J. Aust. Math. Soc. **75** (2003), 247–261

THE L^p–L^q MAPPING PROPERTIES OF CONVOLUTION OPERATORS WITH THE AFFINE ARCLENGTH MEASURE ON SPACE CURVES

YOUNGWOO CHOI

(Received 30 June 2000; revised 5 August 2002)

Communicated by A. H. Dooley

Abstract

The L^p-improving properties of convolution operators with measures supported on space curves have been studied by various authors. If the underlying curve is non-degenerate, the convolution with the (Euclidean) arclength measure is a bounded operator from $L^{3/2}(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$. Drury suggested that in case the underlying curve has degeneracies the appropriate measure to consider should be the affine arclength measure and he obtained a similar result for homogeneous curves $t \mapsto (t, t^2, t^k)$, $t > 0$ for $k \geq 4$. This was further generalized by Pan to curves $t \mapsto (t, t^k, t^l)$, $t > 0$ for $1 < k < l$, $k + l \geq 5$. In this article, we will extend Pan's result to (smooth) compact curves of finite type whose tangents never vanish. In addition, we give an example of a flat curve with the same mapping properties.

2000 *Mathematics subject classification*: primary 42B15; secondary 42B10.
Keywords and phrases: convolution, affine arclength measure, flat curve.

1. Introduction

The L^p-improving properties of convolution operators with measures supported on space curves have been studied by many authors. Oberlin [5] showed that the convolution with the (Euclidean) arclength measure on the curve $t \mapsto (t, t^2, t^3)$, $0 \leq t \leq 1$, maps $L^{3/2}(\mathbb{R}^3)$ boundedly into $L^2(\mathbb{R}^3)$. Later, Pan [6] extended this result to non-degenerate compact curves in $\mathbb{R}^3$.

Drury [3] suggested that in case the underlying curve has degeneracies the appropriate measure to consider should be the affine arclength measure and obtained

This work is based on a research during the author's stay at Department of Mathematics, University of Wisconsin-Madison. The author would like to express his gratitude to both the university and the department for providing excellent environment for research.

a similar result for a class of homogeneous curves. Recall that the affine arclength measure σ_γ on the curve $\gamma : I \to \mathbb{R}^3$ is defined by

$$\int_{\mathbb{R}^3} f \, d\sigma_\gamma = \int_I f\,(\gamma(t))\,\lambda(t)\,dt$$

for $f \in C_0^\infty(\mathbb{R}^3)$, where

$$\lambda(t) = \left| \det \begin{bmatrix} \gamma'(t) \\ \gamma''(t) \\ \gamma'''(t) \end{bmatrix} \right|^{1/6}, \quad t \in I.$$

Associated with σ_γ is the convolution operator T_{σ_γ} given by

$$T_{\sigma_\gamma} f\,(x) = \sigma_\gamma * f\,(x) = \int_{\mathbb{R}^3} f\,(x-y)\,d\sigma_\gamma(y) \tag{1}$$

for $f \in C_0^\infty(\mathbb{R}^3)$. Drury showed that T_{σ_γ} is bounded from $L^{3/2}(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$ for $\gamma(t) = (t, t^2, t^k)$, $t > 0$, if $k \geq 4$. This was further improved by Pan [7, 8]:

PROPOSITION 1.1. *Let $1 < k < l < \infty$, and suppose $k + l \geq 5$. Let σ_γ be the affine arclength measure on the curve $\gamma(t) = (t, t^k, t^l)$, $t > 0$, defined by*

$$\int_{\mathbb{R}^3} f \, d\sigma_\gamma = c \int_0^\infty f\,(\gamma(t)) t^{(k+l-5)/6}\,dt$$

for $f \in C_0^\infty(\mathbb{R}^3)$. Then T_{σ_γ} defined by (1) maps $L^{3/2}(\mathbb{R}^3)$ boundedly into $L^2(\mathbb{R}^3)$.

Notice that all the results mentioned above are sharp. In other words, if we denote by $\mathscr{T}$ the trapezoid with vertices at $(0,0)$, $(1,1)$, $(2/3, 1/2)$, and $(1/2, 1/3)$, then T_{σ_γ} is a bounded operator from $L^p(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ if and only if $(1/p, 1/q) \in \mathscr{T}$. The operator T_{σ_γ} is related to the convolution with the (Euclidean) arclength measure. To be more specific, for $1 < k < l$, we let

$$\mathscr{T}_0^{k,l} = \{(1/p, 1/q) \in \mathscr{T} : 1/p - 1/q \leq 1/(1+k+l)\}.$$

Then an analytic interpolation shows that the convolution operator with the (Euclidean) arclength measure on the curve $t \mapsto (t, t^k, t^l)$, $0 \leq t \leq 1$, is a bounded operator from $L^p(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ at least for $(1/p, 1/q) \in \operatorname{int} \mathscr{T}_0^{k,l}$.

In this paper, we will establish similar results to Proposition 1.1 for curves of finite type with non-vanishing tangents and also an example of flat curve with the same mapping properties will be provided. More precisely, we will prove:

THEOREM 1.2. *Let γ be a smooth compact curve of finite type in $\mathbb{R}^3$. Assume $\gamma'(t)$ doesn't vanish at any point. Let $T_{\sigma_\gamma} f = \sigma_\gamma * f$, where σ_γ is the affine arclength measure on the curve γ. Then T_{σ_γ} maps $L^{3/2}(\mathbb{R}^3)$ boundedly into $L^2(\mathbb{R}^3)$.*

THEOREM 1.3. *For any $t_0 > 0$, the operator $T_{\sigma_\gamma}^{t_0}$ defined by $T_{\sigma_\gamma}^{t_0} f = \sigma_\gamma^{t_0} * f$ is a bounded operator from $L^{3/2}(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$, where $\sigma_\gamma^{t_0}$ is the affine arclength measure on the curve $\gamma : t \mapsto (t, t^2, e^{-1/t})$, $0 < t < t_0$.*

The organization of this paper is as follows. In Section 2, a perturbed version of Proposition 1.1 will be studied. Based on this, we prove Theorem 1.2 in Section 3. In the last section we will prove Theorem 1.3.

