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MINIMAL CONVEX USCOS AND MONOTONE OPERATORS ON SMALL SETS

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ABSTRACT. We generalize the generic single-valuedness and continuity of monotone operators defined on open subsets of Banach spaces of class (S) and Asplund spaces to monotone operators defined on convex subsets of such spaces which may even fail to have non-support points. This yields differentiability theorems for convex Lipschitzian functions on such sets. From a result about minimal convex uscos which are densely single-valued we obtain generic differentiability results for certain Lipschitzian real-valued functions.

1. **Introduction.** Recently, Verona ([Ve], [V-V]), Noll ([No1], [No2]) and Rainwater [Ra] have studied the differentiability properties of Lipschitzian convex functions on sets which may have empty interior. A convex subset C of a normed linear space E has *normal cone* $N_C(x)$ at $x \in C$, defined by

$$N_C(x) := \{x^* \in E^* \mid \langle x^*, c - x\rangle \leq 0 \text{ for all } c \in C\}.$$

The point $x \in C$ is a *non-support point* provided $N_C(x) = \{0\}$. In Section 2 we prove results that generalize the generic single-valuedness and continuity of monotone operators on open subsets of Banach spaces E of class (S) and Asplund spaces to monotone operators defined and bw^* upper semicontinuous on convex subsets of E which may have no non-support points. These results show single-valuedness or continuity modulo elements of the normal cone to the domain. We recall that the bounded weak* topology on E^* is generated by the polars of compact subsets of E and coincides with the weak* topology on w^* compact subsets of E^*. However for our purposes it will be convenient to use a slightly different topology: we denote by bw^* convergence of bounded nets in E^*.

In Section 3 we apply these results to Lipschitzian convex functions on such "small" convex sets and prove generic Gatêaux or Fréchet differentiability (in a natural sense) of such functions.

Since maximal monotone operators on open sets in Banach spaces and Clarke subgradients of Lipschitzian functions are convex-valued weak* uscos (upper semicontinuous compact-valued mappings), we take the abbreviation one stage further and call a multivalued mapping $T: X \longrightarrow E^*$ (more properly $T: X \longrightarrow 2^{E^*}$) *weak* cusco* provided it has nonempty weak* compact convex values and is weak* upper semicontinuous. It is *minimal weak* cusco* provided it does not contain any other weak* cusco with the same

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domain. In Section 4 we prove results about minimal weak* cusco mappings which are densely single-valued and apply these results to locally Lipschitzian real-valued functions to generalize the generic differentiability results in [deB-F-G].

Preiss [Pr] has proved an intricate theorem more general than the following statement. [We denote the weak* closure of a set A by weak* A.]

PREISS' THEOREM. *Let E be a Banach space with an equivalent smooth norm or an Asplund space. If $f: U \longrightarrow \mathbb{R}$ is a locally Lipschitzian function on an open subset U of E then f is Gateâux differentiable on a dense subset D of U and for $x \in U$ the Clarke subgradient of f is*

$$\partial f(x) = \bigcap_{\varepsilon>0} \textit{weak}^* \operatorname{conv}\{\nabla f(y) \mid y \in D \cap B(x,\varepsilon)\}.$$

If E is an Asplund space then the same holds for D being the set of points of Fréchet differentiability of f.

The Fréchet property characterizes Asplund spaces. We turn Preiss' theorem into a definition and say a Banach space E is a *Preiss space* provided every locally Lipschitzian real valued function on an open subset U of E is Gatêaux differentiable on a dense subset D of U such that the formula

$$\partial f(x) = \bigcap_{\varepsilon>0} \text{weak}^* \operatorname{conv}\{\nabla f(y) \mid y \in D \cap B(x,\varepsilon)\}$$

holds for all $x \in U$. It is not known whether every Preiss space has an equivalent smooth norm. (Remark 4.12 added in revision provides a negative answer.) Nor is it known whether a Preiss space must be of class (S), where a Banach space E is of *class* (S) [St] provided every weak* usco from a Baire space into E^* has a selection which is generically weak* continuous. This property is also described by saying the topological space (E^*, weak*) has *type S*. As will be seen later in this section, any Banach space E with rotund dual is of class (S), and is a Preiss space because the norm is smooth. Every Asplund space is a class (S) Preiss space. There are Fréchet normable, hence Asplund, spaces whose duals admit no rotund dual norm [Ph2 p. 90], consequently the same is true of class (S) Preiss spaces. See also Remark 4.12.

Now we give some preliminary results. The first one shows the relationship between minimal weak* cuscos and minimal weak* uscos. For us a set is *generic* if it contains a dense G_δ set and *residual* if it contains a countable intersection of dense open sets. The following basic result appears in Jokl [Jo].

THEOREM 1.1. *If $T: X \longrightarrow E^*$ is weak* usco then $T^*(x) :=$ weak* conv $T(x)$ defines a weak* cusco T^* on X. If T is minimal weak* usco then T^* is minimal weak* cusco.*

This yields a characterization of spaces of class (S).

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THEOREM 1.2. *A Banach space E is of class (S) precisely if every minimal weak* cusco from a Baire space into E^* is generically single-valued.*

PROOF. Suppose that this property holds for weak* cucsos and let T be a weak* usco from a Baire space X into E^* and let T_m be a minimal weak* usco contained in T. By Theorem 1.1, T_m^* is a minimal weak* cusco.

Now T_m^* is single-valued on generic subset G of X. Therefore T_m is single-valued on G and any selection σ of T_m is weak* continuous at each point of G, showing that E is of class (S).

On the other hand suppose E is of class (S). If T is a minimal weak* cusco into E^* then for any minimal weak* usco S contained in T we have $S^*(x) = \text{weak}^* \operatorname{conv} S(x)$ is a minimal weak* cusco by Theorem 1.1 and thus $S^* = T$ and T is single-valued whenever S is. Thus if E is of class (S) and X is Baire then T is generically single-valued. ■

Now we show that any space with rotund dual is of class (S).

THEOREM 1.3. *Let E be a Banach space whose dual norm is strictly convex. Then E is of class (S).*

PROOF. Let T be a minimal weak* cusco from a Baire space X into E^*. By Theorem 1.2 it suffices to show T is generically single-valued.

