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ON THE LOCAL SOLVABILITY OF VECTOR FIELDS WITH CRITICAL POINTS

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To Louis Boutet, with friendship and admiration

Abstract The article discusses the local solvability (or lack thereof) of various classes of smooth, complex vector fields that vanish on some non-empty subset of the base manifold.

Keywords: vector fields; local solvability; hypoellipticity

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Contents

1. Introduction

This article consists of a number of observations about smooth, *complex* vector fields on a smooth manifold $\mathcal{M}$, with special emphasis on their local solvability. The case of a vector field without critical points (i.e. nowhere zero) is well understood. We focus on a vector field L that vanishes on a subset of $\mathcal{M}$, to which we refer as the *critical set* of L (and denote by $\operatorname{Crit} L$). Little is known on this topic. Some results about *linear vector fields* in $\mathbb{R}^n$, i.e. vector fields whose coefficients are *linear* functions, can be found in [**7**, **8**], and in the more recent articles [**15**, **16**]. It is remarkable that the solvability (or lack thereof) of such a restricted class of vector fields is unknown (at least, to the author) in dimension $n \geqslant 3$, beyond the real coefficients case.

Since the general case is a vast *terra incognita* the purpose of this article is to isolate a class of complex vector fields on $\mathcal{M}$ that hold some promise of being manageable. Our choice of such a class is based on a *microlocal* concept: that of *microlocal principal type*, defined by the property that *the differential in phase-space of the symbol of the vector field* (denoted by $\sigma(L)$) *is linearly independent from the symplectic one-form* $\xi \cdot \mathrm{d}x$. A prototype of this class is the rotation vector field $x_1\partial/\partial x_2 - x_2\partial/\partial x_1$ in $\mathbb{R}^2$ (which, by the way, *is not* locally solvable at the origin); a prototype of the complementary class, the vector fields not of principal type, is the radial vector field $\sum_{j=1}^{n} x_j\partial/\partial x_j$ (which, by the way, *is* locally solvable at the origin). In the two-dimensional case, when $\mathcal{M}$ is a surface, the principal type property ensures that the critical points of L are isolated (Corollary 3.20). The same is true when the coefficients are real; moreover, in the latter case, the principal type property requires that the dimension n of the base manifold $\mathcal{M}$ be even (Proposition 3.4).

Off its critical set every vector field is of principal type. At a critical point $\wp$ the differential of the symbol $\sigma(L)$ is equal to the differential of its *linear part* $L_\wp$ (in which every coefficient of L is reduced to the homogeneous part of degree 1 of its Taylor expansion at $\wp$). This linear part (mainly viewed as a vector field in the tangent space $T_\wp\mathcal{M} \cong \mathbb{R}^n$ and obviously an invariant of the vector field L at $\wp$) plays a central role in the present work; and justifies devoting some effort (in the future) to the study of complex linear vector fields in dimensions higher than 2. So far the case of planar complex linear vector fields is completely settled (Theorem 3.21). It reveals the role of the *Meziani invariant* (see (3.25)). Extending the result to higher dimension may require the identification of invariant(s) of the same type.

The last subsection of the paper focuses on the case where $L_\wp$ is *elliptic* in the complement of the origin $T_\wp\mathcal{M}\backslash\{0\}$: in this case (which requires $\dim\mathcal{M} = 2$ and is automatically of principal type) L is locally solvable at $\wp$, in the L^2-sense, actually (Theorem 3.27). This fact points to a class of complex vector fields on surfaces that is very stable and ready for some global analysis: the vector fields L that are elliptic in $\mathcal{M}\backslash \operatorname{Crit} L$ and whose linear part $L_\wp$ is elliptic in the complement of the origin $T_\wp M\backslash\{0\}$, whatever $\wp \in \operatorname{Crit} L$. They all are locally solvable everywhere in $\mathcal{M}$; we tentatively give them the name of *quasi-elliptic*.

If $\wp$ is a critical point of a vector field L of principal type the bicharacteristics that intersect the cotangent space $T^*_\wp\mathcal{M}$ are completely contained in $T^*_\wp\mathcal{M}$; their base projection is the single point $\{\wp\}$. This reveals a noteworthy feature of vector fields with critical points and the contrast they present with the established theory of differential operators $P(x, D)$ of principal type. We refer particularly to Theorem 26.11.3 in [**3**], stating, as a sufficient condition for semiglobal solvability, that $P(x, D)$ satisfy the local solvability condition (P) and that the base projections of *every* one of its 'bicharacteristics' (per force of dimension 1 or 2) escape from every compact subset of the base, i.e. not be 'trapped'. What transpires, so far, in the cases of vector fields with critical points where necessary and sufficient conditions for local solvability have been found (see Theorems 3.8, 3.21 and, as just mentioned, 3.27), is that non-solvability can be equated with the property that *all* bicharacteristics are trapped. It is tempting to conjecture, at least in the case of a real vector field L, that local solvability holds at a critical point $\wp$ if and only if, given an arbitrary neighbourhood $\mathcal{N}(\wp)$, *at least one* integral curve of L escapes from $\mathcal{N}(\wp)$.

2. Basic concepts and known results

2.1. Vector fields with critical points: principal type

Throughout this article $\mathcal{M}$ denotes a smooth (i.e. $\mathcal{C}^\infty$), connected manifold; $\dim \mathcal{M} = n \geqslant 2$. We use standard notation: $\mathcal{C}^\infty(\mathcal{A})$ stands for the space of smooth functions in the open set $\mathcal{A} \subset \mathcal{M}$; $\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{A})$ for the space of test functions (i.e. smooth and compactly supported) in $\mathcal{A}$; $\mathcal{D}'(\mathcal{A})$ for its dual, the space of distributions in $\mathcal{A}$. All functions and distributions are assumed to be complex-valued, unless specified otherwise.

We shall be concerned with a complex vector field L of class $\mathcal{C}^\infty$ in the manifold $\mathcal{M}$, mainly with its local solvability. We shall look at a special class of vector fields with critical points. We denote by $\operatorname{Crit} L$ the set of critical points of L. Most of the time we reason in a local coordinate chart $(\mathcal{U}, x_1, \dots, x_n)$, in which

$$L = \sum_{j=1}^{n} a_j(x)\partial_{x_j}, \tag{2.1}$$

where $\partial_{x_j} = \partial/\partial x_j$ and $a_j \in \mathcal{C}^\infty(\mathcal{U})$. We denote by $\xi_1, \dots, \xi_n$ the coordinates in the cotangent spaces at points of $\mathcal{U}$ with respect to the basis $\mathrm{d}x_1, \dots, \mathrm{d}x_n$. In these local coordinates the *symbol* of L will be the linear functional with respect to ξ,

$$\sigma(L)(x, \xi) = \sum_{j=1}^{n} a_j(x)\xi_j. \tag{2.2}$$

(We omit the customary factors $\sqrt{-1}$ as we shall not make use of the Fourier transform.) The symbol of L is equivariant under changes of local coordinates and thus defines a $\mathcal{C}^\infty$ function in the cotangent bundle $T^*\mathcal{M}$. By the *characteristic set* of L we shall mean the null set of $\sigma(L)$ in the complement of the zero section, $T^*\mathcal{M}\backslash 0$; it will be denoted by $\operatorname{Char} L$.

We shall denote by π the base projection $T^*\mathcal{M}\backslash 0 \to \mathcal{M}$. Obviously, $\overset{-1}{\pi}(\operatorname{Crit} L) \subset \operatorname{Char} L$.

We will now make use of the *symplectic one-form* $\tau = \xi \cdot \mathrm{d}x$. In local coordinates $x_1, \dots, x_n$ we have $\tau = \xi_1\,\mathrm{d}x_1 + \cdots + \xi_n\,\mathrm{d}x_n$. The chain rule shows that this definition is invariant under coordinate changes. The differential $\mathrm{d}\tau$ is the *fundamental symplectic two-form* on $T^*\mathcal{M}$.

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Definition 2.1. The vector field L is said to be of *principal type* at a point $\wp \in \mathcal{M}$ if the differential $\mathrm{d}\sigma(L)$ and the symplectic one-form τ are linearly independent at every point of $\operatorname{Char} L \cap T^*_{\wp}\mathcal{M}$.

Definition 2.1 is relevant when $\wp \in \operatorname{Crit} L$. Indeed, L is of principal type at every point $\wp \notin \operatorname{Crit} L$, actually in a stronger sense than that in Definition 2.1: the differential of $\sigma(L)$ with respect to the 'fibre' variables, $\mathrm{d}_\xi \sigma(L) = \sum_{j=1}^n a_j(x)\,\mathrm{d}\xi_j$ in the local chart $(\mathcal{U}, x_1, \dots, x_n)$, is nowhere zero in $\overset{-1}{\pi}(\wp)$.

At a critical point $\wp$ of L in $\mathcal{U}$ we have $\mathrm{d}\sigma(L) = \mathrm{d}_x\sigma(L)$ and, according to (2.2),

$$\mathrm{d}\sigma(L) \wedge \tau = \sum_{1 \leqslant j < k \leqslant n} \sigma_{j,k}(x,\xi)\,\mathrm{d}x_j \wedge \mathrm{d}x_k, \tag{2.3}$$

where we have used the notation

$$\sigma_{j,k}(x,\xi) = \sum_{i=1}^n \left(\frac{\partial a_i}{\partial x_j}(x)\xi_k - \frac{\partial a_i}{\partial x_k}(x)\xi_j \right)\xi_i. \tag{2.4}$$

For L to be of principal type at $x \in \mathcal{U} \cap \operatorname{Crit} L$ means that, to each $\xi \in \mathbb{R}^n \backslash \{0\}$, there is a pair of indices (j,k), $1 \leqslant j < k \leqslant n$, such that $\sigma_{j,k}(x,\xi) \neq 0$.

Remark 2.2. Obviously, if L is of principal type at $x \in \mathcal{U} \cap \operatorname{Crit} L$ necessarily $\mathrm{d}_x\sigma(L) = \sum_{j=1}^n \xi_j\,\mathrm{d}a_j \neq 0$ whatever $\xi \in \mathbb{R}^n \backslash \{0\}$. In other words, the differentials $\mathrm{d}a_j$ at x must be linearly independent over the field $\mathbb{R}$. In particular, if the vector field L is real this means that x is an *isolated* critical point. If the vector field L is real and of principal type in $\mathcal{M}$ then $\operatorname{Crit} L$ is a discrete set.

