Bull. Aust. Math. Soc. **94** (2016), 278–285
doi:10.1017/S0004972716000095

GENERAL STABILITY OF THE EXPONENTIAL AND LOBAČEVSKIĬ FUNCTIONAL EQUATIONS

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(Received 23 December 2015; accepted 8 January 2016; first published online 8 March 2016)

Abstract

Let S be a semigroup possibly with no identity and $f : S \to \mathbb{C}$. We consider the general superstability of the exponential functional equation with a perturbation ψ of mixed variables

$$|f(x+y) - f(x)f(y)| \le \psi(x,y) \quad \text{for all } x, y \in S.$$

In particular, if S is a uniquely 2-divisible semigroup with an identity, we obtain the general superstability of Lobačevskiĭ's functional equation with perturbation ψ

$$\left| f\left(\frac{x+y}{2}\right)^2 - f(x)f(y) \right| \le \psi(x,y) \quad \text{for all } x, y \in S.$$

2010 *Mathematics subject classification*: primary 39B82.

Keywords and phrases: exponential functional equation, Lobačevskiĭ functional equation, stability.

1. Introduction

Throughout this paper, S is a semigroup and X is a real normed space. As usual, $\mathbb{R}^+$ is the set of nonnegative real numbers, $\mathbb{C}$ the set of complex numbers and $\delta \ge 0$.

A function $m : S \to \mathbb{C}$ is called an *exponential function* if $m(x+y) = m(x)m(y)$ for all $x, y \in S$. The Ulam problem for functional equations goes back to 1940 when Ulam proposed the following problem (later published in [9]): *let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot\,,\cdot)$ such that*

$$d(f(xy), f(x)f(y)) \le \delta.$$

Does there exist a group homomorphism h and $\theta_\delta > 0$ such that

$$d(f(x), h(x)) \le \theta_\delta$$

for all $x \in G_1$?

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (no. 2015R1D1A3A01019573).

This problem was solved affirmatively by Hyers under the assumption that G_2 is a Banach space (see [5, 6]).

As a result of the Ulam problem for the exponential functional equation, it is well known that if $f : S \to \mathbb{C}$ satisfies

$$|f(x+y) - f(x)f(y)| \le \delta$$

for all $x, y \in S$, then f is either a bounded function satisfying $|f(x)| \le \frac{1}{2}(1 + \sqrt{1+4\delta})$ for all $x \in S$, or an exponential function (see [1, 2]). Székelyhidi [8] generalised this result to the case when the difference $f(x+y) - f(x)f(y)$ is bounded for each fixed y (or, equivalently, for each fixed x). In particular, if S is a group, it is proved in [3] that if $f : S \to \mathbb{C}$ satisfies

$$|f(x+y) - f(x)f(y)| \le \phi(y) \text{ or } \phi(x)$$

for all $x, y \in S$ and for some $\phi : S \to [0, \infty)$, then f is either an exponential function or a bounded function satisfying $|f(x)| \le \frac{1}{2}(1 + \sqrt{1 + 4\phi(x)})$ for all $x \in S$ and either $\frac{1}{2}(1 + \sqrt{1 - 4\phi(x)}) \le |f(x)| \le \frac{1}{2}(1 + \sqrt{1 + 4\phi(x)})$ for all $x \in S_0 := \{x \in S : \phi(x) < \frac{1}{4}\}$, or $|f(x)| \le \frac{1}{2}(1 - \sqrt{1 - 4\phi(x)})$ for all $x \in S_0$.

During the Thirty-first International Symposium on Functional Equations, Rassias posed an open problem concerning the behaviour of solutions of the inequality

$$|f(x+y) - f(x)f(y)| \le \theta(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in X$ and for some $\theta > 0, p > 0$ (see [7, page 211] for more detail). To answer this question, Găvruţă investigated the stability of (1.1). As a result, he proved the following theorem in [4] (see also [7, Theorem 9.6]).

THEOREM 1.1. *Assume that $f : X \to \mathbb{C}$ satisfies (1.1). Then either f satisfies*

$$|f(x)| \le \frac{1}{2}(2^p + \sqrt{4^p + 8\theta})\|x\|^p \tag{1.2}$$

for all $x \in X$ with $\|x\| \ge 1$, or f is an exponential function.

