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ARTICLE

On perfect subdivision tilings

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Abstract

For a given graph H, we say that a graph G has a perfect H-subdivision tiling if G contains a collection of vertex-disjoint subdivisions of H covering all vertices of G. Let $\delta_{\mathrm{sub}}(n, H)$ be the smallest integer k such that any n-vertex graph G with minimum degree at least k has a perfect H-subdivision tiling. For every graph H, we asymptotically determined the value of $\delta_{\mathrm{sub}}(n, H)$. More precisely, for every graph H with at least one edge, there is an integer $\mathrm{hcf}_{\xi}(H)$ and a constant $1 < \xi^*(H) \leq 2$ that can be explicitly determined by structural properties of H such that $\delta_{\mathrm{sub}}(n, H) = \left(1 - \frac{1}{\xi^*(H)} + o(1)\right) n$ holds for all n and H unless $\mathrm{hcf}_{\xi}(H) = 2$ and n is odd. When $\mathrm{hcf}_{\xi}(H) = 2$ and n is odd, then we show that $\delta_{\mathrm{sub}}(n, H) = \left(\frac{1}{2} + o(1)\right) n$.

Keywords: Subdivision; tiling; absorption

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1. Introduction

Embedding a large sparse subgraph into a dense graph is one of the most central problems in extremal graph theory. It is well known that any graph G with a minimum degree of at least $\lceil \frac{v(G)}{2} \rceil$ has a Hamiltonian cycle, and hence also a perfect matching if the number of vertices $v(G)$ of G is even. A *perfect matching* of a graph G is a vertex-disjoint collection of edges whose union covers all vertices of G. A natural generalisation of a perfect matching is a perfect H-tiling for a general graph H. We say G has a *perfect H-tiling* if G contains a collection of vertex-disjoint copies of H whose union covers all vertices of G. We note that a perfect H-tiling exists only if $v(G)$ is divisible by $v(H)$. For a positive integer n divisible by $v(H)$, we denote by $\delta(n, H)$ the minimum integer k such that any n-vertex graph G with a minimum degree of at least k has a perfect H-tiling. The celebrated Hajnal–Szemerédi [11] theorem states that for any integer $r \geq 2$, the number $\delta(n, K_r)$ is equal to $\left(1 - \frac{1}{r}\right) n$.

An asymptotic version of the Hajnal–Szemerédi theorem for a general graph H was first proven by Alon and Yuster [2]. They showed that if n is divisible by $v(H)$, then $\delta(n, H) \leq \left(1 - \frac{1}{\chi(H)}\right) n + o(n)$, where $\chi(H)$ is the chromatic number of H. Komlós, Sárközy, and Szemerédi [18] improved the $o(n)$ term in the Alon–Yuster theorem to some constant $C = C(H)$, which settled the conjecture of Alon and Yuster [2]. Another direction for an asymptotic extension of the Hajnal–Szemerédi theorem was proven by Komlós [17]. Komlós showed that for any $\gamma > 0$, there exists $n_0 = n_0(\gamma, H)$ such that if $n \geq n_0$, then any n-vertex graph G whose minimum degree is at least $\left(1 - \frac{1}{\chi_{cr}(H)}\right) n$ contains an H-tiling that covers at least $(1 - \gamma)n$ vertices of G. Here, $\chi_{cr}(H)$ denotes the *critical chromatic number* of H, which is defined as $\chi_{cr}(H) = \frac{(\chi(H)-1)v(H)}{v(H)-\sigma(H)}$, where $\sigma(H)$

is the minimum possible size of a colour class among all colourings of H with $\chi(H)$ colours. Komlós [17] conjectured that the number of uncovered vertices can be reduced to a constant, and this conjecture was confirmed by Shokoufandeh and Zhao [29]. More precisely, the following holds.

Theorem 1.1 (Shokoufandeh and Zhao [29]). *Let H be a graph. Then there exists a constant $C = C(H)$, which only depends on H, such that any graph G on n vertices with a minimum degree of at least $\left(1 - \frac{1}{\chi_{cr}(H)}\right) n$ contains an H-tiling that covers all but at most C vertices of G.*

The almost exact value of $\delta(n, H)$ for every graph H was determined by Kühn and Osthus [23] up to an additive constant depending only on H. To formulate this theorem, we need to define some notations. We say a proper colouring of H is *optimal* if it uses exactly $\chi(H)$ colours. We denote by $\Phi(H)$ the set of all optimal colourings of H. Let $\phi \in \Phi(H)$ be an optimal colouring of H with the size of colour classes being $x_1 \leq \ldots \leq x_{\chi(H)}$. Let $\mathcal{D}(\phi) := \{x_{i+1} - x_i : i \in [\chi(H) - 1]\}$ and $\mathcal{D}(H) := \bigcup_{\phi \in \Phi(H)} \mathcal{D}(\phi)$. We write $\mathrm{hcf}_\chi(H)$ for the highest common factor of all integers in $\mathcal{D}(H)$. (If $\mathcal{D}(H) = \{0\}$, we define $\mathrm{hcf}_\chi(H) = \infty$.) We denote by $\mathrm{hcf}_c(H)$ the highest common factor of all the orders of components in H. For a graph H, we define

$$\mathrm{hcf}(H) := \begin{cases} 1 & \text{if } \chi(H) \geq 3 \text{ and } \mathrm{hcf}_\chi(H) = 1, \\ 1 & \text{if } \chi(H) = 2,\ \mathrm{hcf}_\chi(H) \leq 2,\ \text{and } \mathrm{hcf}_c(H) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We define $\chi^*(H) = \chi_{cr}(H)$ if $\mathrm{hcf}(H) = 1$; otherwise, $\chi^*(H) = \chi(H)$. We are now ready to state the Kühn–Osthus theorem.

Theorem 1.2 (Kühn and Osthus [23]). *Let H be a graph and n be a positive integer divisible by $v(H)$. Then there exists a constant $C = C(H)$ depending only on H such that*

$$\left(1 - \frac{1}{\chi^*(H)}\right) n - 1 \leq \delta(n, H) \leq \left(1 - \frac{1}{\chi^*(H)}\right) n + C.$$

1.1. Main results

Motivated by the Kühn–Osthus theorem on perfect H-tilings, several variations of Theorem 1.2 have been considered. (For instance, see [5, 10, 12–14, 22, 25]).

In this article, we consider a problem related to the concept of perfect H-tilings and subdivision embeddings. A graph H' is a *subdivision* of H if H' is obtained from H by replacing edges of H with vertex-disjoint paths while maintaining their endpoints. Since subdivisions maintain the topological structure of the original graph, various problems related to subdivisions have been posed and extensively studied. In particular, finding a sufficient condition on the host graph to contain a subdivision of H is an important problem in extremal graph theory.

Let H and G be graphs. An *H-subdivision tiling* is a collection of vertex-disjoint unions of subdivisions of H. We say that G has a *perfect H-subdivision tiling* if G has an H-subdivision tiling that covers all vertices of G. A natural question would be to determine the minimum degree threshold for a positive integer n that ensures the existence of a perfect H-subdivision tiling in any n-vertex graph G. We define this minimum degree threshold as follows.

Definition 1.3. *Let H be a graph. We denote the minimum degree threshold for perfect H-subdivision tilings by $\delta_{\mathrm{sub}}(n, H)$, which is the smallest integer k such that any n-vertex graph G with a minimum degree of at least k has a perfect H-subdivision tiling.*

If H has no edges, then a perfect H-subdivision tiling exists if and only if $v(G)$ is divisible by $v(H)$, regardless of the minimum degree $\delta(G)$ of G. Thus, from now on, we only consider graphs with at least one edge.

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For most results on embedding large graphs, including perfect H-tilings and large subdivisions, embedding a graph with more complex structures requires a larger minimum degree threshold. Surprisingly, we observe the opposite phenomenon for perfect subdivision tiling problems. For example, Theorem 1.8 states that $\delta_{\text{sub}}(n, K_r)$ decreases as r increases when $3 \leq r \leq 5$. This is due to the fact that a larger value of r ensures that K_r has a bipartite subdivision with a more asymmetric bipartition.

Since embedding bipartite graphs generally requires a lower minimum degree than non-bipartite graphs, we want to cover most of the vertices of the host graph with subdivisions of H that are bipartite. Suppose every bipartite subdivision of H is, in some sense, balanced. In that case, it cannot be tiled in a highly unbalanced complete bipartite graph that has a lower minimum degree than a balanced complete bipartite graph. For this reason, we need to measure how unbalanced bipartite subdivisions of H can be, as it poses some space barriers to the problem.

For this purpose, we introduce the following two definitions.

Definition 1.4. *Let H be a graph and $X \subseteq V(H)$. We define a function $f_H : 2^{V(H)} \to \mathbb{R}$ as $f_H(X) = \frac{v(H)+e(H[X])+e(H[Y])}{|X|+e(H[Y])}$ where $Y = V(H) \setminus X$.*

Definition 1.5. *Let H be a graph. We define $\xi(H) := \min\{f_H(X) : X \subseteq V(H)\}$.*

Note that subdividing all edges in $H[X]$ and $H[Y]$ yields a bipartite subdivision H' of H, and $f_H(X)$ measures the ratio between $V(H')$ and one part of its bipartition. The smaller the value of f_H, the more unbalanced H' is. Later in Observation 3.2, we will prove that $\xi(H)$ can be used to measure the ratio between the two parts in the most asymmetric bipartite subdivision of H. Note that we always have $1 < \xi(H) \leq 2$. Indeed,

$$f_H(V(H)) \geq \frac{v(H) + e(H)}{v(H)} > 1,$$

and for any partition X, Y of $V(G)$, we have

$$v(H) + e(H[X]) + e(H[Y]) = |X| + e(H[X]) + |Y| + e(H[Y]) \leq 2 \max\{|Z| + e(H[Z]) : Z \in \{X, Y\}\},$$

so, $\min\{f_H(X), f_H(Y)\} \leq 2$.

Another crucial factor is the divisibility issue. Assume that all bipartite subdivisions of H have bipartitions with both parts having the same parity. If G is a complete bipartite graph $K_{a,b}$ with a and b having different parity, then we cannot find perfect H-subdivision tilings in G, as it poses some divisibility barriers to the problem. Hence, we need to introduce the following definitions concerning the difference between the two parts in bipartitions of subdivisions of H and their highest common factor.

Definition 1.6. *Let H be a graph. We define $\mathcal{C}(H) := \{(|X| + e(H[Y])) - (|Y| + e(H[X])) : X \subseteq V(H), Y = V(H) \setminus X\}$. We denote by $\text{hcf}_\xi(H)$ the highest common factor of all integers in $\mathcal{C}(H)$. (If $\mathcal{C}(H) = \{0\}$, we define $\text{hcf}_\xi(H) = \infty$.)*

By considering the space barrier and the divisibility barrier, we introduce the following parameter measuring both obstacles for the problem. We will show that this is the determining factor for $\delta_{\text{sub}}(n, H)$.

Definition 1.7. *Let H be a graph. We define*

$$\xi^*(H) := \begin{cases} \xi(H) & \text{if } \text{hcf}_\xi(H) = 1, \\ \max\left\{\frac{3}{2}, \xi(H)\right\} & \text{if } \text{hcf}_\xi(H) = 2, \\ 2 & \text{otherwise.} \end{cases}$$

We are now ready to state our main theorems. The following theorem gives the asymptotically exact value for $\delta_{\text{sub}}(n, H)$ except in one case, when $\text{hcf}_\xi(H) = 2$.

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Theorem 1.8. *Let H be a graph with $\mathrm{hcf}_\xi(H) \neq 2$. For every $\gamma > 0$, there exists an integer $n_0 = n_0(\gamma, H)$ such that for any $n \geq n_0$, the following holds.*

$$\left(1 - \frac{1}{\xi^*(H)}\right) n - 1 \leq \delta_{\text{sub}}(n, H) \leq \left(1 - \frac{1}{\xi^*(H)} + \gamma\right) n$$

This theorem asymptotically determines $\delta_{\text{sub}}(n, H)$ as long as $\mathrm{hcf}_\xi(H) \neq 2$. If $\mathrm{hcf}_\xi(H) = 2$, then the parity of n is also important. The following theorem asymptotically determines $\delta_{\text{sub}}(n, H)$ for this case.

Theorem 1.9. *Let H be a graph with $\mathrm{hcf}_\xi(H) = 2$. For every $\gamma > 0$, there exists an integer $n_0 = n_0(\gamma, H)$ such that the following holds. For every integer $n \geq n_0$,*

$$\frac{1}{2}n - 1 \leq \delta_{\text{sub}}(n, H) \leq \left(\frac{1}{2} + \gamma\right) n \qquad \text{if } n \text{ is odd,}$$

$$\left(1 - \frac{1}{\xi^*(H)}\right) n - 1 \leq \delta_{\text{sub}}(n, H) \leq \left(1 - \frac{1}{\xi^*(H)} + \gamma\right) n \qquad \text{if } n \text{ is even.}$$

As a direct consequence of Theorems 1.8 and 1.9, we determine the value of $\delta_{\text{sub}}(n, K_r)$ for each $r \geq 2$. By applying our main theorems directly, we have $\delta_{\text{sub}}(n, K_2) = \left(\frac{1}{3} + o(1)\right) n$, and for each $r \in \{3, 4, 5\}$, we have $\delta_{\text{sub}}(n, K_r) = \left(\frac{2}{r+1} + o(1)\right) n$. For the case $r = 7$, if n is even, we have $\delta_{\text{sub}}(n, K_7) = \left(\frac{1}{3} + o(1)\right) n$; otherwise, we have $\delta_{\text{sub}}(n, K_7) = \left(\frac{1}{2} + o(1)\right) n$. Finally, for every $r \geq 8$ and $r = 6$, we have $\delta_{\text{sub}}(n, K_r) = \left(\frac{1}{2} + o(1)\right) n$. This is in contrast to the normal H-tiling problem. This means the determining factors for the minimum degree thresholds of perfect H-tilings and perfect H-subdivision tilings are essentially different. Probably, the most interesting difference between the perfect H-tiling and the perfect H-subdivision tiling is that monotonicity does not hold for subdivision tiling. For a perfect tiling, if H_2 is a spanning subgraph of H_1, then obviously $\delta(n, H_2) \leq \delta(n, H_1)$. However, for perfect subdivision tiling, this does not hold in many cases. For example, our results imply $\delta_{\text{sub}}(n, K_4) = \frac{2}{5}n + o(n) < \delta_{\text{sub}}(n, C_4) = \frac{1}{2}n + o(n)$.