Define $\varphi(x) := \inf\{\, \|x^*\| \mid x^* \in T(x)\}$. Then φ is lower semicontinuous and so continuous at the points of some generic subset G of X. If $x \in G$ and $x^*, y^* \in T(x)$ then it is not possible that $\|x^*\| = \|y^*\| = \varphi(x)$ and $x^* \neq y^*$: for then $(x^*+y^*)/2 \in T(x)$ and has norm less than $\varphi(x)$ by strict convexity, contradicting the definition of $\varphi(x)$. So if $T(x)$ is not a singleton there is $y^* \in T(x)$ with $\|y^*\| > \varphi(x)$. Choose $z \in E$ with $\|z\| = 1$ and $\langle y^*, z\rangle > \varphi(x)$. Then $W := \{w^* \in E^* \mid \langle w^*, z\rangle > (\varphi(x) + \langle y^*, z\rangle)/2\}$ is a weak* open half space with $y^* \in W$. Since $\|w^*\| > \big(\varphi(x) + \langle y^*, z\rangle\big)/2 > \varphi(x)$ for each $w^* \in W$ we see from continuity of φ at x that there is an open set U containing x such that for each $u \in U$ we have $T(u)\setminus W \neq \emptyset$. We define

$$T_1(v) := \begin{cases} T(v)\setminus W & \text{if } v \in U \\ T(v) & \text{if } v \in X\setminus U. \end{cases}$$

We see that T_1 is a weak* cusco on X with $T_1 \leq T$ and $T_1(x) \neq T(x)$. That contradicts minimality at T. ■

Next we note the extension procedure for Lipschitzian (convex) functions.

PROPOSITION 1.4. *Suppose f is a real-valued function on an open convex subset U of a normed linear space E and has Lipschitz constant L on U. Then*

$$\bar{f}(x) := \inf\{f(y) + L\|x - y\| \mid y \in U\}$$

defines an extension of f to E which has Lipschitz constant L on E (and is convex if f is). ■

We will use the following form of Fort's theorem [Fo]. Since his terminology varies from what later became standard we indicate a proof.

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THEOREM 1.5 (FORT). *Let X be a topological space and Y a separable metric space. If $T: X \longrightarrow Y$ is a multivalued upper semicontinuous mapping then there is a residual subset G of X such that T is lower semicontinuous at each point of G.*

OUTLINE OF PROOF. Let $\{U_n\}$ be a countable closed base for the topology of Y and define $G := \bigcap_{n\in\mathbb{N}} G_n$ where

$$G_n := \operatorname{int} T^{-1}(U_n) \cup \big(X \setminus T^{-1}(U_n)\big).$$

Then G_n is open and dense in X so that G is residual. It is routine to check that T is lower semicontinuous at each point of G. ∎

Certain cusco mappings, for example maximal monotone operators on separable spaces, are single-valued except on sets with complements of "measure zero" in an appropriate sense ([Ar], [Ph1]). The following results show that general cuscos do not share such a property, as there are dense G_δ sets which have measure zero.

LEMMA 1.6. *Let X be a metric space and G_n a decreasing sequence of open subsets of X with dense intersection. Let $f(x) := \sum_{n=1}^{\infty} 10^{-n} \sin\big(1/d(x, X\setminus G_n)\big)$ for $x \in G := \bigcap_{n\in\mathbb{N}} G_n$. Then*

(a) f is continuous on G and

(b) no extension of f to X is continuous at any point of $X \setminus G$.

PROOF. (a) Let $x \in G$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that $10^{-N} < \varepsilon$ and let $y \in G$. Then

$$|f(y) - f(x)| \leq \sum_{n=1}^{N} 10^{-n} \big|\sin\big(1/d(x, X\setminus G_n)\big) - \sin\big(1/d(y, X\setminus G_n)\big)\big| + \sum_{n=N+1}^{\infty} 2\, 10^{-n}$$

which is less than ε if $|y - x|$ is small enough.

(b) We let N be the first integer such that $x \notin G_N$.

Since G is dense for each $\delta > 0$ there are $y, z \in B(x, \delta) \cap G$ with

$$10^{-N} \big|\sin\big(1/d(y, X\setminus G_N)\big) - \sin\big(1/d(z, X\setminus G_N)\big)\big| > 10^{-N}.$$

However

$$\sum_{j=N+1}^{\infty} 10^{-j} \big|\sin\big(1/d(y, X\setminus G_j)\big) - \sin\big(1/d(z, X\setminus G_j)\big)\big| < 2 \sum_{j=N+1}^{\infty} 10^{-j} < 10^{-N}/2$$

and

$$\sum_{j=1}^{N-1} 10^{-j} \big|\sin\big(1/d(y, X\setminus G_j)\big) - \sin\big(1/d(z, X\setminus G_j)\big)\big| \longrightarrow 0$$

as $\delta \longrightarrow 0+$ so that $|f(y) - f(z)| > 10^{-N}/2$ for some $y, z \in B(x, \delta) \cap G$. Thus f does not have an extension which is continuous at x. ∎

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THEOREM 1.7. *For f as in Lemma 1.6 define*

$$Tx := \left[\liminf_{y \to x, y \in G} f(y), \limsup_{y \to x, y \in G} f(y)\right]$$

for each $x \in X$. Then T is a minimal cusco which is single-valued at x if and only if $x \in G$.

PROOF. It is easy to check that $T : X \longrightarrow \mathbb{R}$ is a cusco. If T contains a cusco T_1 then $T_1(y)$ must contain $f(y)$ for each $y \in G$ as $T(y) = \{f(y)\}$ for those points. It follows that $T_1 x$ contains $\liminf_{y \to x, y \in G} f(y)$ and $\limsup_{y \to x, y \in G} f(y)$ and by convexity of $T_1(x)$ we have $T(x) \subseteq T_1(x)$ as required. ■

2. **Usco mappings on Baire spaces.** We start with a general result about bw^*-closed multivalued operators, that is, if (x_α, x_α^*) is a net of points in the graph of the operator with (x_α^*) bounded and $x_\alpha \longrightarrow x$ while $x_\alpha^* \longrightarrow x^*$ weak* then (x, x^*) is also in the graph.

