Proceedings of the Edinburgh Mathematical Society (2021) **64**, 200–216
doi:10.1017/S0013091521000092

FOURIER TRANSFORMS RELATED TO $\zeta(S)$

PABLO A. PANZONE

Departamento e Instituto de Matematica, Universidad Nacional del Sur, Av. Alem 1253, 8000 Bahia Blanca, Argentina (pablopanzone@hotmail.com)

(Received 14 February 2020; first published online 30 April 2021)

Abstract Using some formulas of S. Ramanujan, we compute in closed form the Fourier transform of functions related to Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ and other Dirichlet series.

Keywords: Fourier transform; Dirichlet Series; integrals; supported in part by UNS and CONICET

2020 *Mathematics subject classification:* Primary 42A38; 11M99;

1. Introduction and results

B. Riemann stated the following formula in his famous and epoch-making memoir *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* [9, 15]:

If $s = 1/2 + it$, $t \in \mathbb{C}$ *then*

$$\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta\left(\frac{1}{2}+it\right) = \int_0^\infty \Phi(u)\cos(ut)du, \tag{1}$$

where

$$\Phi(u) := 4\sum_{n=1}^{\infty}(2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2})e^{-\pi n^2 e^{2u}}.$$

G. H. Hardy proved in 1914 that an infinite number of zeros of the zeta function are on the critical line. G. Pólya gave a proof of this fact using the above formula [15]. Similar representations hold for other $L-$functions and this fact is well known [10]. A recent paper of D. K. Dimitrov and Y. Xu [7] provides a solution to an old problem of G. Pólya on giving necessary and sufficient conditions on an entire function of order one, which is represented in terms of a Fourier transform, to have only real zeros. In fact, in that paper, there is a necessary and sufficient condition for the Riemann Hypothesis to hold, using the above kernel $\Phi(u)$, in terms of a density condition in $L_1(\mathbb{R})$.

It is worth mentioning that there are other criteria for RH to hold: the classical Nyman–Beurling criterion [5, 8, 11] and its generalizations and refinements due to Báez–Duarte and his collaborators [1, 2].

Also, there is a vast literature concerning the location of the zeros of sine or cosine transforms with contributions of many masters of the classical analysis. The reader may consult the comprehensive paper [6].

From the above, it seems to be of some interest to have other formulas for $\zeta(s)$ or other Dirichlet series as Fourier transforms. To name a few more, there is in fact a beautiful formula due to van der Pol [13] of the sole zeta function as a Fourier transform: *if* $[x]$ *is the largest integer* $\leq x$ *and* $-1/2 < \Im t < 1/2$ *then*

$$\frac{\zeta(\frac{1}{2}+it)}{(\frac{1}{2}+it)} = \int_{-\infty}^{+\infty} (e^{-u/2}[e^u] - e^{u/2})e^{-iut}\mathrm{d}u.$$

Also, S. Ramanujan proved other formulas for $\zeta(s)$ [14, p. 72] and one wonders what other formulas might exist. In the search for other representations for $\zeta(s)$ the author [12] proved, among others, the following formula: *if* $s = \frac{1}{2} + it$ *and* $-3/2 < \Im t < 3/2$, *then*

$$\frac{s(s-1)\Gamma(1-\frac{s}{2})\Gamma(s)}{2^{s-1}\pi^{s/2}}\zeta\left(\frac{1}{2}+it\right)$$
$$= 8\pi \int_0^\infty \left\{ \int_0^\infty \frac{e^{-\pi x e^{-2u}}}{e^{2\pi\sqrt{x}}-1}\{e^{-3u/2} - e^{-7u/2}5\pi x + e^{-11u/2}2\pi^2x^2\}\,\mathrm{d}x \right\} \cos(ut)du. \quad (2)$$

The kernel in the last integral is an even function of u.

The aim of this note is to present, as an application of certain formulas given by S. Ramanujan, Fourier transforms related to the Riemann zeta function and two particular Dirichlet series:

$$L_{\chi_4}(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s},$$

(the Dirichlet series which corresponds to the non-trivial character *mod* 4) and

$$L(t) := \sum_{n=0}^{\infty} \frac{(-1)^n \mathrm{sech}\{\frac{1}{2}\pi(2n+1)\}}{(2n+1)^{it-1}}.$$

We notice that in formula (2), the poles of the Gamma factors turn out to be such that they do not cancel with the trivial zeros of $\zeta(s)$ and this is why the formula holds only in the stated strip (the same phenomenon occurs in Theorems 1 and 5). This is not the case with formula (1) where cancellations occur and therefore, the Fourier integral is an entire function (as in Theorems 2, 3 and 4).

The main results of this note are the following theorems.

Theorem 1. *Set*

$$\psi(x) := \sum_{n=1}^{\infty} e^{-n^2\pi x},$$

https://doi.org/10.1017/S0013091521000092 Published online by Cambridge University Press

$$I(\alpha) := \frac{2}{\sqrt[4]{\pi\alpha}}\psi(1/\alpha) - 2(\pi\alpha)^{3/4}\int_0^\infty \frac{e^{-\pi\alpha x}}{e^{2\pi\sqrt{x}}-1}\,dx,$$
$$\Phi(x) := I(e^x).$$

Then $\Phi(x) = \Phi(-x)$ *if* $0 < x$. *Also if* $-3/4 < \Im t < 3/4$ *then*

$$\int_0^\infty \Phi(x)\cos(tx)\,dx$$
$$= \pi^{it-1/2}\Gamma\left(-it+\frac{1}{4}\right)\left\{1-\frac{1}{\sqrt{\pi}}\Gamma\left(it+\frac{3}{4}\right)\Gamma\left(-it+\frac{3}{4}\right)\right\}\zeta\left(-2it+\frac{1}{2}\right).$$

Theorem 2. *Set*

$$I_4(\alpha) := \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n\sinh(n\alpha\pi)},$$
$$I(\alpha) := I_4(\alpha) - \alpha I_4'(\alpha) - \alpha^2 I_4''(\alpha)$$
$$\Phi(x) := 2I(e^x)$$