As $\xi^*(H)$ is the determining factor for the minimum degree threshold, it is convenient for us to specify the bipartite subdivision achieving the value $\xi^*(H)$. We introduce the following definition.

Definition 1.10. *Let H be a graph. We denote by X_H the subset of $V(H)$, where $f_H(X_H) = \xi(H)$. We define a graph H^* obtained from H by replacing all edges in $H[X_H]$ and $H[V(H) \setminus X_H]$ with paths of length two.*

There can be multiple choices for X_H. Then we fix one choice for X_H so that X_H and H^* are uniquely determined for all H. Note that H^* is a subdivision of H, which is a bipartite graph, and $v(H^*) = v(H) + e(H[X_H]) + e(H[V(H) \setminus X_H])$. Intuitively, $\xi(H)$ seems greater than $\chi_{cr}(H^*)$. Indeed, this intuition is correct. However, Theorem 1.8 cannot be directly deduced by applying Theorem 1.2 for H^*. For example, for a connected graph H, it may satisfy $\chi^*(H^*) = 2$, but $\xi^*(H) < 2$ is possible. Then the minimum degree threshold in Theorem 1.8 is smaller than the one in Theorem 1.2 when $\mathrm{hcf}_\xi(H) = 1$.

However, as we will see in Observation 3.6, for every graph H with at least one edge, the inequality $\xi(H) \geq \chi_{cr}(H^*)$ holds. Hence, if $\mathrm{hcf}_\xi(H) \leq 2$, we may instead use Theorem 1.1 to find an H^*-tiling that covers all but at most a constant number of vertices of G in a graph G with $\delta(G) \geq \left(1 - \frac{1}{\xi(H)} + \gamma\right) n$. To cover the leftover vertices, we use the absorption method. The absorption method was introduced in [27], and since then, it has been used to solve various crucial problems in extremal combinatorics. The main difficulty in applying the absorption method in our setting is that, in many cases, the host graph is not sufficiently dense to guarantee that any vertices can be absorbed in the final step. To overcome this difficulty, we use the regularity lemma

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and an extremal result on the domination number to obtain some control over the vertices that can be absorbed.

2. Preliminaries

We write $[n] = \{1, \ldots, n\}$ for a positive integer n. If we claim a result holds if $\beta \ll \alpha_1, \ldots, \alpha_t$, then it means there exists a function f such that $\beta \le f(\alpha_1, \ldots, \alpha_t)$. We will not explicitly compute these functions. In this paper, we consider $o(1)$ to go to zero as n goes to infinity.

Let G be a graph. We denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively, and we set $v(G) = |V(G)|$ and $e(G) = |E(G)|$. We write $d_G(v)$ for the degree of $v \in V(G)$, and we omit the subscript if the graph G is clear from the context. We denote by $\delta(G)$ and $\Delta(G)$ the minimum degree of G and the maximum degree of G, respectively. For a vertex $v \in V(G)$, we denote by $N_G(v)$ the set of neighbours of v in G which are adjacent to v. We also omit the subscript if G is clear from the context.

For a graph G and a vertex subset $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X. For vertex subsets A and B of G, we denote by $G[A, B]$ the graph where $V(G[A, B]) = A \cup B$ and $E(G[A, B]) = \{uv \in E(G) : u \in A, v \in B\}$. Let v be a vertex of G and X be a vertex subset of G. We denote by $d_G(v; X)$ the degree of v in the induced graph $G[X \cup \{v\}]$.

Let G be a graph and $X \subseteq V(G)$. We denote by $G - X$ the induced graph $G[V(G) \setminus X]$. If X is a single vertex v, we simply denote $G - v$. Similarly, if a graph H is a subgraph of G, then we denote by $G - H$ the induced subgraph $G[V(G) \setminus V(H)]$. Let G' be a subgraph of G and $v \in V(G)$. We denote by $G' + v$ a graph such that $V(G' + v) = V(G') \cup \{v\}$ and $E(G' + v) = E(G') \cup E(G[\{v\}, V(G')])$.

We denote by K_r the complete graph of order r and $K_{n,m}$ the complete bipartite graph with bipartition of sizes n and m. We also denote by C_k and P_k the cycle of length k and the path of length k, respectively. We say H' is a *1-subdivision* of H if H' is obtained from H by replacing all edges of H with vertex-disjoint paths of length two. We denote by H^1 the 1-subdivision of H. We write $\mathrm{Sub}(H)$ for the collection of all subdivisions of H. For a graph $F \in \mathrm{Sub}(H)$, we say a vertex $v \in V(F) \cap V(H)$ is a *branch vertex* of F.

In the proofs of the main theorems, we need to count the number of copies of specific bipartite subgraphs where each bipartition is contained in certain vertex subsets. In order to facilitate this counting, we introduce the notion of embedding. Let H and G be graphs. An *embedding* ϕ of H into G is an injective function from $V(H)$ to $V(G)$ such that for any $uv \in E(H)$, its image $\phi(u)\phi(v)$ is in $E(G)$.

The next simple observation will be used in Section 4.

Observation 2.1. *Let $0 \le \gamma, \frac{1}{n} \ll \frac{1}{t}, d < 1$. Let $\mathcal{B}$ be an n-vertex t-uniform hypergraph with $|E(\mathcal{B})| \ge dn^t$. Let $A \subseteq V(\mathcal{B})$ be a vertex subset of size at most γn. Then we have $|E(\mathcal{B} - A)| \ge \frac{d}{2} n^t$.*

As every vertex is contained in at most n^{t-1} edges, by choosing γ sufficiently small, we can easily verify that Observation 2.1 holds. In the following two subsections, we introduce powerful tools in extremal graph theory, so-called supersaturation, and the regularity lemma.

2.1. The supersaturation

One of the most fundamental problems in extremal graph theory is to determine the maximum number of edges in an n-vertex graph that does not contain a copy of a specific graph as a subgraph. For a graph H and a positive integer n, we denote by $\mathrm{ex}(n, H)$ the *extremal number* of H, defined as the maximum integer k such that there is an n-vertex graph G with k edges not containing H as a subgraph.

A classical theorem of Turán determined the exact value of $\mathrm{ex}(n, K_r)$. For a graph H, we define the *Turán density* of H as $\pi(H) := \lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}}$. The existence of Turán density for every graph H was proved by Katona, Nemetz, and Simonovits [15]. For every graph H, the Erdős–Stone–Simonovits [7, 9] theorem states that $\pi(H) = \left(1 - \frac{1}{\chi(H)-1}\right)$. We note that if a graph H is bipartite, the Turán density satisfies $\pi(H) = 0$.

Assume that an n-vertex graph G has many more edges than $\mathrm{ex}(n, H)$. Intuitively, we can expect there to be a lot of copies of H in G. Indeed, this is true, and this phenomenon is called *supersaturation*, proved by Erdős and Simonovits [8], as follows.

Theorem 2.2 (Supersaturation). *Let $0 < \delta \ll \varepsilon, \frac{1}{h} \leq 1$. For every graph H on h vertices, the following holds. If G is an n-vertex graph with $e(G) \geq (\pi(H) + \varepsilon)\binom{n}{2}$, then G contains at least δn^h copies of H.*

Since the Turán density of a bipartite graph H is zero, Theorem 2.2 leads to the following lemmas.

Lemma 2.3. *Let $0 < \delta \ll \frac{1}{a}, \frac{1}{b}, \varepsilon \leq 1$ and n be a positive integer. For every complete bipartite graph H on bipartition A and B with $|A| = a$ and $|B| = b$, the following holds. Let G be a bipartite graph with bipartition X and Y such that $|X|, |Y| \leq n$. If $e(G) \geq \varepsilon n^2$, then the number of embeddings $\phi : V(H) \to V(G)$ is at least δn^{a+b}, where $\phi(A) \subseteq X$ and $\phi(B) \subseteq Y$.*

Proof. Assume $a \leq b$ and let $v(G) = n'$. Since $e(G) \geq \varepsilon n^2$, we observe that the inequality $\sqrt{2\varepsilon}n \leq n' = |A| + |B| \leq 2n$ holds. Thus, if we choose n sufficiently large, then also n' is sufficiently large, and we have $e(G) \geq \varepsilon n'^2$. By Theorem 2.2, the number of copies of $K_{b,b}$ in G is at least $\delta' n'^{2b}$ for some $\delta' > 0$. We observe that for each copy of H in G, where A is embedded in X, there are at most $(n')^{b-a}$ copies of $K_{b,b}$ in G. Thus, by double counting, the number of embeddings of H in G, where A is embedded in X and B is embedded in Y, is at least $\delta'(n')^{a+b} \geq \delta n^{a+b}$. □

Lemma 2.4. *Let $0 < d \ll \varepsilon, \frac{1}{h} \leq 1$. For every h-vertex bipartite graph H on bipartition A and B, the following holds. Let G be an n-vertex graph with minimum degree at least εn. Let $X \subseteq V(G)$ be a set of at least εn vertices. Then there are at least δn^h distinct embeddings $\phi : V(H) \to V(G)$, where $\phi(A) \subseteq X$.*

Proof. Let $X' := V(G) \setminus X$. If $e(G[X]) \geq \frac{\varepsilon^2}{4} n^2$, by Theorem 2.2, the induced graph $G[X]$ contains at least δn^h distinct copies of H. Thus, in this case, the lemma is proved. We now assume $e(G[X]) \leq \frac{\varepsilon^2}{4} n^2$. Then, by the minimum degree condition of G, the number of edges in the bipartite subgraph $G[X, X']$ is at least $\frac{\varepsilon^2}{4} n^2$. Then, by Lemma 2.3, for the bipartite graph $G[X, X']$, there are at least δn^h distinct desired embeddings. This proves the lemma. □

2.2. The regularity lemma

Szemerédi's regularity lemma [30] is a powerful tool for dealing with large dense graphs. To formulate the regularity lemma, we need to define an ε-regular pair. Let G be a graph and $A, B \subseteq G$. We define the density between A and B as $d_G(A, B) := \frac{e(G[A,B])}{|A||B|}$. The following is the definition of an ε-regular pair.

Definition 2.5. *Let $\varepsilon > 0$ and $d \in [0, 1]$. A pair (A, B) of disjoint subsets of vertices in a graph G is an (ε, d)-regular pair if the following holds. For any subset $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$,*

$$|d_G(A', B') - d| \leq \varepsilon.$$

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If a pair (A, B) is (ε, d)-regular for some $d \in [0, 1]$, we say (A, B) is an ε-regular pair. We also say a pair (A, B) is $(\varepsilon, d+)$-regular if (A, B) is (ε, d')-regular for some $d' \geq d$.

The following is the degree form of the regularity lemma [21, Theorem 1.10].

Lemma 2.6 (Regularity Lemma-degree form). *Let $0 < \frac{1}{T_0} \ll \frac{1}{t_0}, \varepsilon \leq 1$. For every real number $d \in [0, 1]$ and every graph G with order at least T_0, there exists a partition of $V(G)$ into $t+1$ clusters $V_0, V_1, \ldots, V_t$ and a spanning subgraph G_0 of G such that the following holds:*

- $t_0 \leq t \leq T_0$,
- $|V_0| \leq \varepsilon v(G)$,
- $|V_1| = \cdots = |V_t|$,
- *$G_0[V_i]$ has no edge for each $i \in [t]$,*
- *$d_{G_0}(v) \geq d_G(v) - (d + \varepsilon)v(G)$ for all $v \in V(G)$,*
- *for all $1 \leq i < j \leq t$, either the pair (V_i, V_j) is an $(\varepsilon, d+)$-regular pair or the graph $G_0[A, B]$ has no edge.*

The partition in Lemma 2.6 is called an (ε, d)-*regular partition* or simply an ε-*regular partition*. We now define an (ε, d)-*reduced graph* R on the vertex set $\{V_1, \ldots, V_t\}$, which is obtained from Lemma 2.6, by joining the edges for each V_i, V_j if and only if (V_i, V_j) is an $(\varepsilon, d+)$-regular pair. The following lemma shows that the reduced graph inherits the minimum degree condition [24].

Lemma 2.7 ([24], Proposition 9). *For every $\gamma > 0$, there exist $\varepsilon_0 = \varepsilon_0(\gamma)$ and $d_0 = d_0(\gamma) > 0$ such that for every $\varepsilon \leq \varepsilon_0$, $d \leq d_0$, and $\delta > 0$, for every graph G with minimum degree $(\delta + \gamma)n$, the reduced graph R on G obtained by applying Lemma 2.6 with parameters ε and d, the minimum degree of R is at least $\left(\delta + \frac{\gamma}{2}\right)|R|$.*

Reduced graphs are useful for analysing the approximate structures of the host graphs, as they inherit many properties of the host graphs, such as the minimum degree.

The main reason that we need the regularity lemma is to get an efficient absorber for the leftover vertices at the final step of the proofs. To achieve this, we need an additional minimum degree condition for the ε-regular pair. The following lemma guarantees that we can delete low-degree vertices from A and B while preserving the ε-regularity.

Lemma 2.8 ([6], Proposition 12). *Let (A, B) be an (ε, d)-regular pair in a graph G. Let $B' \subseteq B$ with $|B'| \geq \varepsilon|B|$. Then the size of the set $\{a \in A : d_G(a; B') < (d - \varepsilon)|B'|\}$ is at most $\varepsilon|A|$.*

To get more information on the regularity lemma, see the following papers [16, 19–21, 28].