Proposition 2.3. *If L is of principal type at $\wp \in \operatorname{Crit} L$, then there is an open neighbourhood of $\wp$ in $\mathcal{M}$, $\mathcal{N}(\wp)$, such that L is of principal type at every point of $\mathcal{N}(\wp)$.*

Proof. Let the local chart $(\mathcal{U}, x_1, \dots, x_n)$ be centred at $\wp \in \operatorname{Crit} L$ (meaning that $x_1 = \dots = x_n = 0$ at $\wp$). The hypothesis is that $\min_{|\xi|=1} |\sigma_{j,k}(0,\xi)| > 0$ for some pair of indices j, k. We will then have $\min_{|\xi|=1} |\sigma_{j,k}(x,\xi)| > 0$ for every x sufficiently close to 0. □

There is an equivalent definition of principal type in terms of the *Hamiltonian vector field* of $\sigma(L)$. In an open subset $\Omega \subset \overset{-1}{\pi}(\mathcal{U})$ and a function $g \in \mathcal{C}^\infty(\Omega)$ the Hamiltonian vector field of g is given by

$$H_g = \sum_{j=1}^n \frac{\partial g}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial g}{\partial x_j}\frac{\partial}{\partial \xi_j}. \tag{2.5}$$

Assume that g is real-valued; a *null bicharacteristic* of g is an integral curve of H_g in Ω on which g vanishes (g is constant along any integral curve of H_g). We have

$$H_{\sigma(L)} = L - \sum_{j,k=1}^n \frac{\partial a_j}{\partial x_k}\xi_j\frac{\partial}{\partial \xi_k}. \tag{2.6}$$

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It is readily checked that, at critical points of L in $\mathcal{U}$,

$$H_{\sigma(L)} \wedge \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i} = \sum_{1 \leqslant j < k \leqslant n} \sigma_{j,k}(x,\xi) \frac{\partial}{\partial \xi_j} \wedge \frac{\partial}{\partial \xi_k}. \tag{2.7}$$

Comparing with (2.3) shows that, *for L to be of principal type at $\wp \in \operatorname{Crit} L$ it is necessary and sufficient that the Hamiltonian field $H_{\sigma(L)}$ and the radial vector $\sum_{i=1}^{n} \xi_i(\partial/\partial\xi_i)$ be linearly independent at every point of $T^*_\wp \mathcal{M}$.*

Remark 2.4. If all the local coordinates $x_1, \dots, x_n$ vanish at $\wp$ the radial vector field in ξ-space $T^*_\wp\mathcal{M}$, $\sum_{i=1}^{n} \xi_i \partial_{\xi_i}$, is the Hamiltonian field of $-r\partial_r = -\sum_{i=1}^{n} x_i \partial_{x_i}$.

Let the local chart $(\mathcal{U}, x_1, \dots, x_n)$ be centred at $\wp \in \operatorname{Crit} L$. We introduce the vector field

$$L_\wp = \sum_{j,k=1}^{n} a_{j,k} x_k \partial_{x_j}, \tag{2.8}$$

where

$$a_{j,k} = \frac{\partial a_j}{\partial x_k}(0).$$

We might want to view (2.8) as a vector field in $\mathcal{U}$, in which case $L - L_\wp$ vanishes to second order at $\wp$. From this standpoint changing the coordinates x_j might result in an expression of $L_\wp$ whose coefficients are not any more linear. Not so if we view $L_\wp$ as a vector field in $T_\wp\mathcal{M}$ and if we associate to a change of coordinates in $\mathcal{U}$ the tangent linear transformation in $T_\wp\mathcal{M}$. In this sense the following definition is coordinate free.

Definition 2.5. We shall refer to the vector field $L_\wp$ in $T_\wp\mathcal{M}$ as the *linear part* of L at the critical point $\wp$.

We deduce the following proposition immediately from (2.3) and (2.4).

Proposition 2.6. *For L to be of principal type at a point $\wp \in \operatorname{Crit} L$ it is necessary and sufficient that $L_\wp$ be of principal type at every point of $T_\wp\mathcal{M}$.*

Corollary 2.7. *For L to be of principal type in $\mathcal{M}$ it is necessary and sufficient that $L_\wp$ be of principal type at every point of $T_\wp\mathcal{M}$ whatever $\wp \in \operatorname{Crit} L$.*

Let us continue to reason within the local frame $(\mathcal{U}, x_1, \dots, x_n)$ and introduce the Jacobian matrix $A(x)$ with entries

$$a_{i,j}(x) = \frac{\partial a_i}{\partial x_j}(x);$$

$A^{\mathrm{T}}(x)$ will stand for the *transpose* of $A(x)$. With this notation we can rewrite (2.4) as follows:

$$\sigma_{j,k}(x,\xi) = (A^{\mathrm{T}}(x)\xi)_j \xi_k - (A^{\mathrm{T}}(x)\xi)_k \xi_j. \tag{2.9}$$

We see that the quantities $\sigma_{j,k}$ are the components of the 2-vector $\xi \wedge A^{\mathrm{T}}(x)\xi$. We can state the following.

Proposition 2.8. *For L to be of principal type at a critical point $x \in \mathcal{U}$ it is necessary and sufficient that the complex matrix $A^{\mathrm{T}}(x)$ not have any real eigenvector.*

In particular, if L is of principal type at a critical point $x \in \mathcal{U}$ then $\mathbb{R}^n \cap \ker A^{\mathrm{T}}(x) = \{0\}$. This does not mean that $\det A(x) \neq 0$, as shown by the example $L = z\partial_{\bar{z}}$ in the plane.

Example 2.9. The rotation vector field $\partial_\theta = x_1\partial_{x_2} - x_2\partial_{x_1}$ in $\mathbb{R}^2$ is of principal type at the origin. Indeed,

$$\mathrm{d}\sigma(\partial_\theta)|_{x=0} = \xi_2\,\mathrm{d}x_1 - \xi_1\,\mathrm{d}x_2$$

and $\tau \wedge \mathrm{d}\sigma(\partial_\theta)|_{x=0} = (\xi_1^2 + \xi_2^2)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$ does not vanish off the zero section of $T^*\mathbb{R}^2$.

Example 2.10. The radial vector field $r\partial_r = \sum_{j=1}^n x_j\partial_{x_j}$ in $\mathbb{R}^n$ is not of principal type at the origin, since $\mathrm{d}\sigma(r\partial_r)|_{x=0} = \tau$.

A *complex* vector field L can be of principal type and still have a critical set that is a submanifold of positive dimension, as shown in the following.

Example 2.11. The vector field in $\mathbb{R}^3$,

$$L = x_1\partial_{x_2} - x_2\partial_{x_1} + \mathrm{i}x_1\partial_{x_3}, \tag{2.10}$$

is of principal type in $\mathbb{R}^3$. Indeed,

$$A^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & \mathrm{i} \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has the eigenvectors $(0, -\mathrm{i}, 1)$, $(\mathrm{i}, 1, 0)$, $(-\mathrm{i}, 1, 0)$.

A vector field L can be of principal type and still have a critical set that is highly singular, as shown in the following.

Example 2.12. Let the function $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ have only zeros of order greater than 1. The vector field in $\mathbb{R}^3$,

$$L = x_1\partial_{x_2} - x_2\partial_{x_1} + (\mathrm{i}x_1 + \varphi(x_3))\partial_{x_3},$$

is of principal type in $\mathbb{R}^3$. Indeed, the linear part of L at every point $(0, 0, x_3^\circ) \in \mathbb{R}^3$ such that $\varphi(x_3^\circ) = 0$ is given by (2.10). It suffices then to apply Proposition 2.8.

The set $\operatorname{Crit} L$ may have singularities even if L is of class $\mathcal{C}^\omega$ and of principal type.

Example 2.13. Consider the vector field

$$L = x_1\partial_{x_2} - x_2\partial_{x_1} + (x_3^2 - x_4^3 + \mathrm{i}x_1)\partial_{x_3} + (x_3^2 - x_4^3 - \mathrm{i}x_2)\partial_{x_4}$$

in $\mathbb{R}^4$. The transpose of the matrix associated to the linear part of L at the origin,

$$A_0^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & \mathrm{i} & 0 \\ -1 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

has the eigenvectors $(0,-\mathrm{i},1,0)$, $(-\mathrm{i},0,0,1)$, $(\mathrm{i},1,0,0)$, $(-\mathrm{i},1,0,0)$, implying that L is of principal type at 0 (again by Proposition 2.8). We have

$$\operatorname{Crit} L = \{x \in \mathbb{R}^4;\ x_1 = x_2 = x_3^2 - x_4^3 = 0\}.$$

2.2. Basics on local solvability

Below we will make use of the *Sussman foliation* of $\mathcal{M}$ defined by L. Let $\mathfrak{g}(L)$ denote the real Lie algebra generated by the vector fields $\operatorname{Re} L$ and $\operatorname{Im} L$ for the standard commutation bracket. Let us say that two points of $\mathcal{M}$ are L-*connectable* if they can be joined by a continuous path consisting of finitely many arcs of orbits of vector fields belonging to $\mathfrak{g}(L)$; to be L-*connectable* is an equivalence relation among points of $\mathcal{M}$. The main theorem of [**11**] states that every equivalence class for this relation is an immersed submanifold of class $\mathcal{C}^\infty$ without self-intersections (called an L-*leaf* below) having the property that the tangent space at every one of its points contains the 'freezing' of $\mathfrak{g}(L)$ at that point. (In the analytic category the tangent spaces are *equal* to the freezing of $\mathfrak{g}(L)$ at the points. The L-leaves form what is often called the *Nagano foliation* defined by L. See [**9**].) The one-dimensional L-leaves will also be referred to as the *orbits* of L; at every point of every orbit of L, $\operatorname{Re} L$ and $\operatorname{Im} L$ are collinear and at least one of these two vector fields does not vanish. The critical points of L are the zero-dimensional L-leaves.

Definition 2.14. The vector field L is said to be *locally solvable at a point* $\wp \in \mathcal{M}$ if there is an open neighbourhood $\mathcal{N}(\wp)$ of $\wp$ such that to each $f \in \mathcal{C}_c^\infty(\mathcal{N}(\wp))$ there is $u \in \mathcal{D}'(\mathcal{N}(\wp))$ verifying the equation $Lu = f$ in $\mathcal{N}(\wp)$; L is said to be locally solvable in an open subset $\mathcal{A}$ of $\mathcal{M}$ if L is locally solvable at every point of $\mathcal{A}$.

Local solvability theory (see [**10**], [**13**, Chapter VIII] and [**3**, Theorems 26.4.7 and 26.11.3]) tells us that the following holds.

Theorem 2.15. *Assume that the vector field L does not have any critical point in an open subset $\mathcal{A}$ of $\mathcal{M}$. The following properties are equivalent:*

(LocSolv) *L is locally solvable in $\mathcal{A}$;*

(P) $\forall \zeta \in \mathbb{C}$, *the function* $\operatorname{Im}(\zeta\sigma(L))$ *does not change sign along any null bicharacteristic of* $\operatorname{Re}(\zeta\sigma(L))$*;*

(P′) *the dimension of the L-leaves in $\mathcal{A}$ does not exceed 2 and $(1/2\mathrm{i})L \wedge \bar{L}$ does not change sign on any two-dimensional L-leaf in $\mathcal{A}$.*

Implicit in (P′) is the orientability the two-dimensional L-leaves, making sense of the property that $(1/2\mathrm{i})L \wedge \bar{L}$ not change sign on such an L-leaf.