Then $\Phi(x) = \Phi(-x)$ *if* $0 < x$. *Also if* $t \in \mathbb{C}$

$$\int_0^\infty \Phi(x)\cos(xt)\,dx = (1+t^2)\frac{2\Gamma(it)}{\pi^{it}}\left(1-\frac{1}{2^{it}}\right)^2\zeta(it)\zeta(it+1).$$

Moreover if one sets $F(t) := (1+t^2)2\Gamma(it)/\pi^{it}(1-1/2^{it})^2$ *and* $t \in \mathbb{C}$ *then*

$$2\int_0^\infty \Phi(x)\cosh(x/2)\cos(tx)\,dx$$
$$= \zeta\left(it+\frac{1}{2}\right)\big\{F(t+i/2)\zeta(it-1/2) + F(t-i/2)\zeta(it+3/2)\big\},$$

and

$$2i\int_0^\infty \Phi(x)\sinh(x/2)\sin(tx)\,dx$$
$$= \zeta\left(it+\frac{1}{2}\right)\big\{-F(t+i/2)\zeta(it-1/2) + F(t-i/2)\zeta(it+3/2)\big\}.$$

For the next theorem, we need to recall the definition of the Bernoulli numbers B_n. These are defined by the formula

$$\frac{x}{e^x-1} = \sum_{n=0}^\infty \frac{B_n}{n!}x^n.$$

https://doi.org/10.1017/S0013091521000092 Published online by Cambridge University Press

Theorem 3. *Assume that $2 \le n < m$ and that B_n are the Bernoulli numbers. Set*

$$\begin{aligned} I_n^*(\alpha) &:= \alpha \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\pi k \sqrt[n]{\alpha}} - 1}, \\ I(\alpha) &:= I_n^*(\alpha)\frac{B_{2m}}{4m} - I_m^*(\alpha)\frac{B_{2n}}{4n}, \\ \Phi(x) &:= 2I(e^x). \end{aligned}$$

If $t \in \mathbb{C}$ and either n, m are both even numbers or n, m are both odd numbers then

$$\begin{aligned} &\frac{n\Gamma(nit+n)}{(2\pi)^{n(it+1)}}\zeta(nit+n)\zeta(nit-n+1)\frac{B_{2m}}{4m} \\ &\quad - \frac{m\Gamma(mit+m)}{(2\pi)^{m(it+1)}}\zeta(mit+m)\zeta(mit-m+1)\frac{B_{2n}}{4n} \\ &= \begin{cases} \displaystyle\int_0^{\infty} \Phi(x)\cos(tx)\,dx, \\ \displaystyle i\int_0^{\infty} \Phi(x)\sin(tx)\,dx, \end{cases} \end{aligned}$$

respectively.

The next two theorems give sine or cosine transform formulas for certain Dirichlet series.

Theorem 4. *Set*

$$\begin{aligned} I_3(\alpha) &:= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)\cosh\{(2n-1)\alpha\pi/2\}}, \\ I(\alpha) &:= I_3(\alpha)' + \alpha I_3(\alpha)'', \\ \Phi(x) &:= e^x I(e^x), \\ L_{\chi_4}(s) &:= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s}. \end{aligned}$$

If $t \in \mathbb{C}$ then

$$t\frac{2^{it}\Gamma(it+1)}{\pi^{it}}L_{\chi_4}(it)L_{\chi_4}(it+1) = \int_0^{\infty} \Phi(x)\sin(tx)\,dx.$$

Moreover if we set $F(t) := t2^{it}\Gamma(it+1)/\pi^{it}$ and $t \in \mathbb{C}$ then

$$\begin{aligned} &2\int_0^{\infty} \Phi(x)\cosh(x/2)\sin(tx)\,dx \\ &\quad = L_{\chi_4}\left(it+\frac{1}{2}\right)\left\{F(t+i/2)L_{\chi_4}(it-1/2) + F(t-i/2)L_{\chi_4}(it+3/2)\right\}, \end{aligned}$$

https://doi.org/10.1017/S0013091521000092 Published online by Cambridge University Press

and

$$2i\int_0^\infty \Phi(x)\sinh(x/2)\cos(tx)\,dx$$
$$= L_{\chi_4}\left(it+\frac{1}{2}\right)\left\{F(t+i/2)L_{\chi_4}(it-1/2) - F(t-i/2)L_{\chi_4}(it+3/2)\right\}.$$

Theorem 5. *Set*

$$L(t) := \sum_{n=0}^{\infty} \frac{(-1)^n \mathit{sech}\{\frac{1}{2}\pi(2n+1)\}}{(2n+1)^{it-1}}, \quad (t\in\mathbb{C}).$$
$$F(x) := x^2 \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)^3 \mathit{sech}\{\frac{1}{2}\pi(2n+1)\}}{\cosh\{(2n+1)x\} + \cos\{(2n+1)x\}},$$

and

$$\Phi(x) := 2F(e^x\pi/\sqrt{2}).$$

Then $\Phi(x) = \Phi(-x)$ *if* $0 < x$. *If* $\Im t < 2$ *one has*

$$\frac{2^{it/2}}{\pi^{it}}\left\{\int_0^\infty \frac{x^{it+1}}{\cosh x + \cos x}\,dx\right\}L(t) = \int_0^\infty \Phi(x)\cos(tx)dx.$$

The following lemma is a rather trivial result about the zeros of $L(t)$.

Lemma 1. *If* $\Im t \le 1.6$ *then* $L(t) \neq 0$.

Proof. Using the series defining $L(t)$ one has that

$$|L(t)| \ge \operatorname{sech}(\pi/2) - \sum_{n=1}^{\infty} \frac{\operatorname{sech}\{\frac{1}{2}\pi(2n+1)\}}{(2n+1)^{-\Im t-1}} = \operatorname{sech}(\pi/2) - g(\Im t),$$

where $g(\Im t)$ is increasing in $\Im t$. But one has the value $g(1.6) = .369307\cdots$ while $\operatorname{sech}(\pi/2) = .398537\cdots$. □

2. Preliminaries observations

For the proof, we will use the following lemma.