A careful observation shows that the degree p of the upper bound function in (1.2) can be refined to $p/2$. In this paper, using a new approach, we prove the refined stability result for the exponential and Lobačevskiĭ functional equations

$$|f(x+y) - f(x)f(y)| \le \psi(x, y), \tag{1.3}$$

$$\left| f\left(\frac{x+y}{2}\right)^2 - f(x)f(y) \right| \le \psi(x, y) \tag{1.4}$$

for all $x, y \in S$. Since the left-hand sides of (1.3) and (1.4) are symmetric with respect to x and y, without loss of generality we may assume that $\psi(x, y)$ is symmetric. In addition, we assume that $\psi : S \times S \to \mathbb{R}^+$ satisfies the following condition: there exist positive constants a_1, a_2 such that

$$\psi(x+y, z) \le a_1(\psi(x, z) + \psi(y, z)), \tag{1.5}$$

$$\psi(x, y) \le a_2(\psi(x, x) + \psi(y, y)) \tag{1.6}$$

for all $x, y, z \in S$.

Remark 1.2. It is easy to see that if ψ satisfies (1.5) and (1.6), then there exist positive constants c_1, c_2, c_3 such that

$$\psi(2x, z) \le c_1\psi(x, x) + \alpha(z), \tag{1.7}$$
$$\psi(2x + y, z) \le c_2\psi(x, x) + \beta(y, z), \tag{1.8}$$
$$\psi(2x, 2x) \le c_3\psi(x, x) \tag{1.9}$$

for all $x, y, z \in S$, where $\alpha : S \to \mathbb{R}^+, \beta : S \times S \to \mathbb{R}^+$ are appropriately chosen functions. We give examples of ψ satisfying (1.5) and (1.6) later (see Remark 2.3).

As a direct consequence of our main result, it is shown that the upper bound function in (1.2) can be refined in the whole domain by

$$|f(x)| \le \tfrac{1}{2}(\sqrt{2^p} + \sqrt{2^p + 8\theta\|x\|^p}) \tag{1.10}$$

for all $x \in X$. Note that for $\|x\| \ge 1$,

$$\tfrac{1}{2}(\sqrt{2^p} + \sqrt{2^p + 8\theta\|x\|^p}) < \sqrt{2\theta}\sqrt{\|x\|^p} + \sqrt{2^p} < \tfrac{1}{2}(2^p + \sqrt{4^p + 8\theta})\|x\|^p.$$

Thus, the upper bound function in (1.10) is much smaller than that in (1.2) in both degree and coefficient. Further, the degree $p/2$ in (1.10) will be shown to be optimal.

2. Superstability of the exponential functional equation

In this section, we consider the superstability of the exponential functional equation (1.3). Let $S^* = \{x \in S : \psi(x, x) \ne 0\}$. From (1.9), $\sup_{x\in S^*} \psi(2x, 2x)/\psi(x, x) < \infty$. From now on, we set $\mu = \max\{1, \sup_{x\in S^*} \psi(2x, 2x)/\psi(x, x)\}$.

Theorem 2.1. *Assume that $f : S \to \mathbb{C}$ satisfies (1.3). Then either f satisfies*

$$|f(x)| \le \tfrac{1}{2}(\sqrt{\mu} + \sqrt{\mu + 4\psi(x, x)}) \tag{2.1}$$

for all $x \in S$, or f is an exponential function.

Proof. Let $L > 0$ be an arbitrary real number and let $\phi_L(x) = \max\{1, L\psi(x, x)\}$. Then

$$\sup_{x\in S} \frac{\phi_L(2x)}{\phi_L(x)} \le \mu \tag{2.2}$$

for all $L > 0$. Also, it is easy to see that

$$\min\{1, L\}\phi_1(x) \le \phi_L(x) \le \max\{1, L\}\phi_1(x) \tag{2.3}$$

for all $x \in S$ and $L > 0$. From (2.3),

$$\sup_{x\in S} \frac{|f(x)|}{\sqrt{\phi_L(x)}} := M_L < \infty \tag{2.4}$$

for all $L > 0$, or

$$\sup_{x\in S} \frac{|f(x)|}{\sqrt{\phi_L(x)}} = \infty \tag{2.5}$$

for all $L > 0$.