Remark 2.16. In (LocSolv) we could have replaced L by $L - \chi$; the zero-order term $\chi \in \mathcal{C}^\infty(\mathcal{A})$ has no effect on local solvability (provided L has no critical points in $\mathcal{A}$). Indeed, if L is locally solvable in $\mathcal{A}$ we can find a $\mathcal{C}^\infty$ solution of the equation $L\varphi = \chi$ in every suitably small open subset $\mathcal{U}$ of $\mathcal{A}$ [**3**, Theorem 26.11.3]. Whatever $u \in \mathcal{D}'(\mathcal{U})$, $L(u\mathrm{e}^{-\varphi}) = \mathrm{e}^{-\varphi}(L - \chi)u$.

Remark 2.17. Given an open subset $\mathcal{A}$ of $\mathcal{M}$ the space $\boldsymbol{L}^2_{\mathrm{loc}}(\mathcal{A})$ of locally square-integrable functions in $\mathcal{A}$ can be defined by using local coordinates. Let us denote by $\boldsymbol{L}^2_{\mathrm{c}}(\mathcal{A})$ the subspace of $\boldsymbol{L}^2_{\mathrm{loc}}(\mathcal{A})$ consisting of the compactly supported functions. The vector field L is then said to be *locally* L^2*-solvable at a point* $\wp \in \mathcal{M}$ if there is an open set $\mathcal{A} \ni \wp$ such that to each $f \in \boldsymbol{L}^2_{\mathrm{c}}(\mathcal{A})$ there is $u \in \boldsymbol{L}^2_{\mathrm{loc}}(\mathcal{A})$ verifying (*in the distribution sense*) the equation $Lu = f$ in $\mathcal{A}$. It follows from the general result in [**1**] that (LocSolv) in Theorem 2.15 can be replaced by the property that *the vector field* L *is locally* L^2*-solvable in* $\mathcal{A}$ (for a direct proof see [**13**, § VIII.8]).

Remark 2.18. The vector field L can be locally solvable without being of principal type, as is the case for $r\partial_r = \sum_{j=1}^n x_j\partial_{x_j}$ in $\mathbb{R}^n$ (Example 2.10). A different example is $L = a(x)\partial_{x_1}$ when $a \in \mathcal{C}^\omega(\mathbb{R}^n)$ vanishes to high order on a proper analytic subvariety $\boldsymbol{V} \neq \varnothing$. Indeed, the equation $Lu = f \in \mathcal{D}'(\mathbb{R}^n)$ has a solution verifying $\partial_{x_1}u = g$ where $g \in \mathcal{D}'(\mathbb{R}^n)$ is such that $ag = f$. The existence of such a g is proved in [**4**].

Remark 2.19. The hypothesis, in Theorem 2.15, that L has no critical points cannot be replaced by the hypothesis that L is of principal type. The vector field ∂_θ in $\mathbb{R}^2$ is of principal type (Example 2.9) but the existence of a distribution solution u of the equation $\partial_\theta u = f \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^2)$ requires $\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\,\mathrm{d}\theta = 0$ for every $r > 0$.

As before $\operatorname{Crit} L$ stands for the set of critical points of L in the manifold $\mathcal{M}$.

Proposition 2.20. *Suppose* L *is of principal type in* $\mathcal{M}$*. If* L *satisfies condition (P) in* $\mathcal{M}\backslash \operatorname{Crit} L$*, then* L *satisfies condition (P) in* $\mathcal{M}$*.*

Proof. Let $\wp \in \operatorname{Crit} L$ be arbitrary; then $T^*_\wp\mathcal{M} \subset \operatorname{Char} L$ and the Hamiltonian field of $\sigma(L)$ is tangent to $T^*_\wp\mathcal{M}$. This implies that any bicharacteristic of $\operatorname{Re}(\zeta\sigma(L))$ $(0 \neq \zeta \in \mathbb{C})$ that intersects $T^*_\wp\mathcal{M}$ is entirely contained in $\overset{-1}{\pi}(\wp) \subset T^*_\wp\mathcal{M}$ and $\sigma(L) \equiv 0$ on it. The claim ensues. □

Proposition 2.21. *Let* $\wp \in \operatorname{Crit} L$ *be arbitrary and assume that the linear part* $L_\wp$ *is of principal type in the tangent space* $T_\wp\mathcal{M}$*. If* $L_\wp$ *does not satisfy condition (P) in* $T_\wp\mathcal{M}$ *then, given an arbitrary open neighbourhood* $\mathcal{U}$ *of* $\wp$*,* L *does not satisfy condition (P) in* $\mathcal{U}\backslash\{\wp\}$*.*

Proof. The hypothesis that $L_\wp$ is of principal type in the tangent space $T_\wp\mathcal{M}$ allows us to apply Proposition 2.20 to $L_\wp$ (in the place of L): if $L_\wp$ does not satisfy condition (P) in $T_\wp\mathcal{M}$ then it must not satisfy condition (P) in $T_\wp\mathcal{M}\backslash \operatorname{Crit} L_\wp$. Let $(\mathcal{U}, x_1, \dots, x_n)$ be a local chart centred at $\wp$ such that (2.12) is valid. Possibly after a linear change of the coordinates $x_1, \dots, x_n$ we may assume that

$$\operatorname{Crit} L_\wp = \{x \in T_\wp\mathcal{M};\ x_1 = \cdots = x_m = 0\}. \tag{2.11}$$

(When dealing with $L_\wp$ we let x vary in $T_\wp\mathcal{M} \cong \mathbb{R}^n$.) The symbol of $L_\wp$ as a function in $T^*(T_\wp\mathcal{M})$ has the expression (see (2.8))

$$\sigma(L_\wp) = \sum_{j=1}^m \sum_{k=1}^n a_{j,k}x_j\xi_k, \tag{2.12}$$

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where

$$a_{j,k} = \frac{\partial a_k}{\partial x_j}(0).$$

We assume that condition (P) is violated at a point $x^\circ \in T_\wp\mathcal{M}\setminus \operatorname{Crit} L_\wp$, i.e. such that

$$\sum_{k=1}^{n} \left| \sum_{j=1}^{m} a_{j,k} x_j^\circ \right| \neq 0.$$

We can then find $\zeta \in \mathbb{C}$ and an open set U such that $x^\circ \in U \subset T_\wp\mathcal{M}\setminus \operatorname{Crit} L_\wp$ and that

$$\forall x \in U, \quad \sum_{k=1}^{n} \left| \sum_{j=1}^{m} \operatorname{Re}(\zeta a_{j,k}) x_j \right| \neq 0. \tag{2.13}$$

The homogeneity (of degree 1) with respect to x in (2.13) allows us to assume that U is an open cone. (Cones in a vector space are subsets invariant under dilations $x \to \lambda x$, $\lambda > 0$.) Possibly after replacing x° by another point in U we may assume that $\operatorname{Im} \sigma(\zeta L_\wp)$ changes sign at a point $(x^\circ, \xi^\circ) \in \operatorname{Char} L_\wp$ along the null bicharacteristic γ of $\operatorname{Re}(\zeta L_\wp)$ that passes through that point. The bicharacteristic γ is defined by the Hamilton–Jacobi equations

$$\left.\begin{aligned} \frac{\mathrm{d}x_k}{\mathrm{d}t} &= \sum_{j=1}^{m} \operatorname{Re}(\zeta a_{j,k}) x_j, && k = 1, \dots, n, \\ \frac{\mathrm{d}\xi_j}{\mathrm{d}t} &= -\sum_{k=1}^{n} \operatorname{Re}(\zeta a_{j,k}) \xi_k, && j = 1, \dots, m, \\ \frac{\mathrm{d}\xi_k}{\mathrm{d}t} &= 0, && k = m+1, \dots, n, \end{aligned}\right\} \tag{2.14}$$

complemented by the initial value conditions

$$x(0) = x^\circ, \qquad \xi(0) = \xi^\circ. \tag{2.15}$$

Thanks to (2.13) we see that the base projection of γ is a curve that stays within $T_\wp\mathcal{M}\setminus \operatorname{Crit} L_\wp$. Below we denote by $(x(t), \xi(t))$ the solution of (2.14) that satisfies (2.15). It is convenient to introduce the $n \times n$ matrix A with entries $\operatorname{Re}(\zeta a_{j,k})$ if $1 \leqslant j \leqslant m$ and 0 if $m < j \leqslant n$, and $k = 1, \dots, n$. We have

$$x(t) = \mathrm{e}^{tA^{\mathrm{T}}} x^\circ, \qquad \xi(t) = \mathrm{e}^{-tA} \xi^\circ, \tag{2.16}$$

where A^{T} denotes the transpose of A.

The symbol $\sigma(\zeta L_\wp)$ and therefore also the equations (2.14) are invariant under the transformation $(x, \xi) \to (\lambda^{-1} x, \lambda \xi)$. The solution of (2.14) with the initial value $(\lambda^{-1} x^\circ, \lambda \xi^\circ)$ is $(\lambda^{-1} x(t), \lambda \xi(t))$. The bilinearity of $\sigma(L_\wp)$ entails

$$\operatorname{Im} \sigma(\zeta L_\wp)(\lambda^{-1} x(t), \lambda \xi(t)) = \operatorname{Im} \sigma(\zeta L_\wp)(x(t), \xi(t)). \tag{2.17}$$

Let us assume for simplicity that $\operatorname{Im} \sigma(\zeta L_\wp) > 0$ in the semicurve γ^+ described by $(x(t), \xi(t))$, $0 < t < T$, and $\operatorname{Im} \sigma(\zeta L_\wp) < 0$ in the semicurve γ^- described by $(x(t), \xi(t))$,

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$-T < t < 0$. (We are reasoning, here, in an analytic context, $L_\wp$ being a vector field with linear coefficients.) We conclude that, if the number $c > 0$ is suitably small, there are numbers t_1 and t_2, $-T < t_1 < 0 < t_2 < T$, such that

$$-\operatorname{Im}\sigma(\zeta L_\wp)(\lambda^{-1}x(t_1), \lambda\xi(t_1)) = \operatorname{Im}\sigma(\zeta L_\wp)(\lambda^{-1}x(t_2), \lambda\xi(t_2)) = c \tag{2.18}$$

whatever $\lambda > 0$.

