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NONSMOOTH CRITICAL POINT THEORY AND NONLINEAR ELLIPTIC EQUATIONS AT RESONANCE

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Abstract

In this paper we complete two tasks. First we extend the nonsmooth critical point theory of Chang to the case where the energy functional satisfies only the weaker nonsmooth Cerami condition and we also relax the boundary conditions. Then we study semilinear and quasilinear equations (involving the p-Laplacian). Using a variational approach we establish the existence of one and of multiple solutions. In simple existence theorems, we allow the right hand side to be discontinuous. In that case in order to have an existence theory, we pass to a multivalued approximation of the original problem by, roughly speaking, filling in the gaps at the discontinuity points.

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1. Introduction

The purpose of this paper is twofold. First, we want to extend the nonsmooth critical point theory of Chang [7], by replacing the compactness and the boundary conditions. Second, we want to study nonlinear elliptic problems at resonance and establish the existence of solutions and of multiple solutions.

Chang [7], in order to study equations with discontinuities, developed an extension of the classical smooth critical point theory, to nonsmooth locally Lipschitz functionals. The theory of Chang was based on the subdifferential of locally Lipschitz functionals due to Clarke [8]. Using this subdifferential, Chang proposed a generalization of the well-known 'Palais-Smale condition' ((PS)-condition) and through it

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obtained various minimax principles concerning the existence and characterization of critical points for locally Lipschitz functions. As is the case with the 'smooth' theory, we can extend the theory of Chang in two directions. One is to weaken the (PS)-condition, and use a nonsmooth counterpart of the Cerami condition (C-condition; see Cerami [6]). It was shown by Bartolo-Benci-Fortunato [4], that in the smooth case, we can have a deformation theorem and through it minimax principles about critical points using only the weaker C-condition. The other possible generalization, is to relax the boundary conditions, namely allow certain inequalities in the minimax principles to be non-strict. Such generalizations are already well known in the context of the 'smooth' theory (see for example Ghoussoub [9]). In this work we present extensions of the theory of Chang in both the aforementioned directions (see Section 3).

The second task of this paper is to study nonlinear elliptic problems at resonance. In Section 4 and Section 5 we consider equations driven by the p-Laplacian operator ($p \geq 2$) and in Section 6 we deal with semilinear equations ($p = 2$). Moreover, in Section 4 and Section 6, the right hand side nonlinearity $f(z, \cdot)$ is in general discontinuous. On the other hand, in Section 5 the nonlinearity $f(z, \cdot)$ is continuous, but we prove the existence of at least two nontrivial solutions. The proof is based on an abstract multiplicity result under splitting due to Brezis-Nirenberg [5]. In our work the resonance is simple, namely we have that the potential function $F(z, x) = \int_0^x f(z, r)\,dr$ goes to $\pm\infty$ as $|x| \to +\infty$. In this respect our work is similar to that of Ahmad-Lazer-Paul [3] and Rabinowitz [21, Theorem 4.12, page 25]. Both works deal with semilinear equations and have continuous nonlinearities. Strongly resonant problems (that is, $F(z, x)$ having finite limits as $x \to \pm\infty$) were studied by Thews [23], Bartolo-Benci-Fortunato [4], Ward [25] (for semilinear problems with continuous nonlinearity) and Kourogenis-Papageorgiou [16] (for quasilinear problems with discontinuous nonlinearity). Multiplicity results for semilinear resonant problems with continuous right hand side were obtained by Solimini [22], Ahmad [2], Goncalves-Miyagaki [10, 11] and Landesman-Robinson-Rumbos [18]. For quasilinear problems involving the p-Laplacian, existence and multiplicity results were obtained by the authors in a series of papers, see Kourogenis-Papageorgiou [14–17]. Our work here complements and partially extends these works. In particular, Theorem 10 and Theorem 11 extend the existence result of Kourogenis-Papageorgiou [14], where the growth and asymptotic conditions on $f(z, \cdot)$ are more restrictive. Also, Theorem 12 compared to the results of Ahmad-Lazer-Paul [3] and Rabinowitz [24], allows a more general growth condition on the nonlinearity of f, which in the aforementioned works was assumed to be independent of $z \in Z$, continuous and bounded.

In the next section we recall some basic definitions and facts from the critical point theory (smooth and nonsmooth) and the Brezis-Nirenberg abstract multiplicity result. In Section 3 we develop the extensions of the theory of Chang and finally in Section 4, Section 5 and Section 6 we study resonant elliptic problems. In Section 4 and Section 5 the equations are quasilinear involving the p-Laplacian, while in Section 6 the problem is semilinear. Moreover, in Section 4 and Section 6 the nonlinearity is discontinuous and in Section 5 we prove a multiplicity theorem.

2. Mathematical preliminaries

The nonsmooth critical point theory of Chang [7] is based on the subdifferential theory of locally Lipschitz functions due to Clarke [8]. Let X be a Banach space and X^* its topological dual. A function $\phi : X \to \mathbb{R}$ is said to be *locally Lipschitz*, if for every $x \in X$, there exists a neighbourhood U of x and a constant $k > 0$ depending on U such that $|\phi(z) - \phi(y)| \leq k\|z - y\|$ for all $z, y \in U$. For such a function we define a *generalized directional derivative* $\phi^0(x; h)$ at $x \in X$ in the direction $h \in X$ by

$$\phi^0(x; h) = \overline{\lim_{\substack{x' \to x \\ \lambda \downarrow 0}}} \frac{\phi(x' + \lambda h) - \phi(x')}{\lambda}.$$

The function $h \to \phi^0(x; h)$ is sublinear and continuous. By the Hahn-Banach theorem we know that $\phi^0(x; \cdot)$ is the support function of a nonempty, convex and w^*-compact set

$$\partial\phi(x) = \{x^* \in X^* : (x^*, h) \leq \phi^0(x; h) \text{ for all } h \in X\}.$$

The set $\partial\phi(x)$ is called *the generalized* or *Clarke subdifferential* of ϕ at x. If $\phi, \psi : X \to \mathbb{R}$ are locally Lipschitz functions, then $\partial(\phi + \psi)(x) \subseteq \partial\phi(x) + \partial\psi(x)$, while for any $\lambda \in \mathbb{R}$ we have $\partial(\lambda\phi)(x) = \lambda\partial\phi(x)$. Moreover, if $\phi : X \to \mathbb{R}$ is also convex, then this subdifferential coincides with the subdifferential in the sense of convex analysis. If ϕ is strictly differentiable (in particular if $\phi \in C^1(X, \mathbb{R})$, then $\partial\phi(x) = \{\phi'(x)\}$. A point $x \in X$ is a *critical point* of ϕ if $0 \in \partial\phi(x)$. For details and additional results we refer to the monograph of Clarke [8].

It is well known that the smooth critical point theory, uses a compactness-type condition, known as the *Palais-Smale condition* (PS-condition for short). So if $\phi : X \to \mathbb{R}$ is a C^1 function and $c \in \mathbb{R}$, we say that ϕ satisfies the Palais-Smale condition at level c (the $(PS)_c$-condition for short), if every sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $\phi(x_n) \xrightarrow{n \to \infty} c$ and $\phi'(x_n) \xrightarrow{n \to \infty} 0$ has a strongly convergent subsequence. If this is true for every $c \in \mathbb{R}$, then we say that $\phi(\cdot)$ satisfies the PS-condition. In the nonsmooth setting with $\phi : X \to \mathbb{R}$ locally Lipschitz, this condition takes the following form: every sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $\phi(x_n) \xrightarrow{n \to \infty} c$ and $m(x_n) \xrightarrow{n \to \infty} 0$ has a strongly convergent subsequence. Here $m(x) = \inf\{\|x^*\| : x^* \in \partial\phi(x)\}$ and the infimum is actually obtained, because $\partial\phi(x)$ is w^*-compact and the norm $\|\cdot\|$ is w^*-lower semicontinuous. If $\phi \in C^1(X, \mathbb{R})$, then since $\partial\phi(x) = \{\phi'(x)\}$, we see that the above nonsmooth notion is an extension of the original smooth one.

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A weaker form of the (PS)-condition was introduced by Cerami [6], who required that every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\phi(x_n) \to c$ and $(1 + \|x_n\|)\phi'(x_n) \to 0$ as $n \to \infty$, has a strongly convergent subsequence. It was proved by Bartolo-Benci-Fortunato [4], that this weaker condition suffices to prove a deformation theorem and using that derive minimax principles. In the next section we do the same thing in the nonsmooth setting for the theory of Chang [7]. We use the nonsmooth version of Cerami's condition, which says that every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\phi(x_n) \xrightarrow{n\to\infty} c$ and $(1+\|x_n\|)m(x_n) \xrightarrow{n\to\infty} 0$, has a strongly convergent subsequence. In what follows we write $(C)_c$-condition (or simply C-condition if it holds for every level $c \in \mathbb{R}$), for the Cerami condition at level c.

As we already mentioned in the introduction, our multiplicity theorem in Section 5 is based on an abstract multiplicity result in the presence of splitting due to Brezis-Nirenberg [5]. Here we recall the exact statement of this result.

THEOREM 1. *If X is a Banach space, $X = Y \oplus V$ with* $\dim Y < \infty$, $R : X \to \mathbb{R}$ *is a C^1-functional satisfying the Palais-Smale condition ((PS)-condition) such that for some $r > 0$ the following condition hold*

(i) *$R(x) \geq 0$ for $x \in V$, $\|x\| \leq r$;*
(ii) *$R(x) \leq 0$ for $x \in Y$, $\|x\| \leq r$;*
(iii) *R is bounded below and* $\inf_X R < 0$.

Then $R(\cdot)$ has at least two non-zero critical points.