https://doi.org/10.1017/S0004972716000095 Published online by Cambridge University Press

First, we assume that (2.4) holds. Replacing y by x in (1.3) and using the triangle inequality with the result,

$$|f(x)|^2 \leq |f(2x)| + \psi(x,x) \leq |f(2x)| + \frac{1}{L}\phi_L(x) \tag{2.6}$$

for all $x \in S$. Dividing (2.6) by $\phi_L(x)$ and using (2.2) and (2.4),

$$\begin{aligned}\left(\frac{|f(x)|}{\sqrt{\phi_L(x)}}\right)^2 &\leq \frac{|f(2x)|}{\phi_L(x)} + \frac{1}{L} \leq M_L \frac{\sqrt{\phi_L(2x)}}{\phi_L(x)} + \frac{1}{L} \\ &\leq M_L \sqrt{\frac{\phi_L(2x)}{\phi_L(x)}} + \frac{1}{L} \leq M_L \sqrt{\mu} + \frac{1}{L}.\end{aligned} \tag{2.7}$$

Taking the supremum of the left-hand side of (2.7) yields

$$M_L^2 - \sqrt{\mu} M_L - \frac{1}{L} \leq 0. \tag{2.8}$$

By solving the quadratic inequality (2.8),

$$M_L \leq \tfrac{1}{2}(\sqrt{\mu} + \sqrt{\mu + 4/L}). \tag{2.9}$$

From (2.4) and (2.9),

$$|f(x)| \leq \tfrac{1}{2}(\sqrt{\mu} + \sqrt{\mu + 4/L})\sqrt{\max\{1, L\psi(x,x)\}} \tag{2.10}$$

for all $x \in S$ and $L > 0$. Fix $x_0 \in S$. If $\psi(x_0, x_0) > 0$, then we can apply (2.10) with $L := 1/\psi(x_0, x_0)$ to get

$$|f(x)| \leq \frac{1}{2}(\sqrt{\mu} + \sqrt{\mu + 4\psi(x_0,x_0)})\sqrt{\max\left\{1, \frac{\psi(x,x)}{\psi(x_0,x_0)}\right\}}. \tag{2.11}$$

Putting $x = x_0$ in (2.11),

$$|f(x_0)| \leq \tfrac{1}{2}(\sqrt{\mu} + \sqrt{\mu + 4\psi(x_0,x_0)}). \tag{2.12}$$

If $\psi(x_0, x_0) = 0$, then, from (2.10),

$$|f(x_0)| \leq \tfrac{1}{2}(\sqrt{\mu} + \sqrt{\mu + 4/L}) \tag{2.13}$$

for all $L > 0$. Letting $L \to \infty$ in (2.13) yields

$$|f(x_0)| \leq \sqrt{\mu} = \tfrac{1}{2}(\sqrt{\mu} + \sqrt{\mu + \psi(x_0,x_0)}). \tag{2.14}$$

Thus, from (2.12) and (2.14) we get (2.1).

Now we assume that (2.5) holds. Then we can choose $x_n \in S, n = 1, 2, \ldots,$ such that

$$\frac{\sqrt{\psi(x_n, x_n)}}{|f(x_n)|} + \frac{1}{|f(x_n)|} \to 0 \quad \text{as } n \to \infty. \tag{2.15}$$

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Replacing (x, y) by $(x + y, z)$ in (1.3) gives

$$|f(x + y + z) - f(x + y)f(z)| \le \psi(x + y, z) \tag{2.16}$$

for all $x, y, z \in S$ and multiplying by $|f(z)|$ on both sides of (1.3) gives

$$|f(x + y)f(z) - f(x)f(y)f(z)| \le \psi(x, y)|f(z)| \tag{2.17}$$

for all $x, y, z \in S$. Using the triangle inequality with (2.16) and (2.17),

$$|f(x + y + z) - f(x)f(y)f(z)| \le \psi(x + y, z) + \psi(x, y)|f(z)| \tag{2.18}$$

for all $x, y, z \in S$. Replacing both x and y by x_n in (2.18), dividing the result by $|f(x_n)|^2$ and using (1.7),

$$\begin{aligned} \left|\frac{f(2x_n + z)}{f(x_n)^2} - f(z)\right| &\le \frac{\psi(2x_n, z) + \psi(x_n, x_n)|f(z)|}{|f(x_n)|^2} \\ &\le \frac{(c_1 + |f(z)|)\psi(x_n, x_n) + \alpha(z)}{|f(x_n)|^2}. \end{aligned} \tag{2.19}$$