Lemma 2. *Assume that* $I(\alpha)$, $J(\alpha)$ *are defined for real* $\alpha > 0$ *and, for fixed* s, *that* $\alpha^{s-1}I(\alpha)$, $\alpha^{s-1}J(\alpha)$ *are absolutely integrable in* α *on* $[0, \infty]$, $[0, 1]$, *respectively.*

Moreover, assume that they satisfy with $c_0 > 0$:

$$I(\alpha) = \pm c_0 I\left(\frac{1}{\alpha}\right) + J(\alpha).$$

Then

$$\int_0^\infty \alpha^{s-1} I(\alpha)\, d\alpha = \int_1^\infty \frac{I(\alpha)}{\alpha}\left\{\alpha^s \pm c_0\alpha^{-s}\right\} d\alpha + \int_0^1 \alpha^{s-1} J(\alpha)\, d\alpha.$$

Furthermore, if one has $c_0 = 1$, $J \equiv 0$ and $s = it$, $t \in \mathbb{R}$, then setting $\Phi(x) := 2I(e^x)$, the following formula holds

$$\int_0^\infty \alpha^{s-1} I(\alpha)\, d\alpha = \int_0^\infty \Phi(x)\cos(tx)\, dx,$$

in case one takes the plus sign. In case one takes the minus sign, the following formula holds

$$\int_0^\infty \alpha^{s-1} I(\alpha)\, d\alpha = i\int_0^\infty \Phi(x)\sin(tx)\, dx.$$

Proof. One has

$$\begin{aligned}\int_0^1 \alpha^{s-1} I(\alpha)\, \mathrm{d}\alpha &= \pm c_0 \int_0^1 \alpha^{s-1} I(1/\alpha)\, \mathrm{d}\alpha + \int_0^1 \alpha^{s-1} J(\alpha)\, \mathrm{d}\alpha \\ &= \pm c_0 \int_1^\infty \beta^{-s-1} I(\beta)\, \mathrm{d}\beta + \int_0^1 \alpha^{s-1} J(\alpha)\, \mathrm{d}\alpha.\end{aligned}$$

where in the last equality we changed variables $\alpha = 1/\beta$. Inserting this in

$$\int_0^\infty \alpha^{s-1} I(\alpha)\, \mathrm{d}\alpha = \int_0^1 + \int_1^\infty,$$

yields

$$\int_0^\infty \alpha^{s-1} I(\alpha)\, \mathrm{d}\alpha = \int_1^\infty \frac{I(\alpha)}{\alpha}\left\{\alpha^s \pm c_0\alpha^{-s}\right\} \mathrm{d}\alpha + \int_0^1 \alpha^{s-1} J(\alpha)\, \mathrm{d}\alpha,$$

and the first part of the lemma follows.

If $c_0 = 1$, $s = it$, $J \equiv 0$ with $t \in \mathbb{R}$, taking the plus sign and making the change of variable $\alpha = e^x$, one gets

$$\int_0^\infty \alpha^{s-1} I(\alpha)\, \mathrm{d}\alpha = \int_0^\infty \Phi(x)\cos(tx)\, \mathrm{d}x.$$

The other case is similar. □

https://doi.org/10.1017/S0013091521000092 Published online by Cambridge University Press

3. Ramanujan's formulas

We first recall the theta transformation formula: set

$$\psi(x) := \sum_{n=1}^{\infty} e^{-n^2\pi x},$$

$$I_0(x) := \frac{1}{\sqrt[4]{x}}\Big\{2\psi(1/x)+1\Big\}.$$

Lemma 3 (formula (2.6.3) p. 22 of [15]). *If* $0<\alpha$ *then*

$$I_0(\alpha) = I_0(1/\alpha).$$

It is convenient to reformulate some of Ramanujan's formulas. Therefore, we define

$$I_n^*(\alpha) := \alpha\sum_{k=1}^{\infty}\frac{k^{2n-1}}{e^{2\pi k\sqrt[n]{\alpha}}-1}.$$

$$I_1(\alpha) := \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2\sinh^2(\pi n\sqrt{\alpha})} - 2\pi\sqrt{\alpha}\sum_{n=1}^{\infty} n^2\log(1-e^{-2\pi n\sqrt{\alpha}}).$$

$$I_2(\alpha) := \frac{1}{\sqrt[4]{\pi\alpha}}\left\{1+2\pi\alpha\int_0^{\infty}\frac{e^{-\pi\alpha x}}{e^{2\pi\sqrt{x}}-1}\,\mathrm{d}x\right\}.$$

$$I_3(\alpha) := \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)\cosh\{(2n-1)\alpha\pi/2\}}.$$

$$I_4(\alpha) := \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\sinh\{n\alpha\pi\}}.$$

Finally set

$$F(x) := x^2\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^3\mathrm{sech}\{\frac{1}{2}\pi(2n+1)\}}{\cosh\{(2n+1)x\}+\cos\{(2n+1)x\}},$$

$$I_5(\alpha) := F(\alpha\pi/\sqrt{2}).$$

Lemma 4 (Ramanujan). *Assume that* B_n *are the Bernoulli numbers and* $0<\alpha$. *Then if* $2\le n$ *is even*:

$$I_n^*(\alpha) = I_n^*(1/\alpha) + \frac{B_{2n}}{4n}\left(\alpha-\frac{1}{\alpha}\right).$$

Also if $3\le n$ *is odd:*

$$I_n^*(\alpha) = -I_n^*(1/\alpha) + \frac{B_{2n}}{4n}\left(\alpha+\frac{1}{\alpha}\right).$$

Proof. These formulas correspond to Entry 13 p. 261 of [3]. In this cited formula change α by $\pi\sqrt[n]{\alpha}$ and β by $\pi\sqrt[n]{\beta}$. □

https://doi.org/10.1017/S0013091521000092 Published online by Cambridge University Press

Lemma 5 (Ramanujan). *If* $0<\alpha$ *then*

$$I_1(\alpha) = -I_1(1/\alpha) + \frac{\pi^2}{120}\left(\alpha + \frac{1}{\alpha}\right) - \frac{\pi^2}{72},$$
$$I_2(\alpha) = I_2(1/\alpha),$$
$$I_3(\alpha) = -I_3(1/\alpha) + \frac{\pi}{4},$$
$$I_4(\alpha) = I_4(1/\alpha) - \frac{\pi}{12}\left(\alpha - \frac{1}{\alpha}\right),$$
$$I_5(\alpha) = I_5(1/\alpha).$$

Proof. The first formula of the lemma corresponds to Entry 11 p. 323 of [3]: in this entry changing α by $\pi\sqrt{\alpha}$ and β by $\pi\sqrt{\beta}$ yields the first formula.

