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Regularity of the inverse of a Sobolev homeomorphism in space

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Let $\Omega \subset \mathbb{R}^n$ be open. Given a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ of finite distortion with $|Df|$ in the Lorentz space $L^{n-1,1}(\Omega)$, we show that $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ and f^{-1} has finite distortion. A class of counterexamples demonstrating sharpness of the results is constructed.

1. Introduction

Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and let $f : \Omega \to f(\Omega) \subset \mathbb{R}^n$ be a homeomorphism. In this paper we address the issue of the regularity of f^{-1} under regularity assumptions on f. The starting point for us is the following very recent result from [6].

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an open set and $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ be a homeomorphism of finite distortion. Then $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^2)$ and has finite distortion. Moreover,*

$$\int_{f(\Omega)} |Df^{-1}| = \int_{\Omega} |Df|.$$

Above, a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}$ is of (or has) finite distortion if its Jacobian J_f is strictly positive almost everywhere (a.e.) on the set where $|Df|$ does not vanish. Recall that $g \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $1 \leqslant p < \infty$, means that g is locally p-integrable and that the coordinate functions of g have locally p-integrable distributional derivatives. The results are new even in the case when we simply assume that $J_f > 0$ a.e.

One can then expect for an analogue of theorem 1.1 in space. In such a result, one should assume that $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is a homeomorphism of finite distortion, but is not *a priori* clear whether the critical exponent p is 1 as in the plane or some

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larger number. After some experimental computations, the reader should soon be convinced that the critical case should be $p = n - 1$. An example showing that no smaller value of p can work will be given in § 6.

Our first result gives a rather complete analogue of theorem 1.1 in space.

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $f : \Omega \to \mathbb{R}^n$ is a homeomorphism of finite distortion such that $|Df| \in L^{n-1,1}(\Omega)$. Then*

$$f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$$

and has finite distortion. Moreover,

$$\int_{f(\Omega)} |Df^{-1}(y)| \, \mathrm{d}y = \int_{\Omega} |\mathrm{adj}\, Df(x)| \, \mathrm{d}x.$$

Here $L^{n-1,1}(\Omega)$ is a Lorentz space. Recall that

$$\bigcap_{p>n-1} L^p_{\mathrm{loc}}(\Omega) \subset L^{n-1,1}_{\mathrm{loc}}(\Omega) \subset L^{n-1}_{\mathrm{loc}}(\Omega).$$

We do not know whether the conclusion of theorem 1.2 holds if only $|Df| \in L^{n-1}(\Omega)$. Notice that $L^{1,1}(\Omega) = L^1(\Omega)$ and thus theorem 1.2 encompasses theorem 1.1. Our proof of theorem 1.2 is different from the proof of the planar case in [6]; the main new point is the use of the co-area formula.

The assumption that f has finite distortion cannot be dropped from theorem 1.2. Indeed, consider $g(x) = x + u(x)$ on the real line, where u is the usual Cantor ternary function. Let $h = g^{-1}$. Then h^{-1} fails to be absolutely continuous. By setting $f(x) = (h(x_1), x_2, \dots, x_n)$ we obtain a Lipschitz homeomorphism whose inverse fails to be of the class $W^{1,1}_{\mathrm{loc}}$.

Let us now compare our result with known facts. The planar case was explored in [6]. The most relevant work in higher dimensions is [14], because the regularity assumption there is below $W^{1,n}$ and the method involves the co-area formula. It was shown in [14] that if $f \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p > n-1$, is a homeomorphism that satisfies the Lusin (N) and (N^{-1}) conditions, then $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$. It follows from the (N^{-1}) condition that such a mapping f satisfies $J_f \neq 0$ a.e. and therefore either $J_f > 0$ a.e. or $J_f < 0$ a.e. Both the (N) and (N^{-1}) conditions can fail in our setting and, moreover, we can relax the regularity of f to the weaker condition $|Df| \in L^{n-1,1}(\Omega)$. Other related results (see [9,13]) also require that $J_f > 0$ a.e. and the much stronger assumption $f \in W^{1,n}_{\mathrm{loc}}$; this setting guarantees both the (N) condition and that the so-called distributional Jacobian coincides with J_f.

As in [6], it is natural to inquire if a stronger condition than being of finite distortion would result in higher regularity of the inverse. To this end, we consider the inequality

$$|Df(x)|^n \leqslant K(x) J_f(x)$$

to be satisfied almost everywhere in Ω for some measurable function K with $1 \leqslant K(x) < \infty$ a.e. We prove in § 4, under the assumptions of theorem 1.2, that $f^{-1} \in W^{1,n}_{\mathrm{loc}}(\Omega)$ provided that $K \in L^{n-1}(\Omega)$. This conclusion is shown to be sharp in § 6. As in the planar case, there is no interpolation: under the assumptions of theorem 1.2, no better regularity than $W^{1,1}_{\mathrm{loc}}$ is to be expected even when $K \in L^q(\Omega)$

with $q < n-1$ close to $n-1$. On the other hand, we show in § 4 that we gain improved regularity for f^{-1} if we assume that $|Df| \in L^p(\Omega)$ for some $p > n-1$ and that $K \in L^q(\Omega)$ for some $0 < q < n-1$. The formula obtained is shown to be sharp.

The paper is organized as follows. Section 2 fixes notation and introduces some preliminary results. We prove theorem 1.2 in § 3. Section 4 deals with the higher regularity of f^{-1}. We describe a general procedure for producing homeomorphisms of finite distortion in § 5. In the final section, § 6, we then use the general procedure to single out concrete examples that show the sharpness of our results.

2. Preliminaries

Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be the canonical basis in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ we denote by x_i, $i \in \{1, \dots, n\}$, its coordinates, i.e. $x = \sum_{i=1}^n x_i \boldsymbol{e}_i$. We write $\mathbb{H}_i$ for the ith coordinate hyperplane

$$\mathbb{H}_i = \{x \in \mathbb{R}^n : x_i = 0\}$$

and denote by π_i the orthogonal projection to $\mathbb{H}_i$, so that

$$\pi_i(x) = x - x_i \boldsymbol{e}_i, \quad x \in \mathbb{R}^n.$$

Since $\mathbb{H}_i$ is in fact a copy of $\mathbb{R}^{n-1}$, the Hausdorff measure on $\mathbb{H}_i$ can be identified with the Lebesgue measure and we can write $\mathrm{d}z$ instead of $\mathrm{d}\mathcal{H}_{n-1}(z)$ for integration over $\mathbb{H}_i$. The Euclidean norm of $x \in \mathbb{R}^n$ is denoted by $|x|$. The closure and interior of a set A are denoted by $\bar{A}$ and A°, respectively.

Given a square matrix $B \in \mathbb{R}^{n \times n}$, we define the norm $|B|$ as the supremum of $|Bx|$ over all vectors x of unit Euclidean norm. The adjugate $\operatorname{adj} B$ of a regular matrix B is defined by the formula

$$B \operatorname{adj} B = I \det B,$$

where $\det B$ denotes the determinant of B and I is the identity matrix. The operator adj is then continuously extended to $\mathbb{R}^{n \times n}$.

We use the symbol $|E|$ for the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. A mapping $f : \Omega \to \mathbb{R}^n$ is said to satisfy the Luzin condition (N) on E if $|f(A)| = 0$ for every $A \subset E$ such that $|A| = 0$.