After finishing this paper, we learned that, using an ingenious idea by Christ [2], Secco [10] independently obtained a result extending Proposition 1.1 to $1 < k < l$ (possibly $k + l < 5$). She also showed that the convolution operator with the (Euclidean) arclength measure on the curve $t \mapsto (t, t^k, t^l)$, $0 \le t \le 1$, maps $L^p(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ if and only if $(1/p, 1/q) \in \mathcal{T}_0^{k,l}$.

2. Perturbations of homogeneous curves

Recall the following results on plane curves.

LEMMA 2.1 (Littman [4]). *Let I be a compact interval and let $\phi : I \to \mathbb{R}$ be a C^2 function. Suppose there exists a positive constant C such that $|\phi''(t)| \ge C$ for $t \in I$. Then T_e^ϕ given by*

$$T_e^\phi f(x_1, x_2) = \int_I f(x_1 - t, x_2 - \phi(t))\, dt$$

satisfies $\| T_e^\phi f \|_{L^3(\mathbb{R}^2)} \le C' \| f \|_{L^{3/2}(\mathbb{R}^2)}$ with some constant C' depending only on C.

LEMMA 2.2 ([1]). *Let I be a compact interval and let $\phi : I \to \mathbb{R}$ be a C^3 function. Suppose there exist constants C_1 and C_2 with $0 < C_1 < C_2 < \infty$ such that*

(1) *$\phi''(t)$ never vanishes for $t \in I$;*
(2) *$C_1 \phi'(t)^2 \le |\phi(t)\phi''(t)| \le C_2 \phi'(t)^2$ for $t \in I$;*
(3) *$|\phi(t)\phi'''(t)| \le C_2 |\phi'(t)\phi''(t)|$ for $t \in I$;*
(4) *$\displaystyle\int_I \left| \frac{d}{dt} \left\{ \sqrt{\phi(t)\phi''(t)} / \phi'(t) \right\} \right| dt \le C_2$.*

Then T_a^ϕ given by

$$T_a^\phi f(x_1, x_2) = \int_I f(x_1 - t, x_2 - \phi(t)) |\phi''(t)|^{1/3}\, dt$$

satisfies $\| T_a^\phi f \|_{L^3(\mathbb{R}^2)} \le C_3 \| f \|_{L^{3/2}(\mathbb{R}^2)}$ with some constant C_3 depending only on C_1 and C_2.

Suppose $1 < k < l < \infty$ and $k + l \geq 5$. Let

$$\phi(t) = t^k + \phi_1(t) \quad \text{and} \quad \psi(t) = t^l + \psi_1(t).$$

We assume that, for $j = 0, 1, \ldots, 4$,

$$\phi_1^{(j)}(t) = o(t^{k-j}) \quad \text{and} \quad \psi_1^{(j)}(t) = o(t^{l-j})$$

as $t \to 0+$.

Let $\gamma(t) = (t, \phi(t), \psi(t))$ and let $\lambda(t) = |\phi''(t)\psi'''(t) - \phi'''(t)\psi''(t)|^{1/6}$. For $t_0 > 0$ sufficiently small, we consider the operator T_{σ_γ} defined by

$$T_{\sigma_\gamma} f(x) = \int_0^{t_0} f(x - \gamma(t))\lambda(t)\, dt. \tag{2}$$

Then, we have:

PROPOSITION 2.3. *For $t_0 > 0$ sufficiently small, T_{σ_γ} defined by* (2) *maps $L^{3/2}(\mathbb{R}^3)$ boundedly into $L^2(\mathbb{R}^3)$.*

PROOF. We begin with a lemma by Oberlin [5]:

LEMMA 2.4. *If $T^*_{\sigma_\gamma} T_{\sigma_\gamma}$ maps $L^{3/2}(\mathbb{R}^3)$ boundedly into $L^3(\mathbb{R}^3)$, then T_{σ_γ} is a bounded operator from $L^{3/2}(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$.*

PROOF. One has only to observe

$$\begin{aligned} \| T_{\sigma_\gamma} f \|_{L^2}^2 &= \langle T_{\sigma_\gamma} f, T_{\sigma_\gamma} f \rangle = \langle T^*_{\sigma_\gamma} T_{\sigma_\gamma} f, f \rangle \\ &\leq \| T^*_{\sigma_\gamma} T_{\sigma_\gamma} f \|_{L^3} \| f \|_{L^{3/2}} \leq \| T^*_{\sigma_\gamma} T_{\sigma_\gamma} \|_{L^{3/2} \to L^3} \| f \|^2_{L^{3/2}}, \end{aligned}$$

which completes the proof of the lemma. □

According to Lemma 2.4 and by symmetry, it suffices to show that S defined by

$$Sf(x) = \iint_{0<s<t<t_0} f(x - \gamma(t) + \gamma(s))\lambda(t)\lambda(s)\, ds dt$$

is bounded from $L^{3/2}(\mathbb{R}^3)$ into $L^3(\mathbb{R}^3)$ for some $t_0 > 0$. A change of variables

$$\begin{cases} u = t - s; \\ v^* = \phi(t) - \phi(s) \end{cases}$$

https://doi.org/10.1017/S144678870000375X Published online by Cambridge University Press

and fractional integration theorem [11] reduce the proof to verifying the uniform boundedness of $\{\|S'_u\|_{L^{3/2}(\mathbb{R}^2)\to L^3(\mathbb{R}^2)}\}_{0<u<t_0}$ for some $t_0 > 0$, where S'_u are given by

$$S'_u g(x_2, x_3) = u^{2/3} \int_{\phi(u)}^{\phi(t_0)-\phi(t_0-u)} g\big(x_2 - v^*, x_3 - \Psi_u(v^*)\big) \frac{\lambda(t)\lambda(s)\, dv^*}{\phi'(t) - \phi'(s)}$$

with $\Psi_u(v^*) = \psi(t) - \psi(s)$.