The second formula corresponds to formula (13) of [14], p. 56 of the paper *Some definite integrals*, Messenger of Mathematics, XLIV, (1915), pages 10-18.

The third formula is formula (1) p. 129 of [14] of the paper *On certain infinite series*, Messenger of Mathematics, XLV, (1916), pages 11–15. In the cited formula set $t=0$ and change α by $\pi\alpha/2$ and β by $\pi\beta/2$. The fourth formula follows from formula (10) of the same paper with $t=0$, changing α by $\pi\alpha$ and β by $\pi\beta$.

The fifth formula is given in Entry 31 p. 461 of [4]. □

Next, we use the above formulas to define suitable functions $I(\alpha)$ to be used together with Lemma 2.

Corollary 1. *Set*

$$I(\alpha) := \frac{1}{\sqrt[4]{\pi}} I_0(\alpha) - I_2(\alpha).$$

Then if $0<\alpha$

$$I(\alpha) = I(1/\alpha).$$

Corollary 2. *Assume that* $2 \le n < m$ *and that either* n, m *are both even numbers or* n, m *are both odd numbers. Using the above formulas set for* $0<\alpha$

$$I(\alpha) = I_{n,m}(\alpha) := I_n^*(\alpha)\frac{B_{2m}}{4m} - I_m^*(\alpha)\frac{B_{2n}}{4n}. \tag{3}$$

Then in the first case

$$I(\alpha) = I(1/\alpha),$$

and in the second case

$$I(\alpha) = -I(1/\alpha).$$

In any case, there exists positive numbers c, r *(depending on* n, m*) such that* $I(\alpha) = O(e^{-c\sqrt[r]{\alpha}})$ *as* $\alpha \to \infty+$.

https://doi.org/10.1017/S0013091521000092 Published online by Cambridge University Press

Corollary 3. *Set*

$$I(\alpha) := \frac{\pi^2}{10B_6} I_3^*(\alpha) - \frac{\pi}{18} I_3(\alpha) - I_1(\alpha) = \frac{\pi^2 21}{5} I_3^*(\alpha) - \frac{\pi}{18} I_3(\alpha) - I_1(\alpha).$$

Then if $0 < \alpha$

$$I(\alpha) = -I(1/\alpha).$$

There exists positive numbers c, r *such that* $I(\alpha) = O(e^{-c\sqrt[r]{\alpha}})$ *as* $\alpha \to \infty+$.

4. Proof of Theorem 1

In this proof (and in the sequel), we will use the following well-known formula without further notice

$$\int_0^\infty e^{-nx} x^{s-1} \, \mathrm{d}x = \frac{\Gamma(s)}{n^s}, \quad (\Re s > 1). \tag{4}$$

From the definitions of $I_0(\alpha)$ and $I_2(\alpha)$, one sees that

$$\begin{aligned} I(\alpha) &:= \frac{1}{\sqrt[4]{\pi}} I_0(\alpha) - I_2(\alpha) \\ &= \frac{2}{\sqrt[4]{\pi\alpha}} \psi(1/\alpha) - 2(\pi\alpha)^{3/4} \int_0^\infty \frac{e^{-\pi\alpha x}}{e^{2\pi\sqrt{x}} - 1} \, \mathrm{d}x. \end{aligned}$$

and we have, from Corollary 1, that the transformation formula $I(\alpha) = I(1/\alpha)$ holds. Observe that for any negative n, one has $\alpha^n \psi(1/\alpha) \to 0$ as $\alpha \to 0+$. Therefore,

$$I(\alpha) = O(\alpha^{3/4}),$$

as $\alpha \to 0+$, which implies using the transformation formula that

$$I(\alpha) = O(1/\alpha^{3/4}),$$

as $\alpha \to \infty+$. We have proved the following result.

Proposition 1. *The integral*

$$\int_0^\infty I(\alpha) \alpha^{s-1} \, d\alpha,$$

is absolutely convergent for $-3/4 < \Re s < 3/4$.

Next note that if we set $\psi_N(x) := \sum_{n=1}^N e^{-n^2\pi x}$ then $0 \le \psi_N(x) \le \psi(x) = O(1/\sqrt{x})$ if $x \to 0+$. Also $\psi(x) = O(e^{-\pi x})$ as $x \to \infty+$. All these can be used in what follows to interchange the sum and the integral using the dominated Lebesgue's theorem: if

https://doi.org/10.1017/S0013091521000092 Published online by Cambridge University Press

$\Re s < -1/4$ and using (4)

$$\int_0^\infty \frac{1}{\sqrt[4]{\alpha}}\psi(1/\alpha)\alpha^{s-1}\,\mathrm{d}\alpha = \int_0^\infty \psi(\beta)\beta^{-s+1/4-1}\,\mathrm{d}\beta$$
$$= \sum_{n=1}^\infty \int_0^\infty e^{-n^2\pi\beta}\beta^{-s+1/4-1}\,\mathrm{d}\beta$$
$$= \frac{\Gamma(-s+1/4)}{\pi^{-s+1/4}}\sum_{n=1}^\infty \frac{1}{n^{2(-s+1/4)}} = \frac{\Gamma(-s+1/4)}{\pi^{-s+1/4}}\zeta\left(-2s+\frac{1}{2}\right),$$

and if $-3/4 < \Re s < -1/4$

$$\int_0^\infty \alpha^{3/4}\left\{\int_0^\infty \frac{e^{-\pi\alpha x}}{e^{2\pi\sqrt{x}}-1}\,\mathrm{d}x\right\}\alpha^{s-1}\,\mathrm{d}\alpha = \frac{\Gamma(s+\frac{3}{4})}{\pi^{s+3/4}}\int_0^\infty \frac{x^{-s-3/4}}{e^{2\pi\sqrt{x}}-1}\,\mathrm{d}x$$
$$= \frac{\Gamma(s+\frac{3}{4})}{\pi^{s+3/4}}2\int_0^\infty \frac{\tau^{-2s-1/2}}{e^{2\pi\tau}-1}d\tau = \frac{\Gamma(s+\frac{3}{4})}{\pi^{s+3/4}}\frac{2\Gamma(-2s+\frac{1}{2})}{(2\pi)^{-2s+1/2}}\zeta\left(-2s+\frac{1}{2}\right).$$