THEOREM 2.1. *Let X be a Baire topological space and E a normed linear space. Suppose T is a multivalued operator from X to E^* with bw^* closed graph. The following conditions are equivalent*

(a) $X \setminus D(T)$ is nowhere dense in X.

(b) $D(T)$ is generic in X.

(c) There is a dense subset G of X and a locally bounded selection of $T|_G$.

(d) There is a countable disjoint collection $\{U_n \mid n \in \mathbb{N}\}$ of open subsets of x such that $G := \cup\{U_n \mid n \in \mathbb{N}\}$ is dense in X and on G the mapping T_1 defined by

$$T_1(x) := T(x) \cap nB_{E^*} \text{ if } x \in U_n \tag{2.1}$$

is weak usco.*

PROOF. (a)⇒(b) and (d)⇒(c) are trivial.

(b)⇒(d): Let $A_n := \{x \in X \mid T(x) \cap nB_{E^*} \neq \emptyset\}$. Then $D(T) = \cup_{n \in \mathbb{N}} A_n$ and each A_n is closed (because the graph of T bw^* closed). For each nonempty open subset U of X there is a nonempty open subset V of U and $n \in \mathbb{N}$ such that $V_\alpha \subseteq A_n$ (because X is Baire space). Let $\{V_\gamma : \gamma \in \Gamma\}$ be a maximal disjoint collection of open subsets of X with $V \subseteq A_{n(\alpha)}$ for some $n(\alpha) \in \mathbb{N}$ and let $U_n := \cup\{V_\alpha \mid n(\alpha) = n\}$. For contradiction, suppose $G := \cup_{n \in \mathbb{N}} U_n$ is not dense. Then some nonempty open subset U of X does not meet G, and some nonempty open subset V of U is contained in some A_n. Now $\{V_a : \alpha \in \Gamma\} \cup \{V\}$ is a larger collection than our maximal one, a contradiction which shows that G is dense. It is clear that T_1 as defined in (2.1) is weak* usco on G.

(c)⇒(a) If $x \in G$ there is an open neighbourhood U of x such that $T|_{U \cap G}$ has a bounded selection σ. Now we claim that $U \subseteq D(T)$. Indeed, if $y \in U$ and $x_\alpha \in U \cap G$ are such that $x_\alpha \longrightarrow y$ then $\sigma(x_\alpha)$ is a bounded net and so has a bw^* cluster point y^*. Thus (y, y^*) is in the bw^*-closure of the graph of T, showing that $y^* \in T(y)$ and then $y \in D(T)$ as required.

Thus $X \setminus D(T)$ is nowhere dense in X. ■

If we impose suitable conditions on the normed space E we can deduce the existence of selections which are generically continuous.

THEOREM 2.2. *Let X be a Baire topological space and E a normed space. Suppose T is a multivalued operator from X to E^* with $D(T)$ generic in X and the graph of T bw*-closed. Then there are disjoint open subsets U_n, $n \in \mathbb{N}$, of X such that $G := \bigcup_{n\in\mathbb{N}} U_n$ is dense in X, $G \subseteq D(T)$, and there is a selection σ of $T|_{D(T)}$ such that $\|\sigma(x)\| \leq n$ for each $x \in U_n$ and*

(a) *if B_{E^*} has type S in its weak* topology then σ is generically weak* continuous and*

(b) *if E is an Asplund space then σ is generically norm continuous.*

PROOF. We deduce the existence of U_n open, with $G := \bigcup_{n\in\mathbb{N}} U_n$ dense in X and $T_1 := T \cap nB_{E^*}$ weak* usco on U_n from Theorem 2.1. Now S_n defined by $S_n(x) := \frac{1}{n}T(x) \cap B_{E^*}$ is a weak* usco from the Baire space U_n into B_{E^*}. Thus S_n has a selection μ_n which is generically weak* continuous in case (a); and generically norm continuous in case (b) (Rainwater [Ra], Proposition 5). Define $\sigma(x) := n\mu_n(x)$ if $x \in U_n$ and select any element of $T(x)$ if $x \in D(T)\setminus G$. Then σ has the required properties. ∎

COROLLARY 2.3 (Stegall [St]). *A Banach space E is of class (S) if and only if B_{E^*} is of type S in its weak* topology.* ∎

We note two particular cases in which Theorem 2.2 applies.

REMARK 2.4.

(a) If T is a weak* usco then T has weak* closed graph (hence bw* closed). This case was analysed for Asplund spaces by Jokl [Jo].

(b) If T is a maximal monotone operator on the normed linear space E then T has *bw** closed graph. However an example is given in [Fi] of a maximal monotone on a Hilbert space whose graph is not closed in the product of the norm and bounded weak topologies. ∎

For monotone operators defined on convex subsets of E we now investigate the consequences of continuity of selections.

LEMMA 2.5. *Let T be a monotone operator on a Banach space E and let C be a convex subset of E. Suppose D is a dense subset of C and σ is a selection of $T|_D$. Let $x \in D$.*

(a) *If σ is weak* continuous at x then*

$$T(x) \subseteq \sigma(x) + N_C(x).$$

(b) *If σ is norm continuous at x then for each $\varepsilon > 0$ there is $\delta > 0$ such that if $\|y - x\| < \delta$ and $y \in C$ then*

$$T(y) \subseteq B[\sigma(x), \varepsilon] + N_C(y).$$

PROOF. (a) Let $x^* \in T(x)$ and $y \in C$. For each $n \in \mathbb{N}$ let $x_n := x + (y - x)/n$ and

$$y_n \in D \text{ with } \|y_n - x_n\| < 1/n^2.$$

By monotonicity we have $\langle \sigma(y_n) - x^*, y_n - x \rangle \geq 0$, so

$$\langle \sigma(y_n) - x^*, (y-x)/n \rangle \geq \langle \sigma(y_n) - x^*, x_n - y_n \rangle$$
$$\geq -\|\sigma(y_n) - x^*\| \cdot \|x_n - y_n\| \geq -\|\sigma(y_n) - x^*\| / n^2.$$

Since $\sigma(y_n)$ converges weak* to $\sigma(x)$, $\{\|\sigma(y_n)\|\}$ is bounded (E being complete) and thus $\langle \sigma(x) - x^*, y - x \rangle \geq 0$. That shows $T(x) \subseteq \sigma(x) + N_C(x)$.