Now consider

$$\operatorname{Re}\sigma(\zeta L)(x, \xi) = \sum_{j=1}^{m}\sum_{k=1}^{n} \operatorname{Re}(\zeta a_{j,k})x_j\xi_k + \sum_{k=1}^{n} \varphi_k(x)\xi_k, \tag{2.19}$$

where $\varphi_k \in \mathcal{C}^\infty(\mathcal{U})$ vanishes to second order at $\wp$, i.e.

$$\forall x \in U, \quad |\varphi_k(x)| \leqslant \text{const.}|x|^2. \tag{2.20}$$

From (2.13) we derive that

$$\lambda\,\mathrm{d}_\xi \operatorname{Re}\sigma(\zeta L)(\lambda^{-1}x^\circ, \lambda\xi) = \sum_{j=1}^{m}\sum_{k=1}^{n} \operatorname{Re}(\zeta a_{j,k})x_j^\circ + \lambda\sum_{k=1}^{n} \varphi_k(\lambda^{-1}x^\circ) \neq 0. \tag{2.21}$$

From (2.19) we derive that there are positive numbers $\lambda_\circ$ and C, and a smooth function $[\lambda_\circ, +\infty) \ni \lambda \to \xi^*(\lambda) \in \mathbb{R}^n\backslash\{0\}$ such that $|\xi^*(\lambda) - \xi^\circ| < C\lambda^{-1}$ and $\operatorname{Re}\sigma(\zeta L)(\lambda^{-1}x^\circ, \xi^*(\lambda)) = 0$. Since $\sigma(\zeta L)(x, \xi)$ is linear with respect to ξ we have

$$\operatorname{Re}\sigma(\zeta L)(\lambda^{-1}x^\circ, \lambda\xi^*(\lambda)) = 0. \tag{2.22}$$

Consider now the null bicharacteristic $\tilde{\gamma}_\lambda$ of $\operatorname{Re}\sigma(\zeta L)$ through $(\lambda^{-1}x^\circ, \lambda\xi^*(\lambda))$; it is described by the solution $(x(t, \lambda), \xi(t, \lambda))$ of the Hamilton–Jacobi equations associated with $\operatorname{Re}\sigma(\zeta L)$ such that $(x(0, \lambda), \xi(0, \lambda)) = (\lambda^{-1}x^\circ, \lambda\xi^*(\lambda))$. Actually, it is convenient to consider the functions $\tilde{x}(t, \lambda) = \lambda x(t, \lambda)$, $\tilde{\xi}(t, \lambda) = \lambda^{-1}\xi(t, \lambda)$. In vector notation they satisfy the equations

$$\left.\begin{aligned} \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} &= A^{\mathrm{T}}\tilde{x} + \lambda\Phi(\lambda^{-1}\tilde{x}), \\ \frac{\mathrm{d}\tilde{\xi}_j}{\mathrm{d}t} &= -A\tilde{\xi} + \Psi(\lambda^{-1}\tilde{x})\tilde{\xi}, \end{aligned}\right\} \tag{2.23}$$

where $\Phi = (\varphi_1, \dots, \varphi_n)$ and $\Psi(x)$ is an $n \times n$ real matrix whose entries are $\mathcal{C}^\infty$ functions in $\mathcal{U}$ that vanish on $\mathcal{U} \cap \Sigma$. Moreover,

$$(\tilde{x}(0, \lambda), \tilde{\xi}(0, \lambda)) = (x^\circ, \xi^*(\lambda)). \tag{2.24}$$

Note at this point that, if $|t| \leqslant T$ and $\lambda \geqslant \lambda_\circ$, the vectors $\tilde{x}(t, \lambda)$ and $\tilde{\xi}(t, \lambda)$ stay in compact subsets, of $\mathcal{U}$ and $\mathbb{R}^n$ respectively.

By subtracting (2.14) from (2.23) and using (2.16) and (2.24) we obtain, still in vector notation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{x}(t,\lambda)-x(t)) &= A^{\mathrm{T}}(\tilde{x}(t,\lambda)-x(t))+\lambda\Phi(\lambda^{-1}\tilde{x}(t,\lambda)),\\ \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\xi}(t,\lambda)-\xi(t)) &= -A(\tilde{x}(t,\lambda)-x(t))+\Psi(\lambda^{-1}\tilde{x}(t,\lambda))\xi,\\ \tilde{x}(0,\lambda)-x(0)=0, &\qquad \tilde{\xi}(0,\lambda)-\xi(0)=\xi^*(\lambda)-\xi^\circ,\end{aligned}$$

whence

$$\begin{aligned}\tilde{x}(t,\lambda)-x(t) &= \lambda\int_0^t \mathrm{e}^{(t-s)A^{\mathrm{T}}}\Phi(\lambda^{-1}\tilde{x}(s,\lambda))\,\mathrm{d}s,\\ \tilde{\xi}(t,\lambda)-\xi(t) &= \mathrm{e}^{-tA}(\xi^*(\lambda)-\xi^\circ)+\int_0^t \mathrm{e}^{-(t-s)A}\Psi(\lambda^{-1}\tilde{x}(s,\lambda))\tilde{\xi}(s,\lambda)\,\mathrm{d}s.\end{aligned}$$

We get, for some constant $C>0$ and all $|t|<T$,

$$\begin{aligned}|\tilde{x}(t,\lambda)-x(t)| &\leqslant C\lambda^{-1}\sup_{|s|<T}(|\tilde{x}(s,\lambda)|^2),\\ |\tilde{\xi}(t,\lambda)-\xi(t)| &\leqslant C\lambda^{-1}\Big(1+\sup_{|s|<T}|\tilde{\xi}(s,\lambda)|\Big).\end{aligned}$$

We derive that if $\lambda_\circ$ is sufficiently large and if $\lambda\geqslant\lambda_\circ$, then

$$\sup_{|t|<T}|\operatorname{Im}\sigma(\zeta L)(\tilde{x}(t,\lambda),\tilde{\xi}(t,\lambda))-\operatorname{Im}\sigma(\zeta L_\wp)(x(t),\xi(t))|<\tfrac{1}{2}c.$$

Combining this with (2.18) we conclude that

$$\begin{aligned}\operatorname{Im}\sigma(L)(\tilde{x}(t_1,\lambda),\tilde{\xi}(t_1,\lambda)) &< -\tfrac{1}{2}c,\\ \operatorname{Im}\sigma(L)(\tilde{x}(t_2,\lambda),\tilde{\xi}(t_2,\lambda)) &> \tfrac{1}{2}c,\end{aligned}$$

thus proving that $\operatorname{Im}\sigma(L)$ changes sign along $\tilde{\gamma}_\lambda$. □

Corollary 2.22. *Assume that the linear part $L_\wp$ is of principal type in the tangent space $T_\wp\mathcal{M}$. If $L_\wp$ does not satisfy condition (P) in $T_\wp\mathcal{M}$, then L is not locally solvable in $\mathcal{M}\setminus\operatorname{Crit} L$.*

3. Some special classes of vector fields

3.1. Essentially real vector fields

Definition 3.1. We say that a $\mathcal{C}^\infty$ vector field in $\mathcal{M}$ is *essentially real* if L and $\bar{L}$ are collinear at every point of $\mathcal{M}$ (i.e. $L\wedge\bar{L}$ vanishes identically in $\mathcal{M}$).

To say that L is essentially real is equivalent to saying that every L-leaf is either an orbit (a one-dimensional, smooth, immersed submanifold) or a critical point of L.

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Lemma 3.2. *If an essentially real $\mathcal{C}^\infty$ vector field L in $\mathcal{M}$ is of principal type at a critical point $\wp$, then there is $c_\wp \in \mathbb{C}\backslash\{0\}$ such that $c_\wp L_\wp$ is real.*

Proof. Suppose L has the expression (2.1) in a local chart $(\mathcal{U}, x_1, \ldots, x_n)$ centred at $\wp$ and let A be the $n \times n$ matrix with entries

$$a_{j,k} = \frac{\partial a_j}{\partial x_k}(0).$$

We have

$$\frac{1}{2\mathrm{i}} L \wedge \bar{L} = \sum_{1 \leqslant j < k \leqslant n} \operatorname{Im}\left(\sum_{p,q=1}^{n} a_{j,p} \bar{a}_{k,q} x_p x_q \right) \partial_{x_j} \wedge \partial_{x_k} + O(|x|^2).$$

If $L \wedge \bar{L} \equiv 0$ then, for every pair of indices j, k,

$$\forall x \in \mathbb{R}^n, \quad \sum_{p,q=1}^{n} \operatorname{Im}(a_{j,p} \bar{a}_{k,q} + a_{j,q} \bar{a}_{k,p}) x_p x_q = 0. \tag{3.1}$$

Putting $p = q$ in (3.1) yields $a_{j,p} \bar{a}_{k,p} \in \mathbb{R}$ for all pairs j, k, implying that for each $p = 1, \ldots, n$, there is a complex number λ_p such that $a_{j,p} = b_{j,p} \lambda_p$ with $b_{j,p} \in \mathbb{R}$, $j = 1, \ldots, n$, and therefore $\det A = \lambda_1 \cdots \lambda_n \det B$, where $B = (b_{j,p})_{1 \leqslant j,p \leqslant n}$. If $p \neq q$ we derive from (3.1)

$$(b_{j,p} b_{k,q} - b_{j,q} b_{k,p}) \operatorname{Im}(\lambda_p \bar{\lambda}_q) = 0$$

for all $j, k = 1, \ldots, n$. But $\det B \neq 0$ implies that $b_{j,p} b_{k,q} \neq b_{j,q} b_{k,p}$ for some j, k. It ensues that $\lambda_q = s_q \lambda_1$ with $s_q \in \mathbb{R}$ for each $q = 2, \ldots, n$, proving the claim. □

Remark 3.3. The conclusion in Lemma 3.2 is generally not valid if we remove the principal type hypothesis. Example: $L = (x_1 + \mathrm{i} x_2) \partial_{x_1}$ in $\mathbb{R}^2$.

Proposition 3.4. *If an essentially real $\mathcal{C}^\infty$ vector field L in $\mathcal{M}$ is of principal type at some critical point $\wp$ then the following properties hold:*

(1) *$\dim \mathcal{M}$ is even;*

(2) *$\wp$ is an isolated critical point of L;*

(3) *the origin in $T_\wp \mathcal{M}$ is an isolated critical point of $L_\wp$.*

Recall that L is of principal type at a critical point $\wp \in \mathcal{M}$ if and only if the linear part $L_\wp$ is of principal type in $T_\wp \mathcal{M}$ (Proposition 2.6).

