of $\psi(x)$ and a sentence asserting that $\{x \mid \psi(x)\}$

is closed for products, inverses, and H-conjugates. In each finite group the sentence

$$(\forall k)(\pi'_{\mathrm{c}}(k) \wedge (\forall x)(\psi^+(x) \rightarrow \pi(k,x)) \wedge (\forall a \neq 1)(\psi^+(a) \rightarrow (\exists b)(\pi(k,b) \wedge [a,b] \neq 1)) \rightarrow (\forall y)(\pi(k,y) \rightarrow \psi^+(y)))$$

holds, because components of finite groups have no proper non-central normal subgroups. Therefore this sentence holds in G, and our conclusion holds. ⊣

PROPOSITION 4.2. *Let G be a pseudofinite group. The following assertions hold.*

(a) *The non-abelian minimal definable normal subgroups of G are the subgroups $\{x \mid \pi(h,x)\}$ for elements h satisfying $\pi'_{\mathrm{m}}(h)$.*
(b) *The definable components of G are the subgroups $\{x \mid \pi(h,x)\}$ for elements h satisfying $\pi'_{\mathrm{c}}(h)$.*

PROOF. (a) From Lemma 4.1(b), every non-abelian minimal definable normal subgroup has the form $\{x \mid \pi(h,x)\}$ for some h satisfying $\pi'_{\mathrm{m}}(h)$.

Now let $h \in G$ satisfy $\pi'_{\mathrm{m}}(h)$. Then the set $M = \{x \mid \pi(h,x)\}$ is a non-abelian normal subgroup and we must establish minimality. Since in a finite group a non-abelian minimal normal subgroup has no non-trivial element that commutes with all of its conjugates, M contains no non-trivial abelian normal subgroup of G. Let N be a definable normal subgroup contained in M. By Lemma 4.1(b), N contains a subgroup $\{x \mid \pi(k,x)\}$ with k satisfying $\pi'_{\mathrm{m}}(k)$. Therefore $N = M = \{x \mid \pi(k,x)\}$ by Lemma 4.1(a), and minimality follows.

(b) By Lemma 4.1(c) it suffices to show that if H is a definable component of G, defined by a first-order formula $\chi(x)$, then $H = \{x \mid \pi(h,x)\}$ for some h satisfying $\pi'_{\mathrm{c}}(h)$. We may replace χ by its conjunction with a sentence asserting that $\{x \mid \chi(x)\}$ is a non-abelian subgroup that normalizes its conjugates and all of whose nilpotent normal subgroups of class at most 2 lie in its centre. In a finite group F, a set E defined by χ is a subgroup that normalizes its conjugates (so is subnormal in F) and by Lemma 2.5 it contains a component of E, and hence of F; thus, $(\exists h)(\pi'_{\mathrm{c}}(h) \wedge (\forall x)(\pi(h,x) \rightarrow \chi(x)))$ holds in F by Theorem 1.3. Therefore this sentence holds in G and we can find $h \in G$ satisfying $\pi'_{\mathrm{c}}(h)$ such that $H_0 \leqslant H$ where $H_0 = \{x \mid \pi(h,x)\}$. Since H_0 is a definable component we have $H_0 \lhd L \lhd H$ where $L = \bigcap_{h \in H} \mathrm{N}_H(H_0^h)$, and since L is definable and H is a definable component we have $H = \{x \mid \pi(h,x)\}$, as required. ⊣

PROOF OF THEOREM 1.1. This is easy now that we have identified the non-abelian minimal definable normal subgroups and the components as the sets $\{x \mid \pi(h,x)\}$ for elements h satisfying $\pi_{\mathrm{m}}(h)$ and $\pi_{\mathrm{c}}(h)$. Assertion (a) follows from Lemma 4.1(b) and Proposition 4.2(a). Each non-abelian minimal normal subgroup N of a finite group is the direct product of a simple component and its centralizer in N; since this statement can be expressed by a first-order sentence, assertion (b) holds for G. Assertions (c) and (d) also follow from the corresponding assertions for finite groups and Theorem 1.3. ⊣

Let G be a finite or pseudofinite group. Write $\Pi(G) = \{h \mid \pi'(h)\}$ and for each $h \in \Pi(G)$ write $\Gamma_h = \{x \mid \pi(h,x)\}$. Define a pre-order on $\Pi(G)$ by setting $h_1 \leqslant h_2$ if and only if $\pi(h_1,x) \rightarrow \pi(h_2,x)$. Thus, $\Pi(G)$ and the relation $\leqslant$ are definable in G. We note that $1 \in \Pi(G)$ and $1 \leqslant h$ for all $h \in \Pi(G)$.