(b) Let $\delta > 0$ be such that $\|\sigma(y) - \sigma(x)\| \leq \varepsilon$ whenever $\|y - x\| < \delta$. If $\|y - x\| < \delta$ and $y^* \in T(y)$, suppose for contradiction that $y^* \notin B[\sigma(x), \varepsilon] + N_C(y)$. Then we can separate by a weak* continuous functional, that is, there is $z \in E$ with

$$\langle y^* - \sigma(x), z \rangle > \sup \langle \varepsilon B_{E^*} + N_C(y), z \rangle$$
$$= \varepsilon \|z\| + \sup \langle N_C(y), z \rangle$$

which show that $z \in N_C(y)^\circ = \overline{\mathbb{P}(C-y)}$ and $\langle y^* - \sigma(x), z \rangle > \varepsilon \|z\|$. Therefore we can find $c \in C$ with $\langle y^* - \sigma(x), c - y \rangle > \varepsilon \|c - y\|$.

Let $x_n \in D$ with $\|y + (c-y)/n - x_n\| < 1/n^2$. Then $\langle \sigma(x_n) - y^*, x_n - y \rangle \geq 0$ by monotonicity so that

$$\langle \sigma(x_n) - y^*, (c-y)/n \rangle \geq \langle \sigma(x_n) - y^*, y + (c-y)/n - x_n \rangle$$
$$\geq -\|\sigma(x_n) - y^*\| \cdot \|y + (c-y)/n - x_n\|$$
$$\geq -\|\sigma(x_n) - y^*\| / n^2.$$

Now for n large, $\|x_n - x\| < \delta$ so $\sigma(x_n) \in B[\sigma(x), \varepsilon]$. Thus

$$\begin{aligned}\varepsilon \|c - y\| &< \langle y^* - \sigma(x), c - y \rangle \\ &\leq \liminf_{n \to \infty} \big(\langle y^* - \sigma(x_n), c - y \rangle + \|\sigma(x_n) - \sigma(x)\| \cdot \|c - y\| \big) \\ &\leq \liminf_{n \to \infty} \langle y^* - \sigma(x_n), c - y \rangle + \varepsilon \|c - y\| \\ &\leq \liminf_{n \to \infty} \|\sigma(x_n) - y^*\| / n + \varepsilon \|c - y\| \\ &= \varepsilon \|c - y\|\end{aligned}$$

which is a contradiction, showing that $T(y) \subseteq B[\sigma(x), \varepsilon] + N_C(y)$ as required. ∎

THEOREM 2.6. *Let T be a monotone operator on a Banach space E with norm $\times bw^*$ closed graph. Let C be a convex subset of E which is a Baire space in its relative norm topology. If $D(T) \cap C$ is generic in C then there are disjoint relatively open subsets U_n, $n \in \mathbb{N}$, of C with $U := \bigcup_{n \in \mathbb{N}} U_n$ dense in C and a selection σ of $T|_{C \cap D(T)}$ such that $\|\sigma\| \leq n$ on U_n, $U_n \subseteq D(T)$ and*

(a) if E is of class (S) then there is a generic subset G of C such that for each $x \in G$, σ is weak continuous at x and $T(x) \subseteq \sigma(x) + N_C(x)$; while*

(b) if E is an Asplund space there is a generic subset G of C such that for each $x \in G$, σ is norm continuous at x and for each $\varepsilon > 0$ there is $\delta = \delta(x, \varepsilon) > 0$ so that $T(y) \subseteq B[\sigma(x), \varepsilon] + N_C(y)$ whenever $\|y - x\| < \delta$ and $y \in C$.

PROOF. To get the generically continuous selection use Theorem 2.2. Now Lemma 2.5 shows that $T(x) \subseteq \sigma(x) + N_C(x)$ in case (a) and that $T(y) \subseteq B[\sigma(x), \varepsilon] + N_C(y)$ in case (b). ∎

REMARKS 2.7.

(1) Any $\delta \geq 0$ such that $\sigma\big(B(x,\delta)\big) \subseteq B[\sigma(x),\varepsilon]$ will do in (b).

(2) If $N_C(x) = \{0\}$, that is, if x is a *non-support point* of C, then (a) becomes $T(x) = \{\sigma(x)\}$.

(3) Furthermore, the non-support points, $N(C)$, of C form a G_δ convex subset of C [Ph3], so if there are non-support points of C then T is single-valued on a generic subset of $N(C)$ in case (a) and $T|_{N(C)}$ is norm continuous on a generic subset of $N(C)$ in case (b).

(4) Maximal monotone operators and subdifferentials of convex functions satisfy the bw^* closed graph condition. ∎

Now we turn our attention to separable normed linear spaces. These are of class S, but we get generic results that are stronger than those available in nonseparable spaces.

THEOREM 2.8. *Let C be a convex subset of a separable normed linear space E which is a Baire space in the relative norm topology. Then for x in a generic subset of C we have $N_C(x) = (C-x)^\perp$.*

PROOF. Let $T(x) = N_C(x) \cap B_{E^*}$ for each $x \in C$. Then T is norm-weak* upper semicontinuous and B_{E^*} is compact, separable and metrizable in the weak* topology. By Fort's Theorem 1.5, T is weak* lower semicontinuous on a generic subset G of C.

If $x \in G$ let $y^*, z^* \in T(x)$. Then we claim $y^* - z^* \in (C-x)^\perp$. Otherwise there is $c \in C$ so that $\langle y^* - z^*, c-x\rangle \neq 0$ and we may assume $\langle y^* - z^*, c-x\rangle > 0$. If $x_t := x + t(c-x)$ and $x_t^* \in T(x_t)$ then $\langle x_t^* - y^*, c-x\rangle \geq 0$. Thus $x_t^* \notin W := \{z^* \in B_{E^*} \mid \langle z^*, c-x\rangle < \langle y^*, c-x\rangle$. However $T(x) \cap W \neq \emptyset$ and W is weak* open. That contradicts weak* lower semicontinuity of T.