Therefore, using this last two results, one has:

$$\int_0^\infty I(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha$$
$$= 2\pi^{s-1/2}\left\{\Gamma\left(-s+\frac{1}{4}\right) - 2^{2s+1/2}\Gamma\left(s+\frac{3}{4}\right)\Gamma\left(-2s+\frac{1}{2}\right)\right\}\zeta\left(-2s+\frac{1}{2}\right)$$
$$= 2\pi^{s-1/2}\Gamma\left(-s+\frac{1}{4}\right)\left\{1-\frac{1}{\sqrt{\pi}}\Gamma\left(s+\frac{3}{4}\right)\Gamma\left(-s+\frac{3}{4}\right)\right\}\zeta\left(-2s+\frac{1}{2}\right),$$

where in the last equality, we have used Legendre's duplication formula. This identity holds true for $-3/4 < \Re s < 3/4$ using the last proposition and analytic continuation. Now the theorem follows if $s = it$, using Lemma 2 and analytic continuation (we have cancelled the number 2 on both sides of the formula).

5. Proof of Theorem 2

Observe that from the definition of I_4, one has $I_4(\alpha) = O(e^{-2\pi\alpha})$ as $\alpha \to \infty+$. Also from the transformation formula for I_4 given in Lemma 5, one gets that $I_4(\alpha) = O(1/\alpha)$ if $\alpha \to 0+$.

Next note that if we set $I_{4,N}(\alpha) := \sum_{n=1}^{2N}(-1)^{n+1}/n\sinh\{n\alpha\pi\}$ then $0 \le I_{4,N}(\alpha) \le I_4(\alpha)$ if $0 < \alpha$. These can be used in what follows to interchange the sum and the integral

https://doi.org/10.1017/S0013091521000092 Published online by Cambridge University Press

using the dominated Lebesgue's theorem. If $2 < \Re s$ then

$$\begin{aligned}\int_0^\infty I_4(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha &= \int_0^\infty \left\{\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n\sinh(n\alpha\pi)}\right\}\alpha^{s-1}\,\mathrm{d}\alpha\\
&= \sum_{n=1}^\infty \int_0^\infty \frac{(-1)^{n+1}2}{ne^{n\alpha\pi}(1-e^{2n\alpha\pi})}\alpha^{s-1}\,\mathrm{d}\alpha\\
&= \sum_{n=1}^\infty \frac{(-1)^{n+1}2}{n}\int_0^\infty \left\{e^{-\alpha n\pi}+e^{-3\alpha n\pi}+e^{-5\alpha n\pi}+\cdots\right\}\alpha^{s-1}\,\mathrm{d}\alpha\\
&= \sum_{n=1}^\infty \frac{(-1)^{n+1}2\Gamma(s)}{\pi^s n^{s+1}}\left\{\frac{1}{1^s}+\frac{1}{3^s}+\frac{1}{5^s}+\cdots\right\}\\
&= \frac{2\Gamma(s)}{\pi^s}\left(1-\frac{1}{2^s}\right)^2\zeta(s)\zeta(s+1).\end{aligned}$$

The derivative of the transformation formula (i.e. Lemma 5) for I_4 multiplied by α yields

$$\alpha I_4'(\alpha) = -\frac{1}{\alpha}I_4'(1/\alpha) - \frac{\pi}{12}\left(\alpha+\frac{1}{\alpha}\right),$$

and the derivative of this last formula multiplied by α yields

$$\alpha\{I_4'(\alpha)+\alpha I_4''(\alpha)\} = \frac{1}{\alpha}\{I_4'(1/\alpha)+1/\alpha I_4''(1/\alpha)\} - \frac{\pi}{12}\left(\alpha-\frac{1}{\alpha}\right).$$

Subtracting this last formula from the transformation formula for I_4 yields that if we set $I(\alpha) := I_4(\alpha) - \alpha I_4'(\alpha) - \alpha^2 I_4''(\alpha)$ then one has the transformation formula

$$I(\alpha) = I(1/\alpha).$$

Integrating by parts one sees that ($1 < \Re s$)

$$\begin{aligned}\int_0^\infty I_4'(\alpha)\alpha^s\,\mathrm{d}\alpha &= -s\int_0^\infty I_4(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha,\\
\int_0^\infty I_4''(\alpha)\alpha^{s+1}\,\mathrm{d}\alpha &= s(s+1)\int_0^\infty I_4(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha,\end{aligned}$$

which yields

$$\begin{aligned}\int_0^\infty I(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha &= \int_0^\infty \{I_4(\alpha)-\alpha I_4'(\alpha)-\alpha^2 I_4''(\alpha)\}\alpha^{s-1}\,\mathrm{d}\alpha\\
&= (1-s^2)\int_0^\infty I_4(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha.\end{aligned}$$

Note that the first integral in this last formula is defined for all values of s: $I(\alpha)$ has exponential decay at infinity and for $\alpha = 0$ one uses the transformation formula for $I(\alpha)$.

https://doi.org/10.1017/S0013091521000092 Published online by Cambridge University Press

Finally, using the first formula of this section and Lemma 2 with $s = it$ yields

$$\int_0^\infty 2I(e^x)\cos xt\,\mathrm{d}x = (1+t^2)\frac{2\Gamma(it)}{\pi^{it}}\left(1-\frac{1}{2^{it}}\right)^2\zeta(it)\zeta(it+1),$$

which is the desired first result.