Next consider the following nonlinear eigenvalue problem. Here $Z \subseteq \mathbb{R}^N$ is a bounded domain with a C^1-boundary Γ:

$$(1) \qquad \left\{ \begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = \lambda|x(z)|^{p-2}x(z) \text{ almost everywhere on } Z \\ x_{|_\Gamma} = 0. \end{array} \right\}$$

The least real number λ for which (1) has a nontrivial solution is called the first eigenvalue of the negative p-Laplacian $-\Delta_p x = -\operatorname{div}(\|Dx\|^{p-2}Dx)$ with Dirichlet boundary conditions (that is, $(-\Delta_p, W_0^{1,p}(Z))$) and it is denoted by λ_1. This first eigenvalue λ_1 is positive, isolated and simple (that is, the associated eigenfunctions are constant multiples of each other). Moreover, we have the following variational characterization of $\lambda_1 > 0$ via the Rayleigh quotient

$$(2) \qquad \lambda_1 = \min\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z), \;\; x \neq 0\right].$$

This minimum is realized at the normalized eigenfunction u_1 (recall that λ_1 is simple). Note that if u_1 minimizes the Rayleigh quotient, then so does $|u_1|$ and so we infer that the first eigenfunction u_1 does not change sign on Z. Moreover, we can show that $u_1(z) \neq 0$ almost everywhere on Z and so we may assume that $u_1(z) > 0$

almost everywhere on Z (note that by the nonlinear elliptic regularity theory $u_1 \in C^{1,\beta}_{loc}(Z), 0 < \beta < 1$; see Tolksdorf [24]). For details on these facts we refer to Lindqvist [20]. The Ljusternik-Schnirelmann theory gives, in addition to λ_1, a whole strictly increasing sequence of positive numbers $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_k < \cdots$ for which there exist nontrivial solutions for problem (1). In other words, the spectrum $\sigma(-\Delta_p)$ of $(-\Delta_p, W_0^{1,p}(Z))$ contains at least these points $\{\lambda_k\}_{k\geq 1}$. Nothing is known about the possible existence of other points in $\sigma(-\Delta_p) \subseteq [\lambda_1, \infty) \subseteq \mathbb{R}_+$. However, if $Y = \langle u_1 \rangle = \mathbb{R}u_1$ and V is a topological complement (that is, $W_0^{1,p}(Z) = Y \oplus V$), then because λ_1 is isolated, we have

$$\lambda_V^* = \inf\left[\frac{\|Dv\|_p^p}{\|v\|_p^p} : v \in V,\ v \neq 0\right] > \lambda_1, \quad \lambda^* = \sup_V \lambda_V^*. \tag{3}$$

If $p = 2$, then $\lambda^* = \lambda_2$ is the second eigenvalue of $(-\Delta, W_0^{1,p}(Z))$.

3. Abstract nonsmooth critical point theory

In this section we extend Chang's theory to the case where the locally Lipschitz functional satisfies the nonsmooth C-condition and the boundary conditions are relaxed. Throughout this section X is a reflexive Banach space and $\phi : X \to \mathbb{R}$ a locally Lipschitz functional. For each $c \in \mathbb{R}$ we set

$$K_c = \{x \in X : 0 \in \partial\phi(x), \phi(x) = c\} \quad \text{and} \quad \phi^c = \{x \in X : \phi(x) \leq c\}.$$

Recalling that $Gr\partial\phi = \{(x, x^*) \in X \times X^* : x^* \in \partial\phi(x)\}$ is sequentially closed in $X \times X_w^*$ (here X_w^* denotes the space X^* furnished with the weak topology), we see at once that if $\phi(\cdot)$ satisfies the nonsmooth C-condition, then K_c is compact. We start with two auxiliary results which are analogous to [7, Lemma 3.2 and Lemma 3.3.]. In what follows $(K_c)_\delta = \{x \in X : d(x, K_c) < \delta\}$ for $\delta > 0$.

LEMMA 2. *If $\phi : X \to \mathbb{R}$ satisfies the nonsmooth $(C)_c$-condition, then for each $\delta > 0$ there exist $\gamma > 0$ and $0 < \varepsilon$ such that*

$$(1 + \|x\|)m(x) \geq \gamma \quad \textit{for all } x \in (K_c)_\delta^c \textit{ and } c - \varepsilon \leq \phi(x) \leq c + \varepsilon.$$

PROOF. Suppose the result is not true. Then for $\gamma_n, \varepsilon_n \downarrow 0$, we can find $x_n \in (K_c)_\delta^c, \phi(x_n) \to c$ such that $(1 + \|x_n\|)m(x_n) \to 0$. By virtue of the nonsmooth $(C)_c$-condition, we may assume that $x_n \to x$ in X. Therefore, we have $\phi(x) = c$. Moreover, from Chang [7, page 105], we know that if $m(x) \leq \underline{\lim}\, m(x_n) = 0$, then $m(x) = 0$ and so $x \in K_c$, a contradiction (recall that for any $n \geq 1, x_n \in (K_c)_\delta^c$). This proves the lemma. □

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The second lemma gives us a locally Lipschitz vector field which plays the role of a pseudogradient vector field of the smooth case.

LEMMA 3. *If* $\phi : X \to \mathbb{R}$ *satisfies the nonsmooth* $(C)_c$*-condition,* $\delta > 0$ *is given and* $\gamma, \varepsilon > 0$ *are as in Lemma 2, then there exists a locally Lipschitz vector field* $v : \{x \in X : |\phi(x) - c| \leq \varepsilon\} \cap (K_c)^c_\delta \to X$ *such that*

$$\|v(x)\| \leq (1 + \|x\|) \quad \textit{and} \quad (x^*, v(x)) \geq \gamma/2 \quad \textit{for all } x^* \in \partial\phi(x).$$

PROOF. We follow the proof of Lemma 3.3 of Chang [7] with the necessary modifications.

Let $x \in X$ and let $x^* \in \partial\phi(x)$ such that $m(x) = \|x^*\|$. We have $B(0, \|x^*\|) \cap \partial\phi(x) = \emptyset$ (where $B(0, \|x^*\|) = \{z^* \in X^* : \|z^*\| < \|x^*\|\}$). So by the separation theorem, we find $u \in X$ with $\|u\| = 1$ such that $(z^*, u) \leq (x^*, u) \leq (y^*, u)$ for all $z^* \in B(0, \|x^*\|)$ and all $y^* \in \partial\phi(x)$. Recall that $\sup[(z^*, u) : z^* \in B(0, \|x^*\|)] = \|x^*\|$. Hence we obtain $\gamma/(2(1 + \|x\|)) < \|x^*\| \leq (y^*, u)$ for all $y^* \in \partial\phi(x)$. Exploiting the fact that the multifunction $v \to \partial\phi(v)$ is upper semicontinuous from X into X^*_w, for each $x \in \{x \in X : |\phi(x) - c| \leq \varepsilon, x \in (K_c)^c_\delta\}$ we find $\theta > 0$ such that for all $y \in B(x, \theta) = \{y \in X : \|y - x\| < \theta\}$ and all $y^* \in \partial\phi(y)$ we have $\gamma/(2(1 + \|y\|)) < (y^*, u)$. Then $\{B(x, \theta)\}$ is a cover of the set $\{x \in X : |\phi(x) - c| \leq \varepsilon, x \in (K_c)^c_\delta\}$. By paracompactness we find a locally Lipschitz finite refinement $\{U_i\}_{i\in I}$. Let $\{\xi_i\}_{i\in I}$ be a locally Lipschitz partition of unity subordinate to $\{U_i\}_{i\in I}$ and let $v(x) = (1 + \|x\|)\sum_{i\in I} \xi_i(x)u_i$. Evidently, $v : \{x \in X : |\phi(x) - c| \leq \varepsilon, x \in (K_c)^c_\delta\} \to X$ is locally Lipschitz and

$$\|v(x)\| \leq (1 + \|x\|) \quad \text{while} \quad (y^*, v(x)) = (1 + \|x\|)\sum_{i\in I} \xi_i(x)(y^*, u_i) > \gamma/2. \qquad \square$$

The next theorem (the deformation theorem) is the key tool for the nonsmooth critical point theory. It extends Theorem 3.1 of Chang [7].

THEOREM 4. *If* $\phi : X \to \mathbb{R}$ *satisfies the nonsmooth* $(C)_c$*-condition, then for every* $\varepsilon_0 > 0$*, and for every neighbourhood* U *of* K_c *(if* $K_c = \emptyset$*, we take* $U = \emptyset$*), there exist* $0 < \varepsilon < \varepsilon_0$ *and* $\eta : [0, 1] \times X \to X$ *continuous such that for all* $(t, x) \in [0, 1] \times X$ *we have*

(a) $\|\eta(t, x) - x\| \leq e(1 + \|x\|)t$;
(b) $|\phi(x) - c| \geq \varepsilon_0 \Rightarrow \eta(t, x) = x$;
(c) $\eta(\{1\} \times \phi^{c+\varepsilon}) \subseteq \phi^{c-\varepsilon} \cup U$;
(d) $\phi(\eta(t, x)) \leq \phi(x)$;
(e) $\eta(t, x) \neq x \Rightarrow \phi(\eta(t, x)) < \phi(x)$.

PROOF. By the compactness of K_c, we find $\delta > 0$ such that $(K_c)_{3\delta} \subseteq U$. Using Lemma 2, we find $\gamma > 0$ and $0 < \bar{\varepsilon} < \varepsilon_0$ such that $\gamma \leq (1 + \|x\|)m(x)$ for all $x \in (K_c)^c_\delta$ and $c - \bar{\varepsilon} \leq \phi(x) \leq c + \bar{\varepsilon}$. Consider the following two closed sets in X:

$$C_1 = \{x \in X : |\phi(x) - c| \geq \bar{\varepsilon}\} \cup \overline{(K_c)_\delta} \qquad \text{and}$$
$$C_2 = \{x \in X : |\phi(x) - c| \leq \bar{\varepsilon}/2\} \cap (K_c)^c_{2\delta}.$$

Evidently, $C_1 \cap C_2 = \emptyset$ and so we find $\xi : X \to [0, 1]$ a locally Lipschitz function such that $\xi_{|C_1} = 0$ and $\xi_{|C_2} = 1$. Using the vector field $v(x)$ obtained in Lemma 3, we define $L : X \to X$ by

$$L(x) = \begin{cases} -\xi(x)v(x) & \text{if } |\phi(x) - c| \leq \bar{\varepsilon} \text{ and } x \in (K_c)^c_\delta; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $L(\cdot)$ is locally Lipschitz. We also have for $x \in \{x \in X : |\phi(x) - c| \leq \bar{\varepsilon}$, $x \in (K_c)^c_\delta\}$:

$$\|L(x)\| = \xi(x)\|v(x)\| \leq (1 + \|x\|) \qquad \text{and} \tag{4}$$

$$(x^*, L(x)) = -\xi(x)(x^*, v(x)) \leq -\xi(x)\frac{\gamma}{2}. \tag{5}$$

For every fixed $x \in X$, we consider the following Banach space-valued Cauchy problem:

$$\left\{ \begin{array}{l} \dfrac{d}{dt}\eta(x;t) = L(\eta(x;t)) \ \text{ a.e. on } [0, 1] \\ \eta(x;0) = x. \end{array} \right\} \tag{6}$$