Thus if $x \in G$ and $y_1^*, z_1^* \in N_C(x)$ we can take $\lambda > 0$ so that $y^* := \lambda y_1^*$ and $z^* := \lambda z_1^*$ are in B_{E^*} and then $\lambda(y_1^* - z_1^*) \in (C-x)^\perp$. Thus for $x \in G$, $N_C(x) \subseteq (C-x)^\perp$. But we always have $(C-x)^\perp \subseteq N_C(x)$. ∎

Now for monotone operators on such a set C we don't need closedness of the graph and get a stronger conclusion than that of Theorem 2.6(a).

THEOREM 2.9. *Let E be a separable Banach space and C a convex subset of E which is a Baire space in its relative norm topology. Suppose T is a monotone operator from C into E^* with $D(T)$ generic in C. Then for x in a generic subset of C*

$$T(x) - T(x) \subseteq (C-C)^\perp.$$

PROOF. Let S be the norm $\times bw^*$ closure of T. Then S is monotone and by Theorem 2.6 (as E is of class (S)) there are disjoint relatively open subsets U_n of C with $\bigcup_{n\in\mathbb{N}} U_n$ dense in C and a selection σ of $S|_{D(S)\cap C}$ with $\|\sigma(x)\| \leq n$ for all $x \in U_n$, $U_n \subseteq D(S)$ and a generic set G in C so that $S(x) \subseteq \sigma(x) + N_C(x)$ for all $x \in G$. By Theorem 2.8 there is a generic set H in C so that $N_C(x) = (C-x)^\perp$ for all $x \in H$. Thus for all $x \in G \cap H$ we have

$$S(x) \subseteq \sigma(x) + (C-x)^\perp$$

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so that

$$S(x) - S(x) \subseteq (C - x)^{\perp}$$

and in particular $T(x) - T(x) \subseteq (C - x)^{\perp}$ for all x in the generic sebset $G \cap H$ of C, which completes the proof. ∎

3. **Convex Lipschitzian functions.** In this section we apply the result of Section 2 to the monotone operator ∂f where f is a convex Lipschitzian function on C. The results we obtain hold even when the closed affine span of C is less than all of E, otherwise they are similar to those in [No1] and [V-V]. We start with an example to show the need for some Lipschitz assumption.

EXAMPLE 3.1. Let C be the positive cone in ℓ_2 and

$$f(x) := \begin{cases} \sup_{\mathbb{N}} \exp(-nx_n) & x \in C \\ +\infty & x \notin C. \end{cases}$$

Then

(a) f is lower semicontinuous and convex on C;
(b) If $x_n := t/n$ where $t \geq 0$ then $f(x) = e^{-t}$;
(c) $f(C) = (0, 1]$;
(d) $f(x) = 1$ on the dense subset $D := \{y \in C \mid \text{some } y_n = 0\}$ of C.
(e) Hence f is not continuous on C;
(f) If $x \in D$ and $x_n = 0$ then $(-\infty, -n]\delta_n \subseteq \partial f(x)$;
(g) If $x \in C/D = N(C)$ and $f(x) = e^{-nx_n}$ then $-ne^{-nx_n}\delta_n \in \partial f(x)$;
(h) If $x \in C/D$ and $f(x) > e^{-nx_n}$ for all n then $\partial f(x) = \emptyset$;
(i) Hence $\partial f(x) = \emptyset$ on a dense subset of C.
(j) Thus ∂f densely nonempty does not imply ∂f generically nonempty. ∎

If f is a convex function on a convex subset C of E we say $x^* \in E^*$ is a *Gateâux derivative* of f at $x \in C$ provided for every $c \in C$

$$\lim_{t \to 0+} \frac{f\big(x + t(c - x)\big) - f(x)}{t} = \langle x^*, c - x \rangle .$$

We say x^* is a *Fréchet derivative* of f at x provided

$$\lim_{\substack{c \to x \\ c \in C \setminus \{x\}}} \frac{f(c) - f(x) - \langle x^*, c - x \rangle}{\|c - x\|} = 0.$$

THEOREM 3.2. *Let E be a Banach space of class (S) and let C be a G_δ convex subset of E. If $f: C \longrightarrow \mathbb{R}$ is locally Lipschitzian convex on C then there is a dense G_δ subset G of C such that f has a Gateâux derivative at each point of G.*

PROOF. Let $T(x) := \partial f(x)$ (which can be seen to be nonempty using Proposition 1.4) for all $x \in C$. Then T has norm $\times bw^*$ closed graph and Theorem 2.6(a) yields a selection σ of T such that $T(x) \subseteq \sigma(x) + N_C(x)$ for x in a generic subset G of C. We will show that $\sigma(x)$ is a Gatêaux derivative for f at $x \in G$. Now let $c \in C$, and for $0 < t < 1$ let $x_t^* \in T\big(x + t(c-x)\big)$. Then $f\big(x + t(c-x)\big) - f(x) \leq \langle x_t^*, c - x\rangle t$ and if x^* is any weak* cluster point of x_t^* at $t \longrightarrow 0+$ we have $\lim_{t \to 0+}\Big(f\big(x + t(c-x)\big) - f(x)\Big)/t \leq \langle x^*, c-x\rangle$. However, $\sigma(x) \in T(x) = \partial f(x)$ so that

$$\langle \sigma(x), c - x\rangle \leq \lim_{t \longrightarrow 0+}\Big(f\big(x + t(c-x)\big) - f(x)\Big)/t.$$

Also $\langle x^*, c-x\rangle = \langle \sigma(x), c-x\rangle + \langle y^*, c-x\rangle$ for some $y^* \in N_C(x)$ so that $\langle x^*, c-x\rangle \leq \langle \sigma(x), c-x\rangle$. Thus $\langle \sigma(x), c-x\rangle = \lim_{t \to 0+}\Big(f\big(x + t(c-x)\big) - f(x)\Big)/t$ as required. ∎