A homogeneity argument further reduces the proof to establishing the uniform boundedness of $\{\|S_u\|_{L^{3/2}(\mathbb{R}^2)\to L^3(\mathbb{R}^2)}\}_{0<u<t_0}$ for some $t_0 > 0$, where

$$S_u g(x_2, x_3 t) = \int_{\phi(u)/u^k}^{(\phi(t_0)-\phi(t_0-u))/u^k} g\big(x_2 - v, x_3 - \tilde{\Psi}_u(v)\big) J_u(v)\, dv,$$

$$J_u(v) = u^{2/3-(k+l)/3+k} \frac{\lambda(t)\lambda(s)}{\phi'(t) - \phi'(s)},$$

with $\tilde{\Psi}_u(v)$ given by

$$\begin{cases} \phi(t) - \phi(s) = u^k v; \\ \psi(t) - \psi(s) = \Psi_u(u^k v) \equiv u^l \tilde{\Psi}_u(v). \end{cases}$$

To simplify our notation, we introduce $\bar{t} = t/u$ and $\bar{s} = s/u$. Then, we have:

LEMMA 2.5. *There exist $t_0 > 0$, $b > 0$, C_1 and C_2 such that*

(I) *If $0 < u < t_0$ and $0 < \bar{s} \le b$, then*

   (1) $\tilde{\Psi}''_u(v) \ge C_1$;

   (2) $J_u(v) \le C_2$.

(II) *If $0 < u < t_0$ and $\bar{s} \ge b$, then*

   (1) $\tilde{\Psi}_u(v), \tilde{\Psi}'_u(v), \tilde{\Psi}''_u(v) > 0$;

   (2) $C_1 v^{(l-1)/(k-1)-j} \le \tilde{\Psi}^{(j)}_u(v) \le C_2 v^{(l-1)(k-1)-j}$, $j = 0, 1, 2$;

   (3) $|\tilde{\Psi}'''_u(v)| \le C_2 v^{(l-1)/(k-1)-3}$;

   (4) $\displaystyle\int_{\bar{s}\ge b} \left| \frac{d}{dv} \left\{ \tilde{\Psi}_u(v) \tilde{\Psi}''_u(v) / \tilde{\Psi}'_u(v)^2 \right\} \right| dv \le C_2$.

PROOF. Recall the following facts:

FACT 2.6. *Let $k > 0$. Then there exist positive constants C_1 and C_2 depending only on k such that $C_1(t-s)t^{k-1} \le t^k - s^k \le C_2(t-s)t^{k-1}$ whenever $0 < s \le t < \infty$.*

FACT 2.7. *Let $k < 0$. Then there exist positive constants C_1 and C_2 depending only on k such that $C_1(t-s)s^k/t \le s^k - t^k \le C_2(t-s)s^k/t$ whenever $0 < s \le t < \infty$. In particular, if $t/s \ge \delta > 0$, there exists a positive constant C depending only on δ and k such that $(1/C)(t-s)t^{k-1} \le s^k - t^k \le C(t-s)t^{k-1}$.*

Before we proceed, we observe that for any $\epsilon > 0$ there exists $t_0 > 0$ such that if $0 < \bar{s} < \bar{t} < t_0/u$ and $\bar{s} \leq b$,

$$\frac{1}{u^{k-j}}\left|\phi_1^{(j)}(u\bar{t}) - \phi_1^{(j)}(u\bar{s})\right| \leq \epsilon \bar{t}^{k-j-1}, \qquad j = 0, 1, \tag{3}$$

$$\frac{1}{u^{k-j}}\left|\phi_1^{(j)}(u\bar{t}) - \phi_1^{(j)}(u\bar{s})\right| \leq \epsilon\left(\bar{t}^{k-3} + \bar{s}^{k-2}\right), \quad j = 2, \tag{4}$$

$$\frac{1}{u^{k-j}}\left|\psi_1^{(j)}(u\bar{t}) - \psi_1^{(j)}(u\bar{s})\right| \leq \epsilon \bar{t}^{l-j-1}, \qquad j = 0, 1, 2, \tag{5}$$

and that if $0 < \bar{s} < \bar{t} < t_0/u$ and $\bar{s} \geq b$,

$$\frac{1}{u^{k-j}}\left|\phi_1^{(j)}(u\bar{t}) - \phi_1^{(j)}(u\bar{s})\right| \leq \epsilon \bar{t}^{k-j-1}, \quad j = 0, 1, 2, 3, \tag{6}$$

$$\frac{1}{u^{k-j}}\left|\psi_1^{(j)}(u\bar{t}) - \psi_1^{(j)}(u\bar{s})\right| \leq \epsilon \bar{t}^{l-j-1}, \quad j = 0, 1, 2, 3. \tag{7}$$

Write

$$\tilde{\Psi}_u''(v) = \frac{M_1(v) + P_1(v)}{\left\{\left(\phi'(u\bar{t}) - \phi'(u\bar{s})\right)/u^{k-1}\right\}^3},$$

where

$$M_1(v) = (l-1)(\bar{t}^{l-2} - \bar{s}^{l-2})(\bar{t}^{k-1} - \bar{s}^{k-1}) - (k-1)(\bar{t}^{l-1} - \bar{s}^{l-1})(\bar{t}^{k-2} - \bar{s}^{k-2}).$$