3. Extremal examples and properties of $\xi(H)$ and $\chi_{cr}(H^*)$

We first show that for any $\gamma > 0$ and every graph H, there is $n_0 = n_0(\gamma, H)$ such that for all $n \geq n_0$, the inequality $\delta_{\text{sub}}(n, H) \leq \left(\frac{1}{2} + \gamma\right) n$ holds. Note that for any graph H, we have $\left(1 - \frac{1}{\chi^*(H^*)}\right) \leq \frac{1}{2}$ since H^* is bipartite. Then Theorem 1.2 implies $\delta(n, H^*) \leq \left(\frac{1}{2} + \frac{\gamma}{2}\right) n$ if n is sufficiently large and divisible by $v(H^*)$.

Proposition 3.1. *Let $0 < \frac{1}{n} \ll \gamma, \frac{1}{h} \leq 1$. Then for every h-vertex graph H, the following holds:*

$$\delta_{\text{sub}}(n, H) \leq \left(\frac{1}{2} + \gamma\right) n.$$

Proof. Let G be an n-vertex graph and $\delta(G) \geq \left(\frac{1}{2} + \gamma\right) n$. By subdividing one edge of H^* at most $v(H^*)$ more times, we obtain a graph F with a constant number of vertices, $\chi(F) \leq 3$,

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and $v(G) - v(F)$ is divisible by $v(H^*)$. By the Erdős–Stone–Simonovits theorem, we can find a copy of F in G. Since $v(F)$ is bounded, the minimum degree of $G[V(G) \setminus V(F)]$ is greater than $\left(\frac{1}{2} + \frac{\gamma}{2}\right) n$ for $n \geq n_0$, where n_0 depends only on H. Then by Theorem 1.2, there is a perfect H^*-tiling in $G[V(G) \setminus V(F)]$ which yields a perfect H-subdivision tiling in G together with the copy of F. This completes the proof. □

We now prove lower bounds on $\delta_{\text{sub}}(n, H)$ by constructing graphs without perfect H-subdivision tilings. The following observation shows that H^* is a bipartite subdivision with the most unbalanced bipartition. We recall that $\xi(H)$ is the minimum value among all possible values of f_H where $f_H : 2^{V(H)} \to \mathbb{R}$ is a function such that for every $X \subseteq V(G)$, $f_H(X) = \frac{v(H)+e(H[X])+e(H[Y])}{|X|+e(H[Y])}$, where $Y = V(G) \setminus X$.

Observation 3.2. *Let H be a graph and let $F \in \text{Sub}(H)$ be a bipartite graph with bipartition A and B. Then $\frac{|B|}{|A|} \leq \frac{1}{\xi(H)-1}$.*

Proof. We may assume $|A| \leq |B|$. It suffices to show that $\xi(H) \leq \frac{|A|+|B|}{|B|}$. Let U be the branch vertices of F and let $X = A \cap U$ and $Y = B \cap U$. Let $p = |X| + e(H[Y])$ and $q = |Y| + e(H[X])$. Then $f_H(Y) = \frac{p+q}{q}$. Since F is a bipartite subdivision of H with branch set $X \cup Y$, every edge in $H[X, Y]$ is subdivided an even number of times. Thus, there is a non-negative integer c such that $\frac{|A|+|B|}{|B|} = \frac{p+q+2c}{q+c}$. If $\frac{p+q}{q} \geq 2$, then $\frac{|A|+|B|}{|B|} \geq 2$. Since $\xi(H) \leq 2$, in this case, the observation is proved. Otherwise, we may observe that $p < q$. Then $\frac{|A|+|B|}{|B|} = \frac{p+q+2c}{q+c} \geq \frac{p+q}{q}$ holds. By the definition of $\xi(H)$, we have $\xi(H) \leq H(Y) = \frac{p+q}{q}$. This completes the proof. □

By using Observation 3.2, we can show the following proposition.

Proposition 3.3. *For every integer $n > 0$ and every graph H, there is an n-vertex graph G with minimum degree at least $\left\lfloor \left(1 - \frac{1}{\xi(H)}\right) n \right\rfloor - 1$ such that G does not have a perfect H-subdivision tiling.*

Proof. Let G be an n-vertex complete bipartite graph with bipartition X and Y such that $|X| \leq |Y|$. Note that as G is a bipartite graph, every subdivision of H in G is bipartite. By Observation 3.2, if $\frac{|Y|}{|X|} > \frac{1}{\xi(H)-1}$, then G does not have a perfect H-subdivision tiling. Let $|X| = \left\lfloor \left(1 - \frac{1}{\xi(H)}\right) n \right\rfloor - 1$. Then $\delta(G) = \left\lfloor \left(1 - \frac{1}{\xi(H)}\right) n \right\rfloor - 1$ and $\frac{|Y|}{|X|} > \frac{1}{\xi(H)-1}$. Thus, G does not have a perfect H-subdivision tiling. □

The next proposition provides a reason for why $\text{hcf}_\xi(H)$ is the determining factor of the value $\xi^*(H)$.

Proposition 3.4. *Let H be a graph with $\text{hcf}_\xi(H) \neq 1$. Then for every integer $n > 0$, there is an n-vertex graph G with minimum degree at least $\left\lfloor \frac{n}{2} \right\rfloor - 1$ which does not have a perfect H-subdivision tiling except for $\text{hcf}_\xi(H) = 2$ and n is even.*

To prove Proposition 3.4, we need the following claim.

Claim 1. *Let H be a graph with at least one edge and let H' be a bipartite subdivision of H with bipartition A and B. Then $|B| - |A|$ is divisible by $\text{hcf}_\xi(H)$.*

Proof of Claim 1. Let A' and B' be the branch vertices of H contained in A and B, respectively. Let $uv \in E(H)$ be an edge and let $uz_1, \ldots, z_k, v$ be the subdivided path in H' that corresponds to uv. If $u \in A$ and $v \in B$, then k is an even number since H' is a bipartite graph. Thus, we observe that $|B \cap \{z_1, \ldots, z_k\}| - |A \cap \{z_1, \ldots, z_k\}| = 0$. If both u and v are in A, the number k is odd and the equality $|B \cap \{z_1, \ldots, z_k\}| - |A \cap \{z_1, \ldots, z_k\}| = 1$ holds since H' is a bipartite graph. Similarly, if both u and v are in B, then we have $|B \cap \{z_1, \ldots, z_k\}| - |A \cap \{z_1, \ldots, z_k\}| = -1$. Hence, we have the following.

$$|B| - |A| = e(H[A']) - e(H[B']) + |B'| - |A'| = (|B'| + e(H[A'])) - (|A'| + e(H[B'])).$$

Thus, by the definition of $\mathrm{hcf}_\xi(H)$, the number $|B| - |A|$ is divisible by $\mathrm{hcf}_\xi(H)$. This completes the proof. □

Proof of Proposition 3.4. Let $F \in \mathrm{Sub}(H)$ be a bipartite graph with bipartition A and B. Since F is a bipartite subdivision of H, by Claim 1, the difference $|B| - |A|$ is divisible by $\mathrm{hcf}_\xi(H)$. If G is a bipartite graph, and the difference between the sizes of two bipartitions of G is not divisible by $\mathrm{hcf}_\xi(H)$, the graph G does not have a perfect H-subdivision tiling since every subdivision of H in G is bipartite.

First, let n be an even number and $\mathrm{hcf}_\xi(H) > 2$. Let $G = K_{\frac{n}{2}-1, \frac{n}{2}+1}$. Then, the difference between the two bipartitions of G is two which cannot be divisible by $\mathrm{hcf}_\xi(H)$ since $\mathrm{hcf}_\xi(H) > 2$.

Second, let n be an odd number and $\mathrm{hcf}_\xi(H) \neq 1$. Let $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. Then, the difference between the sizes of the two bipartitions of G is 1, which is not divisible by $\mathrm{hcf}_\xi(H)$ as $\mathrm{hcf}_\xi(H) \neq 1$. □

In the following proposition, we consider the case when $\mathrm{hcf}_\xi(H) = 2$ and the order of the host graph is even.

Proposition 3.5. *For every graph H with $\mathrm{hcf}_\xi(H) = 2$ and for every even number n, there is an n-vertex graph G with minimum degree at least $\lfloor \frac{1}{3}n \rfloor - 1$ such that G does not contain a perfect H-subdivision tiling.*

Proof. Let (A, B, C) be a partition of $[n]$ with sizes $|A| = \lfloor \frac{1}{3}n \rfloor - 1$ and $|B| = \lfloor \frac{1}{3}n \rfloor$. Let G be a graph on vertex set $[n]$ such that G is a disjoint union of a clique on C and a complete bipartite graph with bipartition (A, B). Then, the minimum degree of G is $\lfloor \frac{1}{3}n \rfloor - 1$. Let $G_1 = G[A \cup B]$ and $G_2 = G[C]$. Since G_1 and G_2 are disjoint, if G contains a perfect H-subdivision tiling, so does G_1. However, the difference between the sizes of the bipartitions of G_1 is 1, which is not divisible by $\mathrm{hcf}_\xi(H) = 2$. Thus, by Claim 1, the graph G_1 does not contain a perfect H-subdivision tiling since G_1 is a bipartite graph, so every subdivision of H in G is also bipartite. Therefore, G does not have a perfect H-subdivision tiling. □

The next observation shows the relationship between $\xi(H)$ and $\chi_{cr}(H^*)$.

Observation 3.6. *For every graph H, the inequality $\left(1 - \frac{1}{\xi(H)}\right) \geq \left(1 - \frac{1}{\chi_{cr}(H^*)}\right)$ holds.*

Proof. It suffices to show that $\chi_{cr}(H^*) \leq \xi(H)$. We recall that X_H is a subset of $V(H)$ such that $f_H(X_H) = \xi(H)$. Let $Y_H := V(H) \setminus X_H$. Let $A = X_H$ and $B = Y_H$. Since H^* is obtained by replacing all edges in $H[X_H]$ and $H[Y_H]$ with vertex-disjoint paths of length two, we update the set B by adding vertices that were used to replace the edges in $H[X_H]$. Similarly, we update the set A by adding vertices that were used to replace the edges in $H[Y_H]$. Then A and B are independent sets and $|B| = |Y_H| + e(H[X_H])$. Thus, by the definition of $\sigma(H^*)$, the inequality $\sigma(H^*) \leq |Y_H| + e(H[X_H])$ holds. This implies $\chi_{cr}(H^*) \leq \frac{v(H^*)}{v(H^*) - (|Y_H| + e(H[X_H]))} = \frac{v(H) + e(H[X_H]) + e(H[Y_H])}{|X_H| + e(H[Y_H])} = \xi(H)$. □

4. Sub(*H*)-absorbers

We use the absorption method to prove Theorems 1.8 and 1.9. In order to execute the absorption method, we need some specific structures that we call $\mathrm{Sub}(H)$-absorbers.

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Definition 4.1. *Let H and G be graphs and take two subsets $A \subseteq V(G)$ and $X \subseteq V(G) \setminus A$. We say A is a* Sub(H)*-absorber for X if both $G[A]$ and $G[A \cup X]$ have perfect H-subdivision tilings. If $X = \{v\}$, we say A is a* Sub(H)*-absorber for v.*

Our strategy for constructing Sub(H)-absorbers is to collect many vertex-disjoint small subgraphs with good properties, which we call *absorber units*. We define three different types of absorber units, which we call type-1, type-2, and type-3 Sub(H)-absorbers, respectively. We first introduce the type-1 Sub(H)-absorber as follows.

Definition 4.2. *Let H be a graph and H^1 be the 1-subdivision of H. Since H^1 is bipartite, H^1 has a bipartition (U, V) such that the set U consists of all branch vertices. Choose an edge $xy \in E(H^1)$, with $x \in U$, and $y \in V$. Now we replace the edge xy with a path of length three, say $xwzy$ and denote by T^1_H the obtained graph. Let u and v be two new vertices and add edges uy, uw, vx, and vz to T^1_H and we denote by $\hat{T}^1_H$ the obtained graph.*

*Let G be a graph and let two distinct vertices a and $b \in V(G)$. We say a set $A \subseteq V(G) \setminus \{a, b\}$ is a type-*1 Sub(H)*-absorber for $\{a, b\}$ if $|A| = v(T^1_H)$ and there is an embedding $\phi : V(\hat{T}^1_H) \to A \cup \{a, b\}$ such that $\phi(u) = a$ and $\phi(v) = b$.*

We assume that the choice of xy is always the same for all H, so that T^1_H is uniquely determined for all H. A graph T^1_H is a subdivision of H since it is obtained from the 1-subdivision of H by replacing one edge with a vertex-disjoint path of length 3. By the construction of $\hat{T}^1_H$, we can replace a path $xwzy$ with a path $xvzwuy$, so $\hat{T}^1_H$ also contains a spanning subdivision of H. Thus, for a graph G and a and $b \in V(G)$, a set $X \subseteq V(G)$ is a type-1 Sub(H)-absorber for $\{a, b\}$, implying both $G[X]$ and $G[X \cup \{a, b\}]$ have perfect H-subdivision tilings.

Type-1 Sub(H)-absorbers are the most efficient bipartite absorber units. If the host graph is bipartite, then we only obtain bipartite absorber units. Note that a bipartite graph cannot absorb a single vertex, so absorbing two vertices is as good as it gets in a bipartite host graph. Moreover, since T^1_H is bipartite, supersaturation implies that there are many type-1 Sub(H)-absorbers, allowing us to obtain many efficient Sub(H)-absorbers.

If there are many type-1 Sub(H)-absorbers for every pair of vertices in G, it would be ideal to obtain the desired absorbers. However, this is not true in general, so it is useful to define the following auxiliary graph that captures information on vertex pairs that have many type-1 Sub(H)-absorbers.

Definition 4.3. *Let H be a graph and $d > 0$. For a given n-vertex graph G, we define the d-absorbing graph F as the graph on the vertex set $V(G)$ such that $uv \in E(F)$ if and only if the number of type-*1 Sub(H)*-absorbers for $\{u, v\}$ is at least $dn^{v(T^1_H)}$.*

In order to deal with non-bipartite host graphs, we define the following two types of absorber units. Note that these absorbers are able to absorb one vertex.