THEOREM 3.3. *Let E be an Asplund space and let C be a convex G_δ subset of E. For each locally Lipschitzian convex function $f\colon C \longrightarrow \mathbb{R}$ there is a generic subset G of C such that f has a Fréchet derivative at each point of G.*

PROOF. Let $T := \partial f$, whose graph is norm $\times bw^*$ closed in $C \times E^*$. By Theorem 2.6(b), there is a selection σ of T and a generic subset G of C such that for each $x \in G$ and $\varepsilon > 0$ there is $\delta = \delta(x,\varepsilon)$ so that $T(y) \subseteq B[\sigma(x), \varepsilon] + N_C(y)$ whenever $y \in C$ has $\|x - y\| < \delta$. We will show that $\sigma(x)$ is a Fréchet derivative for f at $x \in G$. Indeed if ε and δ are as above and $y \in C$ has $\|x - y\| < \delta$ then

$$\begin{aligned} 0 &\leq f(y) - f(x) - \langle \sigma(x), y - x\rangle \\ &\leq \langle \sigma(y) - \sigma(x), y - x\rangle \leq \varepsilon\|y - x\|. \end{aligned}$$ ∎

PROPOSITION 3.4. *Let C be a convex subset of a normed space E and $f\colon C \longrightarrow \mathbb{R}$ convex and locally Lipschitzian. If D is a dense subset of C and $\tau(x) \in \partial f(x)$ for each $x \in D$ and if τ is locally bounded on C then*

$$f'_+(x; c - x) \leq \limsup_{\substack{y \longrightarrow x \\ y \in D}} \langle \tau(y), c - x\rangle$$

for each x and c in C.

PROOF. Let $x, c \in C$, $n \in \mathbb{N}$ and choose $y_n \in D$ such that $\|y_n - x - (c - x)/n\| < 1/n^2$. Then

$$\begin{aligned} n\big[f\big(x + (c-x)/n\big) - f(x)\big] &\leq n[f(y_n) - f(x)] + n\big[f\big(x + (c-x)/n\big) - f(y_n)\big] \\ &\leq n[f(y_n) - f(x)] + nL\|y_n - x - (c-x)/n\| \end{aligned}$$

for large n, where f has Lipschitz constant L on some neighbourhood of x. Thus for large n we have

$$\begin{aligned} n[f(x + (c-x)/n) - f(x)] &\leq n\langle \tau(y_n), y_n - x\rangle + L/n \\ &= \langle \tau(y_n), c - x\rangle + n\langle \tau(y_n), y_n - x - (c-x)/n\rangle + L/n \\ &\leq \langle \tau(y_n), c - x\rangle + M/n + L/n \end{aligned}$$

where $\|\tau\| \le M$ on some neighbourhood of x. Thus

$$\begin{aligned} f'_+(x, c-x) &= \lim_{n\to\infty} n\big[f\big(x+(c-x)/n\big) - f(x)\big] \\ &\le \lim_{n\to\infty} \langle \tau(y_n), c-x\rangle \\ &\le \limsup_{\substack{y\to x\\ y\in D}} \langle \tau(y), c-x\rangle \end{aligned}$$

as required. ■

Define for $x \in C$ and a locally Lipschitzian convex f on C, $D_f(x) := \bigcap_{\varepsilon>0} bw^*\mathrm{cl}\{\, y^* \mid y^*$ is a Gatêaux derivative of f at $y \in C$ and $\|y-x\| < \varepsilon\}$. Let $L_f(x)$ denote the local Lipschitz constant of f at x.

THEOREM 3.5. *Let C be a convex G_δ subset of a Banach space E of class (S) and let f be a locally Lipschitzian convex function on C. Then for all $x, c \in C$*

$$\begin{aligned} f'_+(x, c-x) &= \sup\{\,\langle x^*, c-x\rangle \mid x^* \in D_f(x)\} \\ &= \sup\{\,\langle x^*, c-x\rangle \mid x^* \in D_f(x), \|x^*\| \le L_f(x)\}, \end{aligned}$$

and

$$\begin{aligned} \partial f(x) &= N_C(x) + \mathit{weak}^* \operatorname{conv} D_f(x) \\ &= N_C(x) + \mathit{weak}^* \operatorname{conv} D_f(x) \cap B[0, L_f(x)]. \end{aligned}$$

If E is an Asplund space then the statements hold with Fréchet derivatives replacing Gatêaux derivatives in the definition of $D_f(x)$.

PROOF. We have $y^* \in \partial f(y)$ whenever y^* is a Gatêaux derivative of f at $y \in C$ so that $D_f(x) \subseteq \partial f(x)$ for all $x \in C$ by bw^* closedness of the graph of ∂f. It follows that

$$D_f(x) + N_C(x) \subseteq \partial f(x) + N_C(x) = \partial f(x)$$

and since $\partial f(x)$ is weak* closed and convex

$$N_C(x) + \text{weak}^* \operatorname{conv} D_f(x) \subseteq \partial f(x).$$

Then

$$\begin{aligned} f'_+(x, c-x) &= \sup\{\,\langle x^*, c-x\rangle \mid x^* \in \partial f(x)\} \\ &\ge \sup\{\,\langle x^*, c-x\rangle \mid x^* \in D_f(x)\} \\ &\ge \sup\{\,\langle x^*, c-x\rangle \mid x^* \in D_f(x), \|x^*\| \le L_f(x)\}. \end{aligned}$$

Now suppose $u \in C$ and let $\varepsilon > 0$. Let g be a convex function on E with Lipschitz constant $\le L_f(u) + \varepsilon$ and $g|U = f|U$ on some neighbourhood U of x, by Proposition 1.4. Then $D_g(u)$ is nonempty (as E is a weak Asplund space or an Asplund space) there is $u^*_\varepsilon \in D_f(u)$ with $\|u^*_\varepsilon\| \le L_f(u) + \varepsilon$. Take a weak* cluster point $\tau(u)$ of u^*_ε as $\varepsilon \to 0^+$ and note that $\tau(u) \in D_f(u) \cap B[0, L_f(u)] \neq \emptyset$. Let x and c belong to C. Then Proposition 3.4 shows that as $\tau(u) \in \partial f(u)$

$$\begin{aligned} f'_+(x, c-x) &\leq \limsup_{\substack{y \to x \\ y \in C}} \langle \tau(y), c-x \rangle \\ &\leq \sup\{ \langle x^*, c-x) \rangle \mid x^* \in D_f(x), \|x^*\| \leq L_f(x) \} \\ &\leq \sup\{ \langle x^*, c-x \rangle \mid x^* \in D_f(x) \} \\ &\leq f'_+(x, c-x) \end{aligned}$$

as above. Thus equality holds throughout.