Since L is locally Lipschitz, problem (6) has a unique solution $\eta(x;\cdot)$. We have

$$\|\eta(x;t) - x\| \leq \int_0^t \|L(\eta(x;s))\|\, ds \leq \int_0^t (1 + \|\eta(x;s)\|)\, ds \qquad \text{(see (4))}$$

$$\leq \int_0^t \|\eta(x;s) - x\|\, ds + (1 + \|x\|)t. \tag{7}$$

By Gronwall's inequality, we have

$$\|\eta(x;t) - x\| \leq e(1 + \|x\|)t$$

and so (a) is proved. Also if $|\phi(x) - c| \geq \bar{\varepsilon}$, then $\xi(x) = 0$ and so $\eta(x;t) = x$. So we have proved (b). Next let $h(t) = \phi(\eta(x;t))$. We know that $h : [0, 1] \to X$ is locally Lipschitz, hence differentiable almost everywhere. Moreover, we have (see [7, page 106])

$$h'(t) \leq \max\left[\left(x^*, \frac{d}{dt}\eta(x;t)\right) : x^* \in \partial\phi(\eta(x;t))\right] \quad \text{a.e. on } T,$$

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$$= \max[(x^*, L(\eta(x;t))) : x^* \in \partial\phi(\eta(x;t))] \quad \text{a.e. on } T$$

$$\Rightarrow \quad h'(t) = \begin{cases} -\xi(x)\gamma/2 & \text{if } |\phi(x) - c| \le \bar{\varepsilon} \text{ and } x \in (K_c)^c_\delta; \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad h(\cdot) \text{ is nonincreasing.}$$

Therefore, we infer that for all $t \in T$ and all $x \in X$, $\phi(\eta(x;t)) \le \phi(x)$. This proves (d). Also if $|\phi(x) - c| \le \bar{\varepsilon}$ and $x \in (K_c)^c_\delta$, we have

$$\phi(x) - \phi(\eta(x;t)) = -\int_0^t h'(s)\, ds \ge \xi(x)\frac{\gamma}{2} > 0$$

$$\Rightarrow \qquad \phi(\eta(x;t)) < \phi(x) \quad \text{if } \eta(x;t) \ne x, \quad \text{which proves (e).}$$

It remains to show conclusion (c) of the theorem. Let $\rho > 0$ such that $(\overline{K_c})_{2\delta} \subseteq B(0, \rho)$. Choose $0 < \varepsilon \le \bar{\varepsilon}$ such that

$$4\varepsilon \le \gamma \qquad \text{and} \qquad 4\varepsilon(1 + \rho)e \le \delta\gamma. \tag{8}$$

We proceed by contradiction. Let $x \in \phi^{c+\varepsilon}$ and suppose that $\phi(\eta(x;t)) > c - \varepsilon$ and $\eta(x;1) \in U^c$. We have

$$c - \varepsilon < \phi(\eta(x;t)) \le c + \varepsilon \qquad \text{for all } t \in [0, 1]. \tag{9}$$

Also it cannot happen that $\eta(\{x\} \times [0, 1]) \cap (K_c)_{2\delta} = \emptyset$. Indeed, if this intersection is empty, from (5) and the properties of $\xi(\cdot)$, we have

$$\frac{\gamma}{2} \le -\int_0^1 h'(s)\, ds = \phi(x) - \phi(\eta(x;1)).$$

But $x \in \phi^{c+\varepsilon}$. So combining this with (9), we obtain

$$\phi(x) - \phi(\eta(x;1)) < 2\varepsilon, \quad \text{then } \gamma < 4\varepsilon,$$

which contradicts the choice of $\varepsilon > 0$ (see (8)). Therefore, we can find $0 \le t_1 < t_2 \le 1$ such that

$$d(\eta(x;t_1), K_c) = 2\delta, \qquad d(\eta(x;t_2), K_c) = 3\delta \qquad \text{and}$$
$$2\delta < d(\eta(x;t), K_c) < 3\delta \quad \text{for all } t_1 < t < t_2.$$

Using once again (5), we have

$$\frac{\gamma}{2}(t_2 - t_1) \le \int_{t_2}^{t_1} h'(s)\, ds = \phi(\eta(x;t_1)) - \phi(\eta(x;t_2)) < 2\varepsilon$$

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and hence

$$t_2 - t_1 < 4\varepsilon/\gamma.$$

Using the last inequality and arguing as in the proof of (a), we obtain

$$\begin{aligned}\delta \le \|\eta(x;t_2) - \eta(x;t_1)\| &\le \int_{t_1}^{t_2} \|L(\eta(x;s))\|\, ds \\ &\le e(1 + \|\eta(x;t_1)\|)(t_2 - t_1) < (1+\rho)\frac{4\varepsilon}{\gamma}e,\end{aligned}$$

which contradicts the choice of $\varepsilon > 0$ (see (8)). This proves (c) and so the proof of the theorem is complete. □

Using Theorem 4 we can derive useful minimax principles for critical points in the nonsmooth setting. We start by introducing a basic notion of critical point theory.

DEFINITION. Let $A, C \subseteq X$. We say that C *links* A, if $A \cap C = \emptyset$ and C is not contractible in $X \setminus A$.

REMARK. The following is a well-known consequence of degree theory. If X is finite dimensional and U is an open bounded neighbourhood of x, then ∂U (= the boundary of U) is not contractible in $X \setminus \{x\}$.

The next abstract minimax principle generates as byproducts the nonsmooth 'Mountain Pass Theorem', 'Saddle Point Theorem' and 'Linking Theorem', under the nonsmooth C-condition.

THEOREM 5. *If $A, C \subseteq X$ are nonempty, A is closed, C links A, Γ_C is the set of all contractions of C, $\phi : X \to \mathbb{R}$ satisfies the nonsmooth $(C)_c$-condition with*

$$c = \inf_{h\in\Gamma_C} \sup_{[0,1]\times C} \phi\circ h < \infty \qquad \text{and} \qquad \sup_C \phi \le \inf_A \phi,$$

then $c \ge \inf_A \phi$ and c is a critical value of ϕ. Moreover, if $c = \inf_A \phi$, then there exists $x \in A$ such that $x \in K_c$.

PROOF. Since, by the hypothesis, C links A, for every $h \in \Gamma_C$ we have $h([0,1] \times C) \cap A \neq \emptyset$. So we infer that $c \ge \inf_A \phi$.

First we assume that $\inf_A \phi < c$. Suppose that $K_c = \emptyset$. Let $U = \emptyset$ and let $\varepsilon > 0$ and $\eta : [0,1] \times X \to X$ be as in Theorem 4. From the definition of c, we find $h \in \Gamma_C$ such that

(10) $$\phi(h(t,x)) \le c + \varepsilon \quad \text{for all } t \in [0,1], x \in C.$$

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Let $H : [0,1] \times C \to X$ be defined by

$$H(t,x) = \begin{cases} \eta(2t,x) & \text{if } 0 \le t \le 1/2; \\ \eta(1, h(2t-1,x)) & \text{if } 1/2 \le t \le 1. \end{cases}$$

It is easy to check that $H \in \Gamma_C$ and for every $x \in C$ we have

$$\phi(H(t,x)) = \phi(\eta(2t,x)) \le \phi(x) \le \sup_C \phi < c \quad \text{for } 0 \le t \le 1/2$$

(see Theorem 4(d)) and

$$\phi(H(t,x)) = \phi(\eta(1, h(2t-1,x))) \le c - \varepsilon < c \quad \text{for } 1/2 \le t \le 1$$

(see Theorem 4(c) and recall that $h(t,x) \in \phi^{c+\varepsilon}$ for all $t \in [0,1], x \in C$; see (10)). So we have contradicted the definition of c. This proves the nonemptiness of K_c when $c > \inf_A \phi$. Next assume that $c = \inf_A \phi$. We need to show that $K_c \cap A \neq \emptyset$. Suppose the contrary and let U be a neighbourhood of K_c with $U \cap A = \emptyset$. Let $\varepsilon > 0$ and $\eta : [0,1] \times X \to X$ be as in Theorem 4. As before, let $h \in \Gamma_C$ such that $\phi(h(t,x)) \le c + \varepsilon$ for all $(t,x) \in [0,1] \times C$. Then we define $H : [0,1] \times C \to X$ by

$$H(t,x) = \begin{cases} \eta(2t,x) & \text{if } 0 \le t \le 1/2 \\ \eta(1, h(2t-1,x)) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Again we can easily verify that $H \in \Gamma_C$. From Theorem 4, we know that for all $0 \le t \le 1/2$ and all $x \in C$, we have

$$\eta(2t,x) = x \quad \text{or} \quad \phi(\eta(2t,x)) < \phi(x) \le \inf_A \phi = c, \quad \text{then}$$
$$\eta(2t,x) \in A^c \quad \text{for all } 0 \le t \le 1/2 \text{ and all } x \in C.$$

For all $1/2 \le t \le 1$ and all $x \in C$, we have from Theorem 4(c)

$$\eta(1, h(2t-1,x)) \in \phi^{c-\varepsilon} \cup U \quad \text{while} \quad (\phi^{c-\varepsilon} \cup U) \cap A = \emptyset.$$

So H is a contraction of C in $X \setminus A$, a contradiction. This proves the theorem. □

As a first consequence of this minimax theorem, we derive an extended version of the nonsmooth 'Mountain Pass Theorem' (see [7, Theorem 3.4]).

