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ON EDGE-PRIMITIVE GRAPHS WITH SOLUBLE EDGE-STABILIZERS

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Abstract

A graph is edge-primitive if its automorphism group acts primitively on the edge set, and 2-arc-transitive if its automorphism group acts transitively on the set of 2-arcs. In this paper, we present a classification for those edge-primitive graphs that are 2-arc-transitive and have soluble edge-stabilizers.

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1. Introduction

In this paper, all graphs are assumed to be finite and simple, and all groups are assumed to be finite.

A graph is a pair $\Gamma = (V, E)$ of a nonempty set V and a set E of 2-subsets of V. The elements in V and E are called the vertices and edges of Γ, respectively. For $v \in V$, the set $\Gamma(v) = \{u \in V \mid \{u, v\} \in E\}$ is called the neighborhood of v in Γ, while $|\Gamma(v)|$ is called the valency of v. We say that the graph Γ has valency d or Γ is d-regular if its vertices have equal valency d. For an integer $s \geq 1$, an s-arc in Γ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $\{v_{i-1}, v_i\} \in E$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. A 1-arc is also called an arc.

Let $\Gamma = (V, E)$ be a graph. A permutation g on V is called an automorphism of Γ if $\{u^g, v^g\} \in E$ for all $\{u, v\} \in E$. All automorphisms of Γ form a subgroup of the symmetric group $\mathrm{Sym}(V)$, denoted by $\mathsf{Aut}\Gamma$, that is called the automorphism group of Γ. The group $\mathsf{Aut}\Gamma$ has a natural action on E, namely, $\{u, v\}^g = \{u^g, v^g\}$ for $\{u, v\} \in E$ and $g \in \mathsf{Aut}\Gamma$. If this action is transitive, that is, for each pair of edges there exists some $g \in \mathsf{Aut}\Gamma$ mapping one edge to the other one, then Γ is called *edge-transitive*.

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Similarly, we may define the *vertex-transitivity*, *arc-transitivity* and *s-arc-transitivity* of Γ. The graph Γ is called *edge-primitive* if $\mathsf{Aut}\Gamma$ acts primitively on E, that is, Γ is edge-transitive and the stabilizer $(\mathsf{Aut}\Gamma)_{\{u,v\}}$ of some (and hence every) edge $\{u, v\}$ in $\mathsf{Aut}\Gamma$ is a maximal subgroup.

The class of edge-primitive graphs includes many famous graphs such as the Heawood graph, Tutte's 8-cage, the Biggs–Smith graph, the Hoffman–Singleton graph, the Higman–Sims graph and the rank-3 graphs associated with the sporadic simple groups M_{22}, J_2, McL, Ru, Suz and Fi_{23} and so on. In 1973, Weiss [34] determined all edge-primitive graphs of valency three. Up to isomorphism, all edge-primitive cubic graphs consist of the complete bipartite graph $\mathsf{K}_{3,3}$ and the first three graphs mentioned above. After that, edge-primitive graphs had received little attention until Giudici and Li [9] systematically investigated the existence and the general structure of such graphs in 2010. Giudici and Li's work has stimulated a lot of progress in the study of edge-primitive graphs; see [8, 10, 11, 17, 21, 24], for example. Also, their work reveals that those graphs associated with almost simple groups play an important role in the study of edge-primitive graphs. This is one of the main motivations of [21] and the present paper.

Let $\Gamma = (V, E)$ be an edge-primitive graph of valency no less than three. Then, as observed in [9], Γ is also arc-transitive. If Γ is 2-arc-transitive, then Praeger's reduction theorems [25, 26] will be effective tools for us to investigate the group-theoretic and graph-theoretic properties of Γ. However, Γ is not necessarily 2-arc-transitive; for example, by the Atlas [3], the sporadic Rudvalis group Ru is the automorphism group of a rank-three graph that is edge-primitive and of valency 2304 but not 2-arc-transitive. Using the O'Nan–Scott theorem for (quasi)primitive groups [25], Giudici and Li [9] gave a reduction theorem on the automorphism group of Γ. They proved that, as a primitive group on E, only four of the eight O'Nan–Scott types for primitive groups may occur for $\mathsf{Aut}\Gamma$, namely SD, CD, PA and AS. They also considered the possible O'Nan–Scott types for $\mathsf{Aut}\Gamma$ acting on V, and presented constructions or examples to verify the existence of corresponding graphs. Then what will happen if we assume further that Γ is 2-arc-transitive? The third author of this paper showed that either $\mathsf{Aut}\Gamma$ is almost simple or Γ is a complete bipartite graph if Γ is 2-arc-transitive; see [21]. This stimulated our interest in classifying those edge-primitive graphs that are 2-arc-transitive.

In this paper, we present a classification result stated as follows.

THEOREM 1.1. *Let $\Gamma = (V, E)$ be a graph of valency $d \geq 6$ and let $G \leq \mathsf{Aut}\Gamma$ be such that G acts primitively on the edge set and transitively on the 2-arc set of Γ. Assume further that G is almost simple and, for $\{u, v\} \in E$, the edge-stabilizer $G_{\{u,v\}}$ is soluble. Then either Γ is $(G, 4)$-arc-transitive or G, $G_{\{u,v\}}$, G_v and d are listed as in Table 1.*

REMARK 1.2. If Γ is edge-primitive and either 4-arc-transitive or of valency less than six, then the edge-stabilizers must be soluble. The reader may find a complete list of such graphs in [10, 11, 17, 34]. For each triple $(G, G_v, G_{\{u,v\}})$ listed in Table 1, the coset

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TABLE 1. Graphs.

G	$G_{\{u,v\}}$	G_v	d	Remark
$PSL_4(2).2$	$2^4{:}S_4$	$2^3{:}SL_3(2)$	7	
$PSL_5(2).2$	$[2^8]{:}S_3^2.2$	$2^6{:}(S_3\times SL_3(2))$	7	
$F_4(2).2$	$[2^{22}]{:}S_3^2.2$	$[2^{20}].(S_3\times SL_3(2))$	7	
$PSL_4(3).2$	$3_+^{1+4}{:}(2S_4\times 2)$	$3^3{:}SL_3(3)$	13	
$PSL_4(3).2^2$	$3_+^{1+4}{:}(2S_4\times \mathbb{Z}_2^2)$	$3^3{:}(SL_3(3)\times\mathbb{Z}_2)$	13	
$PSL_5(3).2$	$[3^8]{:}(2S_4)^2.2$	$3^6.2S_4.SL_3(3)$	13	
S_p	$\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$	$PSL_2(p)$	$p{+}1$	$p\in\{7,11\}$
M_{11}	$3^2{:}Q_8.2$	M_{10}	10	K_{11}
J_1	$\mathbb{Z}_{11}{:}\mathbb{Z}_{10}$	$PSL_2(11)$	12	
$J_3.2$	$\mathbb{Z}_{19}{:}\mathbb{Z}_{18}$	$PSL_2(19)$	20	
$O'N.2$	$\mathbb{Z}_{31}{:}\mathbb{Z}_{30}$	$PSL_2(31)$	32	
B	$\mathbb{Z}_{19}{:}\mathbb{Z}_{18}\times\mathbb{Z}_2$	$PGL_2(19)$	20	
B	$\mathbb{Z}_{23}{:}\mathbb{Z}_{11}\times\mathbb{Z}_2$	$PSL_2(23)$	24	
M	$\mathbb{Z}_{41}{:}\mathbb{Z}_{40}$	$PSL_2(41)$	42	
$PSL_2(19)$	D_{20}	$PSL_2(5)$	6	
$A_6.2, A_6.2^2$	$\mathbb{Z}_5{:}[4], \mathbb{Z}_{10}{:}\mathbb{Z}_4$	$PSL_2(5), PGL_2(5)$	6	$\mathsf{K}_{6,6}$, $G\not\cong S_6$
$PGL_2(11)$	D_{20}	$PSL_2(5)$	6	
$PSL_3(r)$	$3^2{:}Q_8$	$PSL_2(9)$	10	r is a prime with
$PSL_3(r).2$	$3^2{:}Q_8.2$	$PGL_2(9)$		$r\equiv 4,16,31,34 \bmod 45$
$PSU_3(r)$	$3^2{:}Q_8$	$PSL_2(9)$	10	r is a prime with
$PSU_3(r).2$	$3^2{:}Q_8.2$	$PGL_2(9)$		$r\equiv 11,14,29,41 \bmod 45$
$HS.2$	$[5^3]{:}[2^5]$	$PSU_3(5){:}2$	126	
Ru	$[5^3]{:}[2^5]$	$PSU_3(5){:}2$	126	
M_{10}	$\mathbb{Z}_8{:}\mathbb{Z}_2$	$3^2{:}Q_8$	9	K_{10}
$PSL_3(3).2$	$GL_2(3){:}2$	$3^2{:}GL_2(3)$	9	
J_1	$\mathbb{Z}_7{:}\mathbb{Z}_6$	$\mathbb{Z}_2^3{:}\mathbb{Z}_7{:}\mathbb{Z}_3$	8	
$PSL_2(p^f).[o]$	$D_{2(p^f-1)/(2,p-1)}.[o]$	$\mathbb{Z}_p^f{:}\mathbb{Z}_{p^f-1/(2,p-1)}.[o]$	p^f	K_{p^f+1}, $o\mid(2,p-1)f$
$Sz(2^f).o$	$D_{2(2^f-1)}.o$	$\mathbb{Z}_2^f{:}\mathbb{Z}_{2^f-1}.o$	2^f	f is odd, $o\mid f$

graph $\mathsf{Cos}(G, G_v, G_{\{u,v\}})$, see Section 2 for the definition, is both $(G,2)$-arc-transitive and G-edge-primitive.

2. Preliminaries

Let G be a finite group and $H, K \le G$ with $|K:(H\cap K)| = 2$ and $\cap_{g\in G}H^g = 1$, and let $[G:H] = \{Hx \mid x\in G\}$. We define a graph $\mathsf{Cos}(G,H,K)$ on $[G:H]$ such that $\{Hx, Hy\}$ is an edge if and only if $yx^{-1}\in HKH\setminus H$. The group G can be viewed as a subgroup of $\mathsf{Aut}\mathsf{Cos}(G,H,K)$, where G acts on $[G:H]$ by right multiplication. Then $\mathsf{Cos}(G,H,K)$ is G-arc-transitive and, for $x\in K\setminus H$, the edge $\{H, Hx\}$ has stabilizer K in G. Thus, $\mathsf{Cos}(G,H,K)$ is G-edge-primitive if and only if K is maximal in G.