Lemma 4.3. (a) *There is an element $m \in \Pi(G)$ such that $h \leqslant m$ for all $h \in \Pi(G)$.*

(b) *For all $h_1, h_2 \in \Pi(G)$ there exists $h_3 \in \Pi(G)$ with $\Gamma_{h_2}\Gamma_{h_2} = \Gamma_{h_3}$.*

(c) *For each $h \in \Pi(G)$ there are elements $h^\circ \in \Pi(G)$ with $\Gamma_h\Gamma_{h^\circ} = \Gamma_m$ and $[\Gamma_h, \Gamma_{h^\circ}] = \Gamma_h \cap \Gamma_{h^\circ} = 1$. Moreover, $\Gamma_{(h^\circ)^\circ} = \Gamma_h$ for all choices of $h^\circ, (h^\circ)^\circ$.*

(d) *The minimal elements of $\Pi(G) \setminus \{1\}$ are the elements h satisfying $\pi_c(h)$.*

Proof. These assertions hold for finite groups and since they can be expressed using first-order formulae they also hold in pseudofinite groups. ⊣

It is convenient to use the following (dual of a) result of Frink [2].

Lemma 4.4. *Let B be a partially ordered set with a maximum element 1 in which every pair of elements b_1, b_2 has a least upper bound $b_1 \vee b_2$. Suppose that there is an order-reversing bijection $^\circ\colon B \to B$ such that for all $b_1, b_2 \in B$ we have $b_1 \vee b_2 = 1$ if and only if $b_1 \geqslant b_2^\circ$. Then B is a Boolean lattice, with minimum element 1°, with complementation given by $^\circ$ and with meet operation given by $b_1 \wedge b_2 = (b_1^\circ \vee b_2^\circ)^\circ$.*

Proposition 4.5. *The quotient $B(G)$ of $\Pi(G)$ modulo the equivalence relation defined by*

$$h_1 \sim h_2 \quad \textit{if and only if} \quad h_1 \leqslant h_2 \textit{ and } h_2 \leqslant h_1$$

carries the structure (definable in G) of a Boolean lattice.

Proof. This follows directly from the above two lemmas. The set $B(G)$ inherits a partial order from $\Pi(G)$, with minimum element 0 the class containing 1 and maximum element the class containing m, and $h \mapsto \Gamma_h$ induces an order-preserving bijection $B(G) \to \{\Gamma_h \mid \pi'(h)\}$. By Lemma 4.3(b), for all $b_1, b_2 \in B(G)$ there is a least upper bound $b_1 \vee b_2$, and (c) gives an order-reversing bijection $b \mapsto b^\circ$ satisfying the conditions of Lemma 4.4. Therefore $B(G)$ is a Boolean lattice. Finally, the equivalence relation on $\Pi(G)$ and the operations $\vee$ and $^\circ$ are clearly defined by first-order formulae, and the result follows. ⊣

It is worth noting that when $\mathrm{R}(G) = 1$ the family $\{\Gamma_h \mid \pi'(h)\}$ consists of all definable normal subgroups of Γ_m, and lattice operations $^\circ, \vee, \wedge$ in it are precisely the operations $K \mapsto \mathrm{C}_{\Gamma_m}(K)$, and join and intersection of subgroups.

We turn now to Theorem 1.2. For each group G let $q(G)$ be the subgroup generated by all squares.

Proposition 4.6. *Let G be a pseudofinite group with a normal definable component L. Then $q(G)\mathrm{C}_G(L)$ is definable, $q(G)\mathrm{C}_G(L)/L\mathrm{C}_G(L)$ is metabelian, and $G/q(G)\mathrm{C}_G(L)$ is elementary abelian of order at most 32.*

Proof. We have $L = \{x \mid \pi(h, x)\}$ for some $h \in G$ satisfying π_c' and so G satisfies $(\exists h)\tau(h)$ where $\tau(h)$ is the formula

$$\pi_c'(h) \wedge (\forall x \forall y)(\pi(h, x) \leftrightarrow \pi(h, x^y)).$$

We shall show that $q(G)\mathrm{C}_G(L)$ is defined in G by the formula

$$\chi(h, x)\colon (\exists y_1 \cdots \exists y_{15})(\exists s \exists t)$$
$$x = y_1^2 \ldots y_{15}^2 st \wedge \pi(h, s) \wedge (\forall z)(\pi(h, z) \to [z, t] = 1).$$

