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The initial time layer problem and the quasineutral limit in the semiconductor drift-diffusion model

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The classical time-dependent drift-diffusion model for semiconductors is considered for small scaled Debye length (which is a singular perturbation parameter). The corresponding limit is carried out on both the dielectric relaxation time scale and the diffusion time scale. The latter is a quasineutral limit, and the former can be interpreted as an initial time layer problem. The main mathematical tool for the analytically rigorous singular perturbation theory of this paper is the (physical) entropy of the system.

1 Introduction

Quasineutrality is a frequently used modelling assumption in charged particle transport. It has been applied in the first theoretical studies of semiconductor devices [30], but also in other contexts such as the modelling of plasmas [32] and ionic membranes [24].

Formally, quasineutral models are derived in the limit as the ratio of the Debye length to a characteristic length tends to zero. In the semiconductor context, this formalized perturbation approach has been used extensively for the analysis of the qualitative behaviour of semiconductor devices (see Please [21] and Markowich *et al.* [17] for early contributions, and Markowich *et al.* [18] for an overview and further references). All these studies are based on a description of charge transport by the steady state classical drift-diffusion model, posed on bounded domains representing the semiconductor part of a device and boundary conditions modelling contacts and/or insulating segments.

Similarly, for the time-dependent drift-diffusion model the quasineutrality assumption has been used in the early days of semiconductor device theory [15, 14] and later the formal asymptotics have been carried out [22, 23, 27, 28].

The contributions mentioned so far are concerned with formal asymptotic expansions. Rigorous convergence results for the stationary one-dimensional drift-diffusion model are the consequence of a general theory for singularly perturbed ordinary differential equations [34, 29]. Results for multidimensional problems can be found in Markowich [16] and Schmeiser [25].

The justification of the quasineutral limit in time-dependent problems has become the subject of intensive efforts recently. Actually, different transport models have been considered. Kinetic descriptions in the form of Vlasov–Poisson systems are the subject of Grenier [12] and Brenier & Grenier [2], and Euler–Poisson systems have been investigated in Cordier & Grenier [5], Caffarelli *et al.* [4] and Gasser & Marcati [11, 10]. However, all these results are restricted to special situations. In Grenier [12], Brenier & Grenier [2] and Cordier & Grenier [5], the unipolar case involving only one type of charge carriers is considered. The result of Caffarelli *et al.* [4] concerns travelling wave solutions and the quasineutral limit results in Gasser & Marcati [11, 10] are valid under the assumption of an additional damping relaxation mechanism.

The present study is concerned with the time-dependent bipolar drift-diffusion-Poisson system, where no rigorous results on the quasineutral limit have been available up to now. It is our aim to exploit the information gained from an entropy dissipation equation (carrying additional physical information) and to demonstrate the potential of this approach. In this paper, we obtain the desired result under certain restrictive assumptions:

(1) Sign chances in the doping profile and, thus, p-n junctions are excluded.
(2) Jump discontinuities of the doping profile are excluded by our smoothness assumptions.
(3) An insulated piece of semiconductor without contacts is considered on the 'dielectric relaxation time scale'. More general boundary conditions modelling ohmic contacts and insulating segments are admitted on the 'diffusion time scale'.
(4) Charge neutral initial conditions are assumed on the 'diffusion time scale'.

Assumption 1 permits bounds on the carrier densities from below which are an essential ingredient of our analysis. Spatial layer behaviour of the solution is excluded by assumption 2, and by the choice of boundary conditions below. Unfortunately, both assumptions eliminate cases of great practical interest, which will have to be dealt with in future work. Assumption 3 allows for a thermal equilibrium steady state which is important for the entropy approach. Strongly connected with the quasineutral limit is the presence of the two different time scales mentioned. The quasineutral limit, corresponding to the so called 'diffusion time scale' is singular and incompatible with general initial conditions. Initial layers where the solution varies on the fast 'dielectric relaxation time scale' are responsible for the connection between initial conditions and the quasineutral 'outer problem'. These layers are eliminated by assumption 4.

We also consider the problem on the dielectric relaxation time scale. One-dimensional versions have already been studied in Brezzi & Markowich [3], Markowich & Szmolyan [19] and Szmolyan [33]. In the limit as the scaled Debye length tends to zero, the initial layer problem is derived. For matching to the quasineutral problem, its long time behaviour has to be understood. Under assumption 1 and 3 we prove convergence to a quasineutral state. However, a complete characterisation of the asymptotic state in terms of the initial conditions is still missing.

This work is organized as follows. In §2 the model is presented and the formal results are outlined. A priori estimates are collected in § 3. § 4 is concerned with the problem on the fast time scale. The derivation of the initial layer problem is a special case of a result in Gasser [8, 9], where nonlinear diffusion effects are incorporated in the model. The charge neutral limit is the subject of § 5. Here, vanishing results for vanishing doping profiles [8, 9, 13] are extended to doping profiles satisfying assumption 1.