THEOREM 6. *If there exist $x_1 \in X$ and $r > 0$ such that $\|x_1\| > r$,* $\max[\phi(0), \phi(x_1)] \le \inf[\phi(x) : \|x\| = r]$ *and $\phi : X \to \mathbb{R}$ satisfies the nonsmooth $(C)_c$-condition with $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \phi(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}$, then $c \ge \inf[\phi(x) : \|x\| = r]$ and c is a critical value of ϕ. Moreover, if $c = \inf[\phi(x) : \|x\| = r]$, then there exists a critical point x of ϕ with $\phi(x) = c$ and $\|x\| = r$.*

PROOF. We apply Theorem 5 with $A = \{x \in X : \|x\| = r\}$ and $C = \{0, x_1\}$. Clearly C links A and $c < \infty$. Let $\gamma \in \Gamma$ and define

$$h(t,x) = \begin{cases} \gamma(t) & \text{if } x = 0; \\ x_1 & \text{if } x = x_1. \end{cases}$$

Then $h \in \Gamma_C$ (see Theorem 5). So,

$$\inf_{\bar{h}\in\Gamma_C} \sup_{[0,1]\times C} \phi(\bar{h}(t,x)) \le \phi(h(t,x)) \le c. \tag{11}$$

On the other hand, if $h \in \Gamma_C$, then

$$\gamma(t) = \begin{cases} h(2t, 0) & \text{if } 0 \le t \le 1/2; \\ h(2-2t, x_1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

belongs to Γ and so

$$\inf_{h\in\Gamma_C} \sup_{[0,1]\times C} \phi(h(t,x)) \ge c. \tag{12}$$

From (11) and (12), we have $c = \inf_{h\in\Gamma_C} \sup_{[0,1]\times C} \phi(h(t,x))$ and so we can apply Theorem 5 and finish the proof. $\square$

REMARK. In addition to assuming the weaker nonsmooth $(C)_c$-condition (while Chang [7] assumes that ϕ satisfies the nonsmooth PS-condition), here we have proved the nonsmooth mountain pass theorem under relaxed boundary conditions, that is, it can happen that $\max[\phi(0), \phi(x_1)] = \inf[\phi(x) : \|x\| = r]$ (in Chang [7] the left hand side is strictly smaller than the right hand side). Also the choice of 0 as the second point in C was done only for convenience. In fact we can replace 0 by any $x_2 \in X$, provided that the hypothesis $\|x_1\| > r$ is replaced by the condition $\|x_2 - x_1\| > r$.

The next important consequence of Theorem 5, is an extended version of the nonsmooth 'Saddle Point Theorem' (see [7, Theorem 3.3]).

THEOREM 7. *If* $X = Y \oplus V$, *with* $\dim Y < \infty$, *there exists* $r > 0$ *such that*

$$\max[\phi(x) : x \in Y, \|x\| = r] \le \inf[\phi(x) : x \in V]$$

and $\phi : X \to \mathbb{R}$ *satisfies the nonsmooth* $(C)_c$*-condition, where*

$$c = \inf_{\gamma\in\Gamma} \max_{x\in E} \phi(\gamma(x))$$

with $\Gamma = \{\gamma \in C(E, X) : \gamma_{|\partial E} = \text{identity}\}$, $E = \{x \in Y : \|x\| \le R\}$ *and* $\partial E = \{x \in Y : \|x\| = r\}$, *then* $c \ge \inf_V \phi$ *and* c *is a critical value of* ϕ. *Moreover, if* $c = \inf_V \phi$, *then*

$$V \cap K_c \ne \emptyset.$$

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PROOF. In this case we apply Theorem 5 with $A = V$ and $C = \partial E$. Clearly from the compactness of E (recall that by the hypothesis Y is finite dimensional), we have that $c < \infty$. Let $P : X \to Y$ be the projection operator (see Hu-Papageorgiou [12, Proposition IV.7.8, Proposition IV.7.9, pp. 502–503]). First we show that with the aforementioned choices, C links A. Suppose not and let h be a contraction of C in $X \setminus V$. Let $H(t,x) = Ph(t,x)$, which is a contraction of C in $Y \setminus \{0\}$, a contradiction (see the remark following the definition of linking). So indeed C links A.

Next let $\gamma \in \Gamma$ and define $h(t,x) = \gamma((1-t)x)$. Evidently $h \in \Gamma_C$. So we have

$$\inf_{h\in\Gamma_C} \sup_{[0,1]\times C} \phi(h(t,x)) \le \phi(h(t,x)) \le c. \tag{13}$$

Also if $h \in \Gamma_C$ and $h(1,x) = z_1$ for all $x \in C$, then we define

$$\xi(t,x) = \begin{cases} h(t,x) & \text{if } (t,x) \in [0,1] \times C; \\ z_1 & \text{if } (t,x) \in \{1\} \times E \end{cases}$$

which is continuous from $([0,1] \times C) \cup (\{1\} \times E)$ into X. Let $\theta : E \to ([0,1] \times C) \cup (\{1\} \times E)$ be a homeomorphism such that $\theta(C) = \{0\} \times C$. Then we see that $\xi \circ \theta \in \Gamma$ and so

$$c \le \inf_{h\in\Gamma_C} \sup_{[0,1]\times C} \phi(h(t,x)). \tag{14}$$

From (13) and (14) it follows that $c = \inf_{h\in\Gamma_C} \sup_{[0,1]\times C} \phi(h(t,x))$ and so we can apply Theorem 5 and finish the proof. □

REMARK. In this theorem too in addition to assuming a weaker compactness condition of Chang [7] (namely the nonsmooth $(C)_c$-condition), we also use a relaxed boundary condition, namely we do not require that $\sup[\phi(y) : y \in Y, \|y\| = r]$ be strictly smaller than $\inf_V \phi$. In our formulation equality is also possible.

The next theorem is not in Chang [7] and is a nonsmooth generalization of the well-known 'Linking Theorem' of Rabinowitz [21, Theorem 5.3, page 28] with relaxed boundary condition.

THEOREM 8. *If $X = Y \oplus V$ with* $\dim Y < \infty$, *with $0 < r < R$ and $e \in V$ with $\|e\| = 1$ such that*

$$\max[\phi(x) : x \in \partial Q] \le \inf[\phi(x) : x \in \partial B(0,r) \cap V],$$

where $Q = \{x = y + te : y \in Y, t \ge 0, \|x\| \le R\}$ and ∂Q is its boundary in $Y \oplus \mathbb{R}e$, and $\phi : X \to \mathbb{R}$ satisfies the nonsmooth $(C)_c$-condition, where $c = \inf_{\gamma\in\Gamma} \max_{x\in Q} \phi(\gamma(x))$ with $\Gamma = \{\gamma \in C(Q,X) : \gamma_{|\partial Q} = \text{identity}\}$, then $c \ge \inf[\phi(x) : x \in \partial B(0,r) \cap V]$ and c is a critical value. Moreover, if $c = \inf[\phi(x) : x \in \partial B(0,r) \cap V]$, then $K_c \cap (\partial B(0,r) \cap V) \ne \emptyset$.

PROOF. Because Q is compact, it is clear that $c < \infty$. Let $P_1 : X \to Y$ and $P_2 : X \to V$ be the projection operators on Y and V and let $A = \partial B(0, r) \cap V$ and $C = \partial Q$. If $h(t, x)$ is a contraction of C in $X \setminus A$, then $H(t, x) = P_1 h(t, x) + \|P_2 h(t, x)\| e$ is a contraction of C in $(V \oplus \mathbb{R}e) \setminus \{re\}$ which is not possible (see the remark following the definition of linking). Moreover, as in the proof of Theorem 7, we can verify that $c = \inf_{h \in \Gamma_C} \sup_{[0,1] \times C} \phi \circ h$. So we can apply Theorem 5 to finish the proof. □

We conclude this section with a result which is a direct consequence of Corollary 2.3 of Zhong [26] and extends Theorem 3.5 of Chang [7].

THEOREM 9. *If $\phi : X \to \mathbb{R}$ satisfies the nonsmooth C-condition and is bounded below, then there exists $\widehat{x} \in X$ such that $\phi(\widehat{x}) = \inf_X \phi = c$ (and so $\widehat{x} \in K_c$).*

4. Equations at resonance with discontinuities

Let $Z \in \mathbb{R}^N$ be a bounded domain with a $C^{1+\alpha}$-boundary Γ $(0 < \alpha < 1)$. We consider the following quasilinear elliptic resonant problem:

$$(15) \quad \left\{ \begin{array}{ll} -\operatorname{div}\left(\|Dx(z)\|^{p-2} Dx(z)\right) - \lambda_1 |x(z)|^{p-2} x(z) = f(z, x(z)) & \text{a.e. on } Z \\ x_{|\Gamma} = 0, \quad 2 \le p < \infty. & \end{array} \right\}$$

We do not assume that $f(z, \cdot)$ is continuous. It is well known then that (15) need not have a solution. Then we replace (15) by a multivalued equation which approximates it and is obtained by, roughly speaking, filling in the gaps at the discontinuity points of $f(z, \cdot)$. For the resulting elliptic inclusion, we can develop an existence theory based on the abstract results of Section 3. We introduce the following two functions:

$$f_1(z, x) = \varliminf_{x' \to x} f(z, x') = \lim_{\delta \downarrow 0} \operatorname{ess} \inf_{|x'-x|<\delta} f(z, x') \qquad \text{and}$$
$$f_2(z, x) = \varlimsup_{x' \to x} f(z, x') = \lim_{\delta \downarrow 0} \operatorname{ess} \sup_{|x'-x|<\delta} f(z, x').$$

Let $\widehat{f}(z, x) = \{y \in \mathbb{R} : f_1(z, x) \le y \le f_2(z, x)\}$ and consider the following multivalued approximation of (15):

$$(16) \quad \left\{ \begin{array}{ll} -\operatorname{div}\left(\|Dx(z)\|^{p-2} Dx(z)\right) - \lambda_1 |x(z)|^{p-2} x(z) \in \widehat{f}(z, x(z)) & \text{a.e. on } Z \\ x_{|\Gamma} = 0, \quad 2 \le p < \infty. & \end{array} \right\}$$

In the sequel we deal with (16). By a solution of (16), we mean a function $x \in W_0^{1,p}(Z)$ such that $\operatorname{div}(\|Dx(\cdot)\|^{p-2} Dx(\cdot)) \in L^1(Z)$ and $-\operatorname{div}(\|Dx(z)\|^{p-2} Dx(z)) - \lambda_1 |x(z)|^{p-2} x(z) = u(z)$ almost everywhere on Z, with $u \in L^1(Z)$, $f_1(z, x(z)) \le u(z) \le f_2(z, x(z))$ almost everywhere on Z.