Then, we have

$$\begin{aligned}
M_1(v) &= (l-1)(\bar{t}^{l+k-3} + \bar{s}^{l+k-3} - \bar{t}^{l-2}\bar{s}^{k-1} - \bar{t}^{k-1}\bar{s}^{l-2}) \\
&\quad - (k-1)(\bar{t}^{l+k-3} + \bar{s}^{l+k-3} - \bar{t}^{k-2}\bar{s}^{l-1} - \bar{t}^{l-1}\bar{s}^{k-2}) \\
&= (l-k)(\bar{t}^{l+k-3} + \bar{s}^{l+k-3}) + \bar{t}^{l-2}\bar{s}^{k-2}\left((k-1)\bar{t} - (l-1)\bar{s}\right) \\
&\quad + \bar{t}^{k-2}\bar{s}^{l-2}\left((k-1)\bar{s} - (l-1)\bar{t}\right) \\
&= (l-k)(\bar{t}^{l+k-3} + \bar{s}^{l+k-3}) + \bar{t}^{l-2}\bar{s}^{k-2}\left((k-l)\bar{s} + (k-1)\right) \\
&\quad + \bar{t}^{k-2}\bar{s}^{l-2}\left((k-l)\bar{t} - (k-1)\right) \\
&= (l-k)(\bar{t}^{l+k-3} + \bar{s}^{l+k-3} - \bar{t}^{l-2}\bar{s}^{k-1} - \bar{t}^{k-1}\bar{s}^{l-2}) \\
&\quad + (k-1)(\bar{t}^{l-2}\bar{s}^{k-2} - \bar{t}^{k-2}\bar{s}^{l-2}) \\
&= (l-k)(\bar{t}^{l-2} - \bar{s}^{l-2})(\bar{t}^{k-1} - \bar{s}^{k-1}) + (k-1)\bar{t}^{k-2}\bar{s}^{k-2}(\bar{t}^{l-k} - \bar{s}^{l-k}).
\end{aligned}$$

Thus, we obtain $M_1(v) \geq c\max(\bar{t}^{k+l-5}, \bar{s}^{k-2})$. From (3)–(7), we see that

$$P_1(v) \leq \frac{c}{2}\max\left(\bar{t}^{k+l-5}, \bar{s}^{k-2}\right).$$

https://doi.org/10.1017/S144678870000375X Published online by Cambridge University Press

Now, (I) is clear. Simple calculations show (a), (b) and (c) of (II). To verify (d), we write

$$\frac{d}{dv}\frac{\tilde{\Psi}_u(v)\tilde{\Psi}''_u(v)}{\tilde{\Psi}'_u(v)^2} = \frac{M_2(v) + P_2(v)}{\tilde{\Psi}_u(v)^3},$$

where

$$M_2(v) = \frac{l^2}{k^4}\frac{N(v)}{\left\{\left(\phi'(u\bar{t}) - \phi'(u\bar{s})\right)/u^{k-1}\right\}^6}$$

with

$$\begin{aligned} N(v) = {} & l\left(\bar{t}^{k-1} - \bar{s}^{k-1}\right)\left(\bar{t}^{l-1} - \bar{s}^{l-1}\right)^2 A(v) \\ & + (\bar{t}^l - \bar{s}^l)(\bar{t}^{l-1} - \bar{s}^{l-1})(B_1(v) - 3(k-1)B_2(v)) - 2(\bar{t}^l - \bar{s}^l)C(v), \end{aligned}$$

where

$$\begin{aligned} A(v) &= (l-1)\left(\bar{t}^{l-2} - \bar{s}^{l-2}\right)\left(\bar{t}^{k-1} - \bar{s}^{k-1}\right) - (k-1)\left(\bar{t}^{l-1} - \bar{s}^{l-1}\right)\left(\bar{t}^{k-2} - \bar{s}^{k-2}\right); \\ B_1(v) &= \left(\bar{t}^{k-1} - \bar{s}^{k-1}\right)\left\{(l-1)(l-2)\left(\bar{t}^{l-3} - \bar{s}^{l-3}\right)\left(\bar{t}^{k-1} - \bar{s}^{k-1}\right)\right. \\ &\qquad \left. - (k-1)(k-2)\left(\bar{t}^{l-1} - \bar{s}^{l-1}\right)\left(\bar{t}^{k-3} - \bar{s}^{k-3}\right)\right\}; \\ B_2(v) &= \left(\bar{t}^{k-2} - \bar{s}^{k-2}\right)\left\{(l-1)\left(\bar{t}^{l-2} - \bar{s}^{l-2}\right)\left(\bar{t}^{k-1} - \bar{s}^{k-1}\right)\right. \\ &\qquad \left. - (k-1)\left(\bar{t}^{l-1} - \bar{s}^{l-1}\right)\left(\bar{t}^{k-2} - \bar{s}^{k-2}\right)\right\}; \\ C(v) &= \left\{(l-1)\left(\bar{t}^{l-2} - \bar{s}^{l-2}\right)\left(\bar{t}^{k-1} - \bar{s}^{k-1}\right) - (k-1)\left(\bar{t}^{l-1} - \bar{s}^{l-1}\right)\left(\bar{t}^{k-2} - \bar{s}^{k-2}\right)\right\}^2. \end{aligned}$$

And so, we have $N(v) = \bar{t}^{2k+3l-6}\tilde{N}(\mu)$, where $\mu = \bar{s}/\bar{t}$ and

$$\begin{aligned} \tilde{N}(\mu) = {} & l(1-\mu^{k-1})(1-\mu^{l-1})^2\tilde{A}(\mu) \\ & + (1-\mu^l)(1-\mu^{l-1})\left(\tilde{B}_1(\mu) - 3(k-1)\tilde{B}_2(\mu)\right) - 2(1-\mu^l)\tilde{C}(\mu), \end{aligned}$$

with

$$\begin{aligned} \tilde{A}(\mu) &= (l-1)(1-\mu^{l-2})(1-\mu^{k-1}) - (k-1)(1-\mu^{l-1})(1-\mu^{k-2}); \\ \tilde{B}_1(\mu) &= (1-\mu^{k-1})\left\{(l-1)(l-2)(1-\mu^{l-3})(1-\mu^{k-1})\right. \\ &\qquad \left. - (k-1)(k-2)(1-\mu^{l-1})(1-\mu^{k-3})\right\}; \\ \tilde{B}_2(\mu) &= (1-\mu^{k-2})\left\{(l-1)(1-\mu^{l-2})(1-\mu^{k-1})\right. \\ &\qquad \left. - (k-1)(1-\mu^{l-1})(1-\mu^{k-2})\right\}; \\ \tilde{C}(\mu) &= \left\{(l-1)(1-\mu^{l-2})(1-\mu^{k-1}) - (k-1)(1-\mu^{l-1})(1-\mu^{k-2})\right\}^2. \end{aligned}$$