Let F be a finite group that satisfies $(\exists h)\tau(h)$. This sentence asserts that F has a normal component $N = \{x \mid \pi(h, x)\}$. Clearly, $N = q(N)\mathrm{Z}(N)$. Let $S = N/\mathrm{Z}(N)$. An easy application of the 3-lemma shows that $\mathrm{C}_F(N) = \mathrm{C}_F(S)$;

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therefore, $F/\mathrm{C}_F(N)$ is isomorphic to a subgroup H of $\mathrm{Aut}(S)$ containing $\mathrm{Inn}(S)$. By (CFSG) (b)(i) the group $H/\mathrm{Inn}(S)$ satisfies the hypothesis on G in Lemma 2.6 with $n = 5$, and so $q(F) \leqslant XN\mathrm{C}_F(N)$ where X is the set of products of 15 squares in F. Thus, $q(F)\mathrm{C}_F(N) = q(F)N\mathrm{C}_F(N) = \{x \mid \chi(h, x)\}$. Since every product of 16 squares is a product of an element of X and an element of $N\mathrm{C}_F(N)$, the formula

$$\rho_1(h): \quad (\forall x_1 \cdots \forall x_{16})\chi(hx_1^2, \ldots, x_{16}^2)$$

holds in F. Further use of (CFSG) (b)(i) and Lemma 2.6 shows that $F/q(F)\mathrm{C}_F(N)$ is elementary abelian of order at most 32 and $q(F)\mathrm{C}_F(N)/N\mathrm{C}_F(N)$ is metabelian. Therefore, the formulae

$$\begin{aligned}
&\rho_2(h): (\exists t_1 \cdots \exists t_{32})(\forall g) \textstyle\bigvee_{i=1}^{32} \chi(h, t_i^{-1}g),\\
&\rho_3(h): (\forall x_1 \cdots \forall x_4)(\textstyle\bigwedge_{i=1}^{4}(\chi(h, x_i) \to (\exists u \exists v)\\
&\qquad\qquad (\delta_2(x_1, x_2, x_3, x_4) = uv \wedge \pi(h, u) \wedge (\forall w(\pi(h, w) \to [w, t] = 1)))
\end{aligned}$$

hold in F.

Consequently, for finite groups we have $\tau(h) \to \rho_1(h) \wedge \rho_2(h) \wedge \rho_3(h)$. Therefore, this implication holds for G, and G satisfies $\rho_1(h) \wedge \rho_2(h) \wedge \rho_3(h)$. It follows that $q(G)\mathrm{C}_G(L)$ is defined by the formula $\chi(h, x)$, that this subgroup has index at most 32 and that $q(G)\mathrm{C}_G(L)/L\mathrm{C}_G(L)$ is metabelian, as required. ⊣

Proof of Theorem 1.2. Let $L_1, \ldots, L_n$ be the definable components, G_0 the intersection of their normalizers, and P their product. Then G_0 has finite index in G since it is the kernel of the conjugation action on the set of components. Write

$$G_1 = \bigcap_{i=1}^{n} q(G_0)\mathrm{C}_{G_0}(L_i).$$

Since automorphisms of G permute the subgroups L_i, both G_1 and $P\mathrm{C}_G(P)$ are characteristic subgroups of G. By Proposition 4.6 each subgroup $Q_i = q(G_0)\mathrm{C}_{G_0}(L_i)$ is definable and of finite index in G_0; hence, G_1 is definable and of finite index in both G_0 and G. Because each $Q_i/L_i\mathrm{C}_{G_0}(L_i)$ is metabelian, so is G_1/K where $K = \bigcap L_i\mathrm{C}_{G_0}(L_i)$. For each i, conjugation by elements of K induces only inner automorphisms in $L_i/\mathrm{Z}(L_i)$, and so the images of K and P in $\mathrm{Aut}(\prod L_i/\mathrm{Z}(L_i))$ coincide. Therefore, $K = P\mathrm{C}_G(P/\mathrm{Z}(P))$. Another use of the 3-lemma shows that $\mathrm{C}_G(P/\mathrm{Z}(P)) = \mathrm{C}_G(P)$, and so $G_1/P\mathrm{C}_G(P)$ is metabelian, as required.

Finally, suppose also that $\mathrm{R}(G) = 1$. Then $\mathrm{C}_G(P)$ is a definable normal subgroup containing neither components nor a non-trivial abelian normal subgroup of G, and so $\mathrm{C}_G(P) = 1$. The result follows. ⊣

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