Now if there is $x^* \in \partial f(x)$ not in $N_C(x)+$ weak* $\mathrm{conv}\big(D_f(x) \cap B\big[0, L_f(x)\big]\big)$, then using the separation theorem we can find $c \in C$ such that $f'_+(x, c-x) \geq \langle x^*, c - x \rangle > \sup\{ \langle y^*, c-x \rangle \mid y^* \in D_f(x), \|y^*\| \leq L_f(x) \}$ which contradicts $f'_+(x; c-x) = \sup\{ \langle y^*, c-x \rangle \mid y^* \in D_f(x), \|y^*\| \leq L_f(x) \}$. ∎

REMARK 3.6. If x is a non-support point of C then $N_C(x) = \{0\}$ and every element x^* of $D_f(x)$ has $\|x^*\| \leq L_f(x)$.

4. **Convex weak* usco mappings.** We say an usco Ω from a topological space X into (E^*,weak*) is a *weak* cusco* provided $\Omega(x)$ is always a convex set. We say a weak* cusco Ω is *thin* provided Ω is minimal among the weak* cusco mappings and $\Omega(x)$ is a singleton for x belonging to a dense subset of X.

THEOREM 4.1. *Let X be a topological space and E a Banach space. If $H: X \longrightarrow 2^{E^*}$ is locally bounded and $D(H)$ is dense in X then*

$$S(x) := \cap\{ \text{weak}^* \operatorname{conv} H(V) : V \text{ is a neighbourhood of } x \}$$

defines a weak cusco on X.*

PROOF. Since H is locally bounded and $D(H)$ is dense we have some neighbourhood U of x such that weak* conv $H(U)$ is weak* compact, and $H(V)$ nonempty for every neighbourhood V of x. Thus $S(x)$ is weak* compact convex. To see that S is weak* upper semicontinuous let W be a weak* open set containing $S(x)$. Then for some open neighbourhood V of x we have weak* conv $H(V) \subseteq W$, as $S(x)$ is the intersection of weak* compact sets of this form. Thus for $y \in V$ we have $S(y) \subseteq$ weak* conv $H(V) \subseteq W$ as required. ∎

COROLLARY 4.2. *If $T: X \longrightarrow E^*$ is locally bounded and weak* cusco and σ is a selection of $T|_D$ where D is a dense subset of X then*

$$S_\sigma(x) := \cap\{ \text{weak}^* \operatorname{conv} \sigma(V) \mid V \text{ is a neighbourhood of } x \}$$

is a weak cusco contained in T. If T is single-valued on the dense set D then*

$$T_D(x) := \cap\{ \text{weak}^* \operatorname{conv} T(V \cap D) \mid V \text{ is a neighbourhood of } x \}$$

is the minimal weak cusco inside T.*

PROOF. The first statement is obvious. For the second statement suppose H is a weak* cusco inside T. Then $H|_D = T|_D = \sigma$ so that H contains $S_\sigma = T_D$ ∎

THEOREM 4.3. *Let $T: X \longrightarrow E^*$ be a locally bounded weak* cusco. The following statements are equivalent.*

(1) T is thin.

(2) For every dense subset D of X on which T is single-valued we have $T = T_D$, and there is such a dense set.

(3) For some dense subset D of X on which T is single-valued we have $T = T_D$.

PROOF. We have (1) implies (2) by Corollary 4.2 and (2) implies (3) trivially. Now suppose (3). By Corollary 4.2 T_D is a minimal weak* cusco and we are supposing $T = T_D$, so T is minimal and densely single-valued as required. ∎

Without single-valuedness, we have the following characterization of minimal weak* cusco mappings.

PROPOSITION 4.4. *A locally bounded weak* cusco T from X to E^* is a minimal cusco if and only if $T = S_\sigma$ for every selection σ of T restricted to a dense subset D_σ of X.*

PROOF. If T is minimal then Corollary 4.2 shows that $T = S_\sigma$. On the other hand suppose $T = S_\sigma$ for every such selection σ and let T_1 be a minimal weak* cusco contained in T. Then for any selection σ of T_1 (with $D_\sigma = X$) we have $T = S_\sigma$. But Corollary 4.2 shows $S_\sigma \subseteq T_1$. Thus $T = T_1$. ∎

One reason for considering weak* cusco mappings is that monotone operators on open sets are weak* cusco if and only if maximal monotone. Precisely :

PROPOSITION 4.5 [PH2, V-V]. *Let U be an open subset of a normed linear space E and $T: E \longrightarrow E^*$ monotone with $D(T) = U$. If T is weak* cusco then T is maximal monotone on U.*

PROOF. This is Lemma 7.7 in [Ph2]. ∎

COROLLARY 4.6. *For T monotone on an open subset U of a Banach space E the following are equivalent.*

(1) T is maximal monotone on U.

(2) T is weak cusco on U.*

(3) T is minimal weak cusco on U.*

PROOF. [Ph2]. ∎

Also every locally Lipschitzian function $f: U \longrightarrow \mathbb{R}$ yields a weak* cusco ∂f from U to E^*. It is natural to look for conditions which imply that ∂f is thin, especially if E is Asplund or of class (S) because then f will be generically Fréchet or Gatêaux differentiable. Recall that f is *pseudo-regular* on U if the Clarke derivate$[f^0(x;h)]$ and the upper Dini derivate $[f'_+(x;h)]$ agree for x in U.