2 Formal asymptotics

The scaled semiconductor drift-diffusion equations read

$$n_t = \mathrm{div}(\mu_n(\nabla n + nE)), \tag{2.1a}$$

$$p_t = \mathrm{div}(\mu_p(\nabla p - pE)), \tag{2.1b}$$

$$-\lambda^2 \mathrm{div} E = n - p - C, \tag{2.1c}$$

with $x \in \Omega \subset I\!R^d$, Ω bounded with smooth boundary, $t \geqslant 0$ and $E = -\nabla\Phi$. The unknowns n, p, E, Φ are the electron density, the hole density, the electric field and the electric potential, respectively. The given function $C = C(x)$ is the doping profile describing fixed background charges. The dimensionless positive parameters μ_n, μ_p and λ are the scaled mobilities of electrons and holes and the scaled Debye length, respectively.

We consider inital-boundary value problems with initial conditions for the densities,

$$n(t = 0, x) = n_0(x), \qquad p(t = 0, x) = p_0(x), \tag{2.1d}$$

and with mixed boundary conditions

$$\nabla n \cdot \nu = 0, \quad \nabla p \cdot \nu = 0, \quad E \cdot \nu = 0 \quad \text{on} \quad \partial\Omega_N, \tag{2.1e}$$

$$n = n_i e^{\Phi_{bi}}, \quad p = n_i e^{-\Phi_{bi}}, \quad \Phi = \Phi_{bi} \quad \text{on} \quad \partial\Omega_D, \tag{2.1f}$$

where ν is the normal vector along the boundary $\partial\Omega_N$ and the boundary is the union of Dirichlet and Neumann boundaries $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$. The constant n_i denotes the scaled intrinsic carrier density of the semiconductor, and the space dependent built-in potential $\Phi_{bi} = \Phi_{bi}(x)$ is choosen such that

$$n_i e^{\Phi_{bi}} - n_i e^{-\Phi_{bi}} - C = 0 \quad \text{on} \quad \partial\Omega_D. \tag{2.1g}$$

Ω_D models the Ohmic contacts of the device and Ω_N the insulating boundary segments.

We shall be interested in situations where the scaled Debye length λ takes small values, whereas the scaled mobilities are of order 1. Then there are two significant time scales. The nondimensionalization leading to (2.1a) is based on the diffusion timescale. The rescaled time variable $s = \frac{t}{\lambda^2}$ corresponds to the dielectric relaxation time as the reference time. In the following we shall refer to the diffusion and dielectric relaxation time scales as the slow and fast scales, respectively. In particular, from the point of view of the slow scale, the analysis of initial layers has to be based on the fast scale.

It should also be mentioned that in this scaling, a typical value of the doping profile C has been taken as reference density. Since in cases of practical interest the intrinsic density is small by comparison, n_i will typically be a second small parameter. In the present work, however, it is kept fixed as $\lambda \to 0$. The consecutive limits $\lambda \to 0$, $n_i \to 0$ have been carried out formally for transient problems in Schmeiser & Unterreiter [27] and Schmeiser *et al.* [28] and rigorously for stationary problems in Schmeiser [25]. A significant limit occurs for $n_i \to 0$ with $\lambda^2 \ln(1/n_i)$ kept fixed. In Schmeiser [26], it has been shown that under certain additional assumptions the limiting problem is a free boundary problem.

Formally, setting $\lambda = 0$ in the slow equations (2.1a)–(2.1c) we obtain the system

$$n_t = \mathrm{div}(\mu_n(\nabla n + nE)), \tag{2.2a}$$

$$p_t = \mathrm{div}(\mu_p(\nabla p - pE)), \tag{2.2b}$$

$$0 = n - p - C. \tag{2.2c}$$

Because of the singular perturbation character of the problem (the Poisson equation becomes an algebraic equation in the limit) we cannot expect that the full initial and boundary conditions hold for the limiting problem. However, by the conservation form of the continuity equations the property of zero current flux through the Neumann boundary will prevail in the limit:

$$(\nabla n + nE)\cdot \nu = 0, \quad (\nabla p - pE)\cdot \nu = 0 \quad \text{on } \partial\Omega_N. \tag{2.2d}$$

Initial conditions for the slow limiting problem will be discussed below.

In case of no Dirichlet boundary, $\partial\Omega_D = \emptyset$, a necessary solvability condition for the Poisson equation (2.1c) subject to Neumann boundary conditions for the field in (2.1e) is global charge neutrality, $\int_\Omega (n - p - C)dx = 0$. Since in this case the total numbers of electrons and holes are conserved, it is sufficient to require the corresponding condition for the initial data:

$$\int_\Omega (n_0 - p_0 - C)dx = 0. \tag{2.3}$$

Note that in this case the potential is unique only up to an arbitrary additive constant.

Also, by the nonnegativity of the densities, necessary conditions for the validity of the limiting procedure (and in particular, for the local neutrality (2.2c)) in the case of homogenous Neumann boundary conditions are

$$\int_\Omega n\,dx = \int_\Omega n_0\,dx \geqslant \int_\Omega C_+\,dx, \qquad \int_\Omega p\,dx = \int_\Omega p_0\,dx \geqslant \int_\Omega C_-\,dx \tag{2.4}$$

(where we use the standard notation for the positive and negative parts of real-valued functions). It is easily seen that these conditions are equivalent under the global neutrality assumption (2.3). They constitute an additional condition for the initial data for the densities.