Proof. Same notation as in the proof of Lemma 3.2; after division of L by $c_\wp$ we may assume that the matrix A is real. If L is of principal type then (by Proposition 2.8) the transpose A^{T}, and therefore also its transpose, A, must be injective when acting in $\mathbb{R}^n$, whence properties (2) and (3). Furthermore, A^{T} cannot have any real eigenvector, which is impossible when $\dim \mathcal{M}$ is odd. □

Proposition 3.5. *If an essentially real $\mathcal{C}^\infty$ vector field L in $\mathcal{M}$ is of principal type in $\mathcal{M}$, then there is a unique Lipschitz continuous map $\theta\colon \mathcal{M} \to [0,\pi)$ such that $\mathrm{e}^{\mathrm{i}\theta} L$ is a real vector field. The restriction of θ to $\mathcal{M}\setminus \operatorname{Crit} L$ is $\mathcal{C}^\infty$.*

Proof. Let $\wp \in \mathcal{M}\setminus \operatorname{Crit} L$ be arbitrary. If $\bar{L}|_\wp = \zeta L|_\wp$, then necessarily $\zeta = \mathrm{e}^{2\mathrm{i}\theta(\wp)}$ for some $\theta(\wp) \in \mathbb{R}$ and $\mathrm{e}^{\mathrm{i}\theta(\wp)} L|_\wp \in T_\wp\mathcal{M}$; if we require $0 \leqslant \theta < \pi$ then θ is uniquely determined, and thereby smooth in $\mathcal{M}\setminus \operatorname{Crit} L$. Consider now an arbitrary point $\wp \in \operatorname{Crit} L$. Suppose L has the expression (2.1) in the local chart $(\mathcal{U}, x_1, \dots, x_n)$ centred at $\wp$; we then use the same notation as in the proof of Lemma 3.2. After division either by $|c_\wp|$ or by $-|c_\wp|$ we can assume that the coefficients of the vector field $L^\circ = \mathrm{e}^{\mathrm{i}\theta_\wp} L_\wp$ are real and that the constant $\theta_\wp$ satisfies $0 \leqslant \theta_\wp < \pi$. We view provisionally $L_\wp$ as a vector field in $\mathcal{U}$ (in the coordinates x_j) and use the fact that $L - L_\wp$ vanishes at least to second order at $x = 0$. We obtain, for $x \neq 0$,

$$\mathrm{e}^{\mathrm{i}\theta(x)} L = \mathrm{e}^{\mathrm{i}(\theta(x)-\theta_\wp)} \mathrm{e}^{\mathrm{i}\theta_\wp} L_\wp + O(|x|^2).$$

Since both $\mathrm{e}^{\mathrm{i}\theta(x)} L$ and $\mathrm{e}^{\mathrm{i}\theta_\wp} L_\wp$ are real vector fields we deduce that

$$\sin(\theta(x) - \theta_\wp) \biggl| \sum_{k=1}^{n} a_{j,k} x_k \biggr| \leqslant \text{const.} |x|^2, \quad j = 1, \dots, n.$$

Since $\det A \neq 0$ we get $|\theta(x) - \theta_\wp| \leqslant \text{const.} |x|$, which proves that $\theta(x)$ converges to $\theta_\wp$ as $x \to 0$ and that we can extend θ as a Lipschitz continuous function in $\mathcal{M}$, smooth in $\mathcal{M}\setminus \operatorname{Crit} L$. □

Proposition 3.5 yields a simple sufficient condition for the local solvability of L.

Theorem 3.6. *Let L be an essentially real $\mathcal{C}^\infty$ vector field of principal type in $\mathcal{M}$ and let $\wp \in \operatorname{Crit} L$. If $\operatorname{div} L|_\wp \neq 0$, then L is locally L^2-solvable at $\wp$* (see Remark 2.17).

The value of $\operatorname{div} L$ at a critical point $\wp$ is independent of the choice of local coordinates; we have $\operatorname{div} L|_\wp = \operatorname{div} L_\wp|_0 = \operatorname{tr} A$, the *trace* of the matrix A.

Proof. According to Proposition 3.5 we can find a Lipschitz continuous function $\theta\colon \mathcal{M} \to [0,\pi)$ such that the coefficients of $\mathrm{e}^{\mathrm{i}\theta} L$ are real. Assuming that L is given by (2.1) in the local chart $(\mathcal{U}, x_1, \dots, x_n)$ centred at $\wp$, we obtain

$$\operatorname{div}(\mathrm{e}^{\mathrm{i}\theta} L) = \sum_{j=1}^{n} \partial_{x_j} (\mathrm{e}^{\mathrm{i}\theta} a_j) = \mathrm{e}^{\mathrm{i}\theta} \biggl(\operatorname{div} L + \mathrm{i} \sum_{j=1}^{n} a_j \partial_{x_j} \theta \biggr).$$

Since $\partial_{x_j} \theta \in L^\infty$ and $|a_j(x)| \leqslant \text{const.} |x|$ we obtain

$$|\operatorname{div}(\mathrm{e}^{\mathrm{i}\theta} L) - \mathrm{e}^{\mathrm{i}\theta(0)} \operatorname{div} L|_0| \leqslant \text{const.} |x|. \tag{3.2}$$

Integration by parts yields, for every $\varphi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{U})$,

$$2 \operatorname{Re} \int \mathrm{e}^{\mathrm{i}\theta} (L\varphi) \bar{\varphi} \, \mathrm{d}x = - \int |\varphi|^2 \operatorname{div}(\mathrm{e}^{\mathrm{i}\theta} L) \, \mathrm{d}x. \tag{3.3}$$

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Combining (3.2) and (3.3) yields, for some constant $C > 0$ independent of φ,

$$|\operatorname{div} L|_0| \int |\varphi|^2 \, \mathrm{d}x \leqslant 2\left|\operatorname{Re} \int \varphi \overline{L^*(\mathrm{e}^{-\mathrm{i}\theta}\varphi)} \, \mathrm{d}x\right| + C \int |\varphi|^2 |x| \, \mathrm{d}x,$$

where L^* is the adjoint of L. After contracting $\mathcal{U}$ about $\wp$ we may assume that $C \sup_{\mathcal{U}} |x| < \frac{1}{2}|\operatorname{div} L|_0|$; we conclude, by the Cauchy–Schwarz inequality and after substituting $\mathrm{e}^{\mathrm{i}\theta}\varphi$ for φ,

$$\int |\varphi|^2 \, \mathrm{d}x \leqslant C' \left| \int |L^*\varphi|^2 \, \mathrm{d}x \right|, \tag{3.4}$$

where $C' = 16/(|\operatorname{div} L|_0|^2)$; (3.4) entails the existence of a linear map $L^*\mathcal{C}_{\mathrm{c}}^\infty(\mathcal{U}) \ni L^*\varphi \to \varphi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathcal{U})$ bounded for the L^2-norm. It can be extended as a bounded linear map G^* of $L^2(\mathcal{U})$ into itself. The adjoint G of G^* satisfies $LGf = f$ for every $f \in L^2(\mathcal{U})$. □

Remark 3.7. A natural question is whether results such as Theorem 3.6 can be extended to first-order differential operators of the type $L + \chi$, $\chi \in \mathcal{C}^\infty(\mathcal{U})$. In this context the role of $\operatorname{div} L$ must be played by the *subprincipal symbol* of L. The same observation should also apply to those results in the sequel that depend in some way or other on $\operatorname{div} L$.

3.2. Vector fields with linear coefficients

The role of the linear part $L_\wp$ in the study of vector fields of principal type justifies that of vector fields in $\mathbb{R}^n$ with *linear* coefficients; for the sake of brevity we shall refer to these as *linear vector fields*. The next statement (a direct consequence of [**8**, Theorem 1]; see also [**15**, Theorem 2]) characterizes the local solvability of real linear vector fields.

Theorem 3.8. *Consider a real vector field in $\mathbb{R}^n$ of the form*

$$L = \sum_{j,k=1}^{n} a_{j,k} x_j \partial_{x_k} \tag{3.5}$$

($a_{j,k} \in \mathbb{R}$)*. The following properties are equivalent:*

(a) *L is not locally solvable at the origin of $\mathbb{R}^n$;*

(b) *the closure of every orbit of L in $\mathbb{R}^n \backslash \operatorname{Crit} L$ is compact;*

(c) *each point $x^\circ \in \mathbb{R}^n \backslash \operatorname{Crit} L$ lies in a torus $\mathbb{T}_{x^\circ} \subset \mathbb{R}^n \backslash \operatorname{Crit} L$ such that every orbit of L that intersects $\mathbb{T}_{x^\circ}$ is a geodesic of $\mathbb{T}_{x^\circ}$;*

(d) *the real matrix $A = (a_{j,k})_{1 \leqslant j,k \leqslant n}$ is semisimple and its non-zero eigenvalues are purely imaginary.*

When we apply Proposition 3.4 to $L_\wp$ (*assumed to be of principal type*) we have $\operatorname{Crit} L = \{0\}$, $n = 2m$ (Proposition 3.4). Thus, if $L_\wp$ is not locally solvable in $T_\wp\mathcal{M}$ there is a linear change of the coordinates x_i such that, for each $x^\circ \neq 0$,

$$\mathbb{T}_{x^\circ} = \{x \in T_\wp\mathcal{M};\ x_j^2 + x_{m+j}^2 = (x_j^\circ)^2 + (x_{m+j}^\circ)^2,\ j = 1, \dots, m\};$$

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equivalently, every eigenvalue of the semisimple matrix $A = (a_{j,k})_{1 \leqslant j,k \leqslant n}$ is purely imaginary *and* different from zero.

Consider a general (*a priori*, not of principal type or locally solvable) 'linear' vector field (3.5).

Lemma 3.9. *If the real vector field (3.5) is of principal type the same is true of*

$$L' = \sum_{j,k=1}^{n} a_{j,k} x_k \partial_{x_j}. \tag{3.6}$$

Proof. The hypothesis is equivalent to the property that the real matrix A^{T} does not have any real eigenvector, in turn equivalent to the property that A does not have any real eigenvalue. □

In dealing with vector fields of the form (3.5) (whether their coefficients $a_{i,j}$ are real or complex) it makes sense to use spherical coordinates in $\mathbb{R}^n$: $r = (x_1^2 + \cdots + x_n^2)^{1/2}$, $\omega_j = r^{-1} x_j$; of course, $\omega_1^2 + \cdots + \omega_n^2 = 1$. It is convenient to introduce the vector fields

$$X_{j,k} = \omega_j \partial_{\omega_k} - \omega_k \partial_{\omega_j} = x_j \partial_{x_k} - x_k \partial_{x_j}, \quad X_{k,j} = -X_{j,k} \quad (1 \leqslant j < k \leqslant n); \tag{3.7}$$

they span $T\mathbb{S}^{n-1}$ at every point $\omega \in \mathbb{S}^{n-1}$. It is immediately seen that

$$L = Q(\omega) r \partial_r - Z, \tag{3.8}$$

where we use the notation

$$Q(\omega) = \sum_{p,q=1}^{n} a_{p,q} \omega_p \omega_q, \tag{3.9}$$

$$Z = \sum_{1 \leqslant j < k \leqslant n} \sigma^{\flat}_{j,k}(\omega) X_{j,k}, \tag{3.10}$$

$$\sigma^{\flat}_{j,k}(\omega) = \sum_{\ell=1}^{n} (a_{j,\ell} \omega_k - a_{k,\ell} \omega_j) \omega_\ell. \tag{3.11}$$

(Concerning $\sigma^{\flat}_{j,k}$, see (2.4).) The coefficients of Z are complex-valued, cubic polynomials in the variables ω_j since the $\sigma^{\flat}_{j,k}$ are quadratics. The sum in (3.10) is symmetric with respect to the indices j, k.

Let us denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the Hermitian norm and scalar product in the complexified tangent spaces to $\mathbb{S}^{n-1}$ (or to $\mathbb{R}^n$).