THEOREM 4.7. *Suppose f is a locally Lipschitzian, real valued function which is densely Gatêaux differentiable and is pseudo-regular (in the sense that the Clarke and* *upper Dini derivatives agree) on an open subset U of a normed linear space E. Then ∂f is thin on U.*

PROOF. Let

$$T(x) := \cap \left\{ \text{weak}^* \operatorname{conv}\{ \nabla f(y) \mid y \in V\} \;\middle|\; V \text{ is a neighbourhood of } x \right\}$$

so that $T(x) \subset \partial f(x)$ for all $x \in U$. If $\nabla f(x)$ exists then, by pseudo regularity, $\partial f(x) = \{ \nabla f(x)\} = T(x)$ so we see that T is minimal weak* cusco by Theorem 4.3, and that ∂f is densely single-valued. Suppose ∂f is not thin; then $T(x) \neq \partial f(x)$ for some $x \in U$. Thus there are $h \in E$ and $\varepsilon > 0$ such that $f^0(x; h) \geq \varepsilon + \sup\langle T(x), h\rangle$.

Since f is pseudo-regular $f'_+(x; h) = f^0(x; h)$ so there are $t_n \longrightarrow 0+$ with

$$\big(f(x + t_n h) - f(x)\big)/\, t_n \geq \sup\langle T(x), h\rangle + 3\varepsilon / 4.$$

By the one-dimensional mean value theorem for Lipschitzian functions there are $s_n \in\,]0, t_n[$ such that the two-sided directional derivative $f'(x + s_n h; h)$ exists and

$$f'(x + s_n h; h) \geq f(x + t_n h) - f(x)/\, t_n - \varepsilon / 4.$$

Thus $f'(x + s_n h; h) \geq \sup\langle T(x), h\rangle + \varepsilon / 2$. In particular $f'_+(x + s_n h; -h) \leq -\varepsilon / 2 - \sup\langle T(x), h\rangle$ and since f is pseudo regular, $f^0(x + s_n h : -h) \leq -\varepsilon / 2 - \sup\langle T(x), h\rangle$. Let $u_n \longrightarrow 0$ such that $\nabla f(x + s_n h + u_n)$ exists and $\langle \nabla f(x + s_n h + u_n), -h\rangle \leq f^0(x + s_n h; -h) + \varepsilon / 4$. Then $\langle \nabla f(x + s_n h + u_n)$, exists and $\langle \nabla f(x + s_n h + u_n), -h\rangle \leq f^0(x + s_n h; -h) + \varepsilon / 4$. Then $\langle \nabla f(x + s_n h + u_n), h\rangle > \varepsilon + \sup\langle T(x), h\rangle$ which contradicts the definition of T. ∎

COROLLARY 4.8. *If E is a Banach space of class (S) and f is a locally Lipschitzian, real-valued function which is pseudo-regular and densely Gateâux differentiable on an open subset U then f is strictly Gateâux differentiable on a generic subset of U.*

PROOF. By Theorem 4.7, ∂f is minimal weak* cusco so that ∂f is generically single-valued because E has class (S). ∎

If f is pseudo-regular then f is strictly Gateâux differentiable whenever it is Gateâux differentiable. An even weaker property is that f is strictly Gateâux differentiable whenever it is Fréchet differentiable. The following result generalizes Theorem 2.3 of [deB-F-G].

THEOREM 4.9. *Let E be a Preiss space of class (S) (an Asplund space) and let f be a locally Lipschitzian real-valued function on an open subset U of E. If f is strictly Gateâux differentiable whenever it is Gateâux (Fréchet) differentiable then f is strictly Gateâux (strictly Fréchet) differentiable on a generic subset of U.*

PROOF. Since E is a Preiss (Asplund) space we have f is Gateâux (Frechet) differentiable on a dense subset D of U and

$$\partial f(x) = \bigcap_{\varepsilon > 0} \text{weak}^* \operatorname{conv}\{ \nabla f(y) \mid y \varepsilon D, \|y - x\| < \varepsilon\}.$$

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Since $\partial f(y) = \{\nabla f(y)\}$ whenever $y \in D$ by strict Gateâux differentiability of f at these points, Theorem 4.3 shows that ∂f is minimal on U. Since E is of class (S) (Asplund) it follows that ∂f is generically single-valued (single-valued and norm upper-semicontinuous) on U, and generic strict differentiability follows. ∎

If we apply Theorem 4.3 to maximal monotone operators, which are minimal weak* cusco by Proposition 4.5, then we obtain the following result.

COROLLARY 4.10. *If T is a maximal monotone operator on an open subset U of a class (S) space E then T is thin and for any dense subset D of U on which T is single-valued and every $x \in U$ we have*

$$Tx = \bigcap_{\varepsilon>0} weak^* \operatorname{conv} T\big(B(x,\varepsilon) \cap D\big). \qquad ∎$$

In particular we could take $T = \partial f$ for f convex continuous on U. One class of non-convex functions for which our results apply are distance functions on spaces with uniformly Gateâux differentiable norms.

THEOREM 4.11. *Let E be a Banach space whose norm is uniformly Gateâux differentiable and $f(x) = \operatorname{dist}(x, K)$ for a nonempty closed subset K of E. Then ∂f is thin on $E \setminus K$ and for any dense subset D of E on which f is Gateâux differentiable and $x \in E \setminus K$ we have*

$$\partial f(x) = \bigcap_{\varepsilon>0} weak^* \operatorname{conv}\{\nabla f(y) \mid y \in D \cap B(x,\varepsilon)\}.$$

PROOF. The norm of E^* is strictly convex [Day, p. 148] so that E has class (S) by Theorem 1.3. From [B-F-G, p. 522] we see that f is pseudo-regular. Now Theorem 4.7 shows ∂f is thin and Theorem 4.3 shows the formula for $\partial f(x)$. ∎

In [deB-F-G] this result is proved for D the set of *all* Gateâux differentiability points. In [Bo] the application of minimal cuscos to Clarke subgradients and to differentiability of Lipschitz functions is taken considerably further.

REMARK 4.12. Since this paper was originally submitted, Phelps, Preiss and Namioka [PPN] have extended Theorem 1.3. They show, by much deeper arguments, that every Gateâux renormable space is class (S). In contrast, Haydon [Ha] has exhibited Asplund spaces with no equivalent Gateâux differentiable norm.

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