We turn back to the system (2.2a)–(2.2c). To illuminate the character of the system (2.2a)–(2.2c), we introduce the new variable ϕ

$$\phi = -\Phi - \frac{\mu_n - \mu_p}{\mu_n + \mu_p} \ln\left[\mu_n n + \mu_n(n - C)\right], \tag{2.5}$$

where $E = -\nabla\Phi$. Then (2.2a)–(2.2c) becomes a coupled elliptic-parabolic system of the form

$$n_t = \mathrm{div}\left(\frac{2n - C}{\frac{n}{\mu_p} + \frac{n-C}{\mu_n}} \nabla n + \mu_n n \nabla\phi + \frac{\mu_n - \mu_p}{\mu_n + \mu_p} \frac{n\nabla C}{\frac{n}{\mu_p} + \frac{n-C}{\mu_n}} \right), \tag{2.6a}$$

$$0 = \mathrm{div}\left((\mu_n n + \mu_p(n - C))\nabla\phi - \frac{2}{\frac{1}{\mu_n} + \frac{1}{\mu_p}} \nabla C \right), \tag{2.6b}$$

for the variables n and ϕ. The ellipticity of (2.6b) is guaranteed if nonnegativity of both the electron density n and the hole density $n-C$ are required. Let us consider two simple cases. Assume the doping profile to be constant $C = const$ and $\Omega_D = \emptyset$, then $\phi = const$ due to (2.6b) and the homogeneous Neumann boundary conditions deduced from (2.2c) and (2.2d). We obtain the nonlinear heat equation

$$n_t = \operatorname{div}\left(\frac{2n-C}{\frac{n}{\mu_p}+\frac{n-C}{\mu_n}}\nabla n\right). \tag{2.7}$$

If, further, $C \equiv 0$ the limiting problem is the linear heat equation

$$n_t = \operatorname{div}\left(\frac{2}{\frac{1}{\mu_p}+\frac{1}{\mu_n}}\nabla n\right). \tag{2.8}$$

In both cases, the limiting electron density satisfies homogeneous Neumann boundary conditions.

On the fast scale the drift-diffusion equations are given by

$$n_s = \operatorname{div}(\mu_n(\lambda^2\nabla n + nF)), \tag{2.9a}$$

$$p_s = \operatorname{div}(\mu_p(\lambda^2\nabla p - pF)), \tag{2.9b}$$

$$-\operatorname{div}F = n - p - C. \tag{2.9c}$$

Here we use the rescaled field $F = \lambda^2 E$. Correspondingly, the rescaled potential $V = \lambda^2\Phi$ with $F = -\nabla V$ is introduced. For the sake of completeness, we restate the initial conditions

$$n(s=0,x) = n_0(x), \qquad p(s=0,x) = p_0(x), \tag{2.9d}$$

and the boundary conditions

$$(\lambda^2\nabla n + nF)\cdot\nu = 0, \quad (\lambda^2\nabla p - pF)\cdot\nu = 0, \quad F\cdot\nu = 0 \text{ on } \partial\Omega \tag{2.9e}$$

and the solvability condition (2.3). Formally, for $\lambda \to 0$, the following limiting problem is obtained:

$$n_s = \operatorname{div}(\mu_n nF) \tag{2.10a}$$

$$p_s = -\operatorname{div}(\mu_p pF) \tag{2.10b}$$

$$-\operatorname{div}F = n - p - C \tag{2.10c}$$

with the initial conditions

$$n(s=0,x) = n_0(x), \qquad p(s=0,x) = p_0(x), \tag{2.10d}$$

and the boundary condition

$$F\cdot\nu = 0 \quad \text{on } \partial\Omega. \tag{2.10e}$$

On the fast scale we only deal with homogeneous Neumann boundary conditions. The reason lies in the fact that the boundary $\partial\Omega \times \mathbb{R}_t^+$ is characteristic for both hyperbolic limit equations (2.10a),(2.10b) which essentially allows us to treat (2.10) as an initial value problem. The boundary conditions for the densities are lost in the limit due to the singular perturbation character of (2.9a) and (2.9b). Also, no further boundary conditions are needed for the limiting problem, since the boundary is characteristic for (2.10a) and (2.10b).

The problem (2.10) describes an initial time layer. Initial data for the slow limiting problem (2.2) can be derived by going to the limit $s \to \infty$ in (2.10). The layer solution can be matched to the solution of (2.2), if and only if the limit $(n_\infty, p_\infty, F_\infty)$ as $s \to \infty$ of the solution of (2.10) satisfies local charge neutrality:

$$n_\infty - p_\infty - C = 0\,, \qquad F_\infty = 0\,.$$

This, however, can only be expected if the initial data satisfy (2.4). In more general situations it will be shown below that only

$$-\mathrm{div} F_\infty = n_\infty - p_\infty - C = 0\,, \qquad F_\infty(n_\infty + p_\infty) = 0\,,$$

holds. Thus, Ω is split into vacuum regions ($n_\infty = p_\infty = 0$) and charge neutral regions ($n_\infty - p_\infty - C = 0$).