Definition 4.4. *Let H be a graph and (U, V) be a bipartition of H^1 such that U is the set of all branch vertices. Let x, y be the vertices as in Definition 4.2. For consistency, we denote by T^2_H the copy of H^1. Let u be a new vertex and add edges ux, uy to T^2_H and we denote by $\hat{T}^2_H$ the obtained graph.*

*Let G be a graph and let $a \in V(G)$. We say a set $A \subseteq V(G) \setminus \{a\}$ is a type-*2 Sub(H)*-absorber for $\{a\}$ if $|A| = v(T^2_H)$ and there is an embedding $\phi : V(\hat{T}^2_H) \to A \cup \{a\}$ such that $\phi(u) = a$.*

Since T^2_H is isomorphic to H^1, it has a perfect H-subdivision tiling. The graph $\hat{T}^2_H$ contains a spanning subgraph F which is obtained from T^2_H by replacing the edge $xy \in T^2_H$ with the path xuy. Thus, $\hat{T}^2_H$ also has a perfect H-subdivision packing. The next definition is for type-3 Sub(H)-absorbers.

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Definition 4.5. *Let H be a graph. Let two vertex-disjoint graphs H^1 and T^1_H be given. Let x, y, z, w be the vertices of T^1_H as in Definition 4.2. Choose an edge $x'y' \in E(H^1)$. We denote by T^3_H a graph that is obtained from H^1 and T^1_H by adding two edges $x'z$ and $y'w$. Let u be a new vertex and add edges ux, uy to T^3_H. We denote by $\hat{T}^3_H$ the obtained graph. We write $\tilde{T}^3_H$ to denote a bipartite graph that is $T^3_H - \{x, y\}$.*

Let G be a graph and let $a \in V(G)$. We say a set $A \subseteq V(G) \setminus \{a\}$ is a type-3 $\mathrm{Sub}(H)$-absorber for $\{a\}$ if $|A| = v(T^3_H)$ and there is an embedding $\phi : V(\hat{T}^3_H) \to A \cup \{a\}$ such that $\phi(u) = a$.

Note that T^3_H contains vertex-disjoint copies of H^1 and T^1_H whose union covers all vertices of T^3_H. We observe that $\hat{T}^3_H$ contains vertex-disjoint copies of graphs F_1 and F_2 such that F_1 is obtained from H^1 by replacing the edge $x'y'$ of H^1 with $x'wzy'$ and F_2 is obtained from T^1_H by replacing $xwzy$ with xuy. Both F_1 and F_2 are subdivisions of H. Thus, both graphs T^3_H and $\hat{T}^3_H$ have perfect H-subdivision tilings.

We note that $\hat{T}^2_H - u$ and $\hat{T}^3_H - u$ are both bipartite graphs. Moreover, $\hat{T}^2_H$ contains one triangle, and $\hat{T}^3_H$ contains one C_5, and deleting them leaves a bipartite graph. Hence, in graphs containing many C_3 or C_5, we can find many copies of $\hat{T}^2_H$ and $\hat{T}^3_H$, respectively. We will use these properties of $\hat{T}^2_H$ and $\hat{T}^3_H$ to prove a special case for Theorem 1.9.

Since all four graphs T^2_H, $\hat{T}^2_H$, T^3_H, and $\hat{T}^3_H$ have perfect H-subdivision tilings, type-2 and type-3 $\mathrm{Sub}(H)$-absorbers can be used to absorb a single vertex. Although this is an advantage over the type-1 $\mathrm{Sub}(H)$-absorbers, it is not guaranteed that we can always find many type-2 and type-3 absorbers in a graph G if G is close to being bipartite. Indeed, we only use type-1 $\mathrm{Sub}(H)$-absorbers to prove Theorem 1.8. On the other hand, if $\mathrm{hcf}_\xi(H)$ is two, type-1 absorbers are not sufficient to build the desired absorbers. So, the proof of Theorem 1.9 uses all three types of absorber units.

As we mentioned before, not all vertices may have many absorber units for them. So, it would be useful if we could somehow control what vertices remain uncovered. The following structure allows us to 'exchange' the remaining vertices so that we can absorb those vertices better.

Definition 4.6. *Let H be a graph and (U, V) be a bipartition of H^1 such that U consists of the branch vertices. Let $x_0y_0 \in E(H)$ be an arbitrary edge. Then there is a unique vertex $z \in V$ such that $x_0z, y_0z \in E(H^1)$. Let $S_H = H^1 - z$. We now introduce two new vertices u and v and add edges ux_0, uy_0, vx_0, vy_0 to S_H and we denote by $\hat{S}_H$ the obtained graph.*

Let G be a graph and two distinct vertices $a, b \in V(G)$. We say a set $B \subseteq V(G) \setminus \{a, b\}$ is a $\mathrm{Sub}(H)$-exchanger for $\{a, b\}$ if $|B| = v(S_H)$ and there is an embedding $\phi{:}V(\hat{S}_H) \to B \cup \{a, b\}$ such that $\phi(u) = a$ and $\phi(v) = b$.

We assume that the choice of x_0y_0 is always the same for all H, thus S_H is uniquely determined for all H. We note that both $S_H + a$ and $S_H + b$ are subdivisions of H. The following definition is a graph containing information on pairs that have many $\mathrm{Sub}(H)$-exchangers.

Definition 4.7. *Let H be a graph and $d > 0$. For a given n-vertex graph G and $B \subseteq V(G)$, let F be an auxiliary graph on the vertex set $V(G)$ such that $uv \in E(F)$ if and only if $u \notin B, v \in B$ and the number of $\mathrm{Sub}(H)$-exchangers for $\{u, v\}$ is at least $dn^{v(S_H)}$. We say such an auxiliary graph F is a (d, B)-exchanging graph.*

In order to prove the existence of $\mathrm{Sub}(H)$-absorbers, we need to collect several lemmas regarding the following definition.

Definition 4.8. *Let n, t be positive integers, and let $d \in (0, 1)$. Let X, Y be sets such that $|X| \leq n^2$ and $|Y| = n$. We say a collection of pairs $\mathcal{P} \subseteq X \times \binom{Y}{t}$ is an (n, t, d)-family on (X, Y) if $|\{Z \subseteq Y : (x, Z) \in \mathcal{P}\}| \geq dn^t$ for all $x \in X$.*

The following is the key lemma to obtain $\mathrm{Sub}(H)$-absorbers.

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Lemma 4.9. *Let $0 < \frac{1}{n} \ll \beta \ll d, \alpha, \frac{1}{t} \leq 1$. Let $\mathcal{P}$ be an (n, t, d)-family on a pair of sets (X, Y). Then there is $\mathcal{Y} \subseteq \binom{Y}{t}$ which satisfies the following:*

- *$\mathcal{Y}$ is a collection of pairwise disjoint sets,*
- *$|\mathcal{Y}| \leq \frac{\alpha}{t} n$ and $|\mathcal{Y}|$ is even,*
- *for every $x \in X$, the size of the set $\{Z \in \mathcal{Y} : (x, Z) \in \mathcal{P}\}$ is at least βn.*

Proof. Let $c = \min\{\frac{\alpha}{2}, \frac{d}{10}\}$. Consider a random family $\mathcal{T}$ of size t subsets of Y, such that each set in $\binom{Y}{t}$ is selected independently at random with the probability $p = \frac{c}{n^{t-1}}$. Then $|\mathcal{T}|$ has a binomial distribution with an expectation less than or equal to $\frac{\alpha}{2t} n$. By Chebyshev's inequality, with probability $1 - o(1)$, the following holds.

$$|\mathcal{T}| \leq \frac{\alpha}{t} n. \tag{1}$$

For each $x \in X$, let $\mathcal{Z}_x := \{Z \subseteq Y : (x, Z) \in \mathcal{P}\}$. Then the size $|\mathcal{T} \cap \mathcal{Z}_x|$ has a binomial distribution with an expectation $|\mathcal{Z}_x| p \geq cdn$ for every $x \in X$. By Chernoff bound, for each $x \in X$, the inequality $|\mathcal{T} \cap \mathcal{Z}_x| \geq \frac{cd}{2} n$ holds with probability $1 - o(n^{-3})$. Since $|X| \leq n^2$, we have the following with probability $1 - o(1)$.

$$|\mathcal{T} \cap \mathcal{Z}_x| \geq \frac{cd}{2} n \text{ for all } x \in Z. \tag{2}$$

We now count the number of intersecting pairs in $\mathcal{T}$. Let I be a random variable which is the number of intersecting pairs of $\mathcal{T}$. Since we have $|\{(P, Q) : P, Q \subseteq [n],\ P \cap Q \neq ptyset,\ \text{and } |P| = |Q| = t\}| \leq n^{2t-1}$, and each P, Q are chosen to be in $\mathcal{T}$ with probability $p = \frac{c}{n^{t-1}}$, we have $\mathbb{E}(I) \leq c^2 n$. By Markov's inequality, with probability at least $\frac{1}{2}$,

$$|I| \leq 2c^2 n. \tag{3}$$

Thus, for all sufficiently large n, there is a family $\mathcal{T}$ that satisfies all Equations (1) to (3). Let $\mathcal{Y}$ be a collection obtained from $\mathcal{T}$ by removing intersecting pairs from $\mathcal{T}$ and removing at most one arbitrary element to ensure that $|\mathcal{Y}|$ is even.

Then $\mathcal{Y}$ is a collection of pairwise disjoint sets and by (1), we have $|\mathcal{Y}| \leq |\mathcal{T}| \leq \frac{\alpha}{t} n$ and $|\mathcal{Y}|$ is even. By (2) and (3), the size of the intersection $|\mathcal{Y} \cap \mathcal{Z}_x| \geq \frac{cd}{2} n - 2c^2 n - 1 \geq \frac{\alpha d^2}{100} n$ for all $x \in X$. We now set $\beta = \frac{\alpha d^2}{100}$. Then $\mathcal{Y}$ is the desired collection. This completes the proof. □

We remark that Lemma 4.9 is a systematisation of Lemma 2.3 from [27]. One direct application of Lemma 4.9 is the existence of one-side perfect matchings for small sets, which is the following:

Lemma 4.10. *Let $0 < \frac{1}{n} \ll \beta \ll \alpha, \eta < 1$. Let G be an n-vertex graph. Suppose that a vertex subset $B \subseteq V(G)$ has size at least ηn and all vertices $v \in V(G) \setminus B$ satisfy $d(v, B) \geq \eta n$. Then there is a set $B' \subseteq B$ such that $|B'| \leq \alpha n$ and for any $U \subseteq V(G) \setminus B$ with size at most βn, there is a matching on $G[U, B']$ which covers all vertices in U.*

Proof. Let $X = V(G) \setminus B$. Let $\mathcal{P} = \{(x, \{y\}) \in X \times \binom{V(G)}{1} : xy \in E(G[X, B])\}$. Then $\mathcal{P}$ is an $(n, 1, \eta)$-family on $(X, V(G))$. As $\frac{1}{n} \ll \beta \ll \alpha, \eta$, by applying Lemma 4.9 with the parameters α, η, β, and n playing the roles of α, d, β, and n, respectively, we obtain $\mathcal{Y} \subseteq \binom{B}{1}$ satisfying the following. $|\mathcal{Y}| \leq \alpha n$ and $|\{(v, \{b\}) \in \mathcal{P} : \{b\} \in \mathcal{Y}\}| \geq \beta n$ for all $x \in X$. Let $B' = \bigcup_{\{b\} \in \mathcal{Y}} \{b\}$. Then $|B'| \leq \alpha n$ and for any vertex $v \in V(G) \setminus B$, the degree $d(v, B') \geq \beta n$. Thus, for any subset $U \subseteq V(G) \setminus B$ with size at most βn, we can find a matching M greedily in $G[U, B']$ which covers all vertices of U. □

Using Lemma 4.9, we can obtain the following lemma.

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Lemma 4.11. *Let H be an h-vertex graph and $0 < \frac{1}{n}, \beta \ll d, \alpha, \eta, \frac{1}{h} < 1$. Let G be an n-vertex graph and a vertex subset $B \subseteq V(G)$ has size at least ηn. Let F be the (d, B)-exchanging graph of G. Let $U = \{v \in V(G) \setminus B : d_F(v) \geq \frac{\eta}{3} n\}$. Then there exists a set $A \subseteq V(G)$ satisfying the following:*

- *$G[A]$ has a perfect H-subdivision tiling,*
- *$|A| \leq \alpha n$,*
- *for any $U' \subseteq U \setminus A$ with $|U'| \leq \beta n$, there exists a subset $W \subseteq A \cap B$ such that $|W| = |U'|$ and $G[(A \cup U') \setminus W]$ has a perfect H-subdivision tiling.*

Proof. We recall that $\hat{S}_H$ is a graph defined in Definition 4.6 which has two vertices u and v such that both $S_H - u$ and $S_H - v$ contain spanning subdivisions of H.

Let

$$\mathcal{P} = \{(u, Z) \in U \times \binom{V(G)}{v(S_H) + 1} : \exists v \in Z \cap B \text{ s.t. } Z - v \text{ is a Sub}(H)\text{-exchanger for } \{u, v\}\}.$$

For each $u \in U$, the number of pairs $(v, S) \in B \times \binom{V(G)}{v(S_H)}$ such that S is a Sub(H)-exchanger for $\{u, v\}$ is at least $\frac{\eta d}{3} n^{v(S_H)+1}$. Thus, $\mathcal{P}$ is an $(n, v(S_H) + 1, \frac{\eta d}{3(v(S_H)+1)})$-family on $(U, V(G))$.

As $v(S_H) + 1 \leq h^2$ and $\frac{1}{n}, \beta \ll d, \alpha, \eta, \frac{1}{h}$, by applying Lemma 4.9 with $v(S_H) + 1$, α, $\frac{\eta d}{3(v(S_H)+1)}$ and $\beta(v(S_H) + 1)$ playing the roles of t, α, d, and β, respectively, we obtain a collection of vertex-disjoint subsets $\mathcal{Y} \subseteq \binom{V(G)}{v(S_H)+1}$ satisfying the following.