We say that a function $f : \Omega \to \mathbb{R}^n$ has the ACL property or that it is absolutely continuous on almost all lines parallel to coordinate axes if the following happens: for every $i \in \{1, \dots, n\}$ and for almost every $y \in \mathbb{H}_i$ the coordinate functions of f are absolutely continuous on compact subintervals of $\pi_i^{-1}(y) \cap \Omega$.

If $f : \Omega \to \mathbb{R}$ is a measurable function, we define its distribution function $m(\cdot, f)$ by

$$m(\sigma, f) = |\{x : |f(x)| > \sigma\}|, \quad \sigma > 0,$$

and the nonincreasing rearrangement $f^\star$ of f by

$$f^\star(t) = \inf\{\sigma : m(\sigma, f) \leqslant t\}.$$

The Lorentz space $L^{n-1,1}(\Omega)$ is defined as the class of all measurable functions $f : \Omega \to \mathbb{R}$ for which

$$\int_0^\infty t^{1/(n-1)} f^\star(t) \frac{\mathrm{d}t}{t} < \infty.$$

(For an introduction to Lorentz spaces, see, for example, [12].)

Let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ and $E \subset \Omega$ be a measurable set. The multiplicity function $N(f, E, y)$ of f is defined as the number of pre-images of y under f in E. We say that the area formula holds for f on E if

$$\int_E \eta(f(x))|J_f(x)|\,\mathrm{d}x = \int_{\mathbb{R}^n} \eta(y)N(f, E, y)\,\mathrm{d}y \tag{2.1}$$

for any non-negative Borel measurable function on $\mathbb{R}^n$. It is well known that there exists a set $\Omega' \subset \Omega$ of full measure such that the area formula holds for f on Ω'. Also, the area formula holds on each set on which the Luzin condition (N) is satisfied. This follows from [1, §§ 3.1.4, 3.1.8, 3.2.5], namely, it can be found there that Ω can be covered up to a set of measure 0 by countably many sets the restriction to which of f is Lipschitz continuous. For more explicit statements see, for example, [4, 5].

Notice that the area formula holds on the set where f is differentiable (approximate differentiability would be also enough), and thus in particular the image of the set of all critical points has zero measure (this is a version of the Sard theorem).

3. Weak differentiability of the inverse

In what follows, $\Omega \subset \mathbb{R}^n$ will be an open set.

The following co-area formula is crucial for our proof of theorem 1.2.

LEMMA 3.1. *Let h be a continuous mapping with $|Dh| \in L^{n-1,1}(\Omega)$. Suppose that $|h| = 1$ on Ω. Let $E \subset \Omega$ be a measurable set. Then*

$$\int_{\partial B(0,1)} \mathcal{H}_1(\{x \in E : h(x) = z\})\,\mathrm{d}\mathcal{H}_{n-1}(z) = \int_E |\mathrm{adj}\, Dh|\,\mathrm{d}x.$$

Proof. If h is Lipschitz, the formula can be found in [1, §§ 3.2.12]. In the general case, we cover the domain of h up to a set of measure 0 by countably many sets of the type $\{h = h_j\}$ with h_j Lipschitz. It remains to consider the case in which $E = N$ with $|N| = 0$. By the co-area formula [8] applied to $\pi_i \circ u$,

$$\int_{\mathbb{H}_i} \mathcal{H}_1(\{x \in N : \pi_i(h(x)) = y\})\,\mathrm{d}\mathcal{H}_{n-1}(y) = 0.$$

Since this holds for all $i = 1, \dots, n$, we conclude that

$$\mathcal{H}_1(\{x \in N : h(x) = z\}) = 0$$

for $\mathcal{H}_{n-1}$-a.e. $z \in \partial B(0, 1)$. This concludes the proof. □

The following lemma will give us the $W^{1,1}_{\text{loc}}$-regularity of f^{-1}.

LEMMA 3.2. *Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ be a homeomorphism of finite distortion, and suppose that $|Df| \in L^{n-1,1}(\Omega)$. There then exists $g \in L^1(f(\Omega))$ such that for each ball $B = B(y_0, r_0) \subset f(\Omega)$ we have*

$$\int_B |f^{-1}(y) - c|\,\mathrm{d}y \leqslant Cr_0 \int_B g(y)\,\mathrm{d}y, \tag{3.1}$$

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where

$$c = \fint_B f^{-1}(y)\,\mathrm{d}y$$

and $C = C(n)$.

Proof. We fix $y' = f(x') \in B$. Denote

$$h(x) = \frac{f(x) - y'}{|f(x) - y'|}.$$

If $y'' = f(x'') \in B$ and $\mathrm{co}(\{y'', y'\})$ is the line segment connecting y' and y'', then $f^{-1}(\mathrm{co}(\{y'', y'\}))$ is a curve connecting x' and x'' and thus

$$|x'' - x'| \leqslant \mathcal{H}_1(f^{-1}(\mathrm{co}(\{y'', y'\}))). \tag{3.2}$$

We have

$$y \in \mathrm{co}\{y'', y'\} \implies \frac{y - y'}{|y - y'|} = \frac{y'' - y'}{|y'' - y'|}. \tag{3.3}$$

Hence, if $r = |y'' - y'|$, then

$$\begin{aligned} |f^{-1}(y'') - f^{-1}(y')| &\leqslant \mathcal{H}_1(f^{-1}(\mathrm{co}(\{y'', y'\}))) \\ &\leqslant \mathcal{H}_1\biggl(\biggl\{x \in f^{-1}(B) : h(x) = \frac{y'' - y'}{r}\biggr\}\biggr). \end{aligned}$$

Given $r > 0$, using lemma 3.1 we estimate

$$\begin{aligned} \int_{B\cap\partial B(y',r)} & |f^{-1}(y'') - f^{-1}(y')| d\mathcal{H}_{n-1}(y'') \\ &\leqslant \int_{B\cap\partial B(y',r)} \mathcal{H}_1\biggl(\biggl\{x \in f^{-1}(B) : h(x) = \frac{y'' - y'}{r}\biggr\}\biggr)\,\mathrm{d}\mathcal{H}_{n-1}(y'') \\ &\leqslant r^{n-1} \int_{\partial B(0,1)} \mathcal{H}_1(\{x \in f^{-1}(B) : h(x) = z\})\,\mathrm{d}\mathcal{H}_{n-1}(z) \\ &\leqslant r^{n-1} \int_{f^{-1}(B)} |\mathrm{adj}\, Dh(x)|\,\mathrm{d}x \\ &\leqslant Cr^{n-1} \int_{f^{-1}(B)} \frac{|\mathrm{adj}\, Df(x)|}{|f(x) - f(x')|^{n-1}}\,\mathrm{d}x, \end{aligned} \tag{3.4}$$

where the last inequality follows using the chain rule, the formula $|\mathrm{adj}(AB)| \leqslant C|\mathrm{adj}\, A|\,|\mathrm{adj}\, B|$ and the estimate

$$\biggl|\mathrm{adj}\, D\frac{z - y'}{|z - y'|}\biggr| \leqslant \frac{C}{|z - y'|^{n-1}}.$$

There is a set $\Omega' \subset \Omega$ of full measure such that the area formula (2.1) holds for f on Ω'. We define a function $g : f(\Omega) \to \mathbb{R}$ by setting

$$g(f(x)) = \begin{cases} \dfrac{|\mathrm{adj}\, Df(x)|}{J_f(x)} & \text{if } x \in \Omega' \text{ and } J_f(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since f is a mapping of finite distortion, we have

$$|\operatorname{adj} Df(x)| = g(f(x))J_f(x) \text{ a.e. in } \Omega. \tag{3.5}$$