A rigorous treatment of the above asymptotic procedures has to start with an existence and uniqueness analysis of the full problem (2.1) (or, equivalently, (2.9)). Existence and uniqueness of solutions was shown under natural assumptions on the data such as smoothness of $\partial\Omega$, $C \in L^\infty(\Omega)$, $n_0, p_0 \in L^2(\Omega)$ nonnegative (see, e.g., [7], [6], [20]), and will be assumed here. An important tool is the entropy functional [7], [6]

$$e(t) = \int_\Omega \left(n\,(\ln \frac{n}{n_e} - 1) + p\,(\ln \frac{p}{p_e} - 1) + \frac{\lambda^2}{2} |\nabla \Phi - \nabla \Phi_e|^2 \right) dx + e_0, \tag{2.11}$$

where the constant e_0 is such that the entropy $e(t)$ is a nonnegative quantity. The functions n_e and p_e are given by

$$n_e = n_i e^{\Phi_e}, \qquad p_e = n_i e^{-\Phi_e}, \tag{2.12}$$

where Φ_e solves the equilibrium problem

$$\begin{aligned} \lambda^2 \Delta \Phi_e &= n_i e^{\Phi_e} - n_i e^{-\Phi_e} - C \quad \text{in} \quad \Omega \\ \Phi_e &= \Phi_{bi} \quad \text{on} \quad \partial\Omega_D \\ \nabla \Phi_e \cdot \nu &= 0 \quad \text{on} \quad \partial\Omega_N. \end{aligned} \tag{2.13}$$

A straightforward computation gives the entropy dissipation

$$\frac{de}{dt} = -\int_\Omega \left(\mu_n \frac{|\nabla n + nE|^2}{n} + \mu_p \frac{|\nabla p - pE|^2}{p} \right) dx, \tag{2.14}$$

and, thus, the decay of the physical entropy with time follows. A rescaled version for the fast problem is given by

$$\begin{aligned} \mathscr{E}(s) &= \lambda^2 e(\lambda^2 s) \qquad (2.15) \\ &= \int_\Omega \left(\lambda^2 n(\ln \frac{n}{n_e} - 1) + \lambda^2 p(\ln \frac{p}{p_e} - 1) + \frac{1}{2} |F - \lambda^2 \nabla \Phi_e|^2 \right) dx + \lambda^2 e_0, \end{aligned}$$

satisfying

$$\frac{d\mathscr{E}}{ds} = -\int_\Omega \left(\mu_n \frac{|\lambda^2 \nabla n + nF|^2}{n} + \mu_p \frac{|\lambda^2 \nabla p - pF|^2}{p} \right) dx\,. \tag{2.16}$$

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As a first step in our treatment, *a priori* estimates will be given in the following section. The main tool are invariant region arguments needing one of the following assumptions on the doping profile:

(A1) $C = const$,
(A2) $C \geqslant \underline{C} > 0$ in Ω,
(A2') $C \leqslant -\underline{C} < 0$ in Ω.

The main results of this paper are shown for either (A1) or (A2). Note that, assuming global neutrality (2.3), the condition (2.4) for the charge neutral limit is automatically satisfied if one of the above assumptions (A1), (A2), (A2') holds.

3 *A priori* estimates

In the following we collect new and already known estimates on the solutions on the fast and the slow time scale. We start with L^q-estimates, formulated (for convenience) in terms of the fast problem (2.9) (with homogeneous Neumann conditions).

Lemma 3.1 *Let $n_0, p_0 \in L^q(\Omega)$ (and nonnegative), $1 \leqslant q \leqslant \infty$, $C \in L^\infty(\Omega)$. Then, the solution of* (2.9) *satisfies*

$$\|n(s)\|_1 = \|n_0\|_1, \quad \|p(s)\|_1 = \|p_0\|_1, \tag{3.1a}$$

$$\|n(s)\|_{L^q(\Omega)},\ \|p(s)\|_{L^q(\Omega)} \leqslant \exp\left(\overline{\mu}\|C\|_{L^\infty(\Omega)}s\right)\left(\|n_0\|_{L^q(\Omega)} + \|p_0\|_{L^q(\Omega)}\right) \tag{3.1b}$$

with $\overline{\mu} = \max\{\mu_n,\ \mu_p\}$.

Proof The conservation of the L^1-norms (3.1a) is obvious. For $1 < q < \infty$ we compute

$$\frac{1}{q}\frac{d}{ds}\int_\Omega \left(\frac{n^q}{\mu_n} + \frac{p^q}{\mu_p}\right) dx = -\lambda^2(q-1)\int_\Omega \left(n^{q-2}|\nabla n|^2 + p^{q-2}|\nabla p|^2\right) dx$$
$$-\frac{q-1}{q}\int_\Omega (n^q - p^q)(n - p - C)dx\,,$$

with the consequence

$$\frac{d}{ds}\int_\Omega (n^q + p^q)dx \leqslant (q-1)\overline{\mu}\|C\|_{L^\infty(\Omega)}\int_\Omega (n^q + p^q)dx\,.$$

The estimate (3.1b) is now a consequence of the Gronwall lemma. The L^∞ part is obtained by passing to the limit $q \to \infty$. □

On the slow time scale the L^1 norms of n and p are still conserved. Except for a vanishing doping profile and pure homogenous Neumann boundary conditions, the estimate (3.1b) is nonuniform in s, and thus nonuniform in λ for the slow problem. However, uniform estimates can be obtained under additional assumptions, e.g. for constant doping profiles.