In this section we prove an existence theorem for problem (16) under general growth conditions on the discontinuous nonlinearity $f(z,x)$, extending this way an earlier existence theorem by the authors [14]. The precise hypotheses on $f(z,x)$ are the following:

$\mathbf{H(f)_1}$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that

(i) f_1, f_2 are both N-measurable functions (that is, for every $x : Z \to \mathbb{R}$ measurable function, $z \to f_i(z, x(z))$, $i = 1, 2$, are measurable);

(ii) there exist $a_1 \in L^\infty(Z)$ and $c_1 > 0$ such that for almost all $z \in Z$ and all $x \in \mathbb{R}$

$$|f(z,x)| \le a_1(z) + c_1|x|^{p-1};$$

(iii) for some $0 < \mu < p$ we have $\underline{\lim}_{|x|\to\infty}(f(z,x)x - pF(z,x))/|x|^\mu > 0$ uniformly for almost all $z \in Z$ (recall that $F(z,x) = \int_0^x f(z,r)\,dr$, the potential function corresponding to f);

(iv) $\overline{\lim}_{x\to 0} pF(z,x)/|x|^p \le -\lambda_1$ uniformly for almost all $z \in Z$;

(v) there exists $\xi \neq 0$ such that $\int_Z F(z, \xi u_1(z))\,dz \ge 0$.

We have the following existence result.

THEOREM 10. *If hypotheses* $\mathrm{H(f)_1}$ *hold, then problem* (16) *has at least one nontrivial solution.*

PROOF. Let $\phi : W_0^{1,p}(Z) \to \mathbb{R}$ be defined as

$$\phi(x) = \frac{1}{p}\|Dx\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \int_Z F(z, x(z))\,dz.$$

We know that $\phi(\cdot)$ is locally Lipschitz (see [7, page 111]). In what follows let $\psi : W_0^{1,p}(Z) \to \mathbb{R}$ be defined by $\psi(x) = \int_Z F(z, x(z))\,dz$.

CLAIM 1. $\phi(\cdot)$ *satisfies the nonsmooth C-condition.*

Let $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(Z)$ be a sequence such that $|\phi(x_n)| \le M$ for all $n \ge 1$ and $(1 + \|x_n\|)m(x_n) \xrightarrow{n\to\infty} 0$. Let $x_n^* \in \partial\phi(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \ge 1$. We know that

$$x_n^* = A(x_n) - \lambda_1|x_n|^{p-2}x_n - u_n$$

with $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbb{R}^N}\,dz$$

for all $y \in W_0^{1,p}(Z)$ and $u_n \in \partial\psi(x_n)$, $n \geq 1$. From Chang [7] we know that $u_n \in L^q(Z)$ and $f_1(z, x_n(z)) \leq u_n(z) \leq f_2(z, x_n(z))$ almost everywhere on Z. We have

$$|\langle x_n^*, x_n\rangle| \leq \varepsilon_n \quad \text{and} \quad |\phi(x_n)| \leq M, \quad n \geq 1, \text{ with } \varepsilon_n \downarrow 0,$$

$$\text{(17)} \qquad \Rightarrow \quad -\varepsilon_n \leq -\|Dx_n\|_p^p + \lambda_1\|x_n\|_p^p + \int_Z u_n(z)x_n(z)\,dz \leq \varepsilon_n \qquad \text{and}$$

$$\text{(18)} \qquad -pM \leq \|Dx_n\|_p^p - \lambda_1\|x_n\|_p^p - \int_Z pF(z, x(z))\,dz \leq pM.$$

Adding (17) and (18), we obtain

$$\text{(19)} \qquad -\varepsilon_n - pM \leq \int_Z (u_n(z)x_n(z) - pF(z, x_n(z)))\,dz \leq \varepsilon_n + pM.$$

By the hypothesis $H(f)_1$ (iii), given $\varepsilon > 0$, we find $M_1 = M_1(\varepsilon) > 0$ such that for almost all $z \in Z$ and all $|x| \geq M_1$, we have

$$\text{(20)} \qquad f(z, x)x - pF(z, x) \geq (\beta - \varepsilon)|x|^\mu, \quad (\beta - \varepsilon > 0).$$

On the other hand, by the hypothesis $H(f)_1$ (ii), for almost all $z \in Z$ and all $|x| \leq M_1$ we have

$$\text{(21)} \qquad |f(z, x)x - pF(z, x)| \leq a_2(z) \quad \text{with } a_2 \in L^\infty(Z).$$

Therefore, from (20) and (21) we have that for almost all $z \in Z$ and all $z \in \mathbb{R}$

$$f(z, x)x - pF(z, x) \geq (\beta - \varepsilon)|x|^\mu - \widehat{a}_2(z), \quad \widehat{a}_2 \in L^\infty(Z).$$

Thus going back to inequality (19), we obtain $(\beta - \varepsilon)\|x_n\|_\mu^\mu \leq \beta_1$ for some $\beta_1 > 0$, then $\{x_n\}_{n\geq 1} \subseteq L^\mu(Z)$ is bounded.

Next, from the hypothesis $H(f)_1$ (ii), we see that for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$F(z, x) \leq a_3(z) + \eta|x|^q \quad \text{with } a_3 \in L^\infty(Z), \ \eta > 0$$

with $p < q < \min[p^*, p(N + \mu)/N, \mu + p]$. Since $\mu < q < p^*$, we find $0 < \theta < 1$ such that $1/q = (1 - \theta)/\mu + \theta/p^*$. Using the interpolation inequality, we have

$$\text{(22)} \qquad \|x_n\|_q \leq \|x_n\|_\mu^{1-\theta}\|x_n\|_{p^*}^\theta \leq \beta_2\|x_n\|_{p^*}^\theta \leq \beta_3\|x_n\|_{1,p}^\theta \quad \text{for some } \beta_2, \beta_3 > 0$$

(recall that $W_0^{1,p}(Z)$ is embedded continuously in $L^{p^*}(Z)$). Recall, from the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$, that $|\phi(x_n)| \leq M$ for all $n \geq 1$, then

$$\frac{1}{p}\|Dx_n\|_p^p - \frac{\lambda_1}{p}\|x_n\|_p^p - \int_Z F(z, x_n(z))\,dz \leq M, \quad \text{then}$$

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$$\begin{aligned}\frac{1}{p}\|Dx_n\|_p^p &\leq \frac{\lambda_1}{p}\|x_n\|_p^p + \|a_3\|_1 + \eta\|x_n\|_q^q + M \\ &\leq \frac{\lambda_1\beta_4}{p}\|x_n\|_q^p + \|a_3\|_1 + \eta\|x_n\|_q^q + M \quad \text{for some } \beta_4 > 0.\end{aligned}$$

Let $r = q/p > 1$, $r' > 1$ the conjugate exponent (that is, $1/r + 1/r' = 1$) and on $\lambda_1\beta_4/p\|x_n\|_q^p$ apply Young's inequality with $r, r' > 1$. We have

$$\frac{\lambda_1\beta_4}{p}\|x_n\|_q^p \leq \left(\frac{\lambda_1\beta_4}{p}\right)^{r'}\frac{1}{r'} + \frac{p}{q}\|x_n\|_q^q.$$

Thus we can write that

$$\begin{aligned}\frac{1}{p}\|Dx_n\| &\leq \beta_5(\lambda_1) + \beta_6\|x_n\|_q^q \quad \text{for some } \beta_5(\lambda_1), \beta_6 > 0 \\ &\leq \beta_5(\lambda_1) + \beta_7\|x_n\|^{\theta q} \quad \text{for some } \beta_7 > 0 \text{ (see (22))}.\end{aligned} \tag{23}$$

If $p < N$, then since $1/q = (1-\theta)/\mu + \theta/p^*$, then $\theta q(\mu - p^*) = \mu p^* - qp^*$, then $\theta q = (\mu p^* - qp^*)/(\mu - p^*) = p^*(q-\mu)/(p^* - \mu) < \mu < p$, because $q < p(N+\mu)/N$, then $(q-p)N/p < \mu$.

If $p \geq N$, then $p^* = +\infty$ (Sobolev embedding theorem) and so $1/q = (1-\theta)/\mu$, then $q - q\theta = \mu$, then $\theta q = q - \mu < q - (q-p) = p$, because $q < \mu + p$.

Therefore, in both cases we have $\theta q < p$. Using this fact in (23), together with Poincaré's inequality we infer that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. Thus by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ as $n \to \infty$. If we denote by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$, we have

$$\begin{aligned}\langle A(x_n), x_n - x\rangle - \lambda_1 \int_Z |x_n(z)|^{p-2}x_n(z)(x_n - x)(z)\,dz \\ - \int_Z u_n(z)(x_n - x)(z)\,dz \leq \varepsilon_n\|x_n - x\|_{1,p}\end{aligned}$$

with $\varepsilon_n \downarrow 0$. Since $W_0^{1,p}(Z)$ is embedded compactly in $L^p(Z)$ (Sobolev embedding theorem), we have that $x_n \xrightarrow{n\to\infty} x$ in $L^p(Z)$ and so

$$\int_Z |x_n(z)|^{p-2}x_n(z)(x_n - x)(z)\,dz \xrightarrow{n\to\infty} 0 \quad \text{and} \quad \int_Z u_n(z)(x_n - x)(z)\,dz \xrightarrow{n\to\infty} 0.$$

Therefore, we obtain

$$\overline{\lim}\langle A(x_n), x_n - x\rangle \leq 0.$$

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But we know (see for example Kourogenis-Papageorgiou [14]) that A is monotone, semicontinuous, hence maximal monotone and of course pseudomonotone (see Hu-Papageorgiou [12]). Thus we have

$$\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle \quad \text{then} \quad \|Dx_n\|_p \to \|Dx\|_p.$$

We already know that $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$. Since $L^p(Z, \mathbb{R}^N)$ is uniformly convex, we infer $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$ and so $x_n \to x$ in $W_0^{1,p}(Z)$ which proves the claim. □