Since $\tilde{N}(\mu)$ is real-analytic near $\mu = 1$ and $\tilde{N}^{(j)}(1) = 0$, $j = 0, 1, \ldots, 5$, we can conclude $|\tilde{N}(\mu)| \leq C(1-\mu)^6$ for μ sufficiently close to 1. Altogether, we obtain

$$\left|\frac{d}{dv}\frac{\tilde{\Psi}_u(v)\tilde{\Psi}''_u(v)}{\tilde{\Psi}'_u(v)^2}\right| \leq C\bar{t}^{-k}$$

for $\bar{s} \geq b > 0$. In other words,

$$\left| \frac{d}{dv} \frac{\tilde{\Psi}_u(v)\tilde{\Psi}_u''(v)}{\tilde{\Psi}_u'(v)^2} \right| \leq Cv^{-k/(k-1)}$$

whenever $v \geq v_0 > 0$. From $k/(k-1) > 1$ we see (d) and the proof of the lemma is complete. □

To finish the proof of the proposition, we decompose $S_u = S_{u,1} + S_{u,2}$ by

$$S_{u,1}g(x_2, x_3) = \int_{\phi(u)/u^k}^{v_0} g(x_2 - v, x_3 - \tilde{\Psi}_u(v))J_u(v)\,dv,$$

and

$$S_{u,2}g(x_2, x_3) = \int_{v_0}^{(\phi(t_0)-\phi(t_0-u))/u^k} g(x_2 - v, x_3 - \tilde{\Psi}_u(v))J_u(v)\,dv.$$

Then, we have $|S_{u,1}g| \leq CS_{u,1}^0|g|$ and $|S_{u,2}g| \leq CS_{u,2}^0|g|$, where

$$S_{u,1}^0 g(x_2, x_3) = \int_{\phi(u)/u^k}^{v_0} g(x_2 - v, x_3 - \tilde{\Psi}_u(v))\,dv,$$

and

$$S_{u,2}^0 g(x_2, x_3) = \int_{v_0}^{(\phi(t_0)-\phi(t_0-u))/u^k} g(x_2 - v, x_3 - \tilde{\Psi}_u(v))|\tilde{\Psi}_u''(v)|^{1/3}\,dv.$$

Lemma 2.1 and Lemma 2.2 apply to $S_{u,1}^0$ and $S_{u,2}^0$. The proof is now finished. □

REMARK 2.8. An analytic interpolation gives the L^p–L^q boundedness of T_γ given by $T_\gamma f(x) = \int_I f(x - \gamma(t))\,dt$, for $(1/p, 1/q) \in \operatorname{int} \mathcal{T}_0^{k,l}$, where γ is as in Proposition 2.3.

3. Curves of finite type

Recall the following result on non-degenerate curves.

PROPOSITION 3.1 (Pan [6]). *Let $\gamma_1, \gamma_2 \in C^3(I)$. Suppose that for any $t \in I$, $(\gamma_1^{(2)}(t), \gamma_2^{(2)}(t))$ and $(\gamma_1^{(3)}(t), \gamma_2^{(3)}(t))$ span $\mathbb{R}^2$. Then the convolution operator T defined by $Tf(x) = \int_I f(t, \gamma_1(t), \gamma_2(t))\,dt$ maps $L^{3/2}(\mathbb{R}^3)$ boundedly into $L^2(\mathbb{R}^3)$.*

By compactness there are only finitely many degenerate points on the curve. Since $\gamma'(t)$ never vanishes, Proposition 3.1, a partition of unity argument and a re-parametrization around each degenerate point followed by a suitable affine motion in $\mathbb{R}^3$ allow us to assume

$$\gamma(t) = \left(t + \gamma_1(t), t^k + \gamma_2(t), t^l + \gamma_3(t)\right), \quad 0 < t < t_0$$

with $t_0 > 0$ (as small as we want) and $k, l \in \mathbb{N}$, $1 < k < l$. The perturbation terms are smooth and also satisfy

$$\gamma_1^{(j)}(t) = O\left(t^{-j+2}\right), \quad \gamma_2^{(j)}(t) = O\left(t^{k-j+1}\right), \quad \gamma_3^{(j)}(t) = O\left(t^{l-j+1}\right),$$

as $t \to 0+$, for $j = 0, 1, 2, 3, 4$. A re-parametrization $s = t + \gamma_1(t)$ will bring γ into the form which can be handled by Proposition 2.3.

REMARK 3.2. (1) The smoothness assumption on γ can be weakened in Theorem 1.2.

(2) The type set for T_{σ_γ} in Proposition 2.3, Theorem 1.2 and Corollary 3.3 can be identified with $\mathcal{T}$.

(3) Since a real-analytic curve not contained in a hyperplane is of finite type, we have the following:

COROLLARY 3.3. *Let γ be a compact real-analytic space curve with non-vanishing tangents and let $T_{\sigma_\gamma} f = \sigma_\gamma * f$, where σ_γ is the affine arclength measure on the curve γ. Then T_{σ_γ} maps $L^{3/2}(\mathbb{R}^3)$ boundedly into $L^2(\mathbb{R}^3)$.*

(4) As in Drury [3], one can prove the following result on Fourier restriction:

COROLLARY 3.4. *Let $\gamma : I \to \mathbb{R}^3$ be a smooth compact curve of finite type in $\mathbb{R}^3$ with non-vanishing tangents. Then $\|\widehat{f}\|_{L^q(d\sigma_\gamma)} \leq \|f\|_{L^p(\mathbb{R}^3)}$, whenever $6/p + 1/q = 6$ and $1 \leq p < 36/31$.*