1. $|\mathcal{Y}| \leq \frac{\alpha}{v(S_H)+1} n$,
2. $|\{Z \in \mathcal{Y} : (u, Z) \in \mathcal{P}\}| \geq \beta(v(S_H) + 1)n$ for all $u \in U$.

Let $A := \bigcup_{Z \in \mathcal{Y}} Z$. Then $|A| \leq \alpha n$. Let $U' \subseteq U \setminus A$ with size at most βn. By our choice of U and $\mathcal{Y}$, for each vertex $x \in U'$, there are at least $\frac{\beta(v(S_H)+1)}{v(S_H)+1} n = \beta n$ distinct pairs (u, v) which use distinct Sub(H)-exchangers in $\mathcal{Y}$ such that $v \in A \cap B$ and there is $Z \in \mathcal{Y}$, where $v \in Z$ and $Z - v$ is a Sub(H)-exchanger for $\{u, v\}$. Thus, we can greedily exchange all vertices in U' with a set of distinct vertices W, by using Sub(H)-exchangers, where $W \subseteq A \cap B$ and $|W| = |U'|$. □

4.1. Type-1 Sub(H)*-absorbers*

The following lemma is useful to get Sub(H)-absorbers which use type-1 Sub(H)-absorbers.

Lemma 4.12. *Let H be an h-vertex graph and $0 < \frac{1}{n}, \beta \ll d, \alpha, \frac{1}{h} \leq 1$. Let G be an n-vertex graph and F be the d-absorbing graph of G. Then there exists a set $A \subseteq V(G)$ satisfying the following.*

- *$|A| \leq \alpha n$,*
- *$|A|$ is even,*
- *The set A is a* Sub(H)*-absorber for any vertex set U such that $U \subseteq V(G) \setminus A$, $|U| \leq \beta n$, and $F[U]$ has a perfect matching.*

Proof. Let

$$\mathcal{P} = \{(e, Z) \in E(F) \times \binom{V(G)}{v(T_H^1)} : Z \text{ is a type-1 Sub}(H)\text{-absorber for } e\}.$$

Then $\mathcal{P}$ is an $(n, v(T_H^1), d)$-family on $(E(F), V(G))$. By applying Lemma 4.9 with $v(T_H^1)$, α, d, and β playing the roles of t, α, d, and β, respectively, we obtain a collection of vertex-disjoint subsets $\mathcal{Y} \subseteq \binom{V(G)}{v(T_H^1)}$ satisfying the following.

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1. $|\mathcal{Y}| \leq \frac{\alpha}{v(T_H^1)} n$,
2. $|\mathcal{Y}|$ is even,
3. $|\{Z \in \mathcal{Y} : (e, Z) \in \mathcal{P}\}| \geq \beta n$ for all $e \in E(F)$.

Let $A = \bigcup_{Z \in \mathcal{Y}} Z$. Then $|A|$ is even and less than or equal to αn. Let U be a subset of $V(G) \setminus A$ with size at most βn. Assume $F[U]$ has a perfect matching M. Then $e(M) \leq \frac{\beta}{2} n$. By our choice of $\mathcal{Y}$, for each edge $e \in E(M)$, there are at least βn distinct type-1 Sub(H)-absorbers. Thus, A can absorb every pair $e \in E(M)$ greedily. This means A is a Sub(H)-absorber for U. This proves the lemma. □

4.2. Type-2 and type-3 Sub(H)-absorbers

In this subsection, we will show how we can obtain Sub(H)-absorbers via type-2 and type-3 Sub(H)-absorbers, respectively.

Lemma 4.13. *Let H be an h-vertex graph and $0 < \frac{1}{n}, \beta \ll d, \alpha, \eta, \frac{1}{h} \leq 1$. Let $j \in \{2, 3\}$. Let G be an n-vertex graph and let $B \subseteq V(G)$ with size at least ηn such that for all $v \in B$, there are at least $dn^{v(T_H^j)}$ type-j Sub(H)-absorbers for v. Let F_1 and F_2 be the d-absorbing graph and the (d, B)-exchanging graph of G, respectively. Assume that for every $v \in V(G) \setminus B$, at least one of the inequalities $d_{F_1}(v; B) \geq \frac{\eta}{3} n$ or $d_{F_2}(v; B) \geq \frac{\eta}{3} n$ holds. Then, there is a set $A \subseteq V(G)$ with size at most αn such that for all $U \subseteq V(G) \setminus A$ with size at most βn, the set A is a* Sub(H)*-absorber for U.*

Proof. Choose additional constants $\beta_0, \alpha_0, \alpha_1$ so that the following holds:

$$0 < \frac{1}{n}, \beta \ll \beta_0 \ll \alpha_0 \ll \alpha_1 \ll d, \alpha, \eta, \frac{1}{h} \leq 1.$$

We write $R = V(G) \setminus B$, and $X = \{v \in R : d_{F_1}(v; B) \geq \frac{\eta}{3} n\}$. Let $Y = R \setminus X$. We note that for all $v \in Y$, we have $d_{F_2}(v; B) \geq \frac{\eta}{3} n$. We now collect the following two claims.

Claim 1. *There exists a set $A_0 \subseteq V(G)$ satisfying the following. The size of the set A_0 is less than or equal to $\alpha_0 n$, and for any $U \subseteq B \setminus A_0$ with size at most $\beta_0 n$, the set A_0 is a* Sub(H)*-absorber for U.*

Proof of Claim 1. Let

$$\mathcal{P} = \{(x, Z) \in B \times \binom{V(G)}{v(T_H^j)} : Z \text{ is a type-}j \text{ Sub}(H)\text{-absorber for } x\}.$$

Then $\mathcal{P}$ is an $(n, v(T_H^j), d)$-family on $(B, V(G))$. By applying Lemma 4.9 with $v(T_H^j)$, α_0, d, and β_0 playing the roles of t, α, d, and β, respectively, we obtain a collection of pairwise disjoint sets $\mathcal{Y} \subseteq \binom{V(G)}{v(T_H^j)}$ satisfying the following.

1. $|\mathcal{Y}| \leq \frac{\alpha_0}{v(T_H^j)} n$,
2. $|\{Z \in \mathcal{Y} : (x, Z) \in \mathcal{P}\}| \geq \beta_0 n$ for all $x \in B$.

Let $A_0 := \bigcup_{Z \in \mathcal{Y}} Z$. Then $|A_0| \leq \alpha_0 n$. Let $U \subseteq B \setminus A_0$ with size at most $\beta_0 n$. By our choice of $\mathcal{Y}$, for each vertex $x \in U$, there are at least $\beta_0 n$ distinct type-j Sub(H)-absorbers in A_0. Thus, A_0 can absorb all vertices in U greedily. This means A_0 is a Sub(H)-absorber for U. This proves the claim. □

Claim 2. *There exists a set $A_1 \subseteq V(G)$ with size at most $\alpha_1 n$ satisfying the following: for any $U \subseteq (B \cup X) \setminus A_1$ with size at most βn, the set A_1 is a* Sub(H)*-absorber for U.*

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Proof of Claim 2. We fix additional parameters $\beta_0'', \alpha_0'', \beta_0', \alpha_0'$ so that the following holds: $\beta \ll \beta_0'' \ll \alpha_0'' \ll \beta_0, \beta_0' \ll \alpha_0, \alpha_0' \ll \alpha_1$. By Claim 1, there exists a set $A_0 \subseteq B$ such that $|A_0| \leq \alpha_0 n$ and for any $U \subseteq B \setminus A_0$ with size at most $\beta_0 n$, the set A_0 is a Sub(H)-absorber for U. Let $G_1 = G - A_0$, $B_1 = B \setminus A_0$, and $X_1 = X \setminus A_0$. Then $v(G_1) \geq (1 - \alpha_0)n$ and $|B_1| \geq \frac{\eta}{2} n$. By Observation 2.1, we know that for every vertex $v \in B_1$, the number of type-j Sub(H)-absorbers in G_1 is at least $\frac{d}{2} n_1^{v(T_H^j)}$, and for each pair of vertices $(u, v) \in X_1 \times B_1$, where $uv \in E(F_1)$, the number of type-1 Sub(H)-absorbers for $\{u, v\}$ is at least $\frac{d}{2} n_1^{v(T_H^1)}$. Let F_1' be the $\frac{d}{2}$-absorbing graph of G_1. Then for every vertex $v \in X_1$, the degree $d_{F_1'}(v,\ B_1) \geq \frac{\eta}{3} n - \alpha_0 n \geq \frac{\eta}{4} n$.

By Lemma 4.12, there is $A_0' \subseteq V(G_1)$ with size at most $\alpha_0' n$ such that for any $U \subseteq V(G_1) \setminus A_0'$ with size less than $\beta_0' n$ which has a perfect matching in F_1', the set A_0' is a Sub(H)-absorber for U.

Let $G_2 = G_1 - A_0'$, $B_2 = B_1 \setminus A_0'$, and $X_2 = X_1 \setminus A_0'$. Then $v(G_2) \geq (1 - \alpha_0 - \alpha_0')n$ and $|B_2| \geq \frac{\eta}{4} n$. For every $v \in X_2$, the degree $d_{F_1'}(v; B_2) \geq \frac{\eta}{4} n - \alpha_0' n \geq \frac{\eta}{10} n$. Then by Lemma 4.10, there is a set $A_0'' \subseteq B_2$ with size at most $\alpha_0'' n$ such that for any $U \subseteq X_2 \setminus A_0''$ with size less than $\beta_0'' n$, the graph $F_1'[U, A_0'']$ has a matching M which covers all vertices of U.

Now, let $A_1 = A_0 \cup A_0' \cup A_0''$. Then the size $|A_1| \leq (\alpha_0 + \alpha_0' + \alpha_0'')n \leq \alpha_1 n$. Note that since $A_0'' \subseteq B_1$ and $|A_0''| \leq \alpha_0'' n \leq \frac{\beta_0'}{2} n$, the induced graph $G[A_0 \cup A_0'']$ has a perfect H-subdivision tiling. Since $G[A_0']$ also has a perfect H-subdivision tiling and A_0, A_0', A_0'' are vertex-disjoint subsets, $G[A_1]$ has a perfect H-subdivision tiling.

We now claim that A_1 is the desired Sub(H)-absorber. Let $U \subseteq (B \cup X) \setminus A_1$ with size at most βn. Let $U_B = U \cap B$ and $U_X = U \cap X$. Since $|U_X| \leq \beta n \leq \beta_0'' n$, by our choice of A_0'', there is a set $W \subseteq A_0''$ with the same size as U_X such that $F_1'[U_X, W]$ has a perfect matching. Since $|U_X \cup W| \leq 2\beta n \leq \beta_0' n$, the induced graph $G[(A_0' \cup U_X \cup W)]$ has a perfect H-subdivision tiling. Let $W' = A_0'' \setminus W$. We note that $|W' \cup U_B| \leq \beta n \leq \beta_0 n$ and $(W' \cup U_B) \subseteq B_1$, so by our choice of A_1, the induced graph $G[(A_0 \cup W' \cup U_B)]$ has a perfect H-subdivision tiling. Thus, $G[A_1 \cup U]$ also has a perfect H-subdivision tiling. Therefore, the set A_1 is a Sub(H)-absorber for U. □

By Claim 2, there exists a set $A_1 \subseteq V(G)$ with size at most $\alpha_1 n$ such that for any subset U of $(B \cup X) \setminus A_1$ with size at most βn, the set A_1 is a Sub(H)-absorber for U. Let $G_1 = G - A_1$, $B_1 = B \setminus A_1$, $X_1 = X \setminus A_1$, and $Y_1 = Y \setminus A_1$. Then $v(G_1) \geq (1 - \alpha_1)n$ and $|B_1| \geq \frac{\eta}{2} n$. By Observation 2.1, we know that for each pair of vertices $(u, v) \in Y_1 \times B_1$, where $uv \in E(F_2)$, the number of Sub(H)-exchangers for $\{u, v\}$ is at least $\frac{d}{2} n_1^{v(S_H)}$. Let F_2' be the $(\frac{d}{2}, B_1)$-exchanging graph of G_1. Then for every vertex $v \in Y_1$, the degree $d_{F_2'}(v, B_1) \geq \frac{\eta}{3} n - \alpha_1 n \geq \frac{\eta}{4} n$.

By Lemma 4.11, there is a subset A_2 of $V(G_1)$ with size at most $\alpha_1 n$ such that $G_1[A_2]$ has a perfect H-subdivision tiling and satisfies the following: for any $U \subseteq Y_1 \setminus A_2$ with size at most βn, there is a set $W \subseteq (A_2 \cap X_1)$ with the same size as U such that $G_1[(A_2 \cup U) \setminus W]$ also has a perfect H-subdivision tiling.

Now, let $A = A_1 \cup A_2$. Then the inequality $|A| \leq 2\alpha_1 n \leq \alpha n$ holds. Note that both $G[A_1]$ and $G[A_2]$ have perfect H-subdivision tilings and A_1 and A_2 are disjoint subsets of $V(G)$. Thus, $G[A]$ also has a perfect H-subdivision tiling.

We now claim that A is the desired Sub(H)-absorber. Let $U \subseteq V(G) \setminus A$ with size at most βn. Let $U_B = U \cap B$, $U_X = U \cap X$, and $U_Y = U \cap Y$. Since $|U_Y| \leq \beta n$, by our choice of A_2, there is a set $W \subseteq (A_2 \cap X_1)$ with the same size as U_Y such that $G[(A_2 \cup U_Y) \setminus W]$ has a perfect H-subdivision tiling. Note that $|U_B \cup U_X \cup W| = |U| \leq \beta n$ and $(U_B \cup U_X \cup W) \subseteq (B \cup X) \setminus A_1$. Thus, $U_B \cup U_X \cup W$ can be absorbed by A_1. Thus, $G[A \cup U]$ has a perfect H-subdivision tiling. Therefore, A is a Sub(H)-absorber for U. This completes the proof. □

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5. Proofs

5.1. Proof sketch

In this section, we will prove our main results, Theorems 1.8 and 1.9. By Proposition 3.1, Proposition 3.3, Proposition 3.4, and Proposition 3.5, it suffices to prove the following two lemmas.