Hence,

$$\begin{aligned}\int_{f^{-1}(B)} \frac{|\operatorname{adj} Df(x)|}{|f(x)-f(x')|^{n-1}}\,\mathrm{d}x &= \int_{f^{-1}(B)\cap\Omega'} \frac{g(f(x))J_f(x)}{|f(x)-f(x')|^{n-1}}\,\mathrm{d}x \\ &= \int_B \frac{g(y)}{|y-y'|^{n-1}}\,\mathrm{d}y.\end{aligned} \tag{3.6}$$

Using (3.4) and (3.6) we estimate

$$\begin{aligned}|B||f^{-1}(y')-c| &\leqslant \int_B |f^{-1}(y'')-f^{-1}(y')|\,\mathrm{d}y'' \\ &= \int_0^{2r_0}\biggl(\int_{B\cap\partial B(y',r)} |f^{-1}(y'')-f^{-1}(y')|\,\mathrm{d}\mathcal{H}_{n-1}(y'')\biggr)\,\mathrm{d}r \\ &\leqslant C\int_0^{2r_0} r^{n-1}\biggl(\int_B \frac{g(y)}{|y-y'|^{n-1}}\,\mathrm{d}y\biggr)\,\mathrm{d}r \\ &\leqslant Cr_0^n \int_B \frac{g(y)}{|y-y'|^{n-1}}\,\mathrm{d}y.\end{aligned}$$

Integrating with respect to y' and then using Fubini's theorem on the right-hand side (as in the standard proof of the $(1,1)$-Poincaré inequality), we obtain (3.1). It remains to show that $g \in L^1(f(\Omega))$. But, by the area formula for f on Ω' and (3.5) we have

$$\int_{f(\Omega)} g(y)\,\mathrm{d}y = \int_{\Omega'} g(f(x))J_f(x)\,\mathrm{d}x = \int_\Omega |\operatorname{adj} Df(x)|\,\mathrm{d}x < \infty.$$

□

Proof of theorem 1.2. By lemma 3.2 the pair f, g satisfies a $(1,1)$-Poincaré inequality in $f(\Omega)$. From [2, theorem 9] we then deduce that $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$.

Suppose that f^{-1} is not a mapping of finite distortion. Then we can find a set $\tilde{A} \subset f(\Omega)$ such that $|\tilde{A}| > 0$ and for every $y \in \tilde{A}$ we have $J_{f^{-1}}(y) = 0$ and $|Df^{-1}(y)| > 0$. Since f^{-1} satisfies the ACL property we may assume without loss of generality that f^{-1} is absolutely continuous on all lines parallel to coordinate axes that intersect $\tilde{A}$ and that f^{-1} has classical partial derivatives at every point of $\tilde{A}$.

We claim that we can find a Borel set $A \subset \tilde{A}$ such that $|A| > 0$ and $|f^{-1}(A)| = 0$. We divide $\tilde{E} := f^{-1}(\tilde{A})$ into three sets: E_1 consists of the points at which f is differentiable with $J_f \neq 0$, E_2 consists of the points at which f is differentiable with $J_f = 0$ and E_3 consists of the points of non-differentiability for f. From [10] we know that $|E_3| = 0$ and by the Sard theorem we have $|f(E_2)| = 0$. Suppose that $x \in E_1$. Then f^{-1} is differentiable at $f(x)$ and $J_{f^{-1}}(f(x))J_f(x) = 1$. However, this is not possible since $J_{f^{-1}} = 0$ on $\tilde{A}$. It follows that $E_1 = \emptyset$. Now we can find a Borel set $A \subset f(E_3)$ such that $|A| = |f(E_3)|$. From $E_1 = \emptyset$ and $|f(E_2)| = 0$ we obtain $|A| = |\tilde{A}| > 0$.

Clearly, there exists $i \in \{1, \dots, n\}$ such that the subset of A in which

$$\frac{\partial f^{-1}(y)}{\partial y_i} \neq 0$$

has positive measure. Without loss of generality we will assume that

$$\frac{\partial f^{-1}(y)}{\partial y_i} \neq 0 \quad \text{for every } y \in A.$$

Set $E := f^{-1}(A)$. Then $|E| = 0$ as $E \subset E_3$. Since $|Df| \in L^{n-1,1}(\Omega)$, by the co-area formula from [8] we have

$$\int_{\mathbb{H}_i} \mathcal{H}_1(\{x \in E : \pi_i f(x) = z\})\,\mathrm{d}z = 0, \tag{3.7}$$

whereas, by the Fubini theorem,

$$\int_{\mathbb{H}_i} \mathcal{H}_1(A \cap \pi_i^{-1}(z))\,\mathrm{d}z = |A| > 0.$$

Therefore, there exists $z \in \mathbb{H}_i$ with

$$\mathcal{H}_1(E \cap f^{-1}(\pi_i^{-1}(z))) = \mathcal{H}_1(\{x \in E : \pi_i f(x) = z\}) = 0$$

and

$$\mathcal{H}_1(A \cap \pi_i^{-1}(z)) > 0.$$

Applying the one-dimensional area formula to the absolutely continuous mapping

$$t \mapsto f^{-1}(z + t\boldsymbol{e}_i)$$

we obtain

$$\begin{aligned} 0 &< \int_{A\cap\pi_i^{-1}(z)} \left|\frac{\partial f^{-1}}{\partial y_i}(y)\right| \mathrm{d}\mathcal{H}_1(y) \\ &= \int_{\mathbb{R}^n} N(f^{-1}, A \cap \pi_i^{-1}(z), x)\,\mathrm{d}\mathcal{H}_1(x) \\ &= \int_{E\cap f^{-1}(\pi_i^{-1}(z))} N(f^{-1}, A \cap \pi_i^{-1}(z), x)\,\mathrm{d}\mathcal{H}_1(x) \\ &= 0, \end{aligned}$$

which is a contradiction.

We have proven that $f \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ has finite distortion and we are left to verify that

$$\int_{f(\Omega)} |Df^{-1}(y)|\,\mathrm{d}y = \int_{\Omega} |\mathrm{adj}\, Df(x)|\,\mathrm{d}x.$$

To this end, we claim that there is a Borel set $A \subset f(\Omega)$ so that f^{-1} is differentiable on A with $J_{f^{-1}} > 0$ and so that $|Df^{-1}(y)| = 0$ a.e. on $f(\Omega)\setminus A$. Since f^{-1} is a mapping of finite distortion, we have $J_{f^{-1}}(y) = 0 \Rightarrow |Df^{-1}(y)| = 0$ and therefore we can restrict our attention to the set where $J_{f^{-1}} > 0$. We divide $\tilde{A} := \{y : J_{f^{-1}}(y) > 0\}$

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into three sets: A_1 consists of the points y such that f is differentiable at $f^{-1}(y)$ and $J_f(f^{-1}(y)) > 0$, A_2 consists of the points y such that f is differentiable at $f^{-1}(y)$ and $J_f(f^{-1}(y)) = 0$ and A_3 consists of the points such that f is not differentiable at $f^{-1}(y)$. Since A_2 is an image of a set of critical points of f, by the Sard theorem we have $|A_2| = 0$. From (2.1) and the almost everywhere differentiability of f (see [10]) we have

$$\int_{A_3} J_{f^{-1}}(y)\,\mathrm{d}y = |f^{-1}(A_3')| \leqslant |f^{-1}(A_3)| = 0,$$

where A_3' is a subset of full measure in A_3 for which the area formula holds, see the explanation around (2.1). Since $J_{f^{-1}} > 0$ on A_3, this implies that $|A_3| = 0$. We may thus choose a desired Borel set A from within A_1. Since f is differentiable at $f^{-1}(A)$ with $J_f > 0$ we find that f^{-1} is differentiable at A. Notice also that, by the construction of A, $J_f(x) = 0$ a.e. in $\Omega \setminus f^{-1}(A)$. Because f has finite distortion also $Df(x) = 0$ and $\operatorname{adj} Df(x) = 0$ a.e. in $\Omega \setminus f^{-1}(A)$.