Proposition 3.10. *For every* $\omega \in \mathbb{S}^{n-1}$,

$$\|Z|_\omega\|^2 = \sum_{j=1}^{n} \left| \sum_{k=1}^{n} \sigma^{\flat}_{j,k}(\omega) \omega_k \right|^2. \tag{3.12}$$

Proof. We avail ourselves of the following equalities:

$$\begin{aligned}
\|X_{j,k}\|^2 &= \omega_j^2 + \omega_k^2, \\
(X_{j,k}, X_{j,q}) &= \omega_k\omega_q \qquad \text{if } 1 \leqslant j < \min(k,q) \leqslant n,\ k \neq q, \\
(X_{j,k}, X_{p,k}) &= \omega_j\omega_p \qquad \text{if } 1 \leqslant \max(j,p) < k \leqslant n,\ j \neq p, \\
(X_{j,k}, X_{p,j}) &= -\omega_k\omega_p \qquad \text{if } 1 \leqslant p < j < k \leqslant n, \\
(X_{j,k}, X_{k,q}) &= -\omega_j\omega_q \qquad \text{if } 1 \leqslant j < k < q \leqslant n,
\end{aligned}$$

and $(X_{j,k}, X_{p,q}) = 0$ in all other cases. We deduce from this:

$$\begin{aligned}
\|Z|_\omega\|^2 &= \sum_{1\leqslant j<k\leqslant n} \sum_{1\leqslant p<q\leqslant n} \sigma^\flat_{j,k}\overline{\sigma^\flat_{p,q}}(X_{j,k}, X_{p,q}) \\
&= \sum_{1\leqslant j<k\leqslant n} |\sigma^\flat_{j,k}|^2\omega_j^2 + \sum_{1\leqslant j<k\leqslant n} \sum_{\substack{j<\ell\leqslant n\\ \ell\neq k}} \sigma^\flat_{j,k}\overline{\sigma^\flat_{j,\ell}}\omega_k\omega_\ell \\
&\quad + \sum_{1\leqslant j<k\leqslant n} |\sigma^\flat_{j,k}|^2\omega_k^2 + \sum_{1\leqslant j<k\leqslant n} \sum_{\substack{1\leqslant\ell<k\\ \ell\neq j}} \sigma^\flat_{j,k}\overline{\sigma^\flat_{\ell,k}}\omega_j\omega_\ell \\
&\quad - \sum_{2\leqslant j<k\leqslant n} \sum_{1\leqslant \ell<j} \sigma^\flat_{j,k}\overline{\sigma^\flat_{\ell,j}}\omega_k\omega_\ell - \sum_{1\leqslant j<k<n} \sum_{k<\ell\leqslant n} \sigma^\flat_{j,k}\overline{\sigma^\flat_{k,\ell}}\omega_j\omega_\ell.
\end{aligned}$$

Taking $\sigma^\flat_{j,k} = -\sigma^\flat_{k,j}$ into account we get

$$\begin{aligned}
\|Z|_\omega\|^2 &= \sum_{j=1}^n\sum_{k=1}^n |\sigma^\flat_{j,k}|^2\omega_k^2 + \sum_{j=1}^n\sum_{k=1}^n \sum_{\substack{1\leqslant\ell\leqslant n\\ \ell\neq k}} \sigma^\flat_{j,k}\overline{\sigma^\flat_{j,\ell}}\omega_k\omega_\ell \\
&= \sum_{j=1}^n\sum_{k=1}^n\sum_{\ell=1}^n \sigma^\flat_{j,k}\overline{\sigma^\flat_{j,\ell}}\omega_k\omega_\ell,
\end{aligned}$$

whence (3.12). □

Corollary 3.11. *The following properties of a point $\omega \in \mathbb{S}^{n-1}$ are equivalent:*

(i) *ω is a critical point of Z;*

(ii) *$\sigma^\flat_{j,k}(\omega) = 0$ for every pair (j,k), $1 \leqslant j < k \leqslant n$.*

Proof. That (ii) $\Longrightarrow$ (i) is a direct consequence of (3.12). Actually, (3.12) shows that (i) is equivalent to the system of equations

$$\sum_{j=1}^n \sigma^\flat_{j,k}(\omega)\omega_k = 0, \quad j = 1,\dots,n. \tag{3.13}$$

We derive from (3.11):

$$\omega_k\sigma^\flat_{j,p}(\omega) - \omega_j\sigma^\flat_{k,p}(\omega) = \omega_p\sigma^\flat_{j,k}(\omega). \tag{3.14}$$

Since $\sum_{p=1}^{n} \omega_p^2 = 1$ combining (3.13) and (3.14) yields

$$\sigma^\flat_{j,k}(\omega) = \omega_j \sum_{p=1}^{n} \sigma^\flat_{k,p}(\omega)\omega_p - \omega_k \sum_{p=1}^{n} \sigma^\flat_{j,p}(\omega)\omega_p, \tag{3.15}$$

proving that (i) $\Longrightarrow$ (ii). □

Proposition 3.12. *Suppose the vector field (3.8) is real. For L to be of principal type it is necessary and sufficient that the vector field Z (defined in (3.10)) not have any critical point in the sphere $\mathbb{S}^{n-1}$.*

Proof. Lemma 3.9 states that L is of principal type if and only if the vector field (3.6) is of principal type. Applying (2.3) with L' in the place of L equates this property to

$$\mathrm{d}\sigma(L') \wedge \tau|_0 = \sum_{1 \leqslant j < k \leqslant n} \sigma^\flat_{j,k}(\xi)\, \mathrm{d}x_j \wedge \mathrm{d}x_k \neq 0,$$

which means (by Corollary 3.11) that Z nowhere vanishes in $\mathbb{S}^{n-1}$. □

Remark 3.13. Proposition 3.12 provides another proof of property (1) in Proposition 3.4. Indeed, on an even-dimensional sphere every smooth real vector field has critical points.

Remark 3.14. The hypothesis that (3.8) is of principal type does not impose any requirement on the zeros of the quadratic form $Q(\omega)$. It does not preclude that Q vanishes identically, as in the following example (in $\mathbb{R}^{2p}$):

$$L = \sum_{j=1}^{p} \lambda_j (x_j \partial_{x_{p+j}} - x_{p+j}\partial_{x_j}), \quad 0 \neq \lambda_j \in \mathbb{C}. \tag{3.16}$$

Proposition 3.15. *Suppose that the vector field (3.8) is real and of principal type. For L to be locally solvable at the origin it is necessary and sufficient that the quadratic form Q not vanish identically.*

Proof. To say that $Q(\omega) = 0$ for all $\omega \in \mathbb{S}^{n-1}$ is the same as saying that $a_{j,k} = -a_{k,j}$ for all $j, k = 1, \dots, n$, i.e. the matrix A is skew-symmetric (and thus semisimple). Since L is of principal type n must be even (Proposition 3.4) and $\det A \neq 0$. It follows that the eigenvalues of A are purely imaginary and that none is equal to zero. It suffices then to apply Theorem 3.8. □

3.3. Hypoellipticity off the critical points: vector fields of principal type on surfaces

Let L be a complex, smooth vector field in $\mathcal{M}$ and let $\mathcal{A}$ be an open subset of $\mathcal{M}$. We recall that L is said to be *hypoelliptic* in $\mathcal{A}$ if, given any open set $\mathcal{V} \subset \mathcal{A}$ and any distribution u in $\mathcal{A}$, $Lu \in \mathcal{C}^\infty(\mathcal{V}) \Longrightarrow u \in \mathcal{C}^\infty(\mathcal{V})$. The following terminology is also used: L is hypoelliptic at *a point* $\wp \in \mathcal{M}$ if there is an open neighbourhood $\mathcal{U}$ of $\wp$ in which L is hypoelliptic.

The following facts are known.

- If L is hypoelliptic in $\mathcal{A}$ then its transpose L^{T} is locally solvable in $\mathcal{A}$ [**14**, Theorem 52.2].
- If L does not have critical points in $\mathcal{A}$ the hypoellipticity (as well as the local solvability) of L and that of its transpose $L^{\mathrm{T}} = -L - \operatorname{div} L$ are equivalent properties.
- If L is elliptic in $\mathcal{A}$ (see next subsection) then L is hypoelliptic in $\mathcal{A}$.

Remark 3.16. It is proved in [**6**] that if L has analytic coefficients and $\wp$ is a critical point of L then L is not hypoelliptic at $\wp$. It is natural to conjecture that the same is true when the coefficients are just $\mathcal{C}^\infty$.

There is an analogue of Theorem 2.15 for hypoellipticity (on all this see [**12**]).

Theorem 3.17. *Assume that the vector field L does not have any critical point in an open subset $\mathcal{A}$ of $\mathcal{M}$. The following properties are equivalent:*

(He) *L is hypoelliptic in $\mathcal{A}$;*

(Q) $\forall \zeta \in \mathbb{C}$, *the function* $\operatorname{Im}(\zeta\sigma(L))$ *does not change sign along any null bicharacteristic of* $\operatorname{Re}(\zeta\sigma(L))$ *and does not vanish identically on any arc of such a bicharacteristic.*

Proposition 3.18. *Assume that the $\mathcal{C}^\infty$ manifold $\mathcal{M}$ is connected. If L is hypoelliptic in some open subset $\mathcal{A}$ of $\mathcal{M}$ then* $\dim \mathcal{M} \leqslant 2$.

Proof. If L is hypoelliptic in $\mathcal{A}$ then $\mathcal{A} \backslash \mathcal{A} \cap \operatorname{Crit} L$ is open and non-empty; thus we may as well assume that $\mathcal{A} \cap \operatorname{Crit} L = \varnothing$. As pointed out above this demands that L be locally solvable in $\mathcal{A}$, therefore the dimension of the leaves in the foliation defined by the real vector fields $\operatorname{Re} L$ and $\operatorname{Im} L$ cannot exceed 2 (Theorem 2.15). This being the case, there is an open and dense subset $\mathcal{A}'$ of $\mathcal{A}$ in which the dimension of the leaves is locally constant (either 1 or 2); $\mathcal{A}'$ can be covered with analytic coordinates charts $(\mathcal{U}, x_1, \dots, x_n)$ in which

$$L = a_1(x)\partial_{x_1} + a_2(x)\partial_{x_2} \tag{3.17}$$

with $a_1, a_2 \in \mathcal{C}^\omega(\mathcal{U})$. If $n = \dim \mathcal{M} > 2$ any distribution u in $\mathcal{U}$ such that $\partial_{x_j} u = 0$, $j = 1, 2$, is a solution of $Lu = 0$, which proves that L could not be hypoelliptic. □

In the remainder of the article we assume that $\dim \mathcal{M} = 2$: $\mathcal{M}$ is a smooth, connected, orientable *surface*.