CLAIM 2. *There exist $\beta_8, \beta_9 > 0$ such that $\phi(x) \geq \beta_8\|x\|^p - \beta_9\|x\|^\nu$ with $p < \nu \leq p^*$.*

By the hypothesis H(f)$_1$ (iv), given $\varepsilon > 0$ we find $\delta > 0$ such that for almost all $z \in Z$ and all $|x| \leq \delta$ we have

$$F(z, x) \leq \frac{1}{p}(-\lambda_1 + \varepsilon)|x|^p.$$

Also from the hypothesis H(f)$_1$ (ii), we have

$$|F(z, x)| \leq a_4(z) + \eta'|x|^p \quad \text{almost everywhere on } Z$$

with $a_4 \in L^\infty(Z), \eta' > 0$. Therefore, we find $\sigma > 0$ large enough so that for $p < \nu \leq p^*$ we have

$$F(z, x) \leq \frac{1}{p}(-\lambda_1 + \varepsilon)|x|^p + \sigma|x|^\nu \quad \text{almost everywhere on } Z, \text{ for all } x \in \mathbb{R}.$$

Therefore, for every $x \in W_0^{1,p}(Z)$ we have

$$\begin{aligned}\phi(x) &= \frac{1}{p}\|Dx\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \int_Z F(z, x(z))\,dz \\ &\geq \frac{1}{p}\|Dx\|_p^p - \frac{\varepsilon}{\lambda_1 p}\|Dx\|_p^p - \sigma\|x\|_\nu^\nu = \frac{1}{p}\left(1 - \frac{\varepsilon}{\lambda_1}\right)\|Dx\|_p^p - \sigma\|x\|_\nu^\nu.\end{aligned}$$

Choose $\varepsilon > 0$ so that $\varepsilon < \lambda_1$. So from Poincaré's inequality and since $W_0^{1,p}(Z)$ is embedded continuously in $L^\nu(Z)$ (recall that $\nu \leq p^*$), we can find $\beta_8, \beta_9 > 0$ such that

$$\phi(x) \geq \beta_8\|x\|^p - \beta_9\|x\|^\nu \quad \text{for all } x \in W_0^{1,p}(Z).$$

This proves the claim. □

Using Claim 2 we find $r > 0$ small enough such that

$$\inf[\phi(x) : \|x\| = r] > 0.$$

On the other hand, $\phi(0) = 0$ and by the hypothesis H(f)$_1$ (v) for the particular $\xi \neq 0$, we have (see (2))

$$\phi(\xi u_1) = \frac{|\xi|^p}{p\|Du_1\|_p^p} - \lambda_1 \frac{|\xi|^p}{p\|u_1\|_p^p} - \int_Z F(z, \xi u_1(z))\, dz = \int_Z F(z, \xi u_1(z))\, dz \leq 0.$$

So Claim 1 permits the use of Theorem 6 which gives us $x \in W_0^{1,p}(Z)$ such that $\phi(x) > 0$ (hence $x \neq 0$) and $0 \in \partial\phi(x)$.

From the inclusion we have that

$$A(x) - \lambda_1 |x|^{p-2}x - u = 0$$

with $u \in \partial\psi(x)$, hence $f_1(z, x(z)) \leq u(z) \leq f_1(z, x(z))$ almost everywhere on Z. For every $\theta \in C_0^\infty(Z)$ we have

$$\int_Z \|Dx(z)\|^{p-2}(Dx(z), D\theta(z))_{\mathbb{R}^N}\, dz - \lambda_1 \int_Z |x(z)|^{p-2}x(z)\theta(z)\, dz - \int_Z u(z)\theta(z)\, dz = 0, \quad \text{then}$$

$$\left\langle -\operatorname{div}\left(\|Dx\|^{p-2}Dx\right), \theta\right\rangle = \lambda_1 \int_Z |x(z)|^{p-2}x(z)\theta(z)\, dz + \int_Z u(z)\theta(z)\, dz. \tag{24}$$

From the representation theorem for the elements in $W^{-1,q}(Z)$ (see Adams [1]), we see that $\operatorname{div}(\|Dx\|^{p-2}Dx \in W^{-1,q}(Z)$. Note that $C_0^\infty(Z)$ is dense in $W_0^{1,p}(Z)$ and $W^{-1,q}(Z) = W_0^{1,p}(Z)^*$. So from (24) it follows that

$$\left\{ \begin{array}{ll} -\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) - \lambda_1|x(z)|^{p-2}x(z) \in \widehat{f}(z, x(z)) & \text{a.e. on } Z \\ x_{|\Gamma} = 0, \quad 2 \leq p < \infty. & \end{array} \right\}$$

then $x \in W_0^{1,p}(Z)$ is a nontrivial solution of (16). □

5. Multiple solutions for problems at resonance

In this section we consider quasilinear problems at resonance with the Caratheodory right hand side. So we deal with problem (15). Using Theorem 1, we prove the existence of at least two nontrivial solutions. Recall that a function $f : Z \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function if for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for almost all $z \in Z$, $x \to f(z, x)$ is continuous. Recall that a Caratheodory function is jointly measurable, hence N-measurable (see Hu-Papageorgiou [12, Proposition II.1.6, page 42]). The hypotheses on the nonlinearity $f(z, x)$ are the following:

H(f)$_2$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that

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(i) for every $M > 0$, there exists $a_M \in L^\infty(Z)$ such that for almost all $z \in Z$ and all $|x| \le M$ we have $|f(z,x)| \le a_M(z)$;

(ii) there exists $\delta > 0$ such that for almost all $z \in Z$ and all $|x| \le \delta$, we have $F(z,x) \ge 0$;

(iii) there exists $\theta \in L^\infty(Z)$ with $\theta(z) \le 0$ almost everywhere on Z and the inequality is strict on a set of positive Lebesgue measure such that $\overline{\lim}_{|x|\to\infty} pF(z,x)/|x|^p = \theta(z)$ uniformly for almost all $z \in Z$;

(iv) $\lim_{x\to 0} pF(z,x)/|x|^p = 0$ uniformly for almost all $z \in Z$;

(v) there exists $\xi > 0$ such that $\int_Z F(z, \xi u_1(z))\,dz > 0$.

We have the following multiplicity result.

THEOREM 11. *If hypotheses* H(f)$_2$ *hold, then problem* (15) *has at least two nontrivial solutions.*

PROOF. As before the energy functional $\phi : W_0^{1,p}(Z) \to \mathbb{R}$ is defined by

$$\phi(x) = \frac{1}{p}\|Dx\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \int_Z F(z, x(z))\,dz.$$

Now we have $\phi \in C^1(W_0^{1,p}(Z))$. By the hypothesis H(f)$_2$ (iv), we find $1 \ge \delta_1 = \delta_1(\varepsilon) > 0$ such that for almost all $z \in Z$ and all $|x| \le \delta_1$ we have $F(z,x) \le \varepsilon/p|x|^p$. Combining this with H(f)$_2$ (i) we obtain that for almost all $z \in Z$ and all $x \in \mathbb{R}$

$$F(z,x) \le \frac{\varepsilon}{p}|x|^p + \beta_1|x|^\mu \tag{25}$$

with $p < \mu \le p^*$ (recall that p^*, the critical Sobolev exponent, equals $Np/(N-p)$ if $p < N$ and $+\infty$ if $p \ge N$).

Let $W_0^{1,p}(Z) = Y \oplus V$, where $Y = \mathbb{R}u_1$ and V a topological complement.

CLAIM 1. *There exists $r_1 > 0$ such that $\phi(v) \ge 0$ for all $v \in V$, $\|v\| \le r_1$.*

Using (3) and (25), for every $v \in V$, we have

$$\begin{aligned}
\phi(v) &= \frac{1}{p}\|Dv\|_p^p - \frac{\lambda_1}{p}\|v\|_p^p - \int_Z F(z,x(z))\,dz \\
&\ge \frac{1}{p}\|Dv\|_p^p - \frac{\lambda_1}{\lambda_V^* p}\|Dv\|_p^p - \frac{\varepsilon}{p}\|v\|_p^p - \beta_2\|v\|_p^\theta \quad \text{for some } \beta_2 > 0, \\
&\ge \frac{1}{p}\|Dv\|_p^p - \frac{\lambda_1}{\lambda_V^* p}\|Dv\|_p^p - \frac{\varepsilon}{\lambda_V^* p}\|Dv\|_p^p - \beta_3\|Dv\|_p^\theta \quad \text{for some } \beta_3 > 0, \\
&\ge \frac{1}{p}\left(1 - \frac{\lambda_1+\varepsilon}{\lambda_V^*}\right)\|Dv\|_p^p - \beta_3\|Dv\|_p^\theta.
\end{aligned}$$

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Choose $\varepsilon > 0$ so that $\lambda_1 + \varepsilon < \lambda_V^*$. Then we have

$$\phi(v) \geq \beta_4 \|Dv\|_p^p - \beta_3 \|Dv\|_p^\theta$$

for some $\beta_4 > 0$ and all $v \in V$. Since $\theta > p$, by choosing $r_1 > 0$ small enough we see that $\phi(v) \geq 0$ for all $v \in V$, $\|v\| \leq r_1$.

CLAIM 2. *There exists $r_2 > 0$ such that $\phi(tu_1) \leq 0$ for all $|t| \leq r_2$.*

We have

$$\begin{aligned}\phi(tu_1) &= \frac{|t|^p}{p}\|Du_1\|_p^p - \frac{|t|^p}{p}\lambda_1\|u_1\|_p^p - \int_Z F(z, tu_1(z))\,dz \\ &= -\int_Z F(z, tu_1(z))\,dz.\end{aligned}$$

Since $u_1 \in C^1(\overline{Z})$ (see Lieberman [19, Theorem 1]), from the hypothesis $H(f)_2$ (ii) it follows that if $r_2 = \delta/\|u_1\|_\infty$, we have that $\phi(tu_1) \leq 0$ for all $|t| \leq r_2$.