Lemma 5.1. *Let H be an h-vertex graph with $\mathrm{hcf}_\xi(H)=1$. Let $0<\frac{1}{n}\ll\gamma,\frac{1}{h}\le 1$. Let G be an n-vertex graph. Then the following holds: If $\delta(G)\ge\left(1-\frac{1}{\xi(H)}+\gamma\right)n$, then G has a perfect H-subdivision tiling.*

Lemma 5.2. *Let H be an h-vertex graph with $\mathrm{hcf}_\xi(H)=2$. Let $0<\frac{1}{n}\ll\gamma,\frac{1}{h}\le 1$. Let G be an n-vertex graph, where n is even. Then the following holds: If $\delta(G)\ge\left(\max\{\frac{1}{3},1-\frac{1}{\xi(H)}\}+\gamma\right)n$, then G has a perfect H-subdivision tiling.*

The proofs of Lemmas 5.1 and 5.2 will be provided in Subsections 5.2 and 5.3, respectively. The proofs crucially use the absorption method. Our proof strategy is as follows: Let H be a graph and G be a host graph equipped with a minimum degree condition as in the statements of the above lemmas, in which we want to find a perfect H-subdivision tiling. Then, together with Observation 3.6 and Theorem 1.1, we can find an almost perfect H-subdivision tiling in G. Thus, we mainly focus on absorbing the remaining vertices by constructing suitable absorbers for all cases. A detailed proof sketch is provided below.

- **Step 1: Preprocessing.** Apply the regularity lemma with properly chosen parameters on G. After that, collect bad vertices and construct absorbing graphs on good vertices.
- **Step 2: Place the absorber.** Place an efficient $\mathrm{Sub}(H)$-absorber by using the abundant properties of ε-regular pairs.
- **Step 3: Cover almost all vertices.** Find a set $W\subseteq V(G)$ outside of the $\mathrm{Sub}(H)$-absorber obtained from Step 2 such that $G[W]$ has a perfect H-subdivision tiling, W contains all bad vertices, and W covers all but at most a constant number of vertices of G that are not in the $\mathrm{Sub}(H)$-absorber.
- **Step 4: Absorb the uncovered vertices.** Absorb all the vertices that are not covered in Step 3 by using the $\mathrm{Sub}(H)$-absorber obtained from Step 2.

In the above description, we omitted many details. To achieve Step 3, the following lemma would be useful.

Lemma 5.3. *Let H be an h-vertex graph. Let $0<\frac{1}{n}\ll\alpha,\frac{1}{h}\le 1$, and $0<\frac{1}{C}\ll\frac{1}{h}\le 1$. Let G be an n-vertex graph with minimum degree at least $\left(1-\frac{1}{\xi(H)}+\alpha\right)n$. Let X be a subset of $V(G)$ with size at most $\frac{\alpha}{2v(H^*)}n$. Then there is a set $W\subseteq V(G)$ with size at least $n-C$ such that $X\subseteq W$ and $G[W]$ have a perfect H-subdivision tiling. Moreover, if $\mathrm{hcf}_\xi(H)=2$, we can get such a set W whose size is even.*

Proof. We will first find vertex-disjoint copies of H^* covering all vertices of X. Let $X=\{v_1,\dots,v_t\}$ where $t\le\frac{\alpha}{2v(H^*)}n$. Assume for $i<t$, we have $\mathcal{F}_i=\{F_1,\dots,F_i\}$, where $\mathcal{F}_i$ is a collection of vertex-disjoint copies of H^* and $v_j\in V(F_j)$ for each $j\in[i]$, and $G_i=G-\bigcup_{j\in[i]}V(F_i)-X$.

We note that $\left|\bigcup_{j\in[i]}V(F_i)\cup X\right|\le\frac{\alpha}{2}n$ holds as $i<t$. Thus, $\delta(G_i+v_{i+1})\ge\left(1-\frac{1}{\xi(H^*)}+\frac{\alpha}{2}\right)n$. Let Y be the subset of the neighbourhoods of v_{i+1} in G_i+v_{i+1} of size $\left(1-\frac{1}{\xi(H^*)}+\frac{\alpha}{2}\right)n$. Let A and B be the bipartition of H^* and u be a vertex in A. We will embed u to v_{i+1} first and then embed other vertices of H^*-u while all the vertices of B are embedded in Y. By the minimum degree condition on G_{i+1}, Lemma 2.4 enables us to find such embedding, we obtain a new copy of H^*,

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say F_{i+1} containing v_{i+1}. As it is disjoint with $\bigcup_{j\le i} V(F_j)\cup X$, we may add it to enlarge $\mathcal{F}_i$ to $\mathcal{F}_{i+1}$. We iterate this process until we get $\mathcal{F}_t$. Let $F=\bigcup_{i\in\mathcal{F}_t} V(F_i)$. Then $X\subseteq F$ and $|F|=v(H^*)t\le\frac{\alpha}{2}n$ holds.

Now, let $G'=G-F$. Since $|F|\le\frac{\alpha}{2}n$, the minimum degree of G' is at least $\left(1-\frac{1}{\xi(H)}+\frac{\alpha}{2}\right)n$. Thus, by Theorem 1.1 and Observation 3.6, there is a set $F'\subseteq V(G')$ such that $G'[F']$ has a perfect H^*-tiling and $v(G')-|F'|\le C$. Let $W=F\cup F'$. Then W is the desired set.

When $\mathrm{hcf}_\xi(H)=2$, since the order of $v(H^*)$ is even, the size of W is also even. This completes the proof. □

In the proof of Lemma 5.1, we will use the concept of dominating sets to obtain an efficient Sub(H)-absorber. Let G be a graph. We say D is a *dominating set* of G if every vertex $v\in V(G)$ is either contained in D or has a neighbour in D. The domination number of G is defined as the minimum size of a dominating set of G.

The next lemma on domination number, proven by Arnautov [4], and independently by Payan [26], will be used in the proof of Lemma 5.1.

Lemma 5.4 (Arnautov [4], Payan [26]). *Let G be an n-vertex graph with minimum degree δ. Then the domination number of G is bounded above by $\frac{1+\ln(\delta+1)}{\delta+1}n$.*

Later, Alon [1] proved that Lemma 5.4 is asymptotically best possible.

In order to use type-1 Sub(H)-absorbers, we need to find a perfect matching in a d-absorbing graph for a suitable constant d. To achieve this, it would be useful to define a graph as follows.

Definition 5.5. *For a given graph H, consider a family of disjoint unions of bipartite subdivisions of H. Among those, choose a graph $\hat{H}$ with bipartition $(\hat{A},\hat{B})$ with the smallest possible $v(\hat{H})$ such that $||\hat{A}|-|\hat{B}||=\mathrm{hcf}_\xi(H)$. If there are multiple choices of $\hat{H}$, $\hat{A}$, and $\hat{B}$, we fix one choice.*

We note that for every H, we can prove that the bipartite graph $\hat{H}$ exists in the following way. By the definition of $\mathrm{hcf}_\xi(H)$, there exist bipartite graphs $H_1,\dots,H_m$ with bipartitions $(A_1,B_1),\dots,(A_m,B_m)$, respectively, that are bipartite subdivisions of H such that they satisfy the following: there are positive integers $c_1,\dots,c_m$ that satisfy the inequality $\sum_{i\in[m]} c_i(|A_i|-|B_i|)=\mathrm{hcf}_\xi(H)$. By taking a disjoint union of c_i copies of H_i for each $i\in[m]$, we obtain a bipartite graph that is a disjoint union of bipartite subdivisions of H and the difference between the bipartitions is $\mathrm{hcf}_\xi(H)$. Among such bipartite graphs, we can take our desired graph $\hat{H}$.

We use $\hat{H}$ and the bipartition $(\hat{A},\hat{B})$ to find perfect matchings in a d-absorbing graph for using type-1 Sub(H)-absorbers. The following observation shows that complete bipartite graphs with appropriate sizes act like reservoirs for $\hat{H}$.

Observation 5.6. *Let H be a graph with $\mathrm{hcf}_\xi(H)=t$ and assume $\hat{H}$ has a bipartition with sizes h' and $h'+t$. Let b be a positive integer. Then for any non-negative integer $a\le b$, the complete bipartite graph with bipartition sizes $(2h'+t)b-h'a$ and $(2h'+t)b-(h'+t)a$ has a perfect $\hat{H}$-tiling.*

Proof. Let $x=(2h'+t)b-h'a$ and $y=(2h'+t)b-(h'+t)a$. Since $x=h'(b-a)+(h'+t)b$ and $y=(h'+t)(b-a)+h'b$, the complete bipartite graph with bipartition sizes x and y has $2b-a$ vertex-disjoint copies of $K_{h',h'+t}$. Thus, $K_{x,y}$ has a perfect $\hat{H}$-tiling. □

5.2. Proof of Theorem 1.8

In this subsection, we will prove Lemma 5.1, which will complete the proof of Theorem 1.8.

Proof of Lemma 5.1. Let H be an h-vertex graph with $\mathrm{hcf}_\xi(H)=1$. Let $\hat{H}$ be the bipartite graph as in Definition 5.5 with a bipartition $(\hat{A},\hat{B})$ with sizes h' and $h'+1$. Note that h' is bounded above by a constant that depends only on h. We fix positive constants as follows:

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$$0 < \frac{1}{n} \ll \beta \ll \alpha \ll d_2 \ll \frac{1}{T} \ll \frac{1}{t_0} \ll \varepsilon \ll d_1 \ll d \ll \frac{1}{h'}, \frac{1}{C} \ll \gamma, \frac{1}{h} \leq 1.$$

Let $n \geq n_0$ and G be an n-vertex graph with minimum degree at least $\left(1 - \frac{1}{\xi(H)} + \gamma\right) n$. We apply the regularity lemma to G with parameters ε, d, and t_0 playing the roles of ε, d, and t_0, respectively. Let $V_0, V_1, \ldots, V_t$ be an ε-regular partition of G and G_0 be a spanning subgraph of G that satisfies the following conditions:

(R1) $t_0 \leq t \leq T_0$,
(R2) $|V_0| \leq \varepsilon v(G)$,
(R3) $|V_1| = \cdots = |V_t|$,
(R4) $G_0[V_i]$ has no edges for each $i \in [t]$,
(R5) $d_{G_0}(v) \geq d_G(v) - (d + \varepsilon)v(G)$ for all $v \in V(G)$,
(R6) for all $1 \leq i < j \leq t$, either the pair (V_i, V_j) is an (ε, d^+)-regular pair or $G_0[V_i, V_j]$ has no edges.

By (R5), the minimum degree of G_0 is at least $\left(1 - \frac{1}{\xi(H)} + \frac{\gamma}{2}\right) n$. Let R be the (ε, d)-reduced graph on $\{V_1, \ldots, V_t\}$. Then, by Lemma 2.7, $\delta(R) \geq \left(1 - \frac{1}{\xi(H)} + \frac{\gamma}{2}\right) t$. By Lemma 5.4, there is a dominating set $D \subseteq V(R)$ such that

$$|D| \leq \frac{1 + \ln\left(\left(1 - \frac{1}{\xi(H)} + \frac{\gamma}{2}\right) t + 1\right)}{\left(1 - \frac{1}{\xi(H)} + \frac{\gamma}{2}\right) t + 1} t \leq \frac{\varepsilon}{2} t.$$

We can pick a set $D' \subseteq V(R) \setminus D$ greedily such that $R[D, D']$ has a perfect matching M. Let

$$D = \{X_1, \ldots, X_\ell\}, \quad D' = \{Y_1, \ldots, Y_\ell\}, \quad \text{and} \quad E(M) = \{X_1Y_1, \ldots, X_\ell Y_\ell\}.$$

We note that $v(M) = 2\ell \leq \varepsilon t$. Since D is a dominating set of R, there is a partition $\{\mathcal{V}_1, \ldots, \mathcal{V}_\ell\}$ for $V(R) \setminus (D \cup D')$ such that for every $V_{i,j} \in \mathcal{V}_i$, the edge $X_iV_{i,j} \in E(R)$. Let $P_i := \bigcup_{V_{i,j} \in \mathcal{V}_i} V_{i,j} \cup Y_i$ for each $i \in [\ell]$.

For each $V_{i,j} \in \mathcal{V}_i$, where $i \in [\ell]$, let $U_{i,j} = \{u \in V_{i,j} : d_{G_0}(u; X_i) < (d - \varepsilon)|X_i|\}$. By Lemma 2.8, the size of $U_{i,j}$ is at most $\varepsilon|V_{i,j}| \leq \frac{\varepsilon n}{t}$. Let $V'_{i,j} = V_{i,j} \setminus U_{i,j}$. Similarly, for each $i \in [\ell]$, let $U_{i,D'} = \{u \in Y_i : d_{G_0}(u; X_i) < (d - \varepsilon)|X_i|\}$ and let $Y'_i = Y_i \setminus U_{i,D'}$. Then $|U_{i,D'}| \leq \varepsilon|Y_i| \leq \frac{\varepsilon n}{t}$. Finally, for each $i \in [\ell]$, let $U_{i,D} = \{u \in X_i : d_{G_0}(u; Y_i) < (d - \varepsilon)|Y_i|\}$ and let $X'_i = X_i \setminus U_{i,D}$. Then $|U_{i,D}| \leq \varepsilon|X_i| \leq \frac{\varepsilon n}{t}$.

We collect all the bad vertices, say $Z = V_0 \cup \bigcup_{i,j} U_{i,j} \cup \bigcup_i (U_{i,D} \cup U_{i,D'})$. Then $|Z| \leq \varepsilon n + t\frac{\varepsilon n}{t} \leq 2\varepsilon n$. We obtain a graph $G_1 = G_0 - Z$. Then $v(G_1) \geq (1 - 2\varepsilon)n$. We get a set $P'_i = P_i \setminus Z$ for each $i \in [\ell]$. Then for each $i \in [\ell]$, for every $v \in P'_i$, the degree $d_{G_0}(v; X'_i) \geq (d - \varepsilon)|X_i| - \varepsilon|X_i| \geq (d - 2\varepsilon)|X_i| \geq d_1|X_i|$. For the same reason, for each $i \in [\ell]$, for every $u \in X'_i$, the degree $d_{G_0}(u; Y'_i) \geq d_1|Y_i|$.