Assume that $\Gamma = (V,E)$ is a G-edge-primitive graph of valency $d\ge 3$. Then Γ is G-arc-transitive by [9, Lemma 3.4]. Take an edge $\{u,v\}\in E$ and let $H = G_v$ and $K = G_{\{u,v\}}$. Then K is maximal in G, and $H\cap K = G_{uv}$ that has index two in K. Noting that $\cap_{g\in G}H^g$ fixes V pointwise, $\cap_{g\in G}H^g = 1$. Further, $v^g \mapsto G_vg$ for all $g\in G$ gives an isomorphism from Γ to $\mathsf{Cos}(G,H,K)$. Then, by [5, Theorem 2.1], the following lemma holds.

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LEMMA 2.1. *Let $\Gamma = (V, E)$ be a connected graph of valency $d \geq 3$ and $G \leq \mathrm{Aut}\Gamma$. Then Γ is both $(G, 2)$-arc-transitive and G-edge-primitive if and only if $\Gamma \cong \mathrm{Cos}(G, H, K)$ for some subgroups H and K of G satisfying:*

(1) *$|K : (H \cap K)| = 2$, $\cap_{g \in G} H^g = 1$ and K is maximal in G;*
(2) *H acts 2-transitively on $[H : (H \cap K)]$ by right multiplication.*

Let $\Gamma = (V, E)$ be a connected graph of valency at least 3, $\{u, v\} \in E$ and $G \leq \mathrm{Aut}\Gamma$. Assume that Γ is (G, s)-arc-transitive for some $s \geq 1$, that is, G acts transitively on the s-arc set of Γ. Then G_v acts transitively on the neighborhood $\Gamma(v)$ of v in Γ. Let $G_v^{\Gamma(v)}$ be the transitive permutation group induced by G_v on $\Gamma(v)$, and let $G_v^{[1]}$ be the kernel of G_v acting on $\Gamma(v)$. Then $G_v^{\Gamma(v)} \cong G_v/G_v^{[1]}$. Considering the action of G_{uv} on $\Gamma(v)$,

$$(G_v^{\Gamma(v)})_u = G_{uv}^{\Gamma(v)} \cong G_{uv}/G_v^{[1]}.$$

Similarly, $(G_u^{\Gamma(u)})_v = G_{uv}^{\Gamma(u)} \cong G_{uv}/G_u^{[1]}$. Since G is transitive on the arcs of Γ, there is some element in G interchanging u and v. This implies that

$$|G_{\{u,v\}}{:}G_{uv}| = 2 \text{ and } (G_v^{\Gamma(v)})_u \cong (G_u^{\Gamma(u)})_v.$$

Set $G_{uv}^{[1]} = G_u^{[1]} \cap G_v^{[1]}$. Then $G_{uv}^{[1]}$ is the kernel of G_{uv} acting on $\Gamma(u) \cup \Gamma(v)$ and, noting that $G_{uv}/(G_u^{[1]} \cap G_v^{[1]}) \lesssim (G_{uv}/G_u^{[1]}) \times (G_{uv}/G_v^{[1]})$,

$$G_{uv}/G_{uv}^{[1]} = G_{uv}/(G_u^{[1]} \cap G_v^{[1]}) \lesssim (G_v^{\Gamma(v)})_u \times (G_u^{\Gamma(u)})_v.$$

Since $G_v^{[1]} \trianglelefteq G_{uv}$, we know that $G_v^{[1]}$ induces a normal subgroup $(G_v^{[1]})^{\Gamma(u)}$ of $(G_u^{\Gamma(u)})_v$. In particular,

$$G_v^{[1]}/G_{uv}^{[1]} \cong (G_v^{[1]})^{\Gamma(u)} \trianglelefteq (G_u^{\Gamma(u)})_v.$$

Writing $G_v^{[1]}$, G_{uv} and G_v in group extensions, the next lemma follows.

LEMMA 2.2.

(1) $G_v^{[1]} = G_{uv}^{[1]}.(G_v^{[1]})^{\Gamma(u)}$, $(G_v^{[1]})^{\Gamma(u)} \trianglelefteq (G_u^{\Gamma(u)})_v$.
(2) $G_{uv} = (G_{uv}^{[1]}.(G_v^{[1]})^{\Gamma(u)}).(G_v^{\Gamma(v)})_u$, $G_v = (G_{uv}^{[1]}.(G_v^{[1]})^{\Gamma(u)}).G_v^{\Gamma(v)}$.
(3) *If $G_{uv}^{[1]} = 1$, then $G_{uv} \lesssim (G_v^{\Gamma(v)})_u \times (G_u^{\Gamma(u)})_v$.*

By [32], $s \leq 7$ and, if $s \geq 2$, then $G_{uv}^{[1]}$ is a p-group for some prime p; refer to [7]. Thus, Lemma 2.2 yields a fact as follows.

COROLLARY 2.3. *Let $\Gamma = (V, E)$ be a connected $(G, 2)$-arc-transitive graph and $\{u, v\} \in E$. Then $G_{\{u,v\}}$ is soluble if and only if $(G_v^{\Gamma(v)})_u$ is soluble, and G_v is soluble if and only if $G_v^{\Gamma(v)}$ is soluble.*

Choose s maximal, that is, Γ is (G, s)-arc-transitive but not $(G, s+1)$-arc-transitive. In this case, Γ is said to be (G, s)-transitive. If further $G_{uv}^{[1]} \neq 1$, then one can read

TABLE 2. Vertex-stabilizers.

$\mathbf{O}_p(G_v)$	$G_{uv}^{[1]}$	s	n	q	G_v
$\mathbb{Z}_p^{n(n-1)f}$	$\mathbb{Z}_p^{(n-1)^2 f}$	3			$\mathrm{SL}_{n-1}(q)\times\mathrm{SL}_n(q) \trianglelefteq G_v/\mathbf{O}_p(G_v)$
$\mathbb{Z}_p^{nf}$	$\mathbb{Z}_p^{f}$	2			$a.\mathrm{PSL}_n(q) \trianglelefteq G_v/\mathbf{O}_p(G_v)$ with $a \mid q-1$
$\mathbb{Z}_p^{n(n-1)f/2}$	$\mathbb{Z}_p^{(n-1)(n-2)f/2}$	2			$a.\mathrm{PSL}_n(q) \trianglelefteq G_v/\mathbf{O}_p(G_v)$ with $a \mid q-1$
$[q^{20}]$	$[q^{18}]$	3	3	even	$\mathrm{SL}_2(q)\times\mathrm{SL}_3(q) \trianglelefteq G_v/\mathbf{O}_p(G_v)$
$[3^6]$	$\mathbb{Z}_3^4$	2	3	3	$[3^6]{:}\mathrm{SL}_3(3)$
$\mathbb{Z}_2^{n+1}$	$\mathbb{Z}_2^2$	2		2	$\mathbb{Z}_2^{n+1}{:}\mathrm{SL}_n(2)$
$\mathbb{Z}_2^{11}, \mathbb{Z}_2^{14}$	$\mathbb{Z}_2^{8}, \mathbb{Z}_2^{11}$	2	4	2	$\mathbb{Z}_2^{11}{:}\mathrm{SL}_4(2), \mathbb{Z}_2^{14}{:}\mathrm{SL}_4(2)$
$[2^{30}]$	$[2^{26}]$	2	5	2	$[2^{30}]{:}\mathrm{SL}_5(2)$

out the vertex-stabilizer G_v from [31, 33] for $s \geq 4$ and from [29] for $2 \leq s \leq 3$. In particular, we have the following result from [29, 33].

THEOREM 2.4. *Let $\Gamma = (V, E)$ be a connected (G, s)-transitive graph of valency at least 3 and $\{u, v\} \in E$. Assume that $s \geq 2$.*

(1) *If $G_{uv}^{[1]} = 1$, then $s = 2$ or 3.*
(2) *If $G_{uv}^{[1]} \neq 1$, then $G_{uv}^{[1]}$ is a p-group for some prime p, $\mathrm{PSL}_n(q) \trianglelefteq G_v^{\Gamma(v)}$, $|\Gamma(v)| = (q^n-1)/(q-1)$ and $6 \neq s \leq 7$, where $n \geq 2$ and $q = p^f$ for some integer $f \geq 1$; moreover, either:*
   - (i) *$n = 2$ and $s \geq 4$; or*
   - (ii) *$n \geq 3$, $s \leq 3$ and $\mathbf{O}_p(G_v)$ is given as in Table 2, where $\mathbf{O}_p(G_v)$ is the maximal normal p-subgroup of G_v.*

LEMMA 2.5. *Let $\Gamma = (V, E)$ be a connected $(G, 2)$-arc-transitive graph and $\{u, v\} \in E$. If r is a prime divisor of $|\Gamma(v)|$, then $\mathbf{O}_r(G_v^{[1]}) = 1$, $\mathbf{O}_r(G_{uv}) = 1$ and either $\mathbf{O}_r(G_v) = 1$, or $\mathbf{O}_r(G_v) \cong \mathbb{Z}_r^e \cong \mathrm{soc}(G_v^{\Gamma(v)})$ and $|\Gamma(v)| = r^e$ for some integer $e \geq 1$.*

PROOF. Since Γ is $(G, 2)$-arc-transitive, $G_v^{\Gamma(v)}$ is a 2-transitive group and thus G_{uv} is transitive on $\Gamma(v) \setminus \{u\}$. Since $\mathbf{O}_r(G_{uv}) \trianglelefteq G_{uv}$, all $\mathbf{O}_r(G_{uv})$-orbits on $\Gamma(v) \setminus \{u\}$ have the same size. Noting that $|\Gamma(v) \setminus \{u\}|$ is coprime to r, it follows that $\mathbf{O}_r(G_{uv}) \leq G_v^{[1]}$. Since $G_v^{[1]} \trianglelefteq G_{uv}$, we have $\mathbf{O}_r(G_v^{[1]}) \leq \mathbf{O}_r(G_{uv})$ and so $\mathbf{O}_r(G_v^{[1]}) = \mathbf{O}_r(G_{uv})$. Similarly, considering the action of G_{uv} on $\Gamma(u) \setminus \{v\}$, we get $\mathbf{O}_r(G_u^{[1]}) = \mathbf{O}_r(G_{uv})$. Then $\mathbf{O}_r(G_u^{[1]}) = \mathbf{O}_r(G_{uv}) = \mathbf{O}_r(G_v^{[1]}) \leq G_{uv}^{[1]}$. By Theorem 2.4, either $G_{uv}^{[1]} = 1$, or $G_{uv}^{[1]}$ is a nontrivial p-group for a prime divisor p of $|\Gamma(v)| - 1$. It follows that $\mathbf{O}_r(G_u^{[1]}) = \mathbf{O}_r(G_{uv}) = \mathbf{O}_r(G_v^{[1]}) = 1$.