CLAIM 3. *$\phi(\cdot)$ satisfies the (PS)-condition.*

Let $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ be such that $\{\phi(x_n)\}_{n\geq 1}$ is bounded and $\phi'(x_n) \xrightarrow{n\to\infty} 0$. Let $\psi : W_0^{1,p}(Z) \to \mathbb{R}$ be defined by $\psi(x) = \|Dx\|_p^p - \lambda_1\|x\|_p^p - \int_Z \theta(z)|x(z)|^p\,dz$. We show that there exists $\xi > 0$ such that $\psi(x) \geq \xi\|Dx\|_p^p$. Suppose not. Then we can find $\{x_m\}_{m\geq 0} \subseteq W_0^{1,p}(Z)$ with $\|Dx_m\|_p = 1$ such that $\psi(x_m) \downarrow 0$. Using Poincaré's inequality and by passing to a subsequence if necessary, we may assume that $x_m \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_m \to x$ in $L^p(Z)$. Thus we have

$$\begin{aligned}0 = \lim \psi(x_m) &\geq \underline{\lim}\, \|Dx_m\|_p^p - \lambda_1\|x\|_p^p - \int_Z \theta(z)|x(z)|^p\,dz \\ &\geq \|Dx\|_p^p - \lambda_1\|x\|_p^p - \int_Z \theta(z)|x(z)|^p\,dz > 0 \quad \text{(Rayleigh quotient)},\end{aligned}$$

a contradiction. So there exists $\xi > 0$ such that $\psi(x) \geq \xi\|Dx\|_p^p$ for all $x \in W^{1,p}(Z)$. Now by virtue of hypothesis $H(f)_2$(iii), given $\varepsilon > 0$ we can find $M = M(\varepsilon) > 0$ such that for almost all $z \in Z$ and all $|x| > M$, we have $F(z,x) \leq \theta(z)|x|^p/p$. On the other hand, from the hypothesis $H(f)_2$ (i), we know that for almost all $z \in Z$ and all $|x| \leq M$, we have $|F(z,x)| \leq a_M(z)$. Thus we infer that there exists $a_1 \in L^\infty(Z)$ (take for example $a_1(z) = a_M(z) + \|\theta\|_\infty$) such that for almost all $z \in Z$ and all $z \in \mathbb{R}$, we have

$$F(z,x) \leq \frac{1}{p}\theta(z)|x|^p + \varepsilon + a_1(z). \tag{26}$$

Using (26) we have

$$\begin{aligned}\phi(x) &\geq \frac{1}{p}\|Dx\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \frac{1}{p}\int_Z \theta(z)|x(z)|^p\,dz - \beta_5, \quad \text{for some } \beta_5 > 0,\\ &\geq \frac{1}{p}\xi\|Dx\|_p^p - \beta_5.\end{aligned}$$

From the above inequality we see that $\phi(\cdot)$ is coercive. Since $\{\phi(x_n)\}_{n\geq 1}$ is bounded, we must have that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded and so by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$. Then arguing as in the proof of Theorem 10 we have that $x_n \to x$ in $W_0^{1,p}(Z)$, which proves the claim. □

Finally note that $\phi(\cdot)$ being coercive is bounded below, while by virtue of hypothesis H(f)$_2$(v) and since $\|Du_1\|_p^p = \lambda_1\|u_1\|_p^p$, we have that $\inf_{W_0^{1,p}(Z)}\phi < 0$. These facts together with Claim 1, Claim 2 and Claim 3, allow as to use Theorem 1, which gives $x_1 \neq x_2$, $x_1, x_2 \neq 0$, such that $\phi'(x_1) = \phi'(x_2) = 0$. The same argument as in the proof of Theorem 10, shows that $x_1, x_2 \in W_0^{1,p}(Z)$ are nontrivial distinct solutions of (15). □

6. Semilinear problems at resonance

In this section we prove an existence theorem for the semilinear problem (that is, $p = 2$) at resonance with a discontinuous right hand side. So our problem is the following:

$$\text{(27)} \qquad \left\{\begin{array}{ll} -\Delta x(z) - \lambda_1 x(z) = f(z, x(z)) & \text{a.e. on } Z \\ x_{|\Gamma} = 0. & \end{array}\right\}$$

As before (see Section 4), since we do not require $f(z,\cdot)$ to be continuous, by introducing the functions $f_1(z,x) = \underline{\lim}_{x'\to x} f(z,x')$ and $f_2(z,x) = \overline{\lim}_{x'\to x} f(z,x')$, we pass to the following multivalued approximation of (27):

$$\text{(28)} \qquad \left\{\begin{array}{ll} -\Delta x(z) - \lambda_1 x(z) \in \widehat{f}(z, x(z)) & \text{a.e. on } Z \\ x_{|\Gamma} = 0 & \end{array}\right\},$$

where $\widehat{f}(z,x) = \{y \in \mathbb{R} : f_1(z,x) \leq y \leq f_2(z,x)\}$. Our hypotheses on the discontinuous nonlinearity $f(z,x)$ are the following:

H(f)$_3$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that

(i) f_1 and f_2 are both N-measurable functions

(ii) for every $M > 0$, there exists $a_M \in L^2(Z)$ such that for almost all $z \in Z$ and all $|x| \leq M$ we have $|f(z,x)| \leq a_M(z)$;

(iii) $\lim_{|x|\to\infty} f(z,x)/x = 0$ uniformly for almost all $z \in Z$;
(iv) if

$$G_1(z,x) = \begin{cases} \dfrac{2}{x}\displaystyle\int_0^x f(z,x)\,dr - f_1(z,x) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0 \end{cases}$$

and

$$G_2(z,x) = \begin{cases} \dfrac{2}{x}\displaystyle\int_0^x f(z,x)\,dr - f_2(z,x) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0, \end{cases}$$

then $G_1^-(z) = \lim_{x\to-\infty} G_1(z,x)$ and $G_2^+(z) = \lim_{x\to+\infty} G_2(z,x)$ exist uniformly for almost all $z \in Z$, G_1^-, $G_2^+ \in L^2(Z)$ and $\int_Z G_1^-(z)u_1(z)\,dz < 0 < \int_Z G_2^+(z)u_1(z)\,dz$.

We have the following existence theorem.

THEOREM 12. *If hypotheses* H(f)$_3$ *hold, then problem* (28) *has at least one nontrivial solution.*

PROOF. We consider the energy functional $\phi : H_0^1(Z) \to \mathbb{R}$ defined by

$$\phi(x) = \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda_1}{2}\|x\|_2^2 - \int_Z F(z,x(z))\,dz.$$

By virtue of hypothesis H(f)$_3$ (iii), given $\varepsilon > 0$ there exists $M = M(\varepsilon) > 0$ such that for almost all $z \in Z$ and all $|x| > M$ we have $|f(z,x)| \leq \varepsilon|x|$. Combining this with hypothesis H(f)$_3$ (ii) we infer that for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$|f(z,x)| \leq \varepsilon|x| + a_1(z) \tag{29}$$

with $a_1 \in L^2(Z)$. Evidently the same growth condition is satisfied by f_1 and f_2.

CLAIM 1. *The energy functional $\phi(\cdot)$ satisfies the nonsmooth PS-condition.*

Let $\{x_n\}_{n\geq1} \subseteq H_0^1(z)$ be a sequence such that $\{\phi(x_n)\}_{n\geq1}$ is bounded and $m(x_n) \xrightarrow{n\to\infty} 0$. We show that $\{x_n\}_{n\geq1}$ is bounded. Suppose it is not bounded. Then we may assume that $\|x_n\| \xrightarrow{n\to\infty} \infty$. Let $x_n^* \in \partial\phi(x_n)$, $n \geq 1$, such that $m(x_n) = \|x_n^*\|$. We have

$$x_n^* = A(x_n) - \lambda_1 x_n - w_n, \quad n \geq 1,$$

where $A \in \mathscr{L}(H_0^1(Z), H^{-1}(Z))$ is defined by $\langle A(x), y\rangle = \int_Z (Dx(z), Dy(z))_{\mathbb{R}^N}\,dz$ and $w_n \in L^2(Z)$, $f_1(z,x_n(z)) \leq w_n(z) \leq f_2(z,x_n(z))$ almost everywhere on Z. Let $Y = \mathbb{R}u_1$ and $V = Y^\perp$. Then $H_0^1(Z) = Y \oplus V$. We can write that x_n, with $t_n \in R$ and $v_n \in V$. From the choice of the sequence $\{x_n\}_{n\geq1}$

$$|\langle x_n^*, v_n\rangle| \leq \beta_1\|v_n\| \quad \text{for some } \beta_1 > 0,$$

then, since $\int_Z u_1(z)v_n(z)\,dz = 0$,

$$\|Dv_n\|_2^2 - \lambda_1\|v_n\|_2^2 - \int_Z w_n(z)v_n(z)\,dz \le \beta_1\|v_n\|_{1,2},$$

then (using (31))

$$\|Dv_n\|_2^2 - \lambda_1\|v_n\|_2^2 - \varepsilon\|x_n\|_2\|v_n\|_2 - \|a_1\|_2\|v_n\|_2 \le \beta_1\|v_n\|_{1,2},$$

then

$$\left(1 - \frac{\lambda_1}{\lambda_2}\right)\|Dv_n\|_2^2 - \varepsilon\|x_n\|_{1,2}\|v_n\|_{1,2} - \|a_2\|_2\|v_n\|_{1,2} \le \beta_1\|v_n\|_{1,2} \tag{30}$$

since on V we have $\|Dv\|_2^2 \ge \lambda_2\|v\|_2^2$ for all $v \in V$; here $\lambda_2 > 0$ is the second eigenvalue of $\left(-\Delta, H_0^1(Z)\right)$ and $\lambda_2 = \lambda^*$ (see (3)). Divide (30) by $\|v_n\|_{1,2}$ and use Poincaré's inequality to obtain

$$c\left(1 - \frac{\lambda_1}{\lambda_2}\right)\|v_n\|_{1,2} - \varepsilon\|x_n\|_{1,2} - \|a_2\|_2 \le \beta_1$$

with $c > 0$. Now divide this last inequality by $\|x_n\|_{1,2}$. We obtain

$$c\left(1 - \frac{\lambda_1}{\lambda_2}\right)\frac{\|v_n\|_{1,2}}{\|x_n\|_{1,2}} - \varepsilon - \frac{\|a\|_2}{\|x_n\|_{1,2}} \le \frac{\beta_1}{\|x_n\|_{1,2}}.$$