Let $(\mathcal{U}, x_1, x_2)$ be a local chart in $\mathcal{M}$ centred at a critical point $\wp$ of L (given by (3.17)). It is convenient to use the natural polar coordinates in $\mathcal{U}$, $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arg(x_1 + \mathrm{i}x_2)$. We have (see (3.8)–(3.11))

$$L_\wp = Q(\cos\theta, \sin\theta) r\partial_r - \sigma^\flat_{1,2}(\cos\theta, \sin\theta)\partial_\theta, \tag{3.18}$$

where we have used the notation

$$Q(\cos\theta, \sin\theta) = a_{1,1}\cos^2\theta + a_{2,2}\sin^2\theta + (a_{1,2} + a_{2,1})\cos\theta\sin\theta, \tag{3.19}$$

$$\sigma^\flat_{1,2}(\cos\theta, \sin\theta) = -a_{1,2}\sin^2\theta + a_{2,1}\cos^2\theta - (a_{1,1} - a_{2,2})\cos\theta\sin\theta. \tag{3.20}$$

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Note that both (3.19) and (3.20) are periodic functions of period π. The coefficients in (3.18) are related to each other:

$$2Q(\cos\theta,\sin\theta) = \operatorname{div} L_\wp + \partial_\theta[\sigma^\flat_{1,2}(\cos\theta,\sin\theta)]. \tag{3.21}$$

A special feature of the case $n = 2$ is that

$$\sigma^\flat_{1,2}(\cos\theta,\sin\theta) = \sigma_{1,2}(-\sin\theta,\cos\theta), \tag{3.22}$$

with $\sigma_{1,2}$ defined in (2.4); (2.3) and (3.22) directly imply the following.

Proposition 3.19. *Let L be a $\mathcal{C}^\infty$ vector field in the surface $\mathcal{M}$. Its linear part $L_\wp$, given by (3.18), is of principal type if and only if $\sigma^\flat_{1,2}(\cos\theta,\sin\theta) \neq 0$ for all $\theta \in \mathbb{R}$.*

Corollary 3.20. *If L is of principal type in the surface $\mathcal{M}$ then every critical point $\wp$ of L is isolated and the origin in $T_\wp\mathcal{M}$ is an isolated critical point of $L_\wp$.*

Proof. The claim follows then directly from Proposition 3.19 and from the fact that, in a neighbourhood of $\wp$, we have

$$\begin{aligned} L &= L_\wp + O(r^2) \\ &= (Q(\cos\theta,\sin\theta) + rF(r,\theta))r\partial_r - (\sigma^\flat_{1,2}(\cos\theta,\sin\theta) + rG(r,\theta))\partial_\theta \end{aligned} \tag{3.23}$$

with F and G smooth functions in some cylindrical set $[0,\varepsilon) \times \mathbb{S}^1$. □

The principal type property (of L at $\wp$) is equivalent to the fact that the range of the map $[0,\pi) \ni \theta \to \sigma^\flat_{1,2}(\cos\theta,\sin\theta) \in \mathbb{C}$ is an ellipse $\mathfrak{E}$ that does not pass through the origin (but might be reduced to a compact straight-line segment). This property allows us to introduce the following function of θ:

$$K(\theta) = \frac{1}{2\mathrm{i}\pi}\int_0^\theta \frac{\mathrm{d}t}{\sigma^\flat_{1,2}(\cos t,\sin t)}. \tag{3.24}$$

It is proved in [**16**] that the value of $K(\pi)$ is independent of the choice of coordinates in $\mathcal{U}$ centred at $\wp$. The same is true of the quantity

$$\mu_\wp = (\operatorname{div} L_\wp)K(\pi) + \frac{1}{2\pi}\lim_{\varepsilon\to+0}\arg\sigma^\flat_{1,2}(\cos(\pi-\varepsilon),\sin(\pi-\varepsilon)) - \frac{1}{2\pi}\arg\sigma^\flat_{1,2}(1,0), \tag{3.25}$$

first introduced in [**5**] to analyse *normal forms* of planar vector fields elliptic in the complement of a circle. In [**16**] $\mu_\wp$ is called the *Meziani invariant* of $L_\wp$; here we refer to it as the Meziani invariant of L at $\wp$.

The main result in [**16**] (Theorem 3.21 below) concerns (complex) linear vector fields in $\mathcal{M} = \mathbb{R}^2$. The critical set of a linear vector field L (assumed not to vanish everywhere) is either $\{0\}$, when L is of principal type, or a straight line through the origin, when L is not of principal type. The class of linear vector fields of principal type can be further subdivided into two distinct subtypes, called (IN-E) and (OUT-E) respectively in [**16**], depending on whether the origin in the plane lies ‘inside’ or ‘outside’ the convex hull of the ellipse $\mathfrak{E}$. It is then proved that the linear vector field L is of type (IN-E) if and only if $K(\pi)=0$. It ensues from this and from (3.24) that the Meziani invariant of L is equal to ± 1 if L is of type (IN-E) and to $(\operatorname{div} L)K(\pi)$ if L is of type (OUT-E). In [**16**] the value ∞ was assigned to the Meziani invariant of a linear vector field in the plane that is *not* of principal type.

Theorem 3.21. *Let L be a linear vector field in the plane which is locally solvable in $\mathbb{R}^2 \setminus \operatorname{Crit} L$. The following properties are equivalent:*

(1) *L is not locally solvable at the origin;*

(2) *L is essentially real (Definition 3.1) and not locally solvable at the origin (see Theorem 3.8);*

(3) *the origin is an isolated critical point of L and the Meziani invariant of L at the origin vanishes;*

(4) *L is of principal type (OUT-E) and $\operatorname{div} L = 0$.*

3.4. Quasi-elliptic vector fields

Back to a general (connected, orientable) surface $\mathcal{M}$. More can be said when the complex vector field L is elliptic in $\mathcal{M} \setminus \operatorname{Crit} L$ *and* of principal type at every $\wp \in \operatorname{Crit} L$ (hence in $\mathcal{M}$). The reason is that we have at our disposal the normal forms in [**5**] and in [**2**]. We recall the classical definition: L is said to be *elliptic* in $\mathcal{A}$ if its symbol $\sigma(L)$ does not vanish at any point $(x,\xi) \in T^*\mathcal{M}$ such that $x \in \mathcal{A}$, $\xi \neq 0$. This is equivalent to the property that L and $\bar{L}$ are linearly independent, i.e. $L \wedge \bar{L} \neq 0$, at every point of $\mathcal{A}$: in a sense, ellipticity and essential 'reality' of a vector field (Definition 3.1) are diametrically opposed properties.

We shall continue to reason in a local chart $(\mathcal{U}, x_1, x_2)$ centred at $\wp$, with associated polar coordinates r, θ, where L has the expression (3.23). If we assume that L is of principal type at $\wp$ there are positive constants ε and $c_\circ$ such that

$$r < \varepsilon \Longrightarrow |\sigma_{1,2}^\flat(\cos\theta, \sin\theta) + rG(r,\theta)| \geqslant c_\circ. \tag{3.26}$$

Using the notation

$$\Phi(r,\theta) = \frac{Q(\cos\theta, \sin\theta) + rS(r,\theta)}{\sigma_{1,2}^\flat(\cos\theta, \sin\theta) + rF(r,\theta)} \tag{3.27}$$

we see that the vector field

$$L^\flat = -(\sigma_{1,2}^\flat(\cos\theta, \sin\theta) + rG(r,\theta))^{-1} L = \partial_\theta - r\Phi(r,\theta)\partial_r \tag{3.28}$$

is well defined and smooth in the (cylindrical) open set $\Gamma_\varepsilon = \{(r,\theta) \in \mathbb{R} \times \mathbb{S}^1;\ r < \varepsilon\}$. Below $\mu_\wp$ stands for the Meziani invariant of L at $\wp$; its definition (3.25) can be restated by the formula

$$\mu_\wp = \frac{1}{\mathrm{i}\pi} \lim_{\varepsilon \to +0} \int_0^{\pi-\varepsilon} \Phi(0,\theta)\,\mathrm{d}\theta. \tag{3.29}$$

Using (3.28) we can say that L is elliptic in an open set $\mathcal{A} \subset \mathcal{U}\setminus\{\wp\}$ if and only if $\operatorname{Im}\Phi \neq 0$ at every point of $\mathcal{A}$.

Proposition 3.22. *The following properties are equivalent:*

(1) $L_\wp$ *is elliptic in* $T_\wp\mathcal{M}\backslash\{0\}$*;*

(2) $\forall\theta \in [0, 2\pi]$, $\operatorname{Im}(\sigma^\flat_{1,2}\bar{Q})(\cos\theta, \sin\theta) \neq 0$.

In the notation of (3.28), property (2) can be restated by saying that $\operatorname{Im}\Phi(0, \theta) \neq 0$ for all $\theta \in [0, 2\pi]$.

Corollary 3.23. *If the vector field* $L_\wp$ *is elliptic in* $T_\wp\mathcal{M}\backslash\{0\}$ *then* $L_\wp$ *is of principal type.*

Proof. Recall that $L_\wp$ is of principal type if and only if $\sigma^\flat_{1,2}(\cos\theta, \sin\theta) \neq 0$ for every $\theta \in [0, 2\pi]$. □

Corollary 3.24. *If the vector field* $L_\wp$ *is elliptic in* $T_\wp\mathcal{M}\backslash\{0\}$ *then there is an open neighbourhood* $\mathcal{V}$ *of* $\wp$ *such that* L *is elliptic in* $\mathcal{V}\backslash\{\wp\}$*.*

Proof. The hypothesis entails $\operatorname{Im}\Phi(r, \theta) \neq 0$ for small $r > 0$ and for all $\theta \in [0, 2\pi]$. □

Proposition 3.25. *If* $L_\wp$ *is elliptic in* $T_\wp\mathcal{M}\backslash\{0\}$ *then* $\operatorname{Re}\mu_\wp \neq 0$*.*

Proof. We know that $L_\wp$ is of principal type (Corollary 3.23). By (3.29) we have

$$\operatorname{Re}\mu_\wp = -\pi^{-1} \lim_{\varepsilon\to+0} \int_0^{\pi-\varepsilon} \operatorname{Im}\Phi(0, \theta)\,\mathrm{d}\theta. \tag{3.30}$$

The claim, then, follows directly from the hypothesis that $\operatorname{Im}\Phi(0, \theta) \neq 0$ for all $\theta \in [0, 2\pi]$. □

We now apply Theorem 1.3 and Lemma 4.2 from [**2**]. We select a 'cylindrical' set $\Gamma_\varepsilon = \{(r, \theta) \in \mathbb{R}_+ \times [0, 2\pi];\ r < \varepsilon\}$ such that $(r, \theta) \in \Gamma_\varepsilon \Longrightarrow (r\cos\theta, r\sin\theta) \in \mathcal{U}$.