Note that $\mathbf{O}_r(G_v)G_v^{[1]}/G_v^{[1]} \cong \mathbf{O}_r(G_v)/(\mathbf{O}_r(G_v) \cap G_v^{[1]})$. Clearly, $\mathbf{O}_r(G_v) \cap G_v^{[1]} \leq \mathbf{O}_r(G_v^{[1]})$ and we have $\mathbf{O}_r(G_v) \cap G_v^{[1]} = 1$. It follows that $\mathbf{O}_r(G_v) \cong \mathbf{O}_r(G_v)G_v^{[1]}/G_v^{[1]} \trianglelefteq G_v/G_v^{[1]} \cong \mathrm{G}_v^{\Gamma(v)}$. Thus, $\mathbf{O}_r(G_v)$ is isomorphic to a normal r-subgroup of $\mathrm{G}_v^{\Gamma(v)}$. This implies that either $\mathbf{O}_r(G_v) = 1$, or $\mathrm{G}_v^{\Gamma(v)}$ is an affine 2-transitive group of degree r^e for some e. Thus, the lemma follows. □

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Let $a \geq 2$ and $f \geq 1$ be integers. A prime divisor r of $a^f - 1$ is primitive if r is not a divisor of $a^e - 1$ for all $1 \leq e < f$. By Zsigmondy's theorem [37], if $f > 1$ and $a^f - 1$ has no primitive prime divisor, then $a^f = 2^6$, or $f = 2$ and $a = 2^t - 1$ for some prime t. Assume that $a^f - 1$ has a primitive prime divisor r. Then a has order f modulo r. Thus, f is a divisor of $r - 1$ and, if r is a divisor of $a^{f'} - 1$ for some $f' \geq 1$, then f is a divisor of f'. Thus, we have the following lemma.

Lemma 2.6. *Let $a \geq 2$, $f \geq 1$ and $f' \geq 1$ be integers. If $a^f - 1$ has a primitive prime divisor r, then f is a divisor of $r - 1$, and r is a divisor of $a^{f'} - 1$ if and only if f is a divisor of f'. If $f \geq 3$, then $a^f - 1$ has a prime divisor no less than 5.*

We end this section with a fact on finite primitive groups.

Lemma 2.7. *Assume that G is a finite primitive group with a point-stabilizer H. If H has a normal Sylow subgroup $P \neq 1$, then P is also a Sylow subgroup of G.*

Proof. Assume that $P \neq 1$ is a normal Sylow subgroup of H. Clearly, P is not normal in G. Take a Sylow subgroup Q of G with $P \leq Q$. Then $H \leq \langle \mathbf{N}_Q(P), H \rangle \leq \mathbf{N}_G(P) \neq G$. Since H is maximal in G, we have $H = \langle \mathbf{N}_Q(P), H \rangle$ and so $\mathbf{N}_Q(P) \leq H$. It follows that $\mathbf{N}_Q(P) = P$ and hence $P = Q$. Then the lemma follows. □

3. Some restrictions on stabilizers

In Sections 4 and 5, we prove Theorem 1.1 using the result given in [17] that classifies finite primitive groups with soluble point-stabilizers. Let $\Gamma = (V, E)$ be a graph of valency $d \geq 6$, $\{u, v\} \in E$ and $G \leq \mathrm{Aut}\Gamma$. Assume that G is almost simple, $G_{\{u,v\}}$ is soluble and Γ is G-edge-primitive and $(G, 2)$-arc-transitive. Clearly, each nontrivial normal subgroup of G acts transitively on the edge set E. Choose a minimal X among the normal subgroups of G that act primitively on E. By the choice of X, we have $\mathrm{soc}(X) = \mathrm{soc}(G)$, $X_{\{u,v\}} = X \cap G_{\{u,v\}}$, $G = XG_{\{u,v\}}$ and $G/X = XG_{\{u,v\}}/X \cong G_{\{u,v\}}/X_{\{u,v\}}$. Then, considering the restrictions on both $X_{\{u,v\}}$ and X_v caused by the 2-arc-transitivity of Γ, we may work out the pair $(X, X_{\{u,v\}})$ from [17, Theorem 1.1] and then determine the group G and the graph Γ. Thus, we make the following assumptions.

Hypothesis 3.1. Let $\Gamma = (V, E)$ be a G-edge-primitive graph of valency $d \geq 6$ and $\{u, v\} \in E$, where G is an almost simple group with socle T. Assume that:

(1) Γ is $(G, 2)$-arc-transitive and the edge-stabilizer $G_{\{u,v\}}$ is soluble;
(2) G has a normal subgroup X such that $\mathrm{soc}(X) = T$, $X_{\{u,v\}}$ is maximal in X and $(X, X_{\{u,v\}})$ is one of the pairs (G_0, H_0) listed in [17, Tables 14–20].

For the group X in Hypothesis 3.1, we have $1 \neq X_v^{\Gamma(v)} \trianglelefteq G_v^{\Gamma(v)}$. Note that $G_v^{\Gamma(v)}$ is 2-transitive (on $\Gamma(v)$). Then $G_v^{\Gamma(v)}$ is affine or almost simple; see [4, Theorem 4.1B], for example. It follows that $\mathrm{soc}(G_v^{\Gamma(v)}) = \mathrm{soc}(X_v^{\Gamma(v)})$.

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3.1. Assume that G_v is insoluble. Then $G_v^{\Gamma(v)}$ is an almost simple 2-transitive group (on $\Gamma(v)$). Recall that $\mathrm{soc}(G_v^{\Gamma(v)}) = \mathrm{soc}(X_v^{\Gamma(v)})$. Checking the point-stabilizers of almost simple 2-transitive groups (see [16, Table 2.1], for example), since $(G_v^{\Gamma(v)})_u$ is soluble, we conclude that either $X_v^{\Gamma(v)}$ is 2-transitive, or $G_v^{\Gamma(v)} \cong \mathrm{PSL}_2(8).3$ and $d = 28$. (For a complete list of finite 2-transitive groups, the reader may refer to [2, Tables 7.3 and 7.4].)

LEMMA 3.2. *Suppose that Hypothesis 3.1 holds. If $d = 28$, then $G_v^{\Gamma(v)}$ is not isomorphic to* $\mathrm{PSL}_2(8).3$.

PROOF. Suppose that $G_v^{\Gamma(v)} \cong \mathrm{PSL}_2(8).3$ and $d = 28$. Note that $X_{uv}^{[1]} \leq G_{uv}^{[1]} = 1$; see Theorem 2.4. Thus, $X_{uv} \lesssim (X_v^{\Gamma(v)})_u \times (X_u^{\Gamma(u)})_v$ by Lemma 2.2.

Assume that $X_v^{\Gamma(v)} \cong \mathrm{PSL}_2(8)$. Then $(X_v^{\Gamma(v)})_u \cong \mathrm{D}_{18}$, and $X_{uv} \cong \mathrm{D}_{18}$, $(\mathbb{Z}_3 \times \mathbb{Z}_9){:}\mathbb{Z}_2$, $(\mathbb{Z}_9 \times \mathbb{Z}_9){:}\mathbb{Z}_2$ or $\mathrm{D}_{18} \times \mathrm{D}_{18}$. In particular, the unique Sylow 3-subgroup of $X_{\{u,v\}} = X_{uv}.2$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_9$, where $m = 1$, 3 or 9. Checking the primitive groups listed in [17, Tables 14–20], we know that only the pairs $(\mathrm{PSL}_2(q), \mathrm{D}_{2(q\pm1)/(2,q-1)})$ possibly meet our requirements on $X_{\{u,v\}}$, yielding $X_{\{u,v\}} \cong \mathrm{D}_{2(q\pm1)/(2,q-1)}$. Then $\mathrm{D}_{36} \cong X_{\{u,v\}} \cong \mathrm{D}_{2(q\pm1)/(2,q-1)}$. Calculation shows that $q = 37$; however, $\mathrm{PSL}_2(37)$ has no subgroup that has a quotient $\mathrm{PSL}_2(8)$, which is a contradiction.

Now let $X_v^{\Gamma(v)} = G_v^{\Gamma(v)} \cong \mathrm{PSL}_2(8).3$. Then $(X_v^{\Gamma(v)})_u \cong (X_u^{\Gamma(u)})_v \cong \mathbb{Z}_9{:}\mathbb{Z}_6$ and $X_{uv} \lesssim \mathbb{Z}_9{:}\mathbb{Z}_6 \times \mathbb{Z}_9{:}\mathbb{Z}_6$. In particular, a Sylow 2-subgroup of $X_{\{u,v\}} = X_{uv}.2$ is not a cyclic group of order 8 and the unique Sylow 3-subgroup of $X_{\{u,v\}}$ is nonabelian and contains elements of order 9. Since $X_{\{u,v\}} = X_{uv}.2 = X_v^{[1]}.(X_v^{\Gamma(v)})_u.2$ and $X_v^{[1]} \cong (X_v^{[1]})^{\Gamma(u)} \trianglelefteq (X_u^{\Gamma(u)})_v$, we have $|X_{\{u,v\}}| = 2^2 \cdot 3^3$, $2^2 \cdot 3^4$, $2^2 \cdot 3^5$, $2^2 \cdot 3^6$, $2^3 \cdot 3^5$ or $2^3 \cdot 3^6$. Checking Tables 14–20 given in [17], we conclude that $X = \mathrm{G}_2(3).2$ and $X_{\{u,v\}} \cong [3^6]{:}\mathrm{D}_8$. In this case, $X_v^{[1]} \cong \mathbb{Z}_9{:}\mathbb{Z}_6$ and $X_v \cong \mathbb{Z}_9{:}\mathbb{Z}_6.\mathrm{PSL}_2(8).3$; however, X has no such subgroup by the Atlas [3], which is a contradiction. This completes the proof. □

By Lemma 3.2, combining with Theorem 2.4, the next lemma follows from checking the point-stabilizers of finite almost simple 2-transitive groups; refer to [16, Table 2.1].