4. A flat example

In this section, we prove Theorem 1.3. By Corollary 3.3, we have only to show that there exists $t_0 > 0$ such that $T_{\sigma_\gamma}^{t_0}$ is bounded from $L^{3/2}(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$. For simplicity of notation, put $\psi(t) = e^{-1/t}$, and $\lambda(t) = |\psi'''(t)|^{1/6}$. A homogeneity argument reduces the proof to obtaining a uniform estimate on $\{\|S_u\|_{L^{3/2}(\mathbb{R}^2)\mapsto L^3(\mathbb{R}^2)}\}_{0<u<t_0}$ for some $t_0 > 0$, where S_u are the operators given by

$$S_u g(x_2, x_3) = u^{1/3} \int_0^{t_0 - u} g(x_2 - s, x_3 - \psi(s+u) + \psi(s))\lambda(s+u)\lambda(s)\, ds.$$

The following observation suggested by Professor Stephen Wainger will be useful:

https://doi.org/10.1017/S144678870000375X Published online by Cambridge University Press

PROPOSITION 4.1. *Let N be a positive integer. For $s > 0$ we let $\psi(s) = e^{-1/s}$ and for $u, s > 0$ we let $\psi_u(s) = \psi(s+u) - \psi(s)$.*

Then there exists a positive real number t_0 depending only on N such that for $j = 0, 1, \ldots, N$, the following hold

(i) $\psi_u^{(j)}(s) > 0$, *if* $0 < s < t_0$;
(ii) $\psi_u^{(j)}(s) \sim \psi^{(j)}(s+u)$, *if* $0 < u < t_0$, $0 < s < t_0 - u$, *and* $u \geq s^2$;
(iii) $\psi_u^{(j)}(s) \sim u/s^2 \psi^{(j)}(s+u)$, *if* $0 < u < t_0$, $0 < s < t_0 - u$ *and* $u \leq s^2$.

PROOF. (i) One has only to observe that for $j \geq 1$

$$\psi^{(j)}(s) = \frac{1 - Q_j(s)}{s^{2j}} e^{-1/s},$$

with a polynomial $Q_j(s)$ of degree $j - 1$ which vanishes at $s = 0$.
(ii) It suffices to show the existence of $t_0 > 0$ and a constant $C_j \in (0, 1)$ such that

$$\frac{\psi^{(j)}(s)}{\psi^{(j)}(s+u)} \leq C_j,$$

whenever $0 < s < t_0 - u$ and $u \geq s^2$. This amounts to show

$$F(s,u) \equiv \frac{(s+u)^{2j}}{s^{2j}} e^{-1/s+1/(s+u)} \leq C_j.$$

For $0 < t_0 < 1$ to be determined later, let $0 < u < t_0$ and $u \geq s^2$. Suppose $u \leq t_0 s$. Then

$$F(s,u) \leq (1+t_0)^{2j} e^{-1/(u\sqrt{u}(\sqrt{u}+u))} \leq (1+t_0)^{2j} e^{-1/2}.$$

On the other hand, if $u \geq t_0 s$, we have

$$F(s,u) \leq \left(1 + \frac{u}{s}\right)^{2j} e^{-u/st_0} \leq (1+t_0)^{2j} e^{-1}.$$

Here, the last inequality follows from the fact that $G(y) = (1+y)^a e^{-y/t_0}$ is monotone decreasing on $0 < y < \infty$, if $0 < t_0 < 1/a$. It is only a matter of choosing $t_0 > 0$ small enough.
(iii) One needs to estimate

$$J = \frac{\psi^{(j)}(s+u) - \psi^{(j)}(s)}{(u/s^2)\psi^{(j)}(s+u)}.$$

We write $J = J_1 + J_2 + J_3$, where

$$J_1 = \frac{1 - (1 - Q_j(s))/(1 - Q_j(s+u))}{u/s^2}, \quad J_2 = \frac{1 - (1+u/s)^{2j}}{u/s^2} \frac{1 - Q_j(s)}{1 - Q_j(s+u)}$$

https://doi.org/10.1017/S144678870000375X Published online by Cambridge University Press

and

$$J_3 = \left(1 + \frac{u}{s}\right)^{2j} \frac{1 - Q_j(s)}{1 - Q_j(s+u)} \frac{1 - e^{-u/s(s+u)}}{u/s^2}.$$

By continuity of Q_j and from $Q_j(0) = 0$, one can choose $t_0 > 0$ such that

$$|J_1| \le 2t_0^2 \sup_{0 \le s \le t_0} |Q_j'(s)|, \quad |J_2| \le 2^{2j+1} j\, t_0.$$

Since for any $0 \le y \le 1$, $(1 - e^{-1})y \le 1 - e^{-y} \le y$, we have $J_3 \sim 1$, which in turn implies $J \sim 1$. The proof of Proposition 4.1 is now complete. □

Implications of Proposition 4.1 are

(1) $\psi_u(s), \psi_u'(s), \psi_u''(s) > 0$;
(2) $0 < C_1 \le \psi_u(s)\psi_u''(s)/\psi_u'(s)^2 \le C_2 < \infty$;
(3) $\psi_u(s)\psi_u'''(s) \le C_2 \psi_u'(s)\psi_u''(s)$.

The following lemma shows that S_u can be dominated by the convolution operator with the affine arclength measure on the curve $v \mapsto \psi_u(v)$.

LEMMA 4.2. *Let $t_0 > 0$ be sufficiently small and let*

$$R_u(s) = \frac{u^{1/3}\lambda(s+u)\lambda(s)}{\psi_u''(s)^{1/3}}$$

for $0 < u < t_0$ and $0 < s < t_0 - u$. Then there exists a constant C_2 such that $R_u(s) \le C_2$.