Lemma 3.2 *Assume $n_0, p_0 \in L^\infty(\Omega)$, $n_i e^{\Phi_{bi}}, n_i e^{-\Phi_{bi}} \in L^\infty(\partial\Omega_D)$ and (A1). Then the solution*

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of (2.1) *(and of* (2.9)*) satisfies*

$$n, p \in L^\infty(\Omega \times (0, \infty)) \tag{3.2}$$

uniformly in λ.

Proof The result is independent of the choice of the time scale. For definiteness, we shall work with the slow problem (2.1). By expanding the spatial derivatives in the continuity equations and using the Poisson equation, (2.1a), (2.1b) can be written as a system of reaction-diffusion equations with convection:

$$n_t = \mu_n(\Delta n + E \cdot \nabla n) - \frac{\mu_n}{\lambda^2} n(n - p - C), \tag{3.3a}$$

$$p_t = \mu_p(\Delta p - E \cdot \nabla p) + \frac{\mu_p}{\lambda^2} p(n - p - C). \tag{3.3b}$$

We use an invariant region approach. Consider a rectangle $R = [0, \overline{n}] \times [0, \overline{p}]$ in the (n, p)-plane, chosen such that

$$\overline{n} = \overline{p} + C, \qquad (n_0(x), p_0(x)) \in R \text{ for } x \in \Omega. \tag{3.4}$$

It is easily seen that the region R is forward invariant for spatially homogeneous solutions of (3.3). The invariance of R for the full system (3.3) is a consequence of the maximum principle. Thus, the solution $(n(x, t), p(x, t))$ remains in R for all $t > 0$ $(s > 0)$ and $x \in \Omega$, completing the proof. □

For the analysis in § 5 it is crucial that one of the densities is bounded away from zero.

Lemma 3.3 *Assume* (A2) ((A2')) *and* $n_0 \geqslant \underline{C}$ *(*$p_0 \geqslant \underline{C}$*) in* Ω, $n_i e^{\Phi_{bi}} \geqslant \underline{C}$ *(*$n_i e^{-\Phi_{bi}} \geqslant \underline{C}$*) on* $\partial\Omega_D$. *Then the solution of* (2.1) *(and of* (2.9)*) satisfies*

$$n(x, t) \geqslant \underline{C}, \qquad (p(x, t) \geqslant \underline{C}) \tag{3.5}$$

for $(x, t) \in \Omega \times (0, \infty)$.

Proof We apply the same ideas as in the proof of Lemma 3.2 with the invariant region $R = [\underline{C}, \infty) \times [0, \infty)$ $(R = [0, \infty) \times [\underline{C}, \infty))$. □

We repeat that, in the case of doping profiles with changes of sign, uniform estimates of the above type are not known.

4 The limit $\lambda \to 0$ and the large-time asymptotics in the fast problem

We start this section with the limit $\lambda \to 0$ in the fast problem.

Theorem 4.1 *Let* $C \in L^\infty(\Omega)$, $n_0, p_0 \in L^q(\Omega)$ *with* $q > \frac{2d}{d+1}$, *and* $S > 0$. *Then, as* $\lambda \to 0$ *(after extracting subsequences) the solution of* (2.9) *converges to a solution of* (2.10), *where the convergence of the densities is weak in* $L^\infty((0, S); L^q(\Omega))$ *and the convergence of the field*

F is strong in $C([0,S];L^p(\Omega))$ *for*

$$1 < p < \begin{cases} \frac{dq}{d-q} & \textit{for } q < d\,, \\ \infty & \textit{for } d \leqslant q\,. \end{cases}$$

This theorem is a special case of Theorem 2.1 in Gasser [8, 9], where the nonlinear exponents take the values $\gamma_n = \gamma_p = 1$.

Next we investigate the long time behaviour of the initial layer problem (2.10). In particular, we shall prove convergence to a steady state.

The basic entropy equality is obtained by setting $\lambda = 0$ in (2.16):

$$\frac{1}{2}\frac{d}{ds}\int_\Omega |F|^2 dx + \int_\Omega |F|^2(\mu_n n + \mu_p p)dx = 0. \tag{4.1}$$

Now let $s_k > 0$ be a sequence with $s_k \xrightarrow{k\to\infty} \infty$.

We define for $x \in \Omega$ and $s \in [0,S]$ with $S > 0$

$$F^k(x,s) := F(x, s + s_k) \tag{4.2a}$$
$$n^k(x,s) := n(s, s + s_k) \tag{4.2b}$$
$$p^k(x,s) := p(x, s + s_k). \tag{4.2c}$$

Obviously, F^k, n^k, p^k satisfy the IVP on Ω:

$$n^k_s \;=\; \mu_n \mathrm{div}(F^k n^k), \quad n^k(s=0) = n(x, s_k) \tag{4.3a}$$
$$p^k_s = - \mu_p \mathrm{div}(F^k p^k), \quad p^k(s=0) = p(x, s_k) \tag{4.3b}$$
$$-\mathrm{div}F^k \;=\; n^k - p^k - C(x), \quad F^k = -\nabla V^k \tag{4.3c}$$
$$F^k \cdot \nu \;=\; 0 \quad \text{on } \partial\Omega \tag{4.3d}$$

for $s \in [0,S]$. The entropy equality is as above and, after integration with respect to $s \in [0,S]$

$$\frac{1}{2}\int_\Omega |F^k(S)|^2 dx + I^k(S) = \frac{1}{2}\int_\Omega |F(s_k)|^2 dx \tag{4.4}$$