LEMMA 3.3. *Suppose that Hypothesis 3.1 holds and $G_v^{\Gamma(v)}$ is almost simple. Then one of the following holds:*

(1) $G_v^{\Gamma(v)} = X_v^{\Gamma(v)} = \mathrm{PSL}_3(2)$ *or* $\mathrm{PSL}_3(3)$, *and* $d = 7$ *or* 13, *respectively;*
(2) $\mathrm{soc}(X_v^{\Gamma(v)}) = \mathrm{PSL}_2(q)$ *with* $q > 4$, *and* $d = q+1$*;*
(3) $G_{uv}^{[1]} = 1$, $\mathrm{soc}(X_v^{\Gamma(v)}) = \mathrm{PSU}_3(q)$ *with* $q > 2$, *and* $d = q^3+1$*;*
(4) $G_{uv}^{[1]} = 1$, $\mathrm{soc}(X_v^{\Gamma(v)}) = \mathrm{Sz}(q)$ *with* $q = 2^{2n+1} > 2$, *and* $d = q^2+1$*;*
(5) $G_{uv}^{[1]} = 1$, $\mathrm{soc}(X_v^{\Gamma(v)}) = \mathrm{Ree}(q)$ *with* $q = 3^{2n+1} > 3$, *and* $d = q^3+1$.

In particular, Γ is $(X, 2)$-arc-transitive.

Recall that the Fitting subgroup $\mathrm{Fit}(H)$ of a finite group H is the direct product of $\mathbf{O}_r(H)$, where r runs over the set of prime divisors of $|H|$.

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Lemma 3.4. *Suppose that Hypothesis 3.1 holds and (2) or (5) of Lemma 3.3 occurs. Let $q = p^f$ for some prime p. Assume that $X_{uv}^{[1]} = 1$. Then* $\mathrm{Fit}(X_{uv}) = \boldsymbol{O}_p(X_{uv})$ *and either* $\mathrm{Fit}(X_{uv}) = \mathrm{Fit}(X_{\{u,v\}})$ *or* $\mathrm{Fit}(X_{\{u,v\}}) = \mathrm{Fit}(X_{uv}).2$*; in particular, we have* $|\mathrm{Fit}(X_{\{u,v\}}) : \boldsymbol{O}_p(X_{\{u,v\}})| \leq 2$.

Proof. Let r be a prime divisor of $|X_{uv}|$. Then $\mathbf{O}_r(X_{uv})$ is normal in X_{uv}. Since Γ is $(X, 2)$-arc-transitive, X_{uv} acts transitively on $\Gamma(v) \setminus \{u\}$. Thus, all $\mathbf{O}_r(X_{uv})$-orbits (on $\Gamma(v) \setminus \{u\}$) have equal size that is a power of r and a divisor of $|\Gamma(v) \setminus \{u\}|$. Note that $|\Gamma(v) \setminus \{u\}| = d - 1$, which is a power of p. It follows that either $r = p$ or $\mathbf{O}_r(X_{uv}) = 1$. Then $\mathrm{Fit}(X_{uv}) = \mathbf{O}_p(X_{uv})$.

Note that X_{uv} is normal in $X_{\{u,v\}}$ as $|X_{\{u,v\}} : X_{uv}| = 2$. Since $\mathbf{O}_p(X_{uv})$ is a characteristic subgroup of X_{uv}, it follows that $\mathbf{O}_p(X_{uv})$ is normal in $X_{\{u,v\}}$ and so $\mathbf{O}_p(X_{uv}) \leq \mathbf{O}_p(X_{\{u,v\}}) \leq \mathrm{Fit}(X_{\{u,v\}})$. For each odd prime divisor r of $|X_{\{u,v\}}|$, since $|X_{\{u,v\}} : X_{uv}| = 2$, we have $\mathbf{O}_r(X_{\{u,v\}}) \leq X_{uv}$ and so $\mathbf{O}_r(X_{\{u,v\}}) = \mathbf{O}_r(X_{uv})$. It follows that

$$\mathrm{Fit}(X_{\{u,v\}}) = \mathrm{Fit}(X_{uv})\mathbf{O}_2(X_{\{u,v\}}) = \mathbf{O}_p(X_{uv})\mathbf{O}_2(X_{\{u,v\}}).$$

In particular, $\mathbf{O}_p(X_{uv}) = \mathbf{O}_p(X_{\{u,v\}})$ if $p \neq 2$.

It is easily shown that $X_{uv} \cap \mathbf{O}_2(X_{\{u,v\}}) = \mathbf{O}_2(X_{uv})$. If $X_{uv} \geq \mathbf{O}_2(X_{\{u,v\}})$, then $p = 2$, $\mathrm{Fit}(X_{\{u,v\}}) = \mathbf{O}_2(X_{\{u,v\}}) = \mathrm{Fit}(X_{uv})$ and the lemma is true. Assume that $\mathbf{O}_2(X_{\{u,v\}}) \not\leq X_{uv}$. Since $|X_{\{u,v\}} : X_{uv}| = 2$, we have $X_{\{u,v\}} = X_{uv}\mathbf{O}_2(X_{\{u,v\}})$. Then

$$\begin{aligned} 2|X_{uv}| = |X_{\{u,v\}}| &= |X_{uv}||\mathbf{O}_2(X_{\{u,v\}}) : (X_{uv} \cap \mathbf{O}_2(X_{\{u,v\}}))| \\ &= |X_{uv}||\mathbf{O}_2(X_{\{u,v\}}) : \mathbf{O}_2(X_{uv})|, \end{aligned}$$

yielding $|\mathbf{O}_2(X_{\{u,v\}}) : \mathbf{O}_2(X_{uv})| = 2$. If $p = 2$, then $\mathrm{Fit}(X_{\{u,v\}}) = \mathbf{O}_2(X_{\{u,v\}})$ and $\mathrm{Fit}(X_{uv}) = \mathbf{O}_2(X_{uv})$. If $p \neq 2$, then $\mathbf{O}_2(X_{uv}) = 1$, $|\mathbf{O}_2(X_{\{u,v\}})| = 2$ and so $\mathrm{Fit}(X_{\{u,v\}}) = \mathbf{O}_p(X_{uv}) \times \mathbb{Z}_2$. This completes the proof. □

3.2. Assume that Hypothesis 3.1 holds and G_v is soluble. Then $G_v^{\Gamma(v)}$ is an affine 2-transitive group. Let $\mathrm{soc}(G_v^{\Gamma(v)}) = \mathbb{Z}_p^f$. Then $d = p^f$. Recalling that $d \geq 6$, we have $G_{uv}^{[1]} = 1$ by Theorem 2.4 and so $G_{uv} \lesssim (G_v^{\Gamma(v)})_u \times (G_u^{\Gamma(u)})_v$. If G_{uv} is abelian, then Γ is known from [21]. Thus, we assume further that G_{uv} is not abelian. Then $(G_v^{\Gamma(v)})_u$ is nonabelian and so $(G_v^{\Gamma(v)})_u \not\leq \mathrm{GL}_1(p^f)$; in particular, $f > 1$. Since $(G_v^{\Gamma(v)})_u$ is soluble, by [2, Table 7.3], we have the following lemma.

Lemma 3.5. *Suppose that Hypothesis 3.1 holds, G_v is soluble and G_{uv} is not abelian. Let* $\mathrm{soc}(G_v^{\Gamma(v)}) = \mathbb{Z}_p^f$, *where p is a prime. Then $f > 1$ and one of the following holds:*

(1) *$f = 2$ and either* $\mathrm{SL}_2(3) \trianglelefteq (G_v^{\Gamma(v)})_u \leq \mathrm{GL}_2(p)$ *and* $p \in \{3, 5, 7, 11, 23\}$, *or $p = 3$ and* $(G_v^{\Gamma(v)})_u = \mathrm{Q}_8$;
(2) $2_+^{1+4}{:}\mathbb{Z}_5 \leq (G_v^{\Gamma(v)})_u \leq 2_+^{1+4}.(\mathbb{Z}_5{:}\mathbb{Z}_4) < 2_+^{1+4}.\mathrm{S}_5$ *and* $p^f = 3^4$;
(3) $(G_v^{\Gamma(v)})_u \not\leq \mathrm{GL}_1(p^f)$, $(G_v^{\Gamma(v)})_u \leq \Gamma\mathrm{L}_1(p^f)$ *and* $|(G_v^{\Gamma(v)})_u|$ *is divisible by* $p^f - 1$.