Letting $n \to \infty$ in (2.19) and using (2.15),

$$f(z) = \lim_{n\to\infty} \frac{f(2x_n + z)}{f(x_n)^2}. \tag{2.20}$$

Multiplying both sides of (2.20) by $f(w)$ and using (1.3),

$$f(z)f(w) = \lim_{n\to\infty} \frac{f(2x_n + z)f(w)}{f(x_n)^2} = \lim_{n\to\infty} \frac{f(2x_n + z + w) + R(x_n, z, w)}{f(x_n)^2}, \tag{2.21}$$

where $R(x_n, z, w) = f(2x_n + z)f(w) - f(2x_n + z + w)$. Now, using (1.8),

$$|R(x_n, z, w)| \le \psi(2x_n + z, w) \le c_2\psi(x_n, x_n) + \beta(z, w) \tag{2.22}$$

for all $x_n, z, w \in S$. Using (2.15) in (2.22),

$$\frac{R(x_n, z, w)}{f(x_n)^2} \to 0 \quad \text{as } n \to \infty.$$

Thus, from (2.20) and (2.21),

$$f(z)f(w) = \lim_{n\to\infty} \frac{f(2x_n + z + w)}{f(x_n)^2} = f(z + w)$$

for all $z, w \in S$. This completes the proof. □

REMARK 2.2. As a matter of fact, fixing $x \in S$ and taking the infimum of the right-hand side of (2.10) with respect to $L > 0$, we get the inequality (2.1).

REMARK 2.3. In particular, let $S = X$ be a normed space and $p_j, q_j, a_j, j = 1, 2, \dots, m$, be sequences of nonnegative real numbers. Then

$$\psi(x, y) = \sum_{j=1}^{m} a_j \|x\|^{p_j} \|y\|^{q_j}$$

satisfies (1.7) and (1.8) and, if $p = \max\{p_j + q_j : j = 1, 2, \dots, m\}$, then $\mu = 2^p$. Now, as a direct consequence of Theorem 2.1, we obtain the following corollaries.

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COROLLARY 2.4. *Assume that* $f : X \to \mathbb{C}$ *satisfies*

$$|f(x+y) - f(x)f(y)| \le a_1\|x\|^p + a_2\|x\|^{p/2}\|y\|^{p/2} + a_3\|y\|^p$$

for all $x, y \in X$. *Then either* f *satisfies*

$$|f(x)| \le \tfrac{1}{2}(\sqrt{2^p} + \sqrt{2^p + 4(a_1 + a_2 + a_3)\|x\|^p})$$

for all $x \in X$, *or* f *is an exponential function.*

With $a_1 = a_3 = \theta, a_2 = 0$, Corollary 2.4 gives a refined version of Theorem 1.1.

COROLLARY 2.5. *Assume that* $f : X \to \mathbb{C}$ *satisfies (1.1). Then either* f *satisfies*

$$|f(x)| \le \tfrac{1}{2}(\sqrt{2^p} + \sqrt{2^p + 8\theta\|x\|^p})$$

for all $x \in X$, *or* f *is an exponential function.*

REMARK 2.6. In Corollary 2.5, the degree $p/2$ of the upper bound function of a nonexponential function f satisfying (1.1) is optimal in the sense that one cannot replace $\sqrt{\|x\|^p}$ by a function $\sqrt{\|x\|^q}$ of smaller degree with $q < p$. Indeed, let

$$f(x) = \begin{cases} \delta\sqrt{\|x\|^p}, & \|x\| \ge 1, \\ \delta\|x\|^p, & \|x\| < 1. \end{cases} \tag{2.23}$$

If we choose $\delta = \frac{1}{2}(-\lambda + \sqrt{\lambda^2 + 4\theta})$ with $\lambda = \max\{1, 2^{p-1}\}$, then the inequality $\|x+y\|^p \le \max\{1, 2^{p-1}\}(\|x\|^p + \|y\|^p)$ yields

$$|f(x+y) - f(x)f(y)| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. However, f in (2.23) does not satisfy $\sup_{\|x\|\ge 1} |f(x)|/\|x\|^q < \infty$ for $q < p$.