PROOF. There are three cases to consider:

Case I: $u \le s^2$.

$$\begin{aligned} R_u(s) &\preceq C u^{1/3} \frac{e^{-1/(6s)-1/(6(s+u))}/(s(s+u))}{(u/s^2)^{1/3} e^{-1/(3(s+u))}/(s+u)^{4/3}} \\ &= C \left(\frac{s+u}{s}\right)^{1/3} e^{-1/(6s)+1/(6(s+u))} \le C_2. \end{aligned}$$

Case II: $s^2 \le u \le s$.

$$\begin{aligned} R_u(s) &\le C u^{1/3} \frac{(s+u)^{1/3}}{s} e^{-1/(6s)+1/(6(s+u))} \\ &\le C \frac{u^{1/3}(s+u)^{1/3}}{s} e^{-u/(6s(s+u))} \le C \left(\frac{u}{s^2}\right)^{1/3} e^{-u/(12s^2)} \le C_2. \end{aligned}$$

https://doi.org/10.1017/S144678870000375X Published online by Cambridge University Press

Case III: $s \le u$.

$$\begin{aligned} R_u(s) &\le Cu^{2/3}s^{-1}e^{-1/(6s)+1/(6(s+u))} \\ &\le Cu^{2/3}u^{-1}e^{-1/(12u)} \le Cu^{-1/3}e^{-1/(12u)} \le C_2. \end{aligned}$$

Note that the second inequality follows from the following lemma: □

LEMMA 4.3. *Let t_0, u and s be as in Lemma 4.2. Then G_u defined by*

$$G_u(s) = s^{-1}e^{-1/(6s)+1/(6(s+u))}$$

is monotone increasing on $0 < s \le u$.

PROOF. From $\log G_u(s) = -\log s - 1/(6s) + 1/(6(s+u))$, we get

$$\begin{aligned} \frac{G'_u(s)}{G_u(s)} &= -\frac{1}{s} + \frac{1}{6s^2} - \frac{1}{6(s+u)^2} \ge \frac{1}{12s^2} - \frac{1}{6(s+u)^2} \\ &\ge \frac{1}{12s^2} - \frac{1}{24s^2} = \frac{1}{24s^2} > 0, \end{aligned}$$

assuming t_0 is sufficiently small. This finishes the proof of Lemma 4.3 and Lemma 4.2. □

According to Lemma 2.2, it remains to prove that $\rho_u(s)$ has a bounded variation. To see this, one has only to observe:

LEMMA 4.4. *There exists $t_0 > 0$ such that, for $u \in (0, t_0)$, the function*

$$\rho_u(s) = \frac{\psi_u(s)\psi''_u(s)}{\psi'_u(s)^2}$$

is decreasing on $(0, t_0 - u)$.

PROOF.

$$\begin{aligned} \frac{d}{ds}\rho_u(s) = &-\frac{1-2(s+u)}{(1-(1+u/s)^2e^{-u/(s(s+u))})^3}\left\{\mathrm{I}\,\frac{u(2s+u)}{(s(s+u))^2} + \mathrm{II}\right\}e^{-u/(s(s+u))} \\ &-2\left\{\frac{\left(1-\frac{1-2s}{1-2(s+u)}(1+u/s)^4e^{-u/(s(s+u))}\right)(1-e^{-u/(s(s+u))})}{(1-(1+u/s)^2e^{-u/(s(s+u))})^2}\right\}. \end{aligned}$$

Here, we put

$$\mathrm{I} = \frac{1-2s}{1-2(s+u)}\left(1+\frac{u}{s}\right)^4(1-e^{-u/(s(s+u))})\left(1-\left(1+\frac{u}{s}\right)^2e^{-u/(s(s+u))}\right)$$

https://doi.org/10.1017/S144678870000375X Published online by Cambridge University Press

$$+\left(1-\frac{1-2s}{1-2(s+u)}\left(1+\frac{u}{s}\right)^4 e^{-u/(s(s+u))}\right)\left(1-\left(1+\frac{u}{s}\right)^2 e^{-u/(s(s+u))}\right)$$
$$-2\left(1+\frac{u}{s}\right)^2\left(1-e^{-u/(s(s+u))}\right)\left(1-\frac{1-2s}{1-2(s+u)}\left(1+\frac{u}{s}\right)^4 e^{-u/(s(s+u))}\right),$$

and

$$\mathrm{II}=\frac{4u}{(1-2(s+u))^2}\left(1+\frac{u}{s}\right)^4\left(1-e^{-u/(s(s+u))}\right)\left(1-\left(1+\frac{u}{s}\right)^2 e^{-u/(s(s+u))}\right)$$
$$-\frac{4u}{s^2}\frac{1-2s}{1-2(s+u)}\left(1+\frac{u}{s}\right)^3\left(1-e^{-u/(s(s+u))}\right)\left(1-\left(1+\frac{u}{s}\right)^2 e^{-u/(s(s+u))}\right)$$
$$-\frac{4u}{s^2}\left(1+\frac{u}{s}\right)\left(1-e^{-u/(s(s+u))}\right)\left(1-\frac{1-2s}{1-2(s+u)}\left(1+\frac{u}{s}\right)^4 e^{-u/(s(s+u))}\right).$$

We write

$$\begin{aligned}\mathrm{I}&=\left\{\frac{1-2s}{1-2(s+u)}\left(1+\frac{u}{s}\right)^4-2\left(1+\frac{u}{s}\right)^2+1\right\}\\&\quad+\left(1+\frac{u}{s}\right)^2\left\{\frac{1-2s}{1-2(s+u)}\left(1+\frac{u}{s}\right)^4-\frac{2(1-2s)}{1-2(s+u)}\left(1+\frac{u}{s}\right)^2+1\right\}e^{-u/(s(s+u))}\\&=\left\{\frac{1-2s}{1-2(s+u)}\left(1+\frac{u}{s}\right)^4-2\left(1+\frac{u}{s}\right)^2+1\right\}\\&\quad+\left(1+\frac{u}{s}\right)^2\frac{1-2s}{1-2(s+u)}\left\{\left(\left(1+\frac{u}{s}\right)^2-1\right)^2-\frac{2u}{1-2(s+u)}\right\}e^{-u/(s(s+u))}\\&=A+B,\end{aligned}$$

and

$$\begin{aligned}\mathrm{II}&=\frac{4u}{(1-2(s+u))^2}\left(1+\frac{u}{s}\right)^4\left(1-e^{-u/(s(s+u))}\right)\left(1-\left(1+\frac{u}{s}\right)^2 e^{-u/(s(s+u))}\right)\\&\quad-\frac{4u}{s^2}\left(1+\frac{u}{s}\right)\left\{\frac{1-2s}{1-2(s+u)}\left(1+\frac{u}{s}\right)^2-1\right\}\left(1-e^{-u/(s(s+u))}\right)\\&=C+D.\end{aligned}$$