Consider the case (3) in Lemma 3.5. Write

$$\Gamma\mathrm{L}_1(p^f) = \langle \tau, \sigma \mid \tau^{p^f-1} = 1 = \sigma^f, \sigma^{-1}\tau\sigma = \tau^p \rangle.$$

Let $\langle\tau\rangle \cap (G_v^{\Gamma(v)})_u = \langle\tau^m\rangle$, where $m \mid (p^f - 1)$. Then

$$(G_v^{\Gamma(v)})_u/\langle\tau^m\rangle \cong \langle\tau\rangle(G_v^{\Gamma(v)})_u/\langle\tau\rangle \lesssim \langle\sigma\rangle.$$

Set $(G_v^{\Gamma(v)})_u/\langle\tau^m\rangle \cong \langle\sigma^e\rangle$ for some divisor e of f. Then

$$(G_v^{\Gamma(v)})_u \cong \mathbb{Z}_{(p^f-1)/m}.\mathbb{Z}_{f/e}.$$

Choose $\tau^l\sigma^k \in (G_v^{\Gamma(v)})_u$ with $(G_v^{\Gamma(v)})_u = \langle\tau^m\rangle\langle\tau^l\sigma^k\rangle$. Then $(\tau^l\sigma^k)^{f/e} \in \langle\tau^m\rangle$ but $(\tau^l\sigma^k)^j \notin \langle\tau^m\rangle$ for $1 \le j < f/e$. It follows that σ^k has order f/e. Then $\sigma^k = \sigma^{ie}$ for some i with $(i, f/e) = 1$ and then $(\sigma^k)^{i'} = \sigma^e$ for some i'. Thus, replacing $\tau^l\sigma^k$ by a power of it if necessary, we may let $k = e$. Then

$$(G_v^{\Gamma(v)})_u = \langle\tau^m\rangle\langle\tau^l\sigma^e\rangle.$$

Further, $(G_v^{\Gamma(v)})_u = \langle\tau^m\rangle\langle(\tau^m)^i\tau^l\sigma^e\rangle$ for an arbitrary integer i; thus, we may assume further that $0 \le l < m$. By [6, Proposition 15.3], letting $\pi(n)$ be the set of prime divisors of a positive integer n:

(⋇) $\pi(m) \subseteq \pi(p^e - 1)$, $me \mid f$ and $(m, l) = 1$; in particular, $m = 1$ if $l = 0$.

Suppose that X_{uv} is nonabelian. (The case where X_{uv} is abelian is left to Section 5.) Since $X_{uv}^{[1]} \le \mathrm{G}_{uv}^{[1]} = 1$,

$$X_v^{[1]} \trianglelefteq (X_u^{\Gamma(u)})_v \cong (X_v^{\Gamma(v)})_u,\ X_{uv} \lesssim (X_u^{\Gamma(u)})_v \times (X_v^{\Gamma(v)})_u.$$

This yields that $(X_v^{\Gamma(v)})_u$ is nonabelian. Then a limitation on $\pi(|X_{uv}|)$ is given as follows.

Lemma 3.6. *Assume that Lemma 3.5(3) holds and X_{uv} is nonabelian. Then $(X_v^{\Gamma(v)})_u \cong \mathbb{Z}_{m'}.\mathbb{Z}_{f/e'}$, where m' and e' satisfy:*

(1) *$\mathbb{Z}_{m'} \cong (X_v^{\Gamma(v)})_u \cap \langle\tau^m\rangle$, $mm' \mid p^f - 1$, $e \mid e' \mid f$; and*
(2) *$m' > 1$, $e' < f$, $\pi(p^f - 1) \setminus \pi(p^{e'} - 1) \subseteq \pi(m') \subseteq \pi(|X_{uv}|)$.*

Proof. Recall that $(X_v^{\Gamma(v)})_u \trianglelefteq (G_v^{\Gamma(v)})_u = \langle\tau^m\rangle\langle\tau^l\sigma^e\rangle \cong \mathbb{Z}_{(p^f-1)/m}.\mathbb{Z}_{f/e}$. Then

$$(X_v^{\Gamma(v)})_u/((X_v^{\Gamma(v)})_u \cap \langle\tau^m\rangle) \cong (X_v^{\Gamma(v)})_u\langle\tau^m\rangle/\langle\tau^m\rangle \lesssim \mathbb{Z}_{f/e},$$

yielding $(X_v^{\Gamma(v)})_u \cong \mathbb{Z}_{m'}.\mathbb{Z}_{f/e'}$ with m' and e' satisfying (1). Since X_{uv} is nonabelian, $(X_v^{\Gamma(v)})_u$ is nonabelian and so $m' > 1$ and $e' < f$.

By the above (⋇), each prime $r \in \pi(p^f - 1) \setminus \pi(p^{e'} - 1)$ is a divisor of $|\langle\tau^m\rangle| = (p^f - 1)/m$. Let R be the unique subgroup of order r of $\langle\tau^m\rangle$. Then, since R is normal in $(G_v^{\Gamma(v)})_u$, either $R \le (X_v^{\Gamma(v)})_u$ or $R(X_v^{\Gamma(v)})_u = R\times(X_v^{\Gamma(v)})_u$. Suppose that the latter case occurs. Since $e' < f$, we may let $\tau^n\sigma^{e'} \in (X_v^{\Gamma(v)})_u \setminus \langle\tau^m\rangle$. Then $\sigma^{e'}$ centralizes R. Thus, $x^{p^{e'}} = x$ for $x \in R$, yielding $r \mid (p^{e'} - 1)$, which is a contradiction. Then $R \le (X_v^{\Gamma(v)})_u \cap \langle\tau^m\rangle \cong \mathbb{Z}_{m'}$. Noting that m' is a divisor of $|X_{uv}|$, the result follows. □

4. Graphs with insoluble vertex-stabilizers

In this and the next sections, we prove Theorem 1.1. Thus, we let G, T, X and $\Gamma = (V, E)$ be as in Hypothesis 3.1. Our task is to determine which pair (G_0, H_0) listed in [17, Tables 14–20] is a possible candidate for $(X, X_{\{u,v\}})$, and determine whether or not the resulting triple $(G, G_v, G_{\{u,v\}})$ meets the conditions (1) and (2) in Lemma 2.1.

In this section, we deal with the case where G_v is insoluble; that is, X_v is described as in Lemma 3.3. First, by the following lemma, (4) and (5) of Lemma 3.3 are excluded.

Lemma 4.1. *(4) and (5) of Lemma 3.3 do not occur.*

Proof. Suppose that Lemma 3.3(4) or (5) holds. By Theorem 2.4, $X_{uv}^{[1]} = 1$. Then $X_v = X_v^{[1]}.X_v^{\Gamma(v)}$, $X_v^{[1]} \cong (X_v^{[1]})^{\Gamma(u)} \trianglelefteq (X_u^{\Gamma(u)})_v \cong (X_v^{\Gamma(v)})_u$ and $X_{uv} \lesssim (X_v^{\Gamma(v)})_u \times (X_u^{\Gamma(u)})_v$. Set $q = p^f$ with p a prime. Then the pair $(X_v^{\Gamma(v)}, (X_v^{\Gamma(v)})_u)$ is given as follows:

$X_v^{\Gamma(v)}$	$(X_v^{\Gamma(v)})_u$	
$\mathrm{Sz}(q).e$	$p^{f+f}{:}(q-1).e$	e a divisor of f, $p = 2$, odd $f > 1$
$\mathrm{Ree}(q).e$	$p^{f+2f}{:}(q-1).e$	e a divisor of f, $p = 3$, odd $f > 1$.

In particular, $\mathbf{O}_p(X_{\{u,v\}})$ is not abelian.

We next show that none of the pairs (G_0, H_0) in [17, Tables 14–20] gives a desired pair $(X, X_{\{u,v\}})$. Since $\mathbf{O}_p(X_{\{u,v\}})$ is nonabelian, those pairs (G_0, H_0) with $\mathbf{O}_p(H_0)$ abelian are not in our consideration. In particular, $\mathrm{soc}(X)$ is not isomorphic to an alternating group. Also, noting that $X_{\{u,v\}}$ has a subgroup of index 2, those H_0 having no subgroup of index 2 are excluded.

Case 1. Suppose that $\mathrm{soc}(X_v^{\Gamma(v)}) = \mathrm{Ree}(q)$. Then $p = 3$, $\mathbf{O}_3(X_{\{u,v\}})$ is nonabelian and of order 3^{3f}, 3^{4f}, 3^{5f} or 3^{6f} and $|X_{\{u,v\}}|$ is a divisor of $2 \cdot 3^{6f} \cdot (q-1)^2 f^2$ and divisible by $2(q-1)$. Checking the orders of those H_0 given in [17, Table 15], we conclude that $\mathrm{soc}(X)$ is not a sporadic simple group.

Suppose that $\mathrm{soc}(X)$ is a simple exceptional group of Lie type. Checking [17, Table 20], we conclude that $(X, X_{\{u,v\}})$ is one of the pairs $(\mathrm{Ree}(3^t), [3^{3t}]{:}\mathbb{Z}_{3^t-1})$ and $(\mathrm{G}_2(3^t).\mathbb{Z}_{2^{l+1}}, [3^{6t}]{:}\mathbb{Z}_{3^t-1}^2.\mathbb{Z}_{2^{l+1}})$, where 2^l is the 2-part of t. Recall that $|X_{\{u,v\}}|$ is a divisor of $2 \cdot 3^{6f} \cdot (q-1)^2 f^2$ and divisible by $2(q-1)$. It follows that $f = t$, $X = \mathrm{G}_2(q).\mathbb{Z}_{2^{l+1}}$ and $X_{\{u,v\}} \cong [q^6]{:}\mathbb{Z}_{q-1}^2.\mathbb{Z}_{2^{l+1}}$. This implies that $X_v^{[1]} \neq 1$; in fact, $|\mathbf{O}_3(X_v^{[1]})| = q^3$. Thus, $\mathbf{O}_3(X_v) \neq 1$ and X_v has a quotient $\mathrm{Ree}(q).e$. Checking the maximal subgroups of $\mathrm{G}_2(q).\mathbb{Z}_{2^{l+1}}$ (refer to [15, Theorems A and B]) we conclude that $\mathrm{G}_2(q).\mathbb{Z}_{2^{l+1}}$ has no maximal subgroup containing such X_v as a subgroup, which is a contradiction.

Suppose that $\mathrm{soc}(X)$ is a simple classical group over a finite field of order r^t, where r is a prime. Since $f > 1$ is odd, $3^f - 1$ has an odd prime divisor and so $X_{\{u,v\}}$ is not a $\{2, 3\}$-group as $|X_{\{u,v\}}|$ is divisible by $3^f - 1$. Recall that $\mathbf{O}_3(X_{\{u,v\}})$ is nonabelian and of order 3^{3f}, 3^{4f}, 3^{5f} or 3^{6f}. Checking the groups H_0 given in [17, Tables 16–19], we conclude that $\mathrm{soc}(X) = \mathrm{PSL}_n(r^t)$ or $\mathrm{PSU}_n(r^t)$, where $n \in \{3, 4\}$. Take a maximal subgroup M of X such that $X_v \leq M$. Then M has a simple section (that is, a quotient of some subgroup) $\mathrm{Ree}(q)$. Recall that $q > 3$. Checking Tables 8.3–8.6 and 8.8–8.11

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TABLE 3. Graphs for (1) of Lemma 3.3.