Let $u \in X'_i$ and $v \in P'_i$, where $i \in [\ell]$. We write $A_u = N_{G_1}(u) \cap Y'_i$ and $B_v = N_{G_1}(v) \cap X'_i$. We observe that the pair (X_i, Y_i) is an ε-regular pair, and the sizes $|A_u| \geq d_1|Y_i| > \varepsilon|Y_i|$ and $|B_v| \geq d_1|X_i| > \varepsilon|X_i|$. By the definition of ε-regular pair, the number of edges $e_{G_0}(A_u, B_v) \geq (d - \varepsilon)|A_u||B_v| \geq d_1^3|X_i||Y_i| \geq \frac{d_1^4}{T^2}n^2$. By Lemma 2.3, there are at least $d_2(v(T_H^1)!)n^{v(T_H^1)}$ copies of T_H^1 in $G_1[A_u, B_v]$. This means there are at least $d_2 n^{v(T_H^1)}$ distinct type-1 Sub(H)-absorbers for $\{u, v\}$. Let F be a d_2-absorbing graph of G_1. Then the following holds:

$$\text{For each } i \in [\ell], \text{ every pair } (u, v) \in X'_i \times P'_i, uv \text{ is an edge of } F. \tag{4}$$

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By Lemma 4.12, there is a set $A \subseteq V(G_1)$ with size at most αn such that for all subsets U of $V(G_1) \setminus A$ with size at most βn and $F[U]$ has a perfect matching, then A is a Sub(H)-absorber for U. Let us fix such a Sub(H)-absorber A. For each $i \in [\ell]$, let $X_i'' = X_i' \setminus A$ and $Y_i'' = Y_i' \setminus A$. Then we have $|X_i''| \geq |X_i'| - \alpha n \geq (1 - 2\varepsilon)|X_i|$ and, for the same reason, $|Y_i''| \geq (1 - 2\varepsilon)|Y_i|$ for every $i \in [\ell]$.

We now claim that for each $i \in [\ell]$, there is a copy Q_i of a complete balanced bipartite graph $K_{(2h'+1)C,(2h'+1)C}$. We note that the sizes $|X_i''| \geq \varepsilon|X_i|$ and $|Y_i''| \geq \varepsilon|Y_i|$. Since the pair (X_i, Y_i) is an ε-regular pair, the number of edges in $G_1[X_i'', Y_i'']$ is at least $(d - \varepsilon)|X_i''||Y_i''|$. Thus, by the Erdős–Stone–Simonovits theorem, there is at least one copy of $K_{(2h'+1)C,(2h'+1)C}$ in $G_1[X_i'', Y_i'']$. Thus, for each $i \in [\ell]$, there exists such a bipartite graph Q_i. Let $Q := \bigcup_{i\in[\ell]} V(Q_i)$. We note that by Observation 5.6, for each $i \in [\ell]$, the graph Q_i has a perfect $\hat{H}$-tiling. This implies $G[Q]$ has a perfect H-subdivision tiling.

Note that $|Q| = 2(2h' + 1)C\ell \leq \varepsilon n$. Let $G_2 = G_0 - (A \cup Q)$. Then we have $v(G_2) \geq (1 - \alpha - \varepsilon)n$ and $\delta(G_2) \geq \left(1 - \frac{1}{\xi(H)} + \frac{\gamma}{2} - (\alpha + \varepsilon)\right) n \geq \left(1 - \frac{1}{\xi(H)} + \frac{\gamma}{4}\right) n$. By Lemma 5.3, there is a set $W \subseteq V(G_2)$ such that $G[W]$ has a perfect H-subdivision tiling, the set Z is contained in W, and $|V(G_2) \setminus W| \leq C$. Let $J = V(G_2) \setminus W$ with size at most C.

We observe that $V(G) = A \cup Q \cup W \cup J$, and $J \subseteq \left(\bigcup_{i\in[\ell]} P_i' \cup \bigcup_{i\in[\ell]} X_i'\right) \setminus (A \cup Q)$. We recall that A is the Sub(H)-absorber for a small vertex subset which has a perfect matching in F. For each $i \in [\ell]$, let $J_i^1 = J \cap X_i'$ and $J_i^2 = J \cap P_i'$. We denote by J_i the union of J_i^1 and J_i^2. Let c_i be an integer for each $i \in [\ell]$ such that $c_i = |J_i^1| - |J_i^2|$. We define $\sigma(c_i) = 1$ if $c_i > 0$; otherwise, $\sigma(c_i) = 0$.

For each $i \in [\ell]$, if $c_i = 0$, then let S_i^1 and S_i^2 be empty sets. Otherwise, let them be sets $S_i^1 \subseteq (V(Q_i) \cap X_i')$ with size $(h' + 1 - \sigma(c_i))c_i$ and $S_i^2 \subseteq (V(Q_i) \cap Y_i')$ with size $(h' + \sigma(c_i))c_i$. By Observation 5.6, the induced graph $G[Q_i \setminus (S_i^1 \cup S_i^2)]$ has a perfect H-subdivision tiling for each $i \in [\ell]$. For each $i \in [\ell]$, since $|S_i^1| - |S_i^2| = -c_i$, the equality $|J_i^1 \cup S_i^1| = |J_i^2 \cup S_i^2|$ holds. We observe that $(J_i^1 \cup S_i^1) \subseteq X_i'$ and $(J_i^2 \cup S_i^2) \subseteq P_i'$. Thus, by (4), the induced graph $F[J_i \cup S_i^1 \cup S_i^2]$ has a perfect matching and we have $|J_i \cup S_i^1 \cup S_i^2| = |J_i| + |c_i|(2h' + 1)$. Let $S = \bigcup_{i\in[\ell]} S_i^1 \cup S_i^2$. Then, $Q \setminus S = \bigcup_{i\in[\ell]} (V(Q_i) \setminus (S_i^1 \cup S_i^2))$, so $G[Q \setminus S]$ has a perfect H-subdivision tiling.

Since $J \cup S = \bigcup_{i\in[\ell]} (J_i \cup S_i^1 \cup S_i^2)$, the induced graph $F[J \cup S]$ has a perfect matching. Moreover, $|J \cup S| = \sum_{i\in[\ell]} (J_i \cup S_i^1 \cup S_i^2) \leq |J| + (2h' + 1) \sum_{i\in[\ell]} |c_i| \leq (2h' + 2)C \leq \beta n$ holds. Thus, the set A is a Sub(H)-absorber for $J \cup S$. This means the induced graph $G[A \cup J \cup S]$ has a perfect H-subdivision tiling. Together with $G[W]$, $G[Q \setminus S]$, and $G[A \cup J \cup S]$, the graph G has a perfect H-subdivision tiling. This completes the proof. □

5.3. Proof of Theorem 1.9

We now prove Lemma 5.2 to complete the proof of Theorem 1.9. Our purpose is to find a perfect H-subdivision tiling in a certain graph, where $\mathrm{hcf}_\xi(H) = 2$.

In this case, we cannot apply the same idea as the proof of Theorem 1.8. To prove Theorem 1.9 with type-1 Sub(H)-absorbers, we need to find a perfect matching in an absorbing graph, but since $\mathrm{hcf}_\xi(H) = 2$, parity issues make it difficult to obtain a perfect matching. If the host graph G is close to bipartite, we can handle these parity difficulties, but if G is far from bipartite, then we need to use different types of Sub(H)-absorbers. Thus, we divide the cases depending on whether G has many triangles or many C_5s. If it has many triangles, we will use type-2 Sub(H)-absorbers, and if it has many C_5s, then we will use type-3 Sub(H)-absorbers.

Below is the proof of Lemma 5.2, which yields Theorem 1.9.

Proof of Lemma 5.2. Let H be a graph on h vertices with $\mathrm{hcf}_\xi(H) = 2$. Let $\hat{H}$ be the bipartite graph as in Definition 5.5 with a bipartition $(\hat{A}, \hat{B})$ with sizes h' and $h' + 2$ for a positive integer h'.

We note that h' is bounded above by a constant that only depends on h. We fix positive constants as follows:

$$0 < \frac{1}{n} \ll \beta \ll \alpha \ll d_1 \ll \rho \ll \frac{1}{T} \ll \frac{1}{t_0} \ll \varepsilon \ll d \ll \frac{1}{h'}, \frac{1}{C} \ll \gamma, \frac{1}{h} \le 1.$$

Let n be an even number and let G be an n-vertex graph with a minimum degree of at least $\left(1 - \frac{1}{\xi^*(H)} + \gamma\right) n$. We apply the regularity lemma on G with parameters ε, d, and t_0. Let $V_0, V_1, \dots, V_t$ be an ε-regular partition of G and let G_0 be a spanning subgraph of G which satisfies the following:

(R1) $t_0 \le t \le T_0$,
(R2) $|V_0| \le \varepsilon v(G)$,
(R3) $|V_1| = \dots = |V_t|$,
(R4) $G_0[V_i]$ has no edges for each $i \in [t]$,
(R5) $d_{G_0}(v) \ge d_G(v) - (d + \varepsilon)v(G)$ for all $v \in V(G)$,
(R6) for all $1 \le i < j \le t$, either the pair (V_i, V_j) is an $(\varepsilon, d+)$-regular pair or $G_0[A, B]$ has no edges.

By (R5), the minimum degree of G_0 is at least $\left(1 - \frac{1}{\xi^*(H)} + \frac{\gamma}{2}\right) n$. Let $G_1 = G_0 - V_0$, then the size $v(G_1) \ge (1 - \varepsilon)n$ and $\delta(G_1) \ge \delta(G_0) - \varepsilon n \ge \left(\frac{1}{3} + \frac{\gamma}{4}\right) n$. Let R be the (ε, d)-reduced graph on $\{V_1, \dots, V_t\}$. Then by Lemma 2.7, the minimum degree of R is at least $\left(1 - \frac{1}{\xi^*(H)} + \frac{\gamma}{2}\right) t$. We now divide into two cases depending on the structure of R.

Case 1: R contains C_3 or C_5.

Let $j \in \{2, 3\}$. We choose j as 2 if R contains C_3. Otherwise, we choose j as 3. We may assume $V_1, \dots, V_{2j-1}$ forms a C_{2j-1} in R, where $V_iV_{i+1} \in E(R)$ for each $i \in [2j - 1]$. Below, we write V_{2j} to denote V_1. By Lemma 2.8, there exist sets $V'_1, \dots, V'_{2j-1}$ satisfying the following for each $i \in [2j - 1]$:

- $V'_i \subseteq V_i$,
- $|V_i \setminus V'_i| \le 2\varepsilon |V_i|$,
- for every $v \in V'_i$, the degrees $d_{G_0}(v; V'_{i-1}) \ge (d - 3\varepsilon)|V_{i-1}|$ and $d_{G_0}(v; V'_{i+1}) \ge (d - 3\varepsilon)|V_{i+1}|$.

Let

$$B = \{v \in V(G_1) : \text{there are at least } d_1 n^{v(T_H^j)} \text{ distinct type-}j \text{ Sub}(H)\text{-absorbers for } v\}$$

We now claim that $|B| \ge d_1 n$.

Claim 2. $|B| \ge d_1 n$.

Proof of Claim 1. If $j = 2$, by the definition of an ε-regular pair, the following holds. For each $i \in [3]$, for every $v \in V'_i$, the number of edges between the sets $N_{G_1}(v) \cap V'_{i-1}$ and $N_{G_1}(v) \cap V'_{i+1}$ is at least $(d - \varepsilon)(d - 3\varepsilon)^2 |V_{i-1}||V_{i+1}| \ge \frac{(d-\varepsilon)(d-3\varepsilon)^2(1-\varepsilon)^2}{T^2} n^2 \ge \rho n^2$. Then by Theorem 2.2, there are at least $d_1(v(T_H^2)!)n^{v(T_H^2)}$ copies of T_H^2 in $G_1[N_{G_1}(v)]$. This means there are at least $d_1 n^{v(T_H^2)}$ distinct type-2 Sub(H)-absorbers for v. This implies $\bigcup_{i \in [3]} V'_i \subseteq B$. Since $|V'_1| + |V'_2| + |V'_3| \ge \frac{3(1-\varepsilon)(1-2\varepsilon)}{T} n \ge d_1 n$, in this case, the inequality $|B| \ge d_1 n$ holds.

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Now we consider the case $j=3$. We now show that for every $u\in V_3'$, there are many copies of T_H^3 such that each copy forms a $\hat{T}_H^3$ together with u. For each vertex $x\in V_2'$ and $y\in V_4'$, let $N_x=N_{G_1}(x)\cap V_1'$ and $N_y=N_{G_1}(y)\cap V_5'$. By the definition of an ε-regular pair, there are at least $(d-\varepsilon)|N_x||N_y|\geq(d-\varepsilon)(d-3\varepsilon)^2|V_1||V_5|\geq\frac{(d-\varepsilon)(d-3\varepsilon)^2(1-\varepsilon)^2}{T^2}n^2\geq\rho n^2$ edges in $G_1[N_x,N_y]$. By Lemma 2.3, there are at least $\frac{d_1T^2}{(d-3\varepsilon)^2(1-\varepsilon)^2}(v(T_H^3)!)n^{v(\tilde{T}_H^3)}$ distinct copies of $\tilde{T}_H^3$ such that together with x and y, each copy forms a T_H^3. For each $u\in V_3'$, there are at least $(d-3\varepsilon)^2|V_2||V_4|\geq\frac{(d-3\varepsilon)^2(1-\varepsilon)^2}{T^2}n^2$ distinct pairs $(x,y)\in V_2'\times V_4'$ such that ux, $uy\in E(G_1)$. Thus, there are at least $d_1(v(T_H^3)!)n^{v(T_H^3)}$ distinct copies of T_H^3 such that together with u, each copy forms a $\hat{T}_H^3$. This means, for every $u\in V_3'$, there are at least $d_1n^{v(T_H^3)}$ distinct type-3 Sub(H)-absorbers for u. This means $V_3'\subseteq B$. Since $|V_3'|\geq\frac{(1-\varepsilon)(1-2\varepsilon)}{T}n\geq d_1n$, we have $|B|\geq d_1n$. □

Let the two graphs F_1 and F_2 be the d_1-absorbing graph and the (d_1,B)-exchanging graph of G_1, respectively. We denote by X the set of vertices of G_1 that are not elements of B. Assume for a vertex $u\in X$, the inequality $e(G_1[N_{G_1}(u)])>\rho n^2$ holds. If $j=2$, then by Theorem 2.2, u should be contained in B, a contradiction. If $j=3$, then G_1 does not have a triangle, so a contradiction. Thus, for every vertex $u\in X$, the inequality $e(G_1[N_{G_1}(u)])\leq\rho n^2$ holds. We now claim the following.