Next set $F(t) = (1+t^2)2\Gamma(it)/\pi^{it}(1-1/2^{it})^2$ and $\Phi(x) = 2I(e^x)$. Then changing in the last formula t by $t \pm i/2$ with t real yields

$$\int_0^\infty \Phi(x)(\cos tx\cosh x/2 - i\sin tx\sinh x/2)\,\mathrm{d}x = F(t+i/2)\zeta(it-1/2)\zeta(it+1/2),$$
$$\int_0^\infty \Phi(x)(\cos tx\cosh x/2 + i\sin tx\sinh x/2)\,\mathrm{d}x = F(t-i/2)\zeta(it+1/2)\zeta(it+3/2).$$

Adding and subtracting these two formulas yield

$$\begin{aligned}
&2\int_0^\infty \Phi(x)\cosh x/2\cos tx\,\mathrm{d}x \\
&\quad = \zeta\left(it+\frac{1}{2}\right)\{F(t+i/2)\zeta(it-1/2) + F(t-i/2)\zeta(it+3/2)\},\\
&2i\int_0^\infty \Phi(x)\sinh x/2\sin tx\,\mathrm{d}x \\
&\quad = \zeta\left(it+\frac{1}{2}\right)\{-F(t+i/2)\zeta(it-1/2) + F(t-i/2)\zeta(it+3/2)\},
\end{aligned}$$

which proves the theorem using analytic continuation.

6. Proof of Theorem 3

From the definition of $I_n^*(\alpha)$, one has

$$I_n^*(\alpha) = O(e^{-c\alpha}), \tag{5}$$

for some positive constant c (which depends on n) as $\alpha \to \infty+$. Also, from Lemma 4 (its transformation formula) one has that $I_n^*(\alpha) = O(1/\alpha)$ as $\alpha \to 0+$. If one sets $I_{n,N}^*(\alpha) := \sum_{k=1}^N k^{2n-1}/e^{2\pi k\sqrt[n]{\alpha}} - 1$ then $0 \le I_{n,N}^*(\alpha) \le I_n^*(\alpha)$. Therefore, Lebesgue's dominated theorem can be used to interchange the sum and the integral in what follows: if $1 < \Re s$ then

$$\begin{aligned}
\int_0^\infty I_n^*(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha &= \int_0^\infty \sum_{k=1}^\infty \frac{k^{2n-1}}{e^{2\pi k\sqrt[n]{\alpha}}-1}\alpha^s\,\mathrm{d}\alpha\\
&= \frac{n}{(2\pi)^{n(s+1)}}\sum_{k=1}^\infty \int_0^\infty \frac{k^{2n-1}}{e^{k\beta}-1}\beta^{ns+n-1}\,\mathrm{d}\beta
\end{aligned}$$

https://doi.org/10.1017/S0013091521000092 Published online by Cambridge University Press

$$= \frac{n}{(2\pi)^{n(s+1)}}\Gamma(ns+n)\sum_{k=1}^{\infty}\frac{k^{2n-1}}{k^{ns+n}}\left(1+\frac{1}{2^{ns+n}}+\frac{1}{3^{ns+n}}+\cdots\right)$$
$$= \frac{n\Gamma(ns+n)}{(2\pi)^{n(s+1)}}\zeta(ns+n)\zeta(ns-n+1).$$

Therefore, if $2 \le n < m$ and if

$$I(\alpha) := I_n^*(\alpha)\frac{B_{2m}}{4m} - I_m^*(\alpha)\frac{B_{2n}}{4n},$$

where n, m are both even numbers or n, m are both odd numbers then from Corollary 2 one has $I(\alpha) = \pm I(1/\alpha)$. This and formula (5) implies that $\int_0^\infty I(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha$ is absolutely convergent for all values of s.

Next, observe that

$$\int_0^\infty I(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha = \frac{n\Gamma(ns+n)}{(2\pi)^{n(s+1)}}\zeta(ns+n)\zeta(ns-n+1)\frac{B_{2m}}{4m} - \frac{m\Gamma(ms+m)}{(2\pi)^{m(s+1)}}\zeta(ms+m)\zeta(ms-m+1)\frac{B_{2n}}{4n}.$$

The theorem follows setting $s = it$ and using Lemma 2.

7. Proof of Theorem 4

From the definition of I_3, one has $I_3(\alpha) = O(e^{-\pi\alpha/2})$ as $\alpha \to \infty+$. Also from the transformation formula for I_3 given in Lemma 5 one gets that $I_3(\alpha) \to \pi/4$ if $\alpha \to 0+$.

Thus, for $1 < \Re s$, one has

$$\int_0^\infty I_3(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha = \int_0^\infty \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)\cosh\{(2n-1)\alpha\pi/2\}}\alpha^{s-1}\,\mathrm{d}\alpha$$
$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}2\int_0^\infty\left\{\frac{1}{e^{\alpha(2n-1)\pi/2}} - \frac{1}{e^{3\alpha(2n-1)\pi/2}} + \frac{1}{e^{5\alpha(2n-1)\pi/2}} - \cdots\right\}\alpha^{s-1}\,\mathrm{d}\alpha$$
$$= \frac{2\Gamma(s)}{(\pi/2)^s}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\left\{\frac{1}{(2n-1)^s} - \frac{1}{(2n-1)^s3^s} + \frac{1}{(2n-1)^s5^s} - \cdots\right\}$$
$$= \frac{2^{s+1}\Gamma(s)}{\pi^s}L_{\chi_4}(s)L_{\chi_4}(s+1).$$

Lemma 2 yields ($\Re s > 1$)

$$\int_0^\infty I_3(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha = \int_1^\infty \frac{I_3(\alpha)}{\alpha}\left\{\alpha^s - \alpha^{-s}\right\}\mathrm{d}\alpha + \frac{\pi}{4\,s}.$$

https://doi.org/10.1017/S0013091521000092 Published online by Cambridge University Press