Lemma 3.26. *Let* L *be a* $\mathcal{C}^\infty$ *vector field in* $\mathcal{M}$ *and let* $\wp \in \operatorname{Crit} L$*; assume that* $L_\wp$ *is elliptic in* $T_\wp\mathcal{M}\backslash\{0\}$*. Then, for each integer* $N \geqslant 1$ *there is a* $\mathcal{C}^N$ *diffeomorphism* $\mathcal{V} \to \mathcal{V}'$*, with* $\mathcal{V}$ *and* $\mathcal{V}'$ *open neighbourhoods of* $\{0\} \times \mathbb{S}^1$ *in* Γ_ε*, that preserves* $\{0\} \times \mathbb{S}^1$ *and transforms the vector field (3.23) into* $\kappa(r, \theta)L_\circ$*, with both* $\kappa(r, \theta)$ *and* $\kappa(r, \theta)^{-1}$ *belonging to* L^∞ *and*

$$L_\circ = \partial_\theta - \mathrm{i}\mu_\wp r\partial_r \tag{3.31}$$

if $\mu_\wp \in \mathbb{C}\backslash\mathbb{Q}$*; and*

$$L_\circ = \partial_\theta - \mathrm{i}(\mu_\wp + r^{\max(q,2)}b(r, \theta))r\partial_r \tag{3.32}$$

if $\mu_\wp = p/q$ $(0 \neq p \in \mathbb{Z}$ *and* $1 \leqslant q \in \mathbb{Z}_+$ *coprime). In (3.32)* $r^{\max(q,2)}b(r, \theta) \in \mathcal{C}^{N-1}(\mathcal{V}')$*.*

We are now in a position to prove the following.

Theorem 3.27. *Let* L *be a smooth vector field in a surface* $\mathcal{M}$ *and let* $\wp$ *be a critical point of* L*. If* $L_\wp$ *is elliptic in* $T_\wp\mathcal{M}\backslash\{0\}$ *then there is an open neighbourhood* $\mathcal{N}(\wp)$ *of* $\wp$ *such that to every* $f \in L^2_{\mathrm{loc}}(\mathcal{N}(\wp))$ *there is* $u \in L^2_{\mathrm{loc}}(\mathcal{N}(\wp))$ *verifying* $Lu = f$ *in* $\mathcal{N}(\wp)$*.*

Proof. It suffices to prove the following statement.

Let $L_\circ$ be the planar vector field given by (3.31) or by (3.32). Let $\Delta_\varepsilon = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2;\ r < \varepsilon\}$. If $\varepsilon > 0$ is sufficiently small, to every $f \in L^2(\Delta_\varepsilon)$ there is $u \in L^2(\Delta_\varepsilon)$ such that $L_\circ u = f$ in Δ_ε.

Indeed, in local polar coordinates centred at $\wp$ we have $L = -\kappa(r,\theta)L_\circ$.

We denote by $(\cdot,\cdot)$ the Hermitian product and by $\|\cdot\|$ the norm in $L^2(\mathbb{R}^2)$.

Case I: $\mu_\wp \in \mathbb{R}\backslash\{0\}$.

We can assume that the expression of $L_\circ$ is (3.32) with the understanding that $b \equiv 0$ if $\mu_\wp \notin \mathbb{Q}$, or that $\mu_\wp = p/q$, with $0 \neq p \in \mathbb{Z}$ and $0 \neq q \in \mathbb{Z}_+$ coprime. With the notation $q' = \max(q, 2)$ we have

$$\begin{aligned} L_\circ^{\mathrm{T}} &= -L_\circ - \operatorname{div} L_\circ \\ &= -\partial_\theta + \mathrm{i}(\mu_\wp + r^{q'}b(r,\theta))r\partial_r + 2\mathrm{i}(\mu_\wp + r^{q'}\psi(r,\theta)), \end{aligned}$$

where $\psi = \frac{1}{2}(q'+1)b + \frac{1}{2}rb_r$. We have, for an arbitrary $\varphi \in \mathcal{C}_c^\infty(\Delta_\varepsilon)$,

$$\begin{aligned} \|L_\circ^{\mathrm{T}}\varphi\|^2 \geqslant \|\partial_\theta\varphi\|^2 &+ \Big(|\mu_\wp| - \varepsilon^{q'}\sup_{\Delta_\varepsilon}|b|\Big)^2\|r\partial_r\varphi\|^2 + 4\Big(|\mu_\wp| - \varepsilon^{q'}\sup_{\Delta_\varepsilon}|\psi|\Big)^2\|\varphi\|^2 \\ &+ 2\operatorname{Re}(\mathrm{i}\partial_\theta\varphi, (\mu_\wp + r^{q'}b(r,\theta))r\partial_r\varphi + 2(\mu_\wp + r^{q'}\psi(r,\theta))\varphi) \\ &+ 4\operatorname{Re}((\mu_\wp + r^{q'}b(r,\theta))r\partial_r\varphi, (\mu_\wp + r^{q'}\psi(r,\theta))\varphi). \end{aligned} \tag{3.33}$$

Since $q' \geqslant 2$ we can select $\varepsilon > 0$ sufficiently small to derive from (3.33) the estimate

$$\begin{aligned} \|L_\circ^{\mathrm{T}}\varphi\|^2 \geqslant \tfrac{3}{4}\|\partial_\theta\varphi\|^2 + \tfrac{3}{4}\mu_\wp^2\|r\partial_r\varphi\|^2 + \tfrac{31}{8}\mu_\wp^2\|\varphi\|^2& \\ + 2\mu_\wp\operatorname{Im}(r\partial_r\varphi + 2\varphi, \partial_\theta\varphi) + 4\mu_\wp^2\operatorname{Re}(r\partial_r\varphi, \varphi).& \end{aligned} \tag{3.34}$$

We observe that

$$\left.\begin{aligned} \operatorname{Re}(r\partial_r\varphi, \varphi) &= -\|\varphi\|^2, \\ 2\mathrm{i}\operatorname{Im}(r\partial_r\varphi, \partial_\theta\varphi) &= (r\partial_r\varphi, \partial_\theta\varphi) - (\partial_\theta\varphi, r\partial_r\varphi) = -(\varphi, \partial_\theta\varphi) = -\mathrm{i}\operatorname{Im}(\varphi, \partial_\theta\varphi), \end{aligned}\right\} \tag{3.35}$$

whence, by (3.34),

$$\|L_\circ^{\mathrm{T}}\varphi\|^2 \geqslant \tfrac{3}{4}\|\partial_\theta\varphi\|^2 + \tfrac{3}{4}\mu_\wp^2\|r\partial_r\varphi\|^2 - |\mu_\wp|\,|(\varphi, \mathrm{i}\partial_\theta\varphi)| - \tfrac{1}{8}\mu_\wp^2\|\varphi\|^2. \tag{3.36}$$

On the one hand, we get, by applying the Cauchy–Schwarz inequality,

$$|\mu_\wp|\,|(\varphi, \mathrm{i}\partial_\theta\varphi)| \leqslant |\mu_\wp|\,\|\varphi\|\,\|\partial_\theta\varphi\| \leqslant \tfrac{1}{2}(\|\partial_\theta\varphi\|^2 + \mu_\wp^2\|\varphi\|^2).$$

On the other hand, by (3.35) we have $\|\varphi\| \leqslant \|r\partial_r\varphi\|$. Combining these estimates with (3.35) yields

$$|\mu_\wp|\|\varphi\| \leqslant C\|L_\circ^{\mathrm{T}}\varphi\|^2, \tag{3.37}$$

where $C = \sqrt{8}|\mu_\wp|^{-1}$. A standard argument enables us to deduce from (3.37) the L^2-solvability of $L_\circ$ in Δ_ε.

Case II: $\operatorname{Im}\mu_\wp \neq 0$.

Here we have

$$L_\circ^{\mathrm{T}} = -\partial_\theta + \mathrm{i}\mu_\wp r\partial_r + 2\mathrm{i}\mu_\wp,$$

whence

$$\|L_\circ^{\mathrm{T}}\varphi\|^2 = \|\partial_\theta\varphi\|^2 + |\mu_\wp|^2\|r\partial_r\varphi + 2\varphi\|^2 + 2\operatorname{Re}(\mathrm{i}\mu_\wp\partial_\theta\varphi, r\partial_r\varphi + 2\varphi).$$

We derive from this and from (3.35)

$$\|L_\circ^{\mathrm{T}}\varphi\|^2 = \|\partial_\theta\varphi\|^2 + |\mu_\wp|^2\|r\partial_r\varphi + 2\varphi\|^2 + 2(\operatorname{Re}\mu_\wp)(\mathrm{i}\partial_\theta\varphi,\varphi) - 2(\operatorname{Im}\mu_\wp)\operatorname{Re}(\partial_\theta\varphi, r\partial_r\varphi + 2\varphi).$$

The fact that $\operatorname{Re}(\partial_\theta\varphi, r\partial_r\varphi + 2\varphi) = 0$ implies

$$\|L_\circ^{\mathrm{T}}\varphi\|^2 = \|\partial_\theta\varphi\|^2 + |\mu_\wp|^2\|r\partial_r\varphi + 2\varphi\|^2 + 2(\operatorname{Re}\mu_\wp)(\mathrm{i}\partial_\theta\varphi,\varphi).$$

We also derive from (3.35):

$$\|r\partial_r\varphi + 2\varphi\|^2 = \|r\partial_r\varphi\|^2 + 4\|\varphi\|^2 + 4\operatorname{Re}(r\partial_r\varphi,\varphi) = \|r\partial_r\varphi\|^2 \geqslant \|\varphi\|^2,$$

whence

$$\|L_\circ^{\mathrm{T}}\varphi\|^2 \geqslant \|\mathrm{i}\partial_\theta\varphi + 2(\operatorname{Re}\mu_\wp)\varphi\|^2 + (\operatorname{Im}\mu_\wp)^2\|r\partial_r\varphi + 2\varphi\|^2 \geqslant (\operatorname{Im}\mu_\wp)^2\|\varphi\|^2.$$

The sought conclusion ensues. □

Remark 3.28. Contrary to what happens in $\mathcal{M}\setminus\operatorname{Crit} L$ there is no gain of regularity at critical points. Example: a non-smooth solution of the equation $z\partial_{\bar z}h = 0$ is z^{-1}, while a non-smooth solution of the equation $z\partial_{\bar z}h = 1$ is $\mathrm{e}^{-2\mathrm{i}\arg z}$.

Theorem 3.27 shows that, on a surface, a class of complex vector fields with critical points and good local solvability properties is the one now defined.

Definition 3.29. We say that the vector field L on the surface $\mathcal{M}$ (both L and $\mathcal{M}$ of class $\mathcal{C}^\infty$) is *quasi-elliptic* if L is elliptic in $\mathcal{M}\setminus\operatorname{Crit} L$ and $L_\wp$ is elliptic in $T_\wp\mathcal{M}\setminus\{0\}$ whatever $\wp \in \operatorname{Crit} L$.

A corollary of Theorem 3.27 is that *a quasi-elliptic vector field L is locally L^2-solvable everywhere in $\mathcal{M}$.*

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