where

$$\begin{aligned} I^k(S) &:= \int_0^S \int_\Omega |F^k(s)|^2(\mu_n n^k(s) + \mu_p(p^k(s))dx\,ds \\ &= \int_{s_k}^{s_k+S} \int_\Omega |F(s)|^2(\mu_n n(s) + \mu_p p(s))dx\,ds. \end{aligned} \tag{4.5}$$

s-integration of 4.1 shows that the function

$$s \to \int_\Omega |F(s)|^2(\mu_n n(s) + \mu_p p(s))ds$$

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is in $L^1(0,\infty)$. Therefore

$$I^k(S) \xrightarrow{k\to\infty} 0 \tag{4.6}$$

for every (fixed) $S > 0$.

After possible extraction of a subsequence (which we denote as the sequence for the sake of simplicity) we have

$$n(\cdot, s_k) \rightharpoonup n_\infty \quad \text{in } \mathscr{M}^+(\Omega) \text{ weak-*}, \tag{4.7a}$$
$$p(\cdot, s_k) \rightharpoonup p_\infty \quad \text{in } \mathscr{M}^+(\Omega) \text{ weak-*}, \tag{4.7b}$$

(due to the $L^1(\Omega)$-conservation of $n(s)$, $p(s)$),

$$F(\cdot, s_k) \rightharpoonup F_\infty \quad \text{in } L^2(\Omega) \text{ weak}, \tag{4.7c}$$

(due to 4.1) and

$$n^k \rightharpoonup N_\infty \quad \text{in } \mathscr{M}^+(\Omega \times (0,S)) \text{ weak-*}, \tag{4.7d}$$
$$p^k \rightharpoonup P_\infty \quad \text{in } \mathscr{M}^+(\Omega \times (0,S))\text{weak-*}, \tag{4.7e}$$
$$F^k \rightharpoonup W_\infty \quad \text{in } L^2(\Omega \times (0,S)) \text{ weak}. \tag{4.7f}$$

We estimate

$$\int_0^S \int_\Omega |F^k(x,s)| n^k(x,s) dx\, ds \tag{4.8a}$$
$$\leqslant \left(\int_0^S \int_\Omega |F^k(x,s)|^2 n^k(x,s) dx\, ds \right)^{1/2} \left(\int_0^S \int_\Omega n^k(x,s) dx\, ds \right)^{1/2}$$
$$\leqslant \left(\frac{1}{\mu_n} I^k(S) \right)^{1/2} \left(S \int_\Omega n_0 dx \right)^{1/2} \xrightarrow{k\to\infty} 0$$

and analogously

$$\int_0^S \int_\Omega |F^k(x,s)| p^k(x,s) dx\, ds \xrightarrow{k\to\infty} 0. \tag{4.8b}$$

We can now pass to the limit $k \to \infty$ in (4.3a), (4.3b) and obtain

$$\frac{\partial N_\infty}{\partial s} = \frac{\partial P_\infty}{\partial s} = 0,$$

such that

$$N_\infty = n_\infty, \qquad P_\infty = p_\infty, \qquad W_\infty = F_\infty,$$

follows (i.e. the limits of n^k, p^k, F^k are independent of $s \in [0,S]$). We can now prove under additional hypotheses the following result.

Theorem 4.2 *Assume $C \in L^\infty(\Omega)$ and consider a solution of* (2.10) *such that $n, p \in L^\infty(\Omega \times (0,\infty))$. Let $S > 0$. Then for every sequence $s_k \to \infty$ there exists a subsequence $s_{k_l} \to \infty$ and*

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functions $n_\infty, p_\infty \in L^\infty_+(\Omega)$, $F_\infty \in C(\overline{\Omega})$ *satisfying a.e. in* Ω

$$F_\infty(n_\infty + p_\infty) = 0 \tag{4.9}$$

and

$$\begin{aligned} n^{k_l} &\rightharpoonup n_\infty \quad \text{in } L^\infty(\Omega \times (0,S)) \text{ weak-*}, \\ p^{k_l} &\rightharpoonup p_\infty \quad \text{in } L^\infty(\Omega \times (0,S)) \text{ weak-*}, \\ F^{k_l} &\to F_\infty \quad \text{in } C(\overline{\Omega} \times [0,S]). \end{aligned}$$

Note that $(n_\infty, p_\infty, F_\infty)$ is a steady state solution of (2.10) because of (4.9).

Proof The L^q-theory of elliptic equations implies $F(s) \in W^{1,q}(\Omega)$ uniformly as $s \to \infty$ for every $1 \leqslant q < \infty$. Thus we have $n(s)F(s), p(s)F(s) \in L^r(\Omega)$ uniformly for every $1 \leqslant r < \infty$ as $s \to \infty$. From the time-differentiated version of the Poisson equation,

$$-\text{div} F_s = \text{div}(F(\mu_n n + \mu_p p))\,;$$

we then conclude $F_s(s) \in L^{r'}(\Omega)$ uniformly for every $1 \leqslant r' < \infty$ as $s \to \infty$ and, after possible extraction of subsequence,

$$F^{k_l} \to F_\infty \quad \text{in } C(\overline{\Omega} \times [0,S])$$

follows. We use (4.6),

$$\begin{aligned} &\int_0^S \int_\Omega |F^{k_l}| \left(\mu_n n^{k_l} + \mu_p p^{k_l}\right) dx\, ds = I^{k_l}(S) \\ &\xrightarrow{k_l \to \infty} \int_0^S \int_\Omega |F_\infty|(\mu_n n_\infty + \mu_p p_\infty) dx\, ds = 0, \end{aligned}$$

and conclude the assertion. □

Note that the boundedness condition on n and p in $L^\infty(\Omega \times (0,\infty))$ can be somewhat weakened without weakening the assertion. Also, note that, as shown in the previous section, this condition can be verified if the doping profile is constant and if the initial data for n and p are bounded.