Applying (2.1), the fact that f^{-1} satisfies the Luzin condition (N) on A, the formula for the derivative of the inverse mapping and basic linear algebra we deduce that

$$\begin{aligned}\int_{f(\Omega)} |Df^{-1}(y)|\,\mathrm{d}y &= \int_A |Df^{-1}(y)|\,\mathrm{d}y \\ &= \int_{f^{-1}(A)} |Df^{-1}(f(x))| J_f(x)\,\mathrm{d}x \\ &= \int_{f^{-1}(A)} |(Df(x))^{-1}| J_f(x)\,\mathrm{d}x \\ &= \int_{f^{-1}(A)} |\operatorname{adj} Df(x)|\,\mathrm{d}x \\ &= \int_{\Omega} |\operatorname{adj} Df(x)|\,\mathrm{d}x.\end{aligned}$$

□

4. Higher regularity of the inverse mapping

We prove two results on the improved integrability of Df^{-1}. For the sharpness of our conclusions see § 6.

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $f : \Omega \to \mathbb{R}^n$ is a homeomorphism of finite distortion such that $|Df| \in L^{n-1,1}(\Omega)$ and $K \in L^{n-1}(\Omega)$. Then $f^{-1} \in W^{1,n}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ and f^{-1} is a mapping of finite distortion.*

Proof. From theorem 1.2 we already know that $f^{-1} \in W^{1,1}_{\mathrm{loc}}$ and that f^{-1} is a mapping of finite distortion. Therefore, it is enough to prove that $\int_{f(\Omega)} |Df^{-1}|^n$ is finite.

By the proof of theorem 1.2, we only need to consider the integral over the set A where f^{-1} is differentiable with $J_{f^{-1}} > 0$. Arguing as at the end of the proof of

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theorem 1.2 we conclude that

$$\begin{aligned}\int_{f(\Omega)} |Df^{-1}(y)|^n \,\mathrm{d}y &= \int_{f^{-1}(A)} |(Df(x))^{-1}|^n J_f(x)\,\mathrm{d}x \\ &= \int_{f^{-1}(A)} \frac{|\mathrm{adj}\, Df(x)|^n}{J_f(x)^{n-1}}\,\mathrm{d}x \\ &\leqslant \int_{f^{-1}(A)} \frac{|Df(x)|^{(n-1)n}}{J_f(x)^{n-1}}\,\mathrm{d}x \\ &\leqslant \int_{\Omega} K(x)^{n-1}\,\mathrm{d}x.\end{aligned}$$

□

THEOREM 4.2. *Let $p \in (n-1, \infty]$, $1 < q < n$, and set*

$$a = \frac{(q-1)p}{p+q-n}$$

(for $p = \infty$ we set $a = (q-1)$). Let $\Omega \subset \mathbb{R}^n$ be an open set and suppose that $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is a homeomorphism of finite distortion such that $K^a \in L^1(\Omega)$. Then $f^{-1} \in W^{1,q}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$.

Proof. We reason as in the proof of theorem 4.1 and use Hölder's inequality to conclude that

$$\begin{aligned}\int_{f(\Omega)} |Df^{-1}(y)|^q \,\mathrm{d}y &\leqslant \int_{f^{-1}(A)} \frac{|\mathrm{adj}\, Df(x)|^q}{J_f(x)^{q-1}}\,\mathrm{d}x \\ &\leqslant \int_{f^{-1}(A)} |Df(x)|^{n-q} K(x)^{q-1}\,\mathrm{d}x \\ &\leqslant \|Df\|^{n-q}_{L^p(\Omega)} \biggl(\int_{\Omega} K(x)^a\,\mathrm{d}x\biggr)^{(p+q-n)/p}.\end{aligned}$$

□

5. General construction

5.1. Canonical transformation

Let $Q_0 = [-1, 1]^n$ be the unit cube in $\mathbb{R}^n$. If $c, r \in \mathbb{R}^n$, $r_1, \dots, r_n > 0$, we use the notation

$$Q(c, r) := [c_1 - r_1, c_1 + r_1] \times \cdots \times [c_n - r_n, c_n + r_n].$$

for the interval with centre at c and half-edges r_i, $i = 1, \dots, n$. If $Q = Q(c, r)$, the affine mapping

$$\varphi_Q(y) = (c_1 + r_1 y_1, \dots, c_n + r_n y_n)$$

is called the *canonical parametrization* of the interval Q. Let P and P' be concentric intervals, $P = Q(c, r)$, $P' = Q(c, r')$, where

$$0 < r_i < r'_i, \quad i = 1, \dots, n.$$

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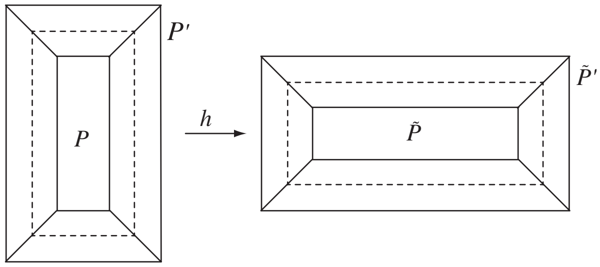

Figure 1. The canonical transformation of $P' \setminus P^\circ$ onto $\tilde{P}' \setminus \tilde{P}^\circ$ for $n = 2$.

We set

$$\varphi_{P,P'}(t, y) = (1 - t)\varphi_P(y) + t\varphi_{P'}(y), \quad t \in [0, 1], \ y \in \partial Q_0.$$

This mapping is called the *canonical parametrization* of the rectangular annulus $P' \setminus P^\circ$. It has the following properties:

(i) $\varphi_{P,P'}(0, y) = \varphi_P(y)$, $y \in \partial Q_0$,

(ii) $\varphi_{P,P'}(1, y) = \varphi_{P'}(y)$, $y \in \partial Q_0$,

(iii) $\varphi_{P,P'}(\,\cdot, y)$ is affine on $[0, 1]$, $y \in \partial Q_0$,

(iv) $\varphi_{P,P'}$ is a bi-Lipschitz homeomorphism of $[0, 1] \times \partial Q_0$ onto $P' \setminus P^\circ$.

Now, we consider two rectangular annuli, $P' \setminus P^\circ$, and $\tilde{P}' \setminus \tilde{P}^\circ$, where $P = Q(c, r)$, $P' = Q(c, r')$, $\tilde{P} = Q(\tilde{c}, \tilde{r})$ and $\tilde{P}' = Q(\tilde{c}, \tilde{r}')$, The mapping

$$h = \varphi_{\tilde{P},\tilde{P}'} \circ (\varphi_{P,P'})^{-1}$$

is called the *canonical transformation* of $P' \setminus P^\circ$ onto $\tilde{P}' \setminus \tilde{P}^\circ$ shown in figure 1.