When $s\le\sqrt{u}$,

$$\begin{aligned}&\frac{B}{(1+u/s)^2[(1-2s)/(1-2(s+u))]e^{-u/(s(s+u))}}\\&\quad=\left(\left(1+\frac{u}{s}\right)^2-1\right)^2-\frac{2u}{1-2(s+u)}\ge\left(\left(1+\sqrt{u}\right)^2-1\right)^2-\frac{2u}{1-t_0}\\&\quad=4u+4u\sqrt{u}+u^2-\frac{2u}{1-t_0}\ge 0\end{aligned}$$

with $t_0 > 0$ chosen sufficiently small. For $u/4 \le s \le \sqrt{u}$, we have $A \ge 4u/s$, and

$$\begin{aligned} A\frac{u(2s+u)}{(s(s+u))^2} + D &\ge \frac{4u}{s^2}\frac{u}{s(s+u)} + D \\ &= \frac{4u}{s^2}\left\{\frac{u}{s(s+u)} - 5\left(\frac{1-2s}{1-2(s+u)}\left(1+\frac{u}{s}\right)^2 - 1\right)\right\} \\ &\ge \frac{4u}{s^2}\left\{\frac{1}{t_0}\frac{u}{s} - 200u - 30\frac{u}{s}\right\} \ge 0. \end{aligned}$$

For $0 < s \le u/4$, we have $A \ge (u/s)^4$ and

$$A\frac{u(2s+u)}{(s(s+u))^2} + D \ge \frac{u}{s^2}\left\{\frac{u^4}{s^3(s+u)} - 64u - 3\left(\frac{u}{s}\right)^2\right\} \ge 0.$$

Therefore, for $s \le \sqrt{u}$, we have

$$\frac{d}{ds}\left\{\frac{\psi_u(s)\psi_u''(s)}{\psi_u'(s)^2}\right\} \le 0.$$

For $s \ge \sqrt{u}$, we have

$$A \ge 4u/s, \quad B \ge -4u, \quad D \ge -24(u/s^2)^2(u/s),$$

and so

$$\begin{aligned} \frac{A}{2} + B &\ge 2u\left(\frac{1}{s} - 1\right) \ge 0, \\ \frac{A}{2}\frac{u(2s+u)}{(s(s+u))^2} + D &\ge \frac{4u^2}{s^3}\frac{1}{s+u}(1-6t_0) \ge 0. \end{aligned}$$

Hence, for $s \ge \sqrt{u}$, we have

$$\frac{d}{ds}\frac{\psi_u(s)\psi_u''(s)}{\psi_u'(s)^2} \le 0.$$

This finishes the proof of Lemma 4.4 and Theorem 1.3. □

REMARK 4.5. We can identify the type set for $T_{\sigma_\gamma}^{t_0}$ with $\mathscr{T}$.

Acknowledgements

The author would like to take this opportunity to express his sincere gratitude to Professor Stephen Wainger and Professor Andreas Seeger for bringing this problem to his attention and giving valuable advice. He would also like to thank the referee for useful suggestions.

References

[1] Y. Choi, 'Convolution operators with the affine arclength measure on plane curves', *J. Korean Math. Soc.* **36** (1999), 193–207.
[2] M. Christ, 'Endpoint bounds for singular fractional integral operators', *Preprint*, 1988.
[3] S. W. Drury, 'Degenerate curves and harmonic analysis', *Math. Proc. Cambridge Philos. Soc.* **108** (1990), 89–96.
[4] W. Littman, 'L^p-L^q-estimates for singular integral operators arising from hyperbolic equations', in: *Proc. Sympos. Pure Math. XXIII* (Amer. Math. Soc., Providence, RI, 1973) pp. 479–481.
[5] D. Oberlin, 'Convolution estimates for some measures on curves', *Proc. Amer. Math. Soc.* **99** (1987), 56–60.
[6] Y. Pan, 'A remark on convolution with measures supported on curves', *Canad. Math. Bull.* **36** (1993), 245–250.
[7] ———, 'Convolution estimates for some degenerate curves', *Math. Proc. Cambridge Philos. Soc.* **116** (1994), 143–146.
[8] ———, 'L^p-improving properties for some measures supported on curves', *Math. Scand.* **78** (1996), 121–132.
[9] F. Ricci and E. M. Stein, 'Harmonic analysis on nilpotent groups and singular integrals. III. Fractional integration along manifolds', *J. Funct. Anal.* **86** (1989), 360–389.
[10] S. Secco, 'Fractional integration along homogeneous curves in $\mathbb{R}^3$', *Math. Scand.* **85** (1999), 259–270.
[11] E. M. Stein, *Singular integrals and differentiability properties of functions* (Princeton University Press, Princeton, NJ, 1970).
[12] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals* (Princeton University Press, Princeton, NJ, 1993).
[13] E. M. Stein and G. Weiss, *Introduction to Fourier analysis in Euclidean spaces* (Princeton University Press, Princeton, NJ, 1971).
[14] T. Tao and J. Wright, 'L^p improving bounds for averages along curves', *J. Amer. Math. Soc.* **16** (2003(, 605–638.
[15] A. Zygmund, *Trigonometric series. Vols. I, II*, 2nd Edition (Cambridge University Press, New York, NY, 1959).

Department of Mathematics
Ajou University
Suwon 442–749
Korea
e-mail: youngwoo@ajou.ac.kr