G	X	$X_{\{u,v\}}$	X_v	s	d
X	$\mathrm{PSL}_4(2).2, \mathrm{S}_8$	$2^4{:}\mathrm{S}_4$	$2^3{:}\mathrm{SL}_3(2)$	2	7
X	$\mathrm{PSL}_5(2).2$	$[2^8]{:}\mathrm{S}_3^2.2$	$2^6{:}(\mathrm{S}_3\times\mathrm{SL}_3(2))$	3	7
X	$\mathrm{F}_4(2).2$	$[2^{22}]{:}\mathrm{S}_3^2.2$	$[2^{20}].(\mathrm{S}_3\times\mathrm{SL}_3(2))$	3	7
$X, X.2$	$\mathrm{PSL}_4(3).2$	$3_+^{1+4}{:}(2\mathrm{S}_4\times 2)$	$3^3{:}\mathrm{SL}_3(3)$	2	13
X	$\mathrm{PSL}_5(3).2$	$[3^8]{:}(2\mathrm{S}_4)^2.2$	$3^6.2\mathrm{S}_4.\mathrm{SL}_3(3)$	3	13

given in [1], we conclude that none of $\mathrm{PSL}_3(r^t)$, $\mathrm{PSL}_4(r^t)$, $\mathrm{PSU}_3(r^t)$ and $\mathrm{PSU}_4(r^t)$ has such maximal subgroups, which is a contradiction.

Case 2. Suppose that $\mathrm{soc}(X_v^{\Gamma(v)}) = \mathrm{Sz}(q)$. Then $q = 2^f$, $|\mathbf{O}_2(X_{\{u,v\}})| = 2^{2f}a, 2^{3f}a$ or $2^{4f}a$, where $f > 1$ is odd and $a = 1$ or 2. Noting that $|X_{\{u,v\}}|$ is divisible by $2(2^f - 1)$, by Lemma 2.6, we conclude that $X_{\{u,v\}}$ is not a $\{2,3\}$-group. Since $X_{\{u,v\}}$ is nonabelian, it follows from [17, Tables 15–20] that either $(X, X_{\{u,v\}})$ is one of $({}^2\mathrm{F}_4(2)', [2^9]{:}5{:}4)$, $(\mathrm{Sz}(2^t), [2^{2t}]{:}\mathbb{Z}_{2^t-1})$ and $(\mathrm{PSp}_4(2^t).\mathbb{Z}_{2^{l+1}}, [2^{4t}]{:}\mathbb{Z}_{2^t-1}^2.\mathbb{Z}_{2^{l+1}})$, or $\mathrm{soc}(X)$ is one of $\mathrm{PSL}_n(r^t)$ and $\mathrm{PSU}_n(r^t)$, where $n \in \{3,4\}$, 2^l is the 2-part of t and r is odd if $n = 4$. The first pair leads to $q = 2^3$ and so $|X_{\{u,v\}}|$ is divisible by 7, which is a contradiction. Checking the maximal subgroups of $\mathrm{soc}(X)$ (refer to [1, Tables 8.3–8.6 and 8.8–8.14]), the groups $\mathrm{PSL}_3(r^t)$, $\mathrm{PSU}_3(r^t)$, $\mathrm{PSL}_4(r^t)$ and $\mathrm{PSU}_4(r^t)$ are excluded as they have no maximal subgroup with a simple section $\mathrm{Sz}(q)$. Thus, $(X, X_{\{u,v\}}) = (\mathrm{PSp}_4(2^t).\mathbb{Z}_{2^{l+1}}, [2^{4t}]{:}\mathbb{Z}_{2^t-1}^2.\mathbb{Z}_{2^{l+1}})$ or $(\mathrm{Sz}(2^t), [2^{2t}]{:}\mathbb{Z}_{2^t-1})$. Note that $|X_{\{u,v\}}|$ is a divisor of $2 \cdot 2^{4f} \cdot (q-1)^2 f^2$ and is divisible by $2^{2f+1}(2^f - 1)$. It follows that $X = \mathrm{PSp}_4(q).\mathbb{Z}_{2^{l+1}}$ and $X_v^{[1]} \cong [q^2]{:}\mathbb{Z}_{q-1}$. However, by [1, Table 8.14], $\mathrm{PSp}_4(q).\mathbb{Z}_{2^{l+1}}$ has no maximal subgroup containing $[q^2]{:}\mathbb{Z}_{q-1}.\mathrm{Sz}(q)$, which is a contradiction. This completes the proof. □

LEMMA 4.2. *Assume that (1) of Lemma 3.3 occurs. Then G, X, $X_{\{u,v\}}$ and X_v are as listed as in Table 3.*

PROOF. Assume first that $X_{uv}^{[1]} = 1$. Then $X_v = X_v^{[1]}.X_v^{\Gamma(v)}$, $X_v^{[1]} \cong (X_v^{[1]})^{\Gamma(u)} \trianglelefteq (X_u^{\Gamma(u)})_v \cong (X_v^{\Gamma(v)})_u$ and $X_{uv} \lesssim (X_v^{\Gamma(v)})_u \times (X_u^{\Gamma(u)})_v$.

Suppose that $X_v^{\Gamma(v)} = \mathrm{PSL}_3(2)$. Then $(X_v^{\Gamma(v)})_u \cong \mathrm{S}_4$ and thus $X_v^{[1]}$ and $X_{\{u,v\}}$ are given as follows:

$X_v^{[1]}$	1	2^2	A_4	S_4
$X_{\{u,v\}}$	$2^2{:}\mathrm{S}_3.2$	$2^4.\mathrm{S}_3.2$	$2^4{:}3^2.[4]$	$2^4{:}\mathrm{S}_3^2.2.$

In particular, $2^2 \le |\mathbf{O}_2(X_{\{u,v\}})| \le 2^5$. Check all possible pairs $(X, X_{\{u,v\}})$ in [17, Tables 14–20]. Noting that $\mathrm{A}_8 \cong \mathrm{PSL}_4(2)$ and $\mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$, we conclude that $X \cong \mathrm{A}_8$, $X_{\{u,v\}} \cong 2^4{:}\mathrm{S}_3^2$ and $X_v^{[1]} \cong \mathrm{A}_4$; or $X = \mathrm{M}_{12}$ with $X_{\{u,v\}} \cong 2_+^{1+4}{:}\mathrm{S}_3$; or $X \cong \mathrm{PSU}_4(2)$ where $X_{\{u,v\}} \cong 2\mathrm{A}_4^2.2$. The group A_8 is excluded as it has no subgroup of the form of $X_v^{[1]}.\mathrm{PSL}_3(2)$. The groups M_{12} and $\mathrm{PSU}_4(2)$ are excluded as their orders are not divisible by $d = 7$.

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TABLE 4. Edge-stabilizers with $d = 7$ or 13.

X_v	$X_{\{u,v\}}$	s	d	p
$2^6.(S_3{\times}SL_3(2))$	$[2^8].S_3^2.2$	3	7	2
$[2^{20}].(S_3{\times}SL_3(2))$	$[2^{22}].S_3^2.2$	3	7	2
$2^3.SL_3(2)$	$[2^5].S_3.2$	2	7	2
$2^4{:}SL_3(2)$	$[2^6].S_3.2$	2	7	2
$3^6.(2A_4{\times}SL_3(3))$	$[3^8].(2A_4{\times}2S_4).2$	3	13	3
$3^6.(2S_4{\times}SL_3(3))$	$[3^8].(2S_4)^2.2$	3	13	3
$3^3.SL_3(3)$	$[3^5].2S_4.2$	2	13	3
$3^3.(2{\times}SL_3(3))$	$[3^5].(2{\times}2S_4).2$	2	13	3
$3^6{:}SL_3(3)$	$[3^8].2S_4.2$	2	13	3

Suppose that $X_v^{\Gamma(v)} = \mathrm{PSL}_3(3)$. Then $(X_v^{\Gamma(v)})_u \cong 3^2{:}2S_4$. Thus, $X_v^{[1]}$ and $X_{\{u,v\}}$ are given as follows:

$X_v^{[1]}$	1	3^2	$3^2{:}2$	3^2Q_8	$3^2{:}2A_4$	$3^2{:}2S_4$
$X_{\{u,v\}}$	$3^2{:}2S_4.2$	$3^4{:}2S_4.2$	$3^4{:}([4].S_4).2$	$3^4{:}Q_8^2.S_3.2$	$3^4{:}(2A_4)^2.[4]$	$3^4{:}(2S_4)^2.2.$

Note that $\mathbf{O}_3(X_{\{u,v\}}) \cong 3^2$ or 3^4. Checking the possible pairs $(X, X_{\{u,v\}})$, we have $X_{\{u,v\}} \cong 3^4{:}2^3.S_4$ and $X = A_{12}$ or $P\Omega_8^+(2)$; in this case, $d = 13$ is not a divisor of $|X|$, which is a contradiction.

Now let $X_{uv}^{[1]}$ be a nontrivial p-group. Then, by Theorem 2.4, X_v and $X_{\{u,v\}}$ are given as shown in Table 4.

Suppose that $p = 2$. Then $|X_{\{u,v\}}|$ is divisible by 9 if and only if $|\mathbf{O}_2(X_{\{u,v\}})| \geq 8$, and $\mathbf{O}_2(X_{\{u,v\}})$ contains no elements of order 8 unless $|\mathbf{O}_2(X_{\{u,v\}})| \geq 2^{22}$. Check the pairs (G_0, H_0) given in [17, Tables 14–20] by estimating $|H_0|$ and $|\mathbf{O}_2(H_0)|$. We conclude that one of the following holds:

(i) $X = \mathrm{PSL}_4(2).2 \cong S_8$ and $X_{\{u,v\}} = 2^4{:}S_4$;
(ii) $X = \mathrm{PSL}_5(2).2$ and $X_{\{u,v\}} = [2^8].S_3^2.2$;
(iii) $X = F_4(2).2$ and $X_{\{u,v\}} = [2^{22}].S_3^2.2$;
(iv) $\mathrm{soc}(X) = \mathrm{PSL}_3(4)$ and $|\mathbf{O}_2(X_{\{u,v\}})| = 2^6$;
(v) $X = \mathrm{PSU}_4(3).2_3$ and $|\mathbf{O}_2(X_{\{u,v\}})| = 2^7$;
(vi) $X = \mathrm{He}.2$ and $X_{\{u,v\}} = [2^8]{:}S_3^2.2$.