Since $\lambda_1 < \lambda_2$, we infer that

$$\overline{\lim}\,\frac{\|v_n\|_{1,2}}{\|x_n\|_{1,2}} \le \frac{\lambda_2}{c(\lambda_2 - \lambda_1)}\varepsilon.$$

Let $\varepsilon \downarrow 0$ to conclude that

$$\frac{\|v_n\|_{1,2}}{\|x_n\|_{1,2}} \xrightarrow{n\to\infty} 0. \tag{31}$$

We also know that $\|x_n\|_{1,2}^2 = t_n^2\|u_1\|_{1,2}^2 + \|v_n\|_{1,2}^2 = t_n^2 + \|v_n\|_{1,2}^2$ and so

$$\frac{t_n^2}{\|x_n\|_{1,2}^2} + \frac{\|v_n\|_{1,2}^2}{\|x_n\|_{1,2}^2} = 1 \qquad \text{hence} \qquad \frac{t_n}{\|x_n\|_{1,2}} \xrightarrow{n\to\infty} \pm 1.$$

Suppose without loss of generality that $t_n/\|x_n\|_{1,2} \xrightarrow{n\to\infty} +1$ (the analysis is the same if $t_n/\|x_n\|_{1,2} \xrightarrow{n\to\infty} -1$). Then $t_n \xrightarrow{n\to\infty} +\infty$ and if $y_n = x_n\|x_n\|_{1,2}$ we have $y_n \xrightarrow{n\to\infty} u_1$ in $H_0^1(Z)$. For $n \ge 1$, let

$$h_n(z) = \begin{cases} \dfrac{F(z, x_n(z))}{x_n(z)} & \text{if } x_n(z) \ne 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned}\langle x_n^*, y_n\rangle - \frac{2\phi(x_n)}{\|x_n\|_{1,2}} &= \int_Z 2h_n(z)y_n(z)\,dz - \int_Z w_n(z)y_n(z)\,dz \\ &\geq \int_Z 2h_n(z)y_n(z)\,dz - \int_{\{y_n>0\}} f_2(z,x_n(z))y_n(z)\,dz \\ &\quad - \int_{\{y_n<0\}} f_1(z,x_n(z))y_n(z)\,dz \\ &= \int_{\{y_n>0\}} G_2(z,x_n(z))y_n(z)\,dz - \int_{\{y_n<0\}} G_1(z,x_n(z))y_n(z)\,dz. \end{aligned} \tag{32}$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ we have that $\langle x_n^*, y_n\rangle \xrightarrow{n\to\infty} 0$ and $2\phi(x_n)/\|x_n\|_{1,2} \leq M/\|x_n\|_{1,2} \xrightarrow{n\to\infty} 0$. In addition, at least for a subsequence, we have $\chi_{\{y_n>0\}} \xrightarrow{n\to\infty} \chi_Z = 1$ almost everywhere on Z. Thus by passing to the limit in (32), we obtain

$$\int_Z G_2^+(z)u_1(z)\,dz \leq 0,$$

which contradicts hypothesis H(f)$_3$ (iv). This proves the boundedness of $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$. So we may assume that $x_n \xrightarrow{w} x$ in $H_0^1(Z)$ as $n \to \infty$ and proceeding as in the previous proofs, we have that $x_n \xrightarrow{n\to\infty} x$ in $H_0^1(Z)$ and so $\phi(\cdot)$ satisfies the nonsmooth PS-condition. □

CLAIM 2. *$\phi(tu_1) \to -\infty$ as $|t| \to \infty$.*

From the hypothesis H(f)$_3$ (iv), given $\varepsilon > 0$, we find $M = M(\varepsilon) > 0$ such that for almost all $z \in Z$ and all $x \leq -M$ we have

$$\begin{aligned} &G_1(z,x) \leq G_1^-(z) + \varepsilon = \theta_\varepsilon^1(z) \\ \Rightarrow\quad &\frac{G_1(z,x)}{x^2} \leq \frac{\theta_\varepsilon^1(z)}{x^2} = \frac{d}{dx}\left(-\frac{\theta_\varepsilon^1(z)}{x}\right). \end{aligned} \tag{33}$$

Note also that from the definition of $G_1(z,x)$, we have

$$\begin{aligned}\frac{G_1(z,x)}{x^2} &\geq \frac{1}{x^2}\left(\frac{2}{x}\int_0^x f(z,r)\,dr - f(z,x)\right) \\ &= \frac{2}{x^3}\int_0^x f(z,r)\,dr - \frac{1}{x^2}f(z,x) \\ &= \frac{d}{dx}\left(-\frac{1}{x^2}\int_0^x f(z,r)\,dr\right) = \frac{d}{dx}\left(-\frac{F(z,x)}{x^2}\right).\end{aligned}$$

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Using this inequality in (33), we obtain

$$\frac{d}{dx}\left(-\frac{F(z,x)}{x^2}\right) \le \frac{d}{dx}\left(-\frac{\theta_\varepsilon^1(z)}{x}\right) \quad \text{a.e. on } Z, \text{ for all } x \le -M.$$

Integrating this inequality on $[y,x]$, $y < x \le -M$, we have

$$-\frac{F(z,x)}{x^2} + \frac{F(z,y)}{y^2} \le \frac{-\theta_\varepsilon^1(z)}{x} + \frac{\theta_\varepsilon^1(z)}{y}. \tag{34}$$

From (29) we know that for almost all $z \in Z$ and all $r \le 0$ we have

$$f(z,x) \le -\varepsilon r + a_1(z), \quad a_1 \in L^2(Z), \quad \text{then}$$

$$\frac{1}{y^2}\int_0^y f(z,r)\,dr \ge -\frac{\varepsilon}{2} + \frac{a_1(z)}{y^2} \quad \text{hence} \quad \varliminf_{y\to-\infty} \frac{F(z,y)}{y^2} \ge -\frac{\varepsilon}{2}.$$

Let $\varepsilon \downarrow 0$ to conclude that

$$\varliminf_{y\to-\infty} \frac{F(z,y)}{y^2} \ge 0.$$

So, if in (34) we let $y \to -\infty$, we obtain that for almost all $z \in A$ and all $x \le -M$ we have

$$\frac{F(z,x)}{x} \le \theta_\varepsilon^1(z), \quad \text{then} \quad \varlimsup_{x\to-\infty} \frac{F(z,x)}{x} \le \theta_\varepsilon^1(z) \quad \text{almost everywhere on } Z.$$

Letting $\varepsilon \downarrow 0$, we have that

$$\varlimsup_{x\to-\infty} \frac{F(z,x)}{x} \le G_1^-(z) \quad \text{almost everywhere on } Z. \tag{35}$$

Similarly, we can show that

$$\varliminf_{x\to+\infty} \frac{F(z,x)}{x} \ge G_2^+(z) \quad \text{almost everywhere on } Z. \tag{36}$$

Suppose the claim was not true. Then we can find $|t_n| \to +\infty$ such that $\phi(t_n u_1) \ge -\gamma$ for some $\gamma > 0$. First assume that $t_n \to -\infty$. We have $(1/t_n)\phi(t_n u_1) \le -\gamma/t_n$. Therefore,

$$\varlimsup \frac{1}{t_n}\phi(t_n u_1) \le 0, \qquad \varlimsup -\frac{1}{t_n}\int_Z F(z, t_n u_1(z))\,dz \le 0,$$

$$\varliminf \frac{1}{t_n}\int_Z F(z, t_n u_1(z))\,dz \ge 0, \qquad \int_Z G_1^-(z) u_1(z)\,dz \ge 0 \quad \text{(see (35))},$$

which contradicts $\mathrm{H(f)}_3$ (iv). Similarly, if $t_n \to +\infty$, we obtain using (36)

$$\int_Z G_2^+(z) u_1(z)\,dz \le 0,$$

which contradicts $\mathrm{H(f)}_3$(iv). Therefore, the claim is true and we have $\phi(tu_1) \xrightarrow{|t|\to\infty} -\infty$.

CLAIM 3. *$\phi(v) \to +\infty$ as $\|v\|_{1,2} \to \infty$, $v \in V$ (hence $\phi_{|_V}$ is bounded below).*

Since for $v \in V$, $\|Dv\|_2^2 \geq \lambda_2 \|v\|_2^2$, $\lambda_2 > \lambda_1$ and using (29) we have for all $v \in V$

$$\begin{aligned}\phi(v) &\geq \frac{1}{2}\|Dv\|_2^2 - \frac{\lambda_1}{2}\|v\|_2^2 - \frac{\varepsilon}{2}\|v\|_2^2 - \|a_1\|_2\|v\|_2 \\ &\geq \frac{1}{2}\left(1 - \frac{\lambda_1}{\lambda_2} - \frac{\varepsilon}{\lambda_2}\right)\|Dv\|_2^2 - \beta\|Dv\|_2 \quad \text{for some } \beta > 0.\end{aligned}$$

Choose $\varepsilon > 0$ so that $\lambda_1 + \varepsilon < \lambda_2$. Then from the above inequality, it is clear that $\phi(v) \to +\infty$ as $\|v\|_{1,2} \to \infty$. Hence $\phi_{|_V}$ is bounded below.

Claim 1, Claim 2 and Claim 3 permit the application of Theorem 7, which gives us $x \in H_0^1(Z)$ such that $0 \in \partial\phi(x)$. As before we conclude that x solves (30). □

REMARK. We know that in this case there exist an orthonormal basis $\{u_m\}_{m\geq 1}$ of $L^2(Z)$ and a sequence of positive real numbers $\{\lambda_m\}_{m\geq 1}$ with $\lambda_m \to +\infty$ such that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_m \leq \cdots$ and $u_m \in H_0^1(Z) \cap C^\infty(Z)$, $m \geq 1$, are solutions of (1) with $p = 2$. Moreover, these higher eigenvalues have variational characterizations similar to (2) (see Kesavan [13]). So, in this case, in contrast to the case $p > 2$, we have full knowledge of the spectrum of $(-\Delta, H_0^1(Z))$. Thus what we did for the resonant at λ_1 problem, we can do it for the problem which is resonant at some higher eigenvalue, using the same approach with minor modifications.

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