3. Superstability of Lobačevskiĭ's functional equation

Using the same argument as in Section 2, we obtain the superstability of Lobačevskiĭ's functional equation. In this section, we assume that S is uniquely 2-divisible (that is, for each $x \in S$, there exists a unique $y \in S$ such that $y + y = x$). In addition to the assumptions (1.5)–(1.7), we assume that $\psi_0(x, y) := \psi(x + y, 0)$ satisfies the same conditions. In this section, we denote

$$\lambda = \max\left\{1, \sup_{x\in S} \frac{\psi(2x, 2x) + \psi(4x, 0)}{\psi(x, x) + \psi(2x, 0)}\right\}.$$

THEOREM 3.1. *Assume that* $f : S \to \mathbb{C}$ *satisfies* (1.4). *Then, if* $f(0) = 0$,

$$|f(x)| \le \sqrt{\psi(2x, 0)} \tag{3.1}$$

for all $x \in S$ *and, if* $f(0) \ne 0$, *then either* f *satisfies*

$$|f(x)| \le \tfrac{1}{2}(|f(0)|\sqrt{\lambda} + \sqrt{|f(0)|^2\lambda + 4(\psi(x, x) + \psi(2x, 0))}) \tag{3.2}$$

for all $x \in S$, *or* $f(x)/f(0)$ *is an exponential function.*

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Proof. Putting $y = 0$ in (1.4),

$$\left|f\left(\frac{x}{2}\right)^2 - f(x)f(0)\right| \le \psi(x,0) \tag{3.3}$$

for all $x \in S$. If $f(0) = 0$, replacing x by $2x$ in (3.3) gives (3.1). If $f(0) \neq 0$, from (1.4) and (3.3), using the triangle inequality and dividing the result by $|f(0)|^2$,

$$|F(x+y) - F(x)F(y)| \le \frac{1}{|f(0)|^2}(\psi(x+y,0) + \psi(x,y))$$

for all $x, y \in S$, where $F(x) = f(x)/f(0)$. By Theorem 2.1,

$$|F(x)| \le \tfrac{1}{2}\left(\sqrt{\lambda} + \sqrt{\lambda + \frac{4}{|f(0)|^2}(\psi(x,x) + \psi(2x,0))}\right) \tag{3.4}$$

for all $x \in S$, or F is an exponential function. Multiplying both sides of (3.4) by $|f(0)|$ gives (3.2). This completes the proof. □

In particular, let $S = X$ be a real normed space. Then we obtain the following result.

Corollary 3.2. *Assume that $f : X \to \mathbb{C}$ satisfies*

$$\left|f\left(\frac{x+y}{2}\right)^2 - f(x)f(y)\right| \le a_1\|x\|^p + a_2\|x\|^{p/2}\|y\|^{p/2} + a_3\|y\|^p$$

for all $x, y \in X$. Then either f satisfies

$$|f(x)| \le \tfrac{1}{2}(|f(0)|\sqrt{2^p} + \sqrt{|f(0)|^2 2^p + 4((2^p+1)a_1 + a_2 + a_3)\|x\|^p})$$

for all $x \in X$, or $f(x)/f(0)$ is an exponential function.

Letting $a_1 = a_3 = \theta, a_2 = 0$ in Corollary 3.2, we obtain the following result.

Corollary 3.3. *Assume that $f : X \to \mathbb{C}$ satisfies*

$$\left|f\left(\frac{x+y}{2}\right)^2 - f(x)f(y)\right| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then either f satisfies

$$|f(x)| \le \tfrac{1}{2}(|f(0)|\sqrt{2^p} + \sqrt{|f(0)|^2 2^p + 8\theta(1 + 2^{p-1})\|x\|^p})$$

for all $x \in X$, or $f(x)/f(0)$ is an exponential function.

Acknowledgement

The author expresses his deep gratitude to the referee for clarifying the proof of Theorem 2.1 and for helpful comments on an earlier version of the paper.

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https://doi.org/10.1017/S0004972716000095 Published online by Cambridge University Press