Case (iv) yields that $X_v \cong 2^3{:}SL_3(2)$ or $2^4{:}SL_3(2)$; however, X has no such subgroup by the Atlas [3]. Similarly, cases (v) and (vi) are excluded. For (i), $G = X$ and Γ is (isomorphic to) the point–plane incidence graph of the projective geometry $\mathrm{PG}(3, 2)$. For (ii), $G = X$ and Γ is (isomorphic to) the line–plane incidence graph of the projective geometry $\mathrm{PG}(4, 2)$. If (iii) holds, then $G = X$ and Γ is the line–plane incidence graph of the metasymplectic space associated with $F_4(2)$; see [30].

Now let $p = 3$. Then $|\mathbf{O}_3(X_{\{u,v\}})| = 3^5$ or 3^8, and $X_{\{u,v\}}$ has no normal Sylow subgroup. Checking all possible pairs $(X, X_{\{u,v\}})$ in [17, Tables 14–20], we know that

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TABLE 5. Graphs for (2) of Lemma 3.3.

G	X	$X_{\{u,v\}}$	X_v	d	Remark
S_p	S_p	$\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$	$PSL_2(p)$	$p+1$	$p \in \{7, 11\}$, Γ bipartite
M_{11}	M_{11}	$3^2{:}Q_8.2$	M_{10}	10	K_{11}
J_1	J_1	$\mathbb{Z}_{11}{:}\mathbb{Z}_{10}$	$PSL_2(11)$	12	
$J_3.2$	$J_3.2$	$\mathbb{Z}_{19}{:}\mathbb{Z}_{18}$	$PSL_2(19)$	20	Γ bipartite
O′N.2	O′N.2	$\mathbb{Z}_{31}{:}\mathbb{Z}_{30}$	$PSL_2(31)$	32	Γ bipartite
B	B	$\mathbb{Z}_{19}{:}\mathbb{Z}_{18}\times\mathbb{Z}_2$	$PGL_2(19)$	20	$X_v <$ Th $<$ B
		$\mathbb{Z}_{23}{:}\mathbb{Z}_{11}\times\mathbb{Z}_2$	$PSL_2(23)$	24	$X_v < Fi_{23} < B$
M	M	$\mathbb{Z}_{41}{:}\mathbb{Z}_{40}$	$PSL_2(41)$	42	see [23] for X_v
$PSL_2(19)$	$PSL_2(19)$	D_{20}	$PSL_2(5)$	6	
$X, X.2$	$PGL_2(9)$	D_{20}	$PSL_2(5)$	6	$K_{6,6}$
$X, X.2$	M_{10}	$\mathbb{Z}_5{:}\mathbb{Z}_4$	$PSL_2(5)$	6	$K_{6,6}$
$PGL_2(11)$	$PGL_2(11)$	D_{20}	$PSL_2(5)$	6	Γ bipartite
$X, X.2$	$PSL_3(r)$	$3^2{:}Q_8$	$PSL_2(9)$	10	r prime, [1, Tables 8.3 and 8.4] $r \equiv 4, 16, 31, 34 \bmod 45$
$X, X.2$	$PSU_3(r)$	$3^2{:}Q_8$	$PSL_2(9)$	10	r prime, [1, Tables 8.5 and 8.6] $r \equiv 11, 14, 29, 41 \bmod 45$

$(X, X_{\{u,v\}})$ is one of the following pairs:

$$(F_4(8).2, 9^4.(2_+^{1+4}{:}S_3^2).2),$$
$$(PSL_5(3).2, [3^8]{:}(2S_4)^2.2),\ (PSL_4(3).2, 3_+^{1+4}{:}(2\times 2S_4)).$$

Note that $\mathbf{O}_3(X_v) \le \mathbf{O}_3(X_{\{u,v\}})$. Then, for the first pair, $\mathbf{O}_3(X_{\{u,v\}}) \cong \mathbb{Z}_9^4$ has no subgroup isomorphic to $\mathbb{Z}_3^6$, which is impossible. For the second pair, $G = X$ and Γ is (isomorphic to) the line–plane incidence graph of the projective geometry PG(4, 3). The last pair implies that $X \cong PGL_4(3)$, $G = X$ or $X.2$, and Γ is (isomorphic to) the line–plane incidence graph of the projective geometry PG(3, 3). This completes the proof. □

LEMMA 4.3. *Assume that Lemma 3.3(2) holds. Then $d = q + 1$ and either Γ is $(X, 4)$-arc-transitive or G, X, $X_{\{u,v\}}$ and X_v are as listed in Table 5.*

PROOF. Let $X_v^{\Gamma(v)} = PSL_2(q).[o]$ and $q = p^f > 4$, where p is a prime and o is a divisor of $(2, p-1)f$. Note that Γ is $(X, 2)$-arc-transitive; see Lemma 3.3. By Theorem 2.4, if $X_{uv}^{[1]} \ne 1$, then Γ is $(X, 4)$-arc-transitive. Thus, we assume next that $X_{uv}^{[1]} = 1$ and then Lemma 3.4 works.

Note that $X_v = X_v^{[1]}.X_v^{\Gamma(v)}$,

$$X_v^{[1]} \cong (X_v^{[1]})^{\Gamma(u)} \trianglelefteq (X_u^{\Gamma(u)})_v \cong (X_v^{\Gamma(v)})_u = p^f{:}(q-1)/(2, q-1).[o]$$

and $X_{uv} \lesssim (X_v^{\Gamma(v)})_u \times (X_u^{\Gamma(u)})_v$. We have $\mathbf{O}_p(X_{\{u,v\}}) = \mathbb{Z}_p^{if}.a$, where $i \in \{1, 2\}$ and a is a divisor of $(2, p)$. It is easily shown that $i = 2$ if and only if $\mathbf{O}_p(X_v^{[1]}) = \mathbb{Z}_p^f$. Combining with Lemma 3.4, we need only consider those pairs (G_0, H_0) in [17, Tables 14–20] that satisfy:

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(a) $\mathbf{O}_p(H_0) = \mathbb{Z}_p^{if}.a$, where $i \in \{1,2\}$ $a \mid (2,p)$; $|\mathrm{Fit}(H_0) : \mathbf{O}_p(H_0)| \le 2$; G_0 has a subgroup, say M_0, such that $|M_0 : (M_0 \cap H_0)| = q+1$, $|H_0 : (M_0 \cap H_0)| = 2$ and M_0 has a simple section $\mathrm{PSL}_2(q)$;

(b) $|H_0{:}\mathbf{O}_p(H_0)|$ is a divisor of $2(q-1)^2 f^2$ and divisible by $q-1$; if $i = 1$, then $|H_0{:}\mathbf{O}_p(H_0)|$ is a divisor of $2(q-1)f$.

Case 1. Assume that $\mathrm{soc}(X)$ is an alternating group. Using [17, Table 14], we have $G = X = \mathrm{S}_p$ and $X_{\{u,v\}} \cong \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$, where $p \in \{7, 11, 17, 23\}$. Then $X_v = \mathrm{PSL}_2(p)$ and $d = p+1$. In particular, Γ is a bipartite graph with two parts being the orbits of A_p on the vertex set V. For $p = 17$ or 23, the group $\mathrm{PSL}_2(p)$ has no transitive permutation representation of degree p and thus it cannot occur as a subgroup of S_p. Therefore, $p = 7$ or 11, and G, X and $X_{\{u,v\}}$ are as listed in Table 5. In fact, X_{uv} and $X_{\{u,v\}}$ are the normalizers of some Sylow p-subgroup in $\mathrm{PSL}_2(p)$ and S_p, respectively. (Note that A_7 can be embedded in $\mathrm{PSL}_4(2)$ acting on the projective points or the hyperplanes of the projective geometry $\mathrm{PG}(3,2)$; see [18, Table III], for example. Then, for $p = 7$, it is easily shown that the resulting graph is the point–plane nonincidence graph of $\mathrm{PG}(3,2)$.)

Case 2. Assume that $\mathrm{soc}(X)$ is a simple sporadic group. By [17, Table 15], with the restrictions (a) and (b), the only pairs (G_0, H_0) are listed as follows:

$$(\mathrm{M}_{11}, 3^2{:}\mathrm{Q}_8.2),\ (\mathrm{J}_1, \mathbb{Z}_{11}{:}\mathbb{Z}_{10}),\ (\mathrm{J}_1, \mathbb{Z}_7{:}\mathbb{Z}_6),\ (\mathrm{J}_3.2, \mathbb{Z}_{19}{:}\mathbb{Z}_{18}),\ (\mathrm{J}_4, \mathbb{Z}_{29}{:}\mathbb{Z}_{28}),$$
$$(\mathrm{O'N}.2, \mathbb{Z}_{31}{:}\mathbb{Z}_{30}),\ (\mathrm{B}, \mathbb{Z}_{19}{:}\mathbb{Z}_{18}\times\mathbb{Z}_2),\ (\mathrm{B}, \mathbb{Z}_{23}{:}\mathbb{Z}_{11}\times\mathbb{Z}_2),$$
$$(\mathrm{M}, \mathbb{Z}_{41}{:}\mathbb{Z}_{40}), (\mathrm{M}, \mathbb{Z}_{47}{:}\mathbb{Z}_{23}\times\mathbb{Z}_2).$$

In particular, $\mathbf{O}_p(H_0)$ is a Sylow p-subgroup of G_0. This yields that $X_v^{[1]} = 1$ and so $\mathrm{soc}(X_v) = \mathrm{PSL}_2(p^f)$.