From now on we consider the case when

$$r_1 = \cdots = r_{n-1} = a, \quad r_n = b,$$
$$r'_1 = \cdots = r'_{n-1} = a', \quad r'_n = b.$$

We will use the notation e.g. $Q(c, (a, b))$ for $Q(c, r)$ with r as above. Let us estimate the action of $\varphi_{P,P'}$ in one of the sides $\{y_i = \pm 1\}$. For $t \in [0, 1]$ fixed we denote

$$a'' = (1 - t)a + ta',$$
$$b'' = (1 - t)b + tb',$$
$$\tilde{a}'' = (1 - t)\tilde{a} + t\tilde{a}',$$
$$\tilde{b}'' = (1 - t)\tilde{b} + t\tilde{b}'.$$

The image of the side has the shape of a pyramidal frustum. We must distinguish two cases, according to the position of the first variable.

CASE A. We will represent the possibilities

$$\begin{matrix} \varphi_{P,P'}(t,1,z_2,\dots,z_n), & \varphi_{P,P'}(t,-1,z_2,\dots,z_n), \\ \vdots & \vdots \\ \varphi_{P,P'}(t,z_1,\dots z_{n-2},1,z_n), & \varphi_{P,P'}(t,z_1,\dots z_{n-2},-1,z_n) \end{matrix}$$

by

$$\varphi(t,z) = \varphi_{P,P'}(t,1,z), \quad z = (z_2,\dots,z_n).$$

The matrix of $D\varphi(t,z)$ is then

$$\begin{pmatrix} a'-a, & 0, & 0, & \dots, & 0 \\ (a'-a)z_2, & a'', & 0, & \dots, & 0 \\ (a'-a)z_3, & 0, & a'', & \dots, & 0 \\ \vdots & & & & \\ (b'-b)z_n, & 0, & 0, & \dots, & b'' \end{pmatrix}$$

and $(D\varphi(t,z))^{-1}$ can be computed as

$$\begin{pmatrix} \dfrac{1}{a'-a}, & 0, & 0, & \dots, & 0 \\ -\dfrac{z_2}{a''}, & \dfrac{1}{a''}, & 0, & \dots, & 0 \\ -\dfrac{z_3}{a''}, & 0, & \dfrac{1}{a''}, & \dots, & 0 \\ \vdots & & & & \\ -\dfrac{z_n}{b''}\dfrac{b'-b}{a'-a}, & 0, & 0, & \dots, & \dfrac{1}{b''} \end{pmatrix}.$$

Also,

$$J_\varphi(t,z) = (a'-a)(a'')^{n-2}b''. \tag{5.1}$$

CASE B. A representative is

$$\varphi(t,z) = ((\varphi_{P,P'})_n(t,z,1), (\varphi_{P,P'})_1(t,z,1), \dots, (\varphi_{P,P'})_{n-1}(t,z,1)),$$
$$z = (z_1,\dots,z_{n-1}).$$

The purpose of the permutation of coordinates is that this leads to a triangular matrix which is easier to handle. The matrix of $D\varphi(t,z)$ is

$$\begin{pmatrix} b'-b, & 0, & 0, & \dots, & 0 \\ (a'-a)z_1, & a'', & 0, & \dots, & 0 \\ (a'-a)z_2, & 0, & a'', & \dots, & 0 \\ \vdots & & & & \\ (a'-a)z_{n-1}, & 0, & 0, & \dots, & a'' \end{pmatrix}$$

and $(D\varphi(t,z))^{-1}$ can be computed as

$$\begin{pmatrix} \dfrac{1}{b'-b}, & 0, & 0, & \dots, & 0 \\ -\dfrac{z_1}{a''}\dfrac{a'-a}{b'-b}, & \dfrac{1}{a''}, & 0, & \dots, & 0 \\ -\dfrac{z_2}{a''}\dfrac{a'-a}{b'-b}, & 0, & \dfrac{1}{a''}, & \dots, & 0 \\ \vdots & & & & \\ -\dfrac{z_{n-1}}{a''}\dfrac{a'-a}{b'-b}, & 0, & 0, & \dots, & \dfrac{1}{a''} \end{pmatrix}.$$

Also, $J_\varphi(t,z) = (b'-b)(a'')^{n-1}$. Let $h = \varphi_{\tilde{P},\tilde{P}'} \circ (\varphi_{P,P'})^{-1}$ be a canonical transformation of $P' \setminus P^\circ$ onto $\tilde{P}' \setminus \tilde{P}^\circ$. We have

$$Dh(\varphi(t,z)) = D\tilde{\varphi}(t,z)(D\varphi(t,z))^{-1}.$$

In case A this is

$$\begin{pmatrix} \dfrac{\tilde{a}'-\tilde{a}}{a'-a}, & 0, & 0, & \dots, & 0 \\ \left(\dfrac{\tilde{a}'-\tilde{a}}{a'-a} - \dfrac{\tilde{a}''}{a''}\right) z_2, & \dfrac{\tilde{a}''}{a''}, & 0, & \dots, & 0 \\ \left(\dfrac{\tilde{a}'-\tilde{a}}{a'-a} - \dfrac{\tilde{a}''}{a''}\right) z_3, & 0, & \dfrac{\tilde{a}''}{a''}, & \dots, & 0 \\ \vdots & & & & \\ \left(\dfrac{\tilde{b}'-\tilde{b}}{a'-a} - \dfrac{\tilde{b}''}{b''}\dfrac{b'-b}{a'-a}\right) z_n, & 0, & 0, & \dots, & \dfrac{\tilde{b}''}{b''} \end{pmatrix} \tag{5.2}$$

and in case B

$$\begin{pmatrix} \dfrac{\tilde{b}'-\tilde{b}}{b'-b}, & 0, & 0, & \dots, & 0 \\ \left(\dfrac{\tilde{a}'-\tilde{a}}{b'-b} - \dfrac{\tilde{a}''}{a''}\dfrac{a'-a}{b'-b}\right) z_1, & \dfrac{\tilde{a}''}{a''}, & 0, & \dots, & 0 \\ \left(\dfrac{\tilde{a}'-\tilde{a}}{b'-b} - \dfrac{\tilde{a}''}{a''}\dfrac{a'-a}{b'-b}\right) z_2, & 0, & \dfrac{\tilde{a}''}{a''}, & \dots, & 0 \\ \vdots & & & & \\ \left(\dfrac{\tilde{a}'-\tilde{a}}{b'-b} - \dfrac{\tilde{a}''}{a''}\dfrac{a'-a}{b'-b}\right) z_{n-1}, & 0, & 0, & \dots, & \dfrac{\tilde{a}''}{a''} \end{pmatrix}. \tag{5.3}$$

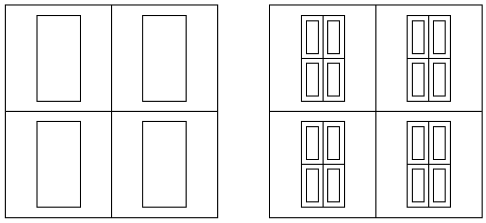

Figure 2. Intervals $Q_{\boldsymbol{v}}$ and $Q'_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathbb{V}^1$ and $\boldsymbol{v} \in \mathbb{V}^2$ for $n = 2$.