Finally, the limit $s \to \infty$ can also be carried out if the total carrier density is bounded away from zero (which can be verified under assumption (A2) or (A2'), see Lemma 3.3).

Theorem 4.3 *Consider a solution of* (2.10) *such that* $F(s = 0) \in L^2(\Omega)$ *and* $n + p \geqslant \underline{C} > 0$. *Let* $s > 0$. *Then,*

$$\|F(\cdot,s)\|_{L^2(\Omega)} \leqslant e^{-Ks} \|F(\cdot,0)\|_{L^2(\Omega)} \tag{4.10}$$

with $K = \underline{C} \min\{\mu_n, \mu_p\}/2$ *holds, and for every sequence* $s_k \to \infty$ *there exists a subsequence* $s_{k_l} \to \infty$ *and nonnegative measures* n_∞, p_∞ *such that a.e. in* Ω

$$n_\infty - p_\infty - C = 0\,, \tag{4.11}$$

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and

$$n^{k_l} \rightharpoonup n_\infty \quad \text{in } \mathscr{M}^+(\Omega \times (0,S)) \text{ weak-*},$$
$$p^{k_l} \rightharpoonup p_\infty \quad \text{in } \mathscr{M}^+(\Omega \times (0,S)) \text{weak-*}.$$

Proof The decay estimate (4.10) is an immediate consequence of the entropy equality (4.1). The remaining results follow from the above. □

5 The limit $\lambda \to 0$ in the slow problem

Here we focus on the quasineutral limit on the slow time scale. An important assumption will be the boundedness uniformly in λ of the entropy functional (2.11). It is easily seen that this implies the assumption of specially prepared initial conditions compatible with the limiting problem:

$$n_0 - p_0 - C = 0.$$

In this case no initial layer occurs.

The first result concerning the vanishing doping profile case has been proved in Gasser [8, 9].

Theorem 5.1 *Assume* $C \equiv 0$. *Let* $e(0) \leqslant c$, $n_0 = p_0 \in L^q(\Omega)$ *with* $1 \leqslant q \leqslant \infty$ *uniformly in* λ. *Let* $T > 0$. *Then, as* $\lambda \to 0$, *for a solution of* (2.1), *the following convergence results hold (after extracting subsequences):*

- $n \rightharpoonup w$, $p \rightharpoonup w$ *in* $L^\infty((0,T); L^q(\Omega))$ *(weak-⋆ convergence in the case* $q = \infty$*)*
- $(n^\lambda - p^\lambda)E^\lambda \to 0$ *strongly in* $L^{2(q+1)/(q+2)}(\Omega \times (0,T))$,

and w *satisfies the heat equation* (2.8) *in* $\mathscr{D}'(\Omega \times [0,T))$ *with initial datum* $w(t=0) = n_0 = p_0$.

Our main result for nonvanishing doping profiles reads

Theorem 5.2 *Let* $n_0, p_0 \in L^1(\Omega)$ *uniformly in* λ, $T > 0$ *and* $C \in H^1(\Omega)$. *Consider a solution of* (2.1) *satisfying* $n + p \geqslant \underline{C} > 0$ *and* $e(0) < c$ *uniformly in* λ. *Then, as* $\lambda \to 0$, (n, p, E) *converges (after extracting subsequences) to a solution of* (2.2) *with initial data* n_0, p_0. *The convergence of* n *and* p *is is strong in* $L^1(\Omega \times (0,T))$, *whereas* E *converges weakly in* $L^2(\Omega \times (0,T))$.

Proof The entropy equality (2.14) gives

$$\frac{\nabla n + nE}{\sqrt{n}}, \quad \frac{\nabla p - pE}{\sqrt{p}} \quad \in L^2(\Omega \times (0,T)) \text{ uniformly in } \lambda. \tag{5.1}$$

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Multiplying the Poisson equation by $(n-p-C)/\lambda^2$ we obtain

$$\begin{aligned}
&\int_\Omega \left((n+p)|E|^2 + \frac{(n-p-C)^2}{\lambda^2}\right) dx \\
&= \int_\Omega (-\nabla C - (\nabla n + nE) - (\nabla p - pE)) \cdot E\, dx \\
&\leqslant \int_\Omega \left(\frac{|\nabla C|}{\sqrt{n+p}} + \frac{|\nabla n + nE|}{\sqrt{n}} + \frac{|\nabla p - pE|}{\sqrt{p}}\right) \sqrt{n+p}|E| dx. \qquad (5.2)
\end{aligned}$$