Claim 2. *For every $u\in X$ and $v\in B$, the edge uv is contained in F_1 or F_2.*

Proof of Claim 2. Fix a vertex $v\in B$ and let $U=N_{G_1}(v)$. We partition the set X into X_1 and X_2 as follows. Let $X_1=\{u\in X:|N_{G_1}(u)\cap U|\leq\frac{\gamma}{10}n\}$ and $X_2=X\setminus X_1$.

Let $u\in X$. Assume $u\in X_1$. Let $Z_1=N_{G_1}(u)$ and let $Y=V(G_1)\setminus(Z_1\cup U)$. We denote by $U_1=U\setminus Z_1$. Since $\delta(G_1)\geq\left(\frac{1}{3}+\frac{\gamma}{4}\right)n$ and $|Z_1\cap U|\leq\frac{\gamma}{10}n$, the inequality $|Y|\leq\left(\frac{1}{3}-\frac{2\gamma}{5}\right)n$ holds. Thus, for every $u'\in Z_1$, the degree $G_1[u';Z_1\cup U]$ is at least $\frac{\gamma}{2}n$. Since $u\in X$, we know that $e(G_1[Z_1])\leq\rho n^2$. These imply that $e(G_1[Z_1,U_1])\geq\frac{\gamma}{2}n\left(\frac{1}{3}+\frac{\gamma}{4}\right)n-2\rho n^2\geq\frac{\gamma}{10}n^2$ holds. Then by Lemma 2.3, there are at least $d_1(v(T_H^1)!)n^{v(T_H^1)}$ copies of T_H^1 in $G_1[Z_1,U_1]$ such that together with u and v, each copy of T_H^1 forms a $\hat{T}_H^1$. Thus, there are at least $d_1n^{v(T_H^1)}$ distinct type-1 Sub(H)-absorbers for $\{u,v\}$. This means $uv\in E(F_1)$.

We now assume $u\in X_2$. Let $Z_2=N_{G_1}(u)$. The size $|Z_2\cap U|\geq\frac{\gamma}{10}n$ causes $u\in X_2$. Since $\delta(G_1)\geq\left(\frac{1}{3}+\frac{\gamma}{4}\right)n$, by Lemma 2.4, there are at least $d_1(v(S_H)!)n^{v(S_H)}$ copies of S_H in $G_1-\{u,v\}$ such that together with u and v, each copy of S_H forms a $\hat{S}_H$. This means there are at least $d_1n^{v(S_H)}$ distinct Sub(H)-exchangers for $\{u,v\}$, so $uv\in E(F_2)$.

Thus, we conclude that for every $u\in X$ and $v\in B$, the edge uv is contained in F_1 or F_2. □

Claim 2 implies for all $u\in X$, we have $d_{F_1}(u;B)\geq\frac{|B|}{2}\geq\frac{d_1}{2}n$ or $d_{F_2}(u;B)\geq\frac{|B|}{2}\geq\frac{d_1}{2}n$. We are now ready to apply Lemma 4.13. By Lemma 4.13, there is a set $A\subseteq V(G_1)$ with size at most αn such that for any set $U\subseteq V(G_1)\setminus A$ with size at most βn, the set A is a Sub(H)-absorber for U. Let us fix such a Sub(H)-absorber A.

Let $G_2=G_0-A$. Then $v(G_2)\geq(1-\alpha)n$ and $\delta(G_2)\geq\delta(G_0)-\alpha n\geq\left(1-\frac{1}{\xi(H)}+\frac{\gamma}{4}\right)n$. Then by Lemma 5.3, there is a set $W\subseteq V(G_2)$ such that $V_0\subseteq W$, the induced graph $G_2[W]$ has a perfect H-subdivision tiling, and $|V(G_2)\setminus W|\leq C$. Let $J=V(G_2)\setminus W$. We observe that $V(G)=A\cup W\cup J$ and $J\subseteq V(G_1)\setminus A$. Since $|J|\leq C\leq\beta n$, by our choice of A, the induced graph $G_1[A\cup J]$ has a perfect H-subdivision tiling. Together with $G[W]$, the graph G has a perfect H-subdivision tiling. This completes the proof of Case 1.

Case 2: R is a $\{C_3,C_5\}$-free graph.

In this case, we only use the type-1 Sub(H)-absorbers. The proof is similar to the proof of Theorem 1.8, but fortunately, it will be simpler.

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By (R4) and (R6), the graph G_1 is also a $\{C_3, C_5\}$-free graph. A classical result of Andrásfai [3] states that every graph on n-vertices with a minimum degree of at least $\frac{2}{2k+1}n+1$ either contains an odd cycle with a length of at most $2k-1$ or is bipartite. Since $\delta(G_1) \geq \left(\frac{1}{3}+\frac{\gamma}{4}\right)n > \frac{2}{7}n+1$ and G_1 is a $\{C_3, C_5\}$-free graph, we can deduce that G_1 is a bipartite graph.

Let (X, Y) be the bipartition of G_1. Then we have $\delta(G_1) \leq |X| \leq n-\delta(G_1)$, implying $2\delta(G_1) - |X| \geq \frac{3\gamma}{4}n$. Let $x \in X$ and $y \in Y$. We write $N_x = N_{G_1}(x)$ and $N_y = N_{G_1}(y)$. Since G_1 is bipartite, we have $N_x \subseteq Y$ and $N_y \subseteq X$. Then

$$e(G_1[N_x, N_y]) \geq (\delta(G_1) - |X \setminus N_y|)|N_x| \geq (2\delta(G_1) - |X|)\delta(G_1) \geq \frac{\gamma}{4}n^2.$$

By Lemma 2.3, there are at least $d_1(v(T_H^1)!)n^{v(T_H^1)}$ copies of T_H^1 in $G_1[N_x, N_y]$ such that together with $\{x, y\}$, each copy of T_H^1 forms a $\hat{T}_H^1$. This means for every $x \in X$ and $y \in Y$, there are at least $d_1 n^{v(T_H^1)}$ distinct type-1 Sub(H)-absorbers for $\{x, y\}$.

Let F be the d_1-absorbing graph for G_1. Then by the previous argument, all pairs xy with $x \in X$ and $y \in Y$ are edges of F. By Lemma 4.12, there is a set $A \subseteq V(G_1)$ such that $|A|$ is even, $|A|$ is at most αn, and it satisfies the following:

> The set A is a Sub(H)-absorber for any set $U \subseteq V(G_1) \setminus A$ satisfying $|U| \leq \beta n$ and $F[U]$ has a perfect matching.

We now consider a graph $G_2 = G_1 - A$. Then $V(G_2) \geq (1-\varepsilon-\alpha)n$ and $\delta(G_2) \geq \delta(G_1) - \alpha n \geq \left(\frac{1}{3}+\frac{\gamma}{10}\right)n$. Let $X' = X \setminus A$ and $Y' = Y \setminus A$. Since $e(G_2[X', Y']) \geq \frac{1}{6}n^2$, by the Erdős–Stone–Simonovits theorem, there is a subgraph Q in G_1 which is isomorphic to $K_{(2h'+2)C,(2h'+2)C}$. We note that by Observation 5.6, the graph Q has a perfect $\hat{H}$-tiling, so it has a perfect H-subdivision tiling.

Let $G_3 = G_0 - (A \cup V(Q))$. Then $v(G_3) \geq (1-\alpha)n - (2h'+2)C$ and $\delta(G_3) \geq \Big(1 - \frac{1}{\xi(H)} + \frac{\gamma}{10}\Big)n$. Then by Lemma 5.3, there is a set $W \subseteq V(G_3)$ such that $V_0 \subseteq W$ and $G[W]$ has a perfect H-subdivision tiling, $|V(G_3) \setminus W|$ is at most C, and the size of W is even. Let $J = V(G_3) \setminus W$. We observe that $V(G) = A \cup V(Q) \cup W \cup J$. Note that all n, $|A|$, and $|W|$ are even, so is $|J|$. Moreover, $J \subseteq V(G_1) \setminus (A \cup V(Q))$ and $|J| \leq C$.

Let $J_X = J \cap X$ and $J_Y = J \cap Y$. We may assume $|J_X| - |J_Y| = 2c$ for some non-negative integer $c \leq \frac{C}{2}$. Let us pick two sets $Q_1 \subseteq V(Q) \cap X$ and $Q_2 \subseteq V(Q) \cap Y$ such that $|Q_1| = ch'$ and $|Q_2| = c(h'+2)$. Then $(J_X \cup Q_1) \subseteq X$ and $(J_Y \cup Q_2) \subseteq Y$. Since $|Q_1| - |Q_2| = -2c$, the equality $|J_X \cup Q_1| = |J_Y \cup Q_2|$ holds. We showed that F contains a complete bipartite graph on the bipartition (X, Y) as a subgraph, and the set $J \cup Q_1 \cup Q_2$ has a perfect matching in F. Moreover, we have $|J \cup Q_1 \cup Q_2| \leq (2h'+2)C \leq \beta n$. Thus, by our choice of A, the induced graph $G[A \cup J \cup Q_1 \cup Q_2]$ has a perfect H-subdivision tiling. Let $Q' = Q - (Q_1 \cup Q_2)$. By Observation 5.6, the graph Q' has a perfect $\hat{H}$-tiling, so it has a perfect H-subdivision tiling. Together with $G[W]$, $G[A \cup J \cup Q_1 \cup Q_2]$, and $G[V(Q')]$, the graph G has a perfect H-subdivision tiling. This completes the proof of Lemma 5.2. □

6. Concluding remarks and open problems

In this article, we determined an asymptotically tight minimum degree threshold that ensures the existence of a perfect H-subdivision tiling for every graph H. In many cases, a minimum degree threshold for perfect H-subdivision tilings is much smaller than for perfect H-tilings since perfect H-subdivision tilings are allowed to use not only H but also subdivisions of H. To prove that the weaker minimum degree suffices, we developed new approaches using the absorption method combined with the regularity lemma and domination numbers.

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Both Theorems 1.8 and 1.9 are tight up to $o(n)$ terms. We conjecture that our results hold with the minimum degree condition sharp up to additive constants depending only on H.

Conjecture 6.1. *Let H be a graph that is not a disjoint union of isolated vertices. Then there exists a constant C_H depending only on H such that the following holds for all $n > 0$.*

If $\mathrm{hcf}_\xi(H) \neq 2$,

$$\delta_{\mathrm{sub}}(n, H) \leq \left(1 - \frac{1}{\xi^*(H)}\right) n + C_H.$$

Otherwise,

$$\delta_{\mathrm{sub}}(n, H) \leq \frac{1}{2} n + C_H \qquad \text{if } n \text{ is odd},$$
$$\delta_{\mathrm{sub}}(n, H) \leq \left(1 - \frac{1}{\xi^*(H)}\right) n + C_H \qquad \text{if } n \text{ is even}.$$

Our main results, Theorems 1.8 and 1.9, imply that perfect H-subdivision tilings and perfect H-tilings behave differently in many cases. In particular, the numbers $\delta_{\mathrm{sub}}(n, K_r)$ behave irregularly for small values of r. In contrast to perfect K_r-tilings, the number $\lim_{n\to\infty} \frac{\delta_{\mathrm{sub}}(n,K_r)}{n}$ does not strictly increase as r increases. For another example, consider a tree T. According to Theorem 1.2, we have $\delta(n, T) = \frac{1}{2}n + O(1)$. However, for $\delta_{\mathrm{sub}}(n, T)$, Theorem 1.8 implies that there is a positive constant c_T such that $\delta_{\mathrm{sub}}(n, T) \leq \left(\frac{1}{2} - c_T + o(1)\right) n$ since $\mathrm{hcf}_\xi(T) = 1$.

Moreover, as we saw before, monotonicity does not hold for subdivision tilings. For $H_2 \subseteq H_1$ with H_2 being a spanning subgraph, $\delta(n, H_2) \leq \delta(n, H_1)$ is trivial, but $\delta_{\mathrm{sub}}(n, H_2) \leq \delta_{\mathrm{sub}}(n, H_1)$ is not true as $\delta_{\mathrm{sub}}(n, K_4) = \frac{2}{5}n + o(n) < \delta_{\mathrm{sub}}(n, C_4) = \frac{1}{2}n + o(n)$.

A perfect H-subdivision tiling is a special case of tiling a host graph by several kinds of graphs. Let $\mathcal{F}$ be a collection of graphs and let $\mathrm{hcf}(\mathcal{F})$ be the highest common factor of the sizes of graphs in $\mathcal{F}$. For each positive integer n divisible by $\mathrm{hcf}(\mathcal{F})$, we define $\delta(n, \mathcal{F})$ to be the smallest integer k such that any n-vertex graph G with a minimum degree of at least k has a perfect tiling that uses only graphs in $\mathcal{F}$. In this notation, we estimated an asymptotically tight value for $\delta(n, \mathrm{Sub}(H))$ for every graph H. Thus, we suggest the following problem.

Problem 6.2. *Let $\mathcal{F}$ be a collection of graphs and n be a positive integer that is divisible by* $\mathrm{hcf}(\mathcal{F})$*. Determine the number $\delta(n, \mathcal{F})$.*

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