REMARK 5.1. We observe that the inverse of a canonical transformation between two rectangular annuli is again canonical and a superposition of two canonical transformations is again a canonical transformation if the domain of the outer transformation coincides with the range of the inner transformation.

5.2. Construction of a Cantor set

By $\mathbb{V}$ we denote the set of 2^n vertices of the cube $[-1, 1]^n$. The sets $\mathbb{V}^k = \mathbb{V} \times \cdots \times \mathbb{V}$, $k \in \mathbb{N}$, will serve as the sets of indices for our construction. If $\boldsymbol{w} \in \mathbb{V}^k$ and $v \in \mathbb{V}$, then the concatenation of $\boldsymbol{w}$ and v is denoted by $\boldsymbol{w} \wedge v$.

LEMMA 5.2. *Let $n \geqslant 2$. Suppose that we are given two sequences of positive real numbers $\{a_k\}_{k \in \mathbb{N}_0}$, $\{b_k\}_{k \in \mathbb{N}_0}$,*

$$a_0 = b_0 = 1, \tag{5.4}$$

$$a_k < a_{k-1}, b_k < b_{k-1}, \quad \text{for } k \in \mathbb{N}. \tag{5.5}$$

There then exist unique systems $\{Q_{\boldsymbol{v}}\}_{\boldsymbol{v} \in \bigcup_{k \in \mathbb{N}} \mathbb{V}^k}$, $\{Q'_{\boldsymbol{v}}\}_{\boldsymbol{v} \in \bigcup_{k \in \mathbb{N}} \mathbb{V}^k}$ of intervals

$$Q_{\boldsymbol{v}} = Q(c_{\boldsymbol{v}}, (2^{-k} a_k, 2^{-k} b_k)), \qquad Q'_{\boldsymbol{v}} = Q(c_{\boldsymbol{v}}, (2^{-k} a_{k-1}, 2^{-k} b_{k-1})) \tag{5.6}$$

such that

$$Q'_{\boldsymbol{v}}, \boldsymbol{v} \in \mathbb{V}^k \ \textit{are non-overlapping for fixed } k \in \mathbb{N}, \tag{5.7}$$

$$Q_{\boldsymbol{w}} = \bigcup_{v \in \mathbb{V}} Q'_{\boldsymbol{w} \wedge v} \quad \textit{for each } \boldsymbol{w} \in \mathbb{V}^k, \ k \in \mathbb{N}, \tag{5.8}$$

$$c_v = \tfrac{1}{2} v, \quad v \in \mathbb{V}, \tag{5.9}$$

$$c_{\boldsymbol{w} \wedge v} = c_{\boldsymbol{w}} + \sum_{i=1}^{n-1} 2^{-k} a_k v_i \boldsymbol{e}_i + 2^{-k} b_k v_n \boldsymbol{e}_n, \tag{5.10}$$

$$\boldsymbol{w} \in \mathbb{V}^k, \quad k \in \mathbb{N}, \quad v = (v_1, \dots, v_n) \in \mathbb{V}.$$

Proof. The centres and edge lengths of the intervals are given by (5.6), (5.9) and (5.10) (see figure 2). The properties (5.7) and (5.8) are evidently satisfied. □

REMARK 5.3. The construction leads to the Cantor set

$$E = \bigcap_{k \in \mathbb{N}} \bigcup_{\boldsymbol{v} \in \mathbb{V}^k} Q_{\boldsymbol{v}},$$

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which is clearly a product of n one-dimensional Cantor sets, say $E = E_a \times E_a \times \cdots \times E_a \times E_b$.

5.3. Construction of a mapping

The following theorem will enable us to construct various examples connected with the theory of mappings of finite distortion. A similar construction was used in [6, example 7.1] for $n = 2$ for specific sequences. The usual constructions of the type in [3, 7, 11] based on 'cubical' Cantor constructions are not suitable for us.

Theorem 5.4. *Let $n \geqslant 2$. Suppose that we are given four sequences of positive real numbers $\{a_k\}_{k\in\mathbb{N}_0}$, $\{b_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{a}_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{b}_k\}_{k\in\mathbb{N}_0}$, such that*

$$a_0 = b_0 = \tilde{a}_0 = \tilde{b}_0 = 1, \tag{5.11}$$

$$a_k < a_{k-1}, \quad b_k < b_{k-1}, \quad \tilde{a}_k < \tilde{a}_{k-1}, \quad \tilde{b}_k < \tilde{b}_{k-1}, \quad \text{for } k \in \mathbb{N}. \tag{5.12}$$

Let the systems $\{Q_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbb{N}}\mathbb{V}^k}$, $\{Q'_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbb{N}}\mathbb{V}^k}$ of intervals be as in lemma 5.2, and similarly systems $\{\tilde{Q}_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbb{N}}\mathbb{V}^k}$, $\{\tilde{Q}'_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\bigcup_{k\in\mathbb{N}}\mathbb{V}^k}$ of intervals be associated with the sequences $\{\tilde{a}_k\}$ and $\{\tilde{b}_k\}$. There then exists a unique sequence $\{f_k\}$ of bi-Lipschitz homeomorphisms of Q_0 onto itself such that:

(i) *f_k maps each $Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathbb{V}^m$, $m = 1, \dots, k$, onto $\tilde{Q}'_{\boldsymbol{v}} \setminus \tilde{Q}_{\boldsymbol{v}}$ canonically;*

(ii) *f_k maps each $Q_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathbb{V}^k$, onto $\tilde{Q}_{\boldsymbol{v}}$ affinely.*

Moreover,

$$|f_k - f_{k+1}| \lesssim 2^{-k}, \qquad |f_k^{-1} - f_{k+1}^{-1}| \lesssim 2^{-k}. \tag{5.13}$$

The sequence f_k converges uniformly to a homeomorphism f of Q_0 onto Q_0.

Proof. The mapping f_k is uniquely determined by its properties. Since the change from f_k to f_{k+1} proceeds only within the intervals $Q_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathbb{V}^k$, and similarly for the inverse, and since the diameters of these intervals are at most $C2^{-k}$, the sequence f_k converges uniformly to a continuous mapping and the same holds for the sequence f_k^{-1}. Hence, the limit mapping is a homeomorphism. □

The construction above is symmetric, i.e. the inverse of the constructed mapping can be constructed by the same procedure if we replace the corresponding sequences. Also, the construction behaves well with respect to superposition. Namely, the following follows easily from remark 5.1.

Remark 5.5. Let $n \geqslant 2$ and suppose that we are given four sequences of positive real numbers $\{a_k\}_{k\in\mathbb{N}_0}$, $\{b_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{a}_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{b}_k\}_{k\in\mathbb{N}_0}$ which satisfy the assumptions of the previous theorem. Denote the mapping constructed in the proof of theorem 5.4 by f and by g the map which is constructed by the same procedure with the role of a_k, b_k and $\tilde{a}_k$, $\tilde{b}_k$ interchanged. Then $g = f^{-1}$.

