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A NOTE ON THE PAPER BY H.J. BIERENS: "COMPLEX UNIT ROOTS AND BUSINESS CYCLES: ARE THEY REAL?"

Ignacio Díaz-Emparanza
Universidad del País Vasco–Euskal Herriko Unibertsitatea

This note discusses the concept of aliasing and its use in the paper by H.J. Bierens (2001, *Econometric Theory* 17, 962–983), in the framework of a second-order autoregression with complex unit roots. The condition on the range of the angular frequency ϕ is extended to $\phi \in (0,2\pi) - \{\pi\}$.

Bierens (2001) studies the asymptotic properties of autoregressive moving average (ARMA) processes with complex-conjugate unit roots in the autoregressive (AR) lag polynomial and derives a nonparametric test of the complex unit root hypothesis against the stationarity hypothesis based on the standardized periodogram. The article is really a valuable work, and what I want to point out here is only a marginal note that does not affect the main results of the paper. In Section 2.1, Bierens presents the structure of an AR(2) process with complex roots. I do not agree with the interpretation of the aliasing problem in the second paragraph of Section 2.1.

The process being considered is (1): $y_t = 2\cos(\phi)y_{t-1} - y_{t-2} + \mu + u_t$ where u_t is $i.i.d.(0,\sigma^2)$, μ is a constant, and $\phi \in (0,\pi)$. It is assumed that y_t is observable for $t = 1,\dots,n$. As mentioned in the paper, the AR lag polynomial $\Phi(L) = 1 - 2\cos(\phi)L + L^2$ has two roots on the complex unit circle, $\exp(i\phi) = \cos(\phi) + i\sin(\phi)$ and its complex-conjugate $\exp(-i\phi) = \cos(\phi) - i\sin(\phi)$ (assuming $\sin(\phi) \neq 0$). In the paragraph mentioned, the author says:

> Note that (1) generates a persistent cycle of $2\pi/\phi$ periods. If $\phi \in (\pi,2\pi)$, the cycle length is less than two periods. Such short cycles are unlikely to occur in macroeconomic time series, and if they occur, they are difficult, if not impossible, to distinguish from random variation. This is the reason for only considering the case $\phi \in (0,\pi)$.

If we had continuous data, each root would imply a different cycle. So, in this case we would have a cycle of $2\pi/\phi$ periods (caused by the root $\cos(\phi) +$

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$i\sin(\phi)$) and another cycle (its alias) of $2\pi/(2\pi - \phi)$ periods.[1] For example, if $\phi = \pi/2$, we have a cycle of $2\pi/(\pi/2) = 4$ periods and another cycle of $2\pi/(2\pi - \pi/2) = \frac{4}{3}$ periods. But both cycles imply the same cyclical behavior in a time series observed at discrete points (for a detailed explanation of this problem, see, e.g., Hamilton, 1994, p. 161). When we work with time series data, normally we only have the values of the variables observed at discrete points, and we do not know what the structure of our variables would be if they could be observed continuously. We do not have such information, and so we cannot decide if the relevant cycle for our variable is the one of period $2\pi/\phi$ or the one of period $2\pi/(2\pi - \phi)$. This is the problem, and this is why the only solution we have is to consider both cycles together because we cannot determine, based on discrete time series data, which is the relevant frequency for our variables.

So, what happens in Bierens's paper with frequencies $\phi \in (\pi, 2\pi)$? They are also being considered in his analysis, given that we may define $\phi' = 2\pi - \phi$, then $\phi' \in (0, \pi)$ and

$$\exp(i\phi) = \cos(\phi) + i\sin(\phi) = \cos(\phi') - i\sin(\phi') = \exp(-i\phi'),$$
$$\exp(-i\phi) = \cos(\phi) - i\sin(\phi) = \cos(\phi') + i\sin(\phi') = \exp(i\phi').$$

Frequencies $\phi \in (\pi, 2\pi)$ are already included in Bierens's analysis through the root $\exp(-i\phi)$. So, condition $\phi \in (0, \pi)$ may be relaxed to $\phi \in (0, 2\pi) - \{\pi\}$, and the aforementioned paragraph of Section 2.1 might (or should) be written:

> Note that (1) generates two persistent cycles of $2\pi/\phi$ and $2\pi/(2\pi - \phi)$ periods. The cycle length corresponding to one of the roots—$\exp(-i\phi)$ if $\phi \in (0, \pi)$ and $\exp(i\phi)$ if $\phi \in (\pi, 2\pi)$—is less than two periods. Such short cycles are unlikely to occur in macroeconomic time series, but they are impossible to distinguish from their alias[2] based only on the observed discrete data. This is the reason for working with the two cycles together.

NOTES

1. This is because the root $\exp(-i\phi) = \cos(\phi) - i\sin(\phi)$ may be expressed in an equivalent form as $\exp[i(2\pi - \phi)] = \cos(2\pi - \phi) + i\sin(2\pi - \phi)$.

2. The alias is the cycle corresponding to root $\exp(i\phi)$ if $\phi \in (0, \pi)$ and $\exp(-i\phi)$ if $\phi \in (\pi, 2\pi)$.

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Bierens, H.J. (2001) Complex unit roots and business cycles: Are they real? *Econometric Theory* 17, 962–983.

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