Observe that the integral in the right-hand side of the last formula is defined for all values of s. Therefore, multiplying the last formula by s, setting $s = it$ and integrating by parts

$$\begin{aligned} s\int_0^\infty \alpha^{s-1} I_3(\alpha)\,\mathrm{d}\alpha &= 2\int_1^\infty I_3(\alpha)\left\{\frac{-t\sin(t\log\alpha)}{\alpha}\right\}\mathrm{d}\alpha + \frac{\pi}{4} \\ &= -2I_3(1) - 2\int_1^\infty I_3(\alpha)'\cos(t\log\alpha)\,\mathrm{d}\alpha + \frac{\pi}{4} \\ &= -2\int_1^\infty I_3(\alpha)'\cos(t\log\alpha)\,\mathrm{d}\alpha, \end{aligned}$$

where we have used the fact that $I_3(1) = \pi/8$, which follows from the transformation formula for I_3. Integrating by parts again

$$-2\int_1^\infty I_3(\alpha)'\cos(t\log\alpha)\,\mathrm{d}\alpha = \frac{2}{t}\int_1^\infty \left\{\alpha I_3(\alpha)'\right\}'\sin(t\log\alpha)\,\mathrm{d}\alpha.$$

Therefore, we have proved that for all values of $s = it$

$$s\frac{2^{s+1}\Gamma(s)}{\pi^s}L_{\chi_4}(s)L_{\chi_4}(s+1) = \frac{2}{t}\int_1^\infty \left\{\alpha I_3(\alpha)'\right\}'\sin(t\log\alpha)\,\mathrm{d}\alpha,$$

or

$$t\frac{2^{s}\Gamma(s+1)}{\pi^s}L_{\chi_4}(s)L_{\chi_4}(s+1) = \int_1^\infty \left\{I_3(\alpha)' + \alpha I_3(\alpha)''\right\}\sin(t\log\alpha)\,\mathrm{d}\alpha.$$

The first result in the theorem follows making the change of variable $\alpha = e^x$.

Next set $F(t) = t2^{it}\Gamma(it+1)/\pi^{it}$ and $\Phi(x) = e^x\{I_3(e^x)' + e^x I_3(e^x)''\}$. Then changing in the last formula t by $t \pm i/2$ with t real yields

$$\int_0^\infty \Phi(x)(\sin tx\cosh x/2 + i\cos tx\sinh x/2)\,\mathrm{d}x = F(t+i/2)L_{\chi_4}(it-1/2)L_{\chi_4}(it+1/2),$$

$$\int_0^\infty \Phi(x)(\sin tx\cosh x/2 - i\cos tx\sinh x/2)\,\mathrm{d}x = F(t-i/2)L_{\chi_4}(it+1/2)L_{\chi_4}(it+3/2).$$

Adding and subtracting these two formulas yield

$$\begin{aligned} &2\int_0^\infty \Phi(x)\cosh x/2\sin tx\,\mathrm{d}x \\ &\quad = L_{\chi_4}\left(it+\frac{1}{2}\right)\left\{F(t+i/2)L_{\chi_4}(it-1/2) + F(t-i/2)L_{\chi_4}(it+3/2)\right\}, \\ &2i\int_0^\infty \Phi(x)\sinh x/2\cos tx\,\mathrm{d}x \\ &\quad = L_{\chi_4}\left(it+\frac{1}{2}\right)\left\{F(t+i/2)L_{\chi_4}(it-1/2) - F(t-i/2)L_{\chi_4}(it+3/2)\right\}, \end{aligned}$$

which proves the theorem using analytic continuation.

8. Proof of Theorem 5

Observe that $I_5(\alpha)$ has exponential decay at infinity and recall that by Lemma 5

$$I_5(\alpha) = I_5(1/\alpha),$$

if $0 < \alpha$. Therefore, the integral $\int_0^\infty I_5(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha$ converges absolutely for all values of s.

Therefore, one may use Lemma 2 (here $a_n := \pi^2/2(-1)^n(2n+1)^3\mathrm{sech}\{\frac{1}{2}\pi(2n+1)\}$) with $s = it$, $t \in \mathbb{R}$ which yields:

$$\begin{aligned}\int_0^\infty & \alpha^{s-1} I(\alpha)\,\mathrm{d}\alpha \\ &= \sum_{n=0}^\infty a_n \int_0^\infty \alpha^{s-1} \frac{\alpha^2}{\cosh\left\{(2n+1)\alpha\pi/\sqrt{2}\right\} + \cos\left\{(2n+1)\alpha\pi/\sqrt{2}\right\}}\,\mathrm{d}\alpha,\end{aligned}$$

where the change of variables $\beta = (2n+1)\alpha\pi/\sqrt{2}$ yields

$$\int_0^\infty \Phi(x)\cos(tx)\,\mathrm{d}x = \sum_{n=0}^\infty a_n \left\{\frac{\sqrt{2}}{\pi(2n+1)}\right\}^{s+2} \int_0^\infty \frac{\beta^{s+1}}{\cosh\beta + \cos\beta}\,\mathrm{d}\beta.$$

This last integral is well defined for $-2 < \Re s$. The theorem follows cleaning this last expression and using analytic continuation.

9. Final remarks

If one sets

$$I(\alpha) := \frac{\pi^2 21}{5} I_3^*(\alpha) - \frac{\pi}{18} I_3(\alpha) - I_1(\alpha),$$

then by Corollary 3 one has $I(\alpha) = I(1/\alpha)$. Set $\Phi(x) := 2I(e^x)$.

The following result holds.