Remark 5.6. Let $n \geqslant 2$ and suppose that we are given six sequences of positive real numbers $\{a_k\}_{k\in\mathbb{N}_0}$, $\{b_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{a}_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{b}_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{\tilde{a}}_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{\tilde{b}}_k\}_{k\in\mathbb{N}_0}$. Suppose that f is constructed by theorem 5.4 applied to $\{a_k\}_{k\in\mathbb{N}_0}$, $\{b_k\}_{k\in\mathbb{N}_0}$ in the domain and $\{\tilde{a}_k\}_{k\in\mathbb{N}_0}$ $\{\tilde{b}_k\}_{k\in\mathbb{N}_0}$ in the range; g is constructed by theorem 5.4 applied to

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$\{\tilde{a}_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{b}_k\}_{k\in\mathbb{N}_0}$ in the domain and $\{\tilde{\tilde{a}}_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{\tilde{b}}_k\}_{k\in\mathbb{N}_0}$ in the range, and finally h is constructed by theorem 5.4 applied to $\{a_k\}_{k\in\mathbb{N}_0}$, $\{b_k\}_{k\in\mathbb{N}_0}$ in the domain and $\{\tilde{\tilde{a}}_k\}_{k\in\mathbb{N}_0}$, $\{\tilde{\tilde{b}}_k\}_{k\in\mathbb{N}_0}$ in the range. Then $h = g\circ f$.

6. Construction of counterexamples

We begin by showing the sharpness of the regularity of f^{-1} obtained from theorem 4.2.

EXAMPLE 6.1. Let $p \in (n-1,\infty)$, $1 < q < n$, $\varepsilon > 0$ and set $a = (q-1)p/(p+q-n)$. There is a homeomorphism $f : Q_0 \to Q_0$ of finite distortion such that $f \in W^{1,p}(Q_0, Q_0)$ and $K^a \in L^1(Q_0)$, but $f^{-1} \notin W^{1,q+\varepsilon}_{\mathrm{loc}}(Q_0, Q_0)$.

Proof. Let $\alpha \geqslant \beta > 0$, $\delta \geqslant \gamma > 0$. Use theorem 5.4 for

$$a_k = \frac{1}{(k+1)^\alpha},\quad b_k = \frac{1}{(k+1)^\beta},\quad \tilde{a}_k = \frac{1}{(k+1)^\gamma}\quad\text{and}\quad \tilde{b}_k = \frac{1}{(k+1)^\delta}$$

to obtain the sequence $\{f_k\}$ and the limit mapping f. For fixed $t \in [0,1]$ we denote

$$\begin{aligned} a''_k &= (1-t)a_k + ta_{k-1},\\ b''_k &= (1-t)b_k + tb_{k-1},\\ \tilde{a}''_k &= (1-t)\tilde{a}_k + t\tilde{a}_{k-1},\\ \tilde{b}''_k &= (1-t)\tilde{b}_k + t\tilde{b}_{k-1}. \end{aligned}$$

Since

$$\frac{1}{k^\omega} - \frac{1}{(k+1)^\omega} \sim \frac{1}{k^{\omega+1}}\quad\text{for every } \omega > 0,$$

it is easy to check that

$$\begin{aligned} \frac{\tilde{a}_{k-1}-\tilde{a}_k}{a_{k-1}-a_k} &\sim \frac{\tilde{a}''_k}{a''_k} \sim k^{\alpha-\gamma},\\ \frac{\tilde{b}_{k-1}-\tilde{b}_k}{b_{k-1}-b_k} &\sim \frac{\tilde{b}''_k}{b''_k} \sim k^{\beta-\delta},\\ \frac{\tilde{b}_{k-1}-\tilde{b}_k}{a_{k-1}-a_k} &\sim \frac{\tilde{b}''_k}{a''_k} \sim k^{\alpha-\delta},\\ \frac{\tilde{a}_{k-1}-\tilde{a}_k}{b_{k-1}-b_k} &\sim \frac{\tilde{a}''_k}{b''_k} \sim k^{\beta-\gamma},\\ \frac{b_{k-1}-b_k}{a_{k-1}-a_k} &\sim \frac{b''_k}{a''_k} \sim k^{\alpha-\beta}. \end{aligned}$$

Let us fix $\boldsymbol{v} \in \mathbb{V}^k$ and write $Q = Q_{\boldsymbol{v}}$, $Q' = Q'_{\boldsymbol{v}}$. Let $\varphi_{Q,Q'}$ be the canonical parametrization of $Q'\setminus Q$ and S be one of the sides of Q_0. We will estimate Df in the pyramidal frustum $F := \varphi_{Q,Q'}([0,1]\times S)$. In case A we have (see (5.2))

$$|Df| \sim \max\{k^{\alpha-\delta}, k^{\beta-\delta}\} \leqslant k^{\alpha-\gamma},$$

and in case B (see (5.3))

$$|Df| \lesssim \max\{k^{\alpha-\gamma}, k^{\beta-\delta}, k^{\beta-\gamma}\} = k^{\alpha-\gamma}.$$

In both cases we have

$$|Df^{-1}| \gtrsim k^{\delta-\beta}, \qquad J_f \sim k^{(n-1)\alpha-(n-1)\gamma+\beta-\delta},$$

and therefore

$$K_f = \frac{|Df|^n}{J_f} \lesssim k^{\alpha-\gamma-\beta+\delta}.$$

Let φ be the canonical parametrization of F (which can be viewed as the restriction of $\varphi_{Q,Q'}$ to $[0,1] \times S$). From (5.1) we have

$$J_\varphi(t,z) \sim 2^{-kn} k^{-(n-1)\alpha-\beta-1}.$$

Thus,

$$\begin{aligned} \int_F |Df|^p \,\mathrm{d}x &= \int_{[0,1]\times S} |Df(\varphi(t,z))|^p J_\varphi(t,z)\,\mathrm{d}t\,\mathrm{d}z \\ &\lesssim 2^{-kn} \frac{(k^{\alpha-\gamma})^p}{k^{(n-1)\alpha+\beta+1}}, \end{aligned} \tag{6.1}$$

$$\int_{f(F)} |Df^{-1}|^{q+\varepsilon} \,\mathrm{d}x \gtrsim 2^{-kn} \frac{(k^{\delta-\beta})^{q+\varepsilon}}{k^{(n-1)\gamma+\delta+1}},$$

$$\int_F K^a \,\mathrm{d}x \lesssim 2^{-kn} \frac{(k^{\alpha-\gamma-\beta+\delta})^a}{k^{(n-1)\alpha+\beta+1}}.$$

We also estimate (recall that $Q = Q_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathbb{V}^k$)

$$\int_Q |Df_k|^p \,\mathrm{d}x \lesssim 2^{-kn} \frac{(k^{\alpha-\gamma})^p}{k^{(n-1)\alpha+\beta+1}}. \tag{6.2}$$

Now, we need to distinguish two cases. First suppose that $p \leqslant n$. Since $a < p$, we can choose $\eta > 0$ sufficiently small such that

$$(n-1)\eta < \varepsilon\left(\frac{1}{a} - \frac{1}{p}\right) \quad \text{and} \quad \eta < 1 + \frac{1}{a} - \frac{n}{p}. \tag{6.3}$$

Set

$$\alpha = \frac{1}{p}, \quad \beta = 1 - \frac{n-1}{p}, \quad \gamma = \eta \quad \text{and} \quad \delta = 1 + \frac{1}{a} - \frac{n}{p}.$$