Using the lower bound on $n+p$ and the bound (5.1) we conclude, by the Cauchy–Schwarz inequality,

$$\frac{n-p-C}{\lambda}, \quad \sqrt{n}E, \quad \sqrt{p}E, \quad \frac{\nabla n}{\sqrt{n}}, \quad \frac{\nabla p}{\sqrt{p}} \in L^2(\Omega \times (0,T)) \qquad (5.3)$$

uniformly in λ. Therefore, we have

$$\sqrt{n}, \sqrt{p} \in L^2((0,T); H^1(\Omega)) \hookrightarrow L^2((0,T); L^{2^*}(\Omega)) \qquad (5.4)$$

and

$$\nabla n, \nabla p, nE, pE \in L^1((0,T); L^r(\Omega)), \quad r = \frac{1}{\frac{1}{2^*} + \frac{1}{2}} = \frac{d}{d-1}. \qquad (5.5)$$

The equations (2.1a) and (2.1b) give

$$\frac{\partial n}{\partial t}, \frac{\partial p}{\partial t} \in L^1((0,T); W^{-1,1}(\Omega)). \qquad (5.6)$$

Combining the last two estimates (5.5) and (5.6) we conclude by standard compactness results [31] strong convergence of n and p in $L^1(\Omega \times (0,T))$.

On the other hand, interpolating between (5.4) and the charge conservation (3.1a) (on the slow time scale)

$$\sqrt{n}, \sqrt{p} \in L^2((0,T); L^{2^*}(\Omega)) \cap L^\infty((0,T); L^2(\Omega)), \qquad (5.7)$$

we obtain $\sqrt{n}, \sqrt{p} \in L^{2(d+2)/d}((0,T)\times\Omega)$ and strong convergence of n and p in $L^{(d+2)/d-}(\Omega \times (0,T))$. In a similar way, we conclude strong convergence of $\sqrt{n}$ and $\sqrt{p}$ in $L^{2(d+2)/d-}(\Omega \times (0,T))$. Weak convergence of E in $L^2(\Omega \times (0,T))$ is obtained from the lower bound on $n+p$,

$$|E| \leqslant \frac{1}{\sqrt{\underline{C}}} \sqrt{n+p}\,|E|, \qquad (5.8)$$

and the estimate (5.3). This allows us to conclude weak convergence of $nE = \sqrt{n}(\sqrt{n}E)$ in $L^1((0,T); L^{d/(d-1)}(\Omega))$, and the same for pE. Also, (5.3) implies the quasineutrality (2.2c) in the limit. This concludes the proof of the theorem. □

The situation is much more delicate if the uniform bound from below for the sum of the densities is removed. To show the diffuculties arising in this case, we consider the example of a unipolar one-dimensional model:

$$n_t = (n_x + nE)_x, \qquad (5.9)$$

$$-\lambda^2 E_x = n - C, \qquad (5.10)$$

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with boundary conditions

$$n_x(0,t) = n_x(1,t) = E(0,t) = E(1,t) = 0. \tag{5.11}$$

For the doping profile we assume $C \geqslant 0$, but allow for zeroes of C. Simple manipulations show that the field satisfies a Burgers equation with a strong relaxation term:

$$\lambda^2 \left(E_t - \left(\frac{E^2}{2} \right)_x - E_{xx} \right) = -CE - C_x, \tag{5.12}$$

$$E(0,t) = E(1,t) = 0. \tag{5.13}$$

Multiplication of the differential equation by E, integration with respect to x, t and using the estimate

$$\left| \int_0^1 C_x E \, dx \right| \leqslant \frac{1}{2} \int_0^1 CE^2 dx + \frac{1}{2} \int_0^1 \frac{C_x^2}{C} dx,$$

gives

$$\int_0^T \int_0^1 CE^2 dx \leqslant 4 \int_0^T \int_0^1 (\sqrt{C}\,)_x^2 dx + \lambda^2 \int_0^1 E(t=0)^2 dx.$$

Assuming $\sqrt{C} \in H^1((0,1))$ and uniform boundedness (with respect to λ) of $\lambda E(t=0)$ in $L^2((0,1))$, uniform boundedness of E in $L^2(B \times (0,T))$ follows for subintervals B of $(0,1)$, where the doping profile is bounded away from zero. Obviously this is enough to pass to the limit $\lambda \to 0$ in the equation (5.12) in such subintervals (in the distributional sense). There a weak limit of the field exists and is equal to $-(\ln C)_x$. Thus, if C has zeros, the field will in general not be bounded as $\lambda \to 0$.

6 Conclusions

This paper represents a first step in the rigorous mathematical analysis of the small Debye length regime for transient drift diffusion semiconductor equations. The main new ingredient in the analysis is the physically motivated entropy/entropy-dissipation technique. It turns out that the entropy and the entropy dissipation provide essential uniform bounds which facilitate the passage to the limit $\lambda \to 0$. Although we only treat a model problem here (semiconductor devices without p-n and other abrupt junctions) we believe that the tools developed here also apply to the analysis of more complicated and realistic small Debye length limit problems, such as hydrodynamic and fully kinetic models, where intuitive formal asymptotic techniques are not so readily available. A next objective in this research program is the justification of the fully formal asymptotics (matching of the two time scales/slow and fast spatial scales at the p-n junctions) for the bipolar drift diffusion equations.

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