Lemma 6. *If $t \in \mathbb{C}$ and $L_{\chi_4}(s) := \sum_{n=1}^\infty \frac{(-1)^{n-1}}{(2n-1)^s}$ then*

$$\begin{aligned} i\int_0^\infty & \Phi(x)\sin(tx)\,dx \\ &= \frac{\pi^2 63}{5}\frac{\Gamma(3it+3)}{(2\pi)^{3(it+1)}}\zeta(3it+3)\zeta(3it-2) \\ &\quad - \frac{\pi}{9}\frac{2^{it}\Gamma(it)}{\pi^{it}} L_{\chi_4}(it)L_{\chi_4}(it+1) - \frac{2\Gamma(2it)}{(2\pi)^{2it}}\{1-2it\}\zeta(2it-1)\zeta(2it+2).\end{aligned}$$

https://doi.org/10.1017/S0013091521000092 Published online by Cambridge University Press

Proof. As above and with some care, one may prove that for $1 < \Re s$

$$\begin{aligned}
&\int_0^\infty I_1(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha \\
&\quad = \int_0^\infty \left\{\frac{1}{4}\sum_{n=1}^\infty \frac{1}{n^2\sinh^2(\pi n\sqrt{\alpha})} - 2\pi\sqrt{\alpha}\sum_{n=1}^\infty n^2\log(1-e^{-2\pi n\sqrt{\alpha}})\right\}\alpha^{s-1}\,\mathrm{d}\alpha \\
&\quad = \sum_{n=1}^\infty \frac{1}{n^2}\int_0^\infty \{e^{-2\pi n\sqrt{\alpha}} + 2e^{-4\pi n\sqrt{\alpha}} + 3e^{-6\pi n\sqrt{\alpha}} + \cdots\}\alpha^{s-1}\,\mathrm{d}\alpha \\
&\quad - 2\pi\sqrt{\alpha}\sum_{n=1}^\infty n^2\int_0^\infty \left\{e^{-2\pi n\sqrt{\alpha}} + \frac{e^{-4\pi n\sqrt{\alpha}}}{2} + \frac{e^{-6\pi n\sqrt{\alpha}}}{3} + \cdots\right\}\alpha^{s-1}\,\mathrm{d}\alpha \\
&= 2\Gamma(2s)\sum_{n=1}^\infty \frac{1}{n^2}\left\{\frac{1}{(2\pi n)^{2s}} + \frac{2}{(4\pi n)^{2s}} + \frac{3}{(6\pi n)^{2s}} + \cdots\right\} \\
&\quad - 2\pi 2\Gamma(2s+1)\sum_{n=1}^\infty n^2\left\{\frac{1}{(2\pi n)^{2s+1}} + \frac{1}{2(4\pi n)^{2s+1}} + \frac{1}{3(6\pi n)^{2s+1}} + \cdots\right\} \\
&\quad = \frac{2\Gamma(2s)}{(2\pi)^{2s}}\zeta(2s-1)\zeta(2s+2) - 2\pi\frac{2\Gamma(2s+1)}{(2\pi)^{2s+1}}\zeta(2s-1)\zeta(2s+2) \\
&\quad = \frac{2\Gamma(2s)}{(2\pi)^{2s}}\{1-2s\}\zeta(2s-1)\zeta(2s+2).
\end{aligned}$$

We have already proved that $(1 < \Re s)$

$$\int_0^\infty I_3(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha = \frac{2^{s+1}\Gamma(s)}{\pi^s}L_{\chi_4}(s)L_{\chi_4}(s+1)$$

and

$$\int_0^\infty I_n^*(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha = \frac{n\Gamma(ns+n)}{(2\pi)^{n(s+1)}}\zeta(ns+n)\zeta(ns-n+1).$$

Next, observe that $I(\alpha)$ has exponential decay at infinity and using Corollary 3, one has that $I(\alpha) = -I(1/\alpha)$ if $0 < \alpha$. Therefore, the following integral is absolutely convergent for all values of s

$$\begin{aligned}
&\int_0^\infty I(\alpha)\alpha^{s-1}\,\mathrm{d}\alpha \\
&\quad = \frac{\pi^2 21}{5}\frac{3\Gamma(3s+3)}{(2\pi)^{3(s+1)}}\zeta(3s+3)\zeta(3s-3+1) \\
&\quad - \frac{\pi}{18}\frac{2^{s+1}\Gamma(s)}{\pi^s}L_{\chi_4}(s)L_{\chi_4}(s+1) - \frac{2\Gamma(2s)}{(2\pi)^{2s}}\{1-2s\}\zeta(2s-1)\zeta(2s+2).
\end{aligned}$$

Setting $s = it$ and using Lemma 2 finishes the proof. □

Acknowledgements. We would like to express our deep thankfulness to an anonymus referee for his/her very careful reading of the paper and for many valuable suggestions.

References

1. L. Báez-Duarte, A strengthening of the Nyman-Beurling criterion for the Riemann Hypothesis, *Atti Acad. Naz. Lincei Rend. Lincei Mat. Appl.* **14** (2003), 5–11.
2. L. Báez-Duarte, M. Balazard, B. Landreau and E. Saias, Notes sur la fonction ζ de Riemann III, *Adv. Math.* **149** (2000), 130–144.
3. B. Berndt, *Ramanujan's Notebooks, Part II* (Springer, 1989).
4. B. Berndt, *Ramanujan's Notebooks, Part V* (Springer, 1997).
5. A. Beurling, A closure problem related to the Riemann zeta-function, *Proc. Nat. Acad. Sci.* **41** (1955), 312–314.
6. Dimitar K. Dimitrov and Peter K. Rusev, Zeros of entire Fourier Transforms, *East J. Approx.* **17**(1) (2011), 1–110.
7. Dimitar K. Dimitrov and Y. Xu, Wronskians of Fourier and Laplace transforms, *Trans. Amer.Math. Soc.* **372** (2019), 4107–4125.
8. W. F. Donoghue, *Distributions and Fourier Transforms* (Academic Press, 1969).
9. H. M. Edwards, *Riemann's Zeta Function* (Dover Publications, 2001).
10. H. Iwaniec and E. Kowalski, *Analytic Number Theory* (AMS, 2004).
11. B. Nyman, *On Some Groups and Semigroups of Translations*, Thesis (Uppsala, 1950).
12. P. A. Panzone, Formulas for the Riemann Zeta-function and certain Dirichlet series, *Int. Trans. Special Funct.* **29** (2018), 893–908.
13. B. van der Pol, An electro-mechanical investigation of the Riemann zeta function in the critical strip, *Bull. Amer. Math. Soc.* **53** (1947), 976–981.
14. S. Ramanujan, *Collected Papers* (Cambridge University Press, Cambridge, 1927). reprinted by Chelsea, New York, 1962; reprinted by the AMS (Providence, RI, 2000).
15. E. C. Titchmarsh, *The theory of Riemann Zeta-Function*, edited and with a preface by D. R. Heath-Brown (Oxford Science Publications, 1986).