It is easy to check that all these expressions are positive and, moreover, that $\alpha \geqslant \beta$ and $\gamma \leqslant \delta$, so that with the help of (6.3) it is not difficult to verify that

$$\left.\begin{aligned} (n-1)\alpha + \beta + p(\gamma - \alpha) &= p\eta > 0, \\ (n-1)\alpha + \beta + a(\beta - \alpha + \gamma - \delta) &= a\eta > 0, \\ (n-1)\gamma + \delta + (q+\varepsilon)(\beta - \delta) &= (n-1)\eta + \varepsilon\left(\frac{1}{p} - \frac{1}{a}\right) < 0. \end{aligned}\right\} \tag{6.4}$$

We consider $k < m$. From (6.2) and (6.4) we infer that

$$\begin{aligned}\int_{Q_0} |Df_k - Df_m|^p \,\mathrm{d}x &\lesssim \int_{\{f_k \neq f_m\}} (|Df_k|^p + |Df_m|^p)\,\mathrm{d}x \\ &\lesssim \sum_{\boldsymbol{v}\in\mathbb{V}^k} \int_{Q_{\boldsymbol{v}}} |Df_k|^p \,\mathrm{d}x + \sum_{j=k+1}^{m} \sum_{\boldsymbol{v}\in\mathbb{V}^j} \int_{Q'_{\boldsymbol{v}}\setminus Q_{\boldsymbol{v}}} |Df|^p \,\mathrm{d}x \\ &\quad + \sum_{\boldsymbol{v}\in\mathbb{V}^m} \int_{Q_{\boldsymbol{v}}} |Df_m|^p \,\mathrm{d}x \\ &\lesssim \sum_{j=k}^{m} \frac{(j^{\alpha-\gamma})^p}{j^{(n-1)\alpha+\beta+1}} \lesssim k^{-p\eta} \to 0.\end{aligned}$$

It follows that the sequence $\{f_k\}$ converges to f in $W^{1,p}$ and, in particular, $f \in W^{1,p}(Q_0, \mathbb{R}^n)$. From (6.4) we also find that

$$\int_{Q_0} |K|^a \lesssim \sum_{k\in\mathbb{N}} \frac{(k^{\alpha-\gamma-\beta+\delta})^a}{k^{1+(n-1)\alpha+\beta}} < \infty,$$

$$\int_{Q_0} |Df^{-1}|^{q+\varepsilon} \gtrsim \sum_{k\in\mathbb{N}} \frac{k^{(q+\varepsilon)(\delta-\beta)}}{k^{1+(n-1)\gamma+\delta}} = \infty.$$

Now let us return to the second case, i.e. $p > n$. In this case we set

$$\alpha = \frac{1}{n}, \quad \beta = \frac{1}{n}, \quad \gamma = \frac{1}{n} - \frac{1}{p} + \eta \quad \text{and} \quad \delta = \frac{1}{n} + \frac{1}{a} - \frac{1}{p},$$

where η is chosen sufficiently small to fulfil $\eta < 1/a$ and (6.3). The computations in this case are similar to the computations above and we therefore leave them to the reader. □

We deduce that the p-integrability of K, $p < n-1$, guarantees no improvement on the regularity of f^{-1} if we only assume that $|Df| \in L^{n-1,1}(\Omega)$.

Corollary 6.2. *Let $0 < \delta < 1$. There exists a homeomorphism $f : Q_0 \to Q_0$ of finite distortion such that $|Df| \in L^{n-1,1}(Q_0)$ and $K^{n-1-\delta} \in L^1(Q_0)$, but $f^{-1} \notin W^{1,1+\delta}(Q_0, Q_0)$.*

Proof. Set $q = 1 + \frac{1}{2}\delta$ and $\varepsilon = \frac{1}{2}\delta$. We can clearly find $\eta > 0$ sufficiently small such that for $p = n - 1 + \eta$ we have

$$a = \frac{(q-1)p}{p+q-n} > n - 1 - \delta;$$

therefore the statement easily follows from the previous example. □

Our final example shows the criticality of the exponent $p = n - 1$ in a strong sense.

Example 6.3. Let $n \geqslant 3$ and $\varepsilon > 0$. There exists a mapping of finite distortion $f : Q_0 \to Q_0$ such that $f \in W^{1,n-1-\varepsilon}(Q_0, Q_0)$ and $K \in L^{n-1-\varepsilon}(Q_0)$, but $f^{-1} \notin W^{1,1}_{\mathrm{loc}}(Q_0, Q_0)$.

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Proof. Choose $l \in \mathbb{N}$ big enough such that

$$\varepsilon l > 2(n-1-\varepsilon) \tag{6.5}$$

and set

$$a_k = \frac{1}{(k+1)^l}, \quad b_k = \frac{1}{2} + \frac{1}{2(k+1)}, \quad \tilde{a}_k = \frac{1}{2} + \frac{1}{2(k+1)}, \quad \tilde{b}_k = \frac{1}{k+1}.$$

We can use theorem 5.4 to obtain our mapping f. Similarly, as in example 6.1 we easily obtain

$$\begin{aligned}\int_{Q_0} |Df|^{n-1-\varepsilon} &\leqslant C \sum_{k\in\mathbb{N}} \frac{1}{k^{1+(n-1)l}} k^{(n-1-\varepsilon)l} \\ &\leqslant C \sum_{k\in\mathbb{N}} \frac{1}{k^{1+l\varepsilon}} < \infty\end{aligned}$$

and

$$\begin{aligned}\int_{Q_0} K^{n-1-\varepsilon} &\leqslant C \sum_{k\in\mathbb{N}} \frac{1}{k^{1+(n-1)l}} \left(\frac{k^{ln}}{k^{l-1}k^{l(n-2)}k^{-1}}\right)^{n-1-\varepsilon} \\ &\quad + C \sum_{k\in\mathbb{N}} \frac{1}{k^{1+(n-1)l+1}} \left(\frac{k^{ln}}{k^{l(n-1)}}\right)^{n-1-\varepsilon} \\ &\leqslant C \sum_{k\in\mathbb{N}} \frac{1}{k^{1+\varepsilon l-2(n-1-\varepsilon)}} < \infty.\end{aligned}$$

We claim that f^{-1} does not satisfy the ACL property and therefore

$$f^{-1} \notin W^{1,1}_{\mathrm{loc}}(Q_0, Q_0).$$

It is clear from the construction in theorem 5.4 that there are Cantor sets E_a, E_b, $E_{\tilde{a}}$ and $E_{\tilde{b}}$ such that f^{-1} maps the Cantor set $E_{\tilde{a}} \times E_{\tilde{a}} \times \cdots \times E_{\tilde{a}} \times E_{\tilde{b}}$ onto the Cantor set $E_a \times E_a \times \cdots \times E_a \times E_b$. Clearly, $\mathcal{H}_1(E_a) = \mathcal{H}_1(E_{\tilde{b}}) = 0$, $\mathcal{H}_1(E_b) > 0$, $\mathcal{H}_1(E_{\tilde{a}}) > 0$ and it is not difficult to check that for every $\tilde{y} \in [-1,1]^{n-1}$ such that $\tilde{y} \in E_{\tilde{a}} \times \cdots \times E_{\tilde{a}}$ there exists $y \in E_a \times \cdots \times E_a$ such that $f^{-1}(\{\tilde{y}\} \times E_{\tilde{b}}) = \{y\} \times E_b$. It follows that $f^{-1}(\tilde{y}, \cdot)$ does not satisfy the Luzin condition (N) and therefore cannot be absolutely continuous there. Since $\mathcal{H}_{n-1}(E_{\tilde{a}} \times \cdots \times E_{\tilde{a}}) > 0$, we see that f^{-1} does not satisfy the ACL property. □

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