If $(X, X_{\{u,v\}})$ is one of $(\mathrm{J}_1, \mathbb{Z}_7{:}\mathbb{Z}_6)$, $(\mathrm{J}_4, \mathbb{Z}_{29}{:}\mathbb{Z}_{28})$ and $(\mathrm{M}, \mathbb{Z}_{47}{:}\mathbb{Z}_{23}\times\mathbb{Z}_2)$, then $X_v = \mathrm{PSL}_2(p)$ for $p = 7$, 29 and 47, respectively; however, by the Atlas [3] and [36, Tables 5.6 and 5.11], X has no subgroup $\mathrm{PSL}_2(p)$, which is a contradiction. Thus, G, X and $X_{\{u,v\}}$ are as listed in Table 5. (Note that the Monster M has a maximal subgroup $\mathrm{PSL}_2(41)$ by [23].)

Case 3. Assume that $\mathrm{soc}(X)$ is a simple group of Lie type over a finite field of order r^t, where r is a prime. We first show that $r \ne p$.

Suppose that $r = p$. Then, by (a), either $\mathbf{O}_p(H_0)$ is abelian or $r = p = 2$. For $r = p > 2$, noting that $|H_0|$ has a divisor $q-1$, there does not exist H_0 in [17, Tables 16–20] such that $\mathbf{O}_p(H_0)$ is abelian. Thus, we have $r = p = 2$. Recalling that $p^f > 4$ and $|H_0/\mathbf{O}_p(H_0)|$ is divisible by $2^f - 1$, it follows from Lemma 2.6 that $H_0/\mathbf{O}_p(H_0)$ is not a $\{2,3\}$-group. Checking those H_0 given in [17, Tables 16–20], we conclude that (G_0, H_0) is one of the following pairs:

$$(\mathrm{PSL}_2(2^t), \mathbb{Z}_2^t{:}\mathbb{Z}_{2^t-1}),\ (\mathrm{PSL}_3(2^t), [2^{3t}]{:}[(2^t-1)^2/(3, 2^t-1)].2),$$
$$(\mathrm{PSU}_3(2^t), [2^{3t}]{:}\mathbb{Z}_{(2^{2t}-1)/(3,2^t+1)}),$$
$$(\mathrm{PSp}_4(2^t).\mathbb{Z}_{2^{l+1}}, [2^{4t}]{:}\mathbb{Z}_{2^t-1}^2.\mathbb{Z}_{2^{l+1}}),\ \text{where } 2^l \text{ is the 2-part of } t,$$
$$(\mathrm{Sz}(2^t), [2^{2t}]{:}\mathbb{Z}_{2^t-1}),\ ({}^3\mathrm{D}_4(2), [2^{11}]{:}(\mathbb{Z}_7\times\mathrm{S}_3)),\ ({}^2\mathrm{F}_4(2)', [2^9]{:}5{:}4).$$

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First, the pair $(\mathrm{Sz}(2^t), [2^{2t}]{:}\mathbb{Z}_{2^t-1})$ is excluded as $\mathrm{Sz}(2^t)$ has no subgroup with a section $\mathrm{PSL}_2(2^f)$. For the last two pairs, we have $f = 5$ and 4, respectively, which yields that $2^f - 1$ is not a divisor of $|H_0|$, which is a contradiction. For the three pairs after the first one, we have $t < f$ and thus G_0 has no maximal subgroup with a section $\mathrm{PSL}_2(2^f)$, which is a contradiction. Suppose finally that $(X, X_{\{u,v\}}) = (\mathrm{PSL}_2(2^t), \mathbb{Z}_2^t{:}\mathbb{Z}_{2^t-1})$. Then $3 \le f < t \le 2f + 1$. Noting that $2^f - 1$ is a divisor of $2^t - 1$, it follows that f is a divisor of t and so $t = 2f$. Then $\mathbf{O}_2(X_{\{u,v\}}) = 2^{2f}$, yielding $|\mathbf{O}_2(X_v^{[1]})| = 2^f$. Thus, $\mathbf{O}_2(X_v) \ne 1$ and X_v has a section $\mathrm{PSL}_2(2^f)$. Check the subgroups of $\mathrm{PSL}_2(2^{2f})$; refer to [12, Hauptsatz II.8.27]. We conclude that $\mathrm{PSL}_2(2^{2f})$ has no subgroup isomorphic to X_v, which is a contradiction.

We assume that $r \ne p$ from now on.

Subcase 3.1. We first deal with those pairs (G_0, H_0) such that H_0 is included in some infinite families in [17, Tables 16–20]. Note that $r \ne p$ and we consider only those H_0 having subgroups of index 2. It follows that either $H_0/\mathrm{Fit}(H_0)$ is a $\{2, 3\}$-group, or $G_0 = \mathrm{E}_8(q')$ and $|H_0| = 30(q'^8 \pm q'^7 \mp q'^5 - q'^4 \mp q'^3 \pm q' + 1)$, where $q' = r^t$. Suppose the latter case occurs. It is easily shown that $q'^8 \pm q'^7 \mp q'^5 - q'^4 \mp q'^3 \pm q' + 1$ is divisible by some primitive prime divisor s of $q'^{15} - 1$ or of $q'^{30} - 1$. Noting that $s \ge 17$, we know that H_0 has a normal cyclic Sylow s-subgroup. It follows from (a) that $17 \le p = s = q'^8 \pm q'^7 \mp q'^5 - q'^4 \mp q'^3 \pm q' + 1$. In particular, $\mathbf{O}_p(H_0) = \mathbb{Z}_p$ and $f = 1$. By (b), $|H_0|$ is divisible by $p - 1$ and then 30 is divisible by $p - 1$. This implies that $30 = p - 1 = q'^8 \pm q'^7 \mp q'^5 - q'^4 \mp q'^3 \pm q'$, which is impossible. Therefore, $H_0/\mathrm{Fit}(H_0)$ is a $\{2, 3\}$-group.

By (a), $\mathrm{Fit}(H_0)$ a $\{2, p\}$-group. Then $|H_0|$ has no prime divisor other than 2, 3 and p. Since $p^f - 1$ is a divisor of $|H_0|$, by Lemma 2.6, we have $f < 3$. Recall that $(X_u^{\Gamma(u)})_v \cong (X_v^{\Gamma(v)})_u = p^f{:}(q - 1)/(2, q - 1).[o]$ and $X_{uv} \lesssim (X_v^{\Gamma(v)})_u \times (X_u^{\Gamma(u)})_v$, where o is a divisor of $(2, q - 1)f$. Then $X_{uv}/\mathbf{O}_p(X_{uv})$ has an abelian Hall $2'$-subgroup. Note that $X_{uv}\mathbf{O}_p(X_{\{u,v\}})/\mathbf{O}_p(X_{\{u,v\}}) \cong X_{uv}/(\mathbf{O}_p(X_{\{u,v\}} \cap X_{uv}) = X_{uv}/\mathbf{O}_p(X_{uv})$ and also $|X_{\{u,v\}} : X_{uv}\mathbf{O}_p(X_{\{u,v\}})| \le 2$. It follows that $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ has an abelian Hall $2'$-subgroup. Thus, as a possible candidate for $X_{\{u,v\}}$, the quotient of H_0 over $\mathbf{O}_p(H_0)$ has abelian Hall $2'$-subgroups. In particular, $H_0/\mathbf{O}_p(H_0)$ has no section A_4.

Considering the restrictions on H_0, r and f, we conclude that (G_0, H_0) can only be one of the following pairs:

$$(\mathrm{PSL}_2(r^t), \mathbb{Z}_{(r^t\pm1)/(2,r^t-1)}{:}\mathbb{Z}_2),\ (\mathrm{PSL}_3(r^t), [(r^t - 1)^2/(3, r^t - 1)].\mathrm{S}_3),$$
$$(\mathrm{PSU}_3(r^t), [(r^t + 1)^2/(3, r^t + 1)].\mathrm{S}_3);$$
$$(\mathrm{PSp}_4(2^t).\mathbb{Z}_{2^{l+1}}, \mathbb{Z}_{2^t\pm1}^2.[2^{l+4}]),\ (\mathrm{PSp}_4(2^t).\mathbb{Z}_{2^{l+1}}, \mathbb{Z}_{2^{2t}+1}.[2^{l+3}]),\ t \ge 3;$$
$$(\mathrm{Sz}(2^t), \mathbb{Z}_{2^t-1}{:}\mathbb{Z}_2),\ (\mathrm{Sz}(2^t), \mathbb{Z}_{2^t\pm\sqrt{2^{t+1}}+1}{:}\mathbb{Z}_4),\ t \ge 3;$$
$$(\mathrm{Ree}(3^t), \mathbb{Z}_{3^t\pm\sqrt{3^{t+1}}+1}.\mathbb{Z}_6),\ (\mathrm{Ree}(3^t), \mathbb{Z}_{3^t+1}.\mathbb{Z}_6),\ t \ge 3;$$
$$(\mathrm{G}_2(3^t).\mathbb{Z}_{2^{l+1}}, \mathbb{Z}_{3^t\pm1}^2.[3 \cdot 2^{l+3}]),\ (\mathrm{G}_2(3^t).\mathbb{Z}_{2^{l+1}}, \mathbb{Z}_{3^{2t}\pm3^t+1}.[3 \cdot 2^{l+2}]),\ t \ge 2;$$
$$({}^3\mathrm{D}_4(r^t), \mathbb{Z}_{r^{4t}-r^{2t}+1}{:}\mathbb{Z}_4),\ ({}^2\mathrm{F}_4(2^t), \mathbb{Z}_{2^{2t}\pm\sqrt{2^{3t+1}}+2^t\pm\sqrt{2^{t+1}}+1}.\mathbb{Z}_{12}),\ t \ge 3;$$
$$(\mathrm{F}_4(2^t).\mathbb{Z}_{2^{l+1}}, \mathbb{Z}_{2^{4t}-2^{2t}+1}.[3 \cdot 2^{l+3}]),\ t \ge 2,$$

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where the power 2^l appearing means the 2-part of t. Recall that $|\mathrm{Fit}(H_0) : \mathbf{O}_p(H_0)| \le 2$ and $|H_0 : \mathbf{O}_p(H_0)|$ is divisible by $p^f - 1$. This allows us determine the values of p^f and r^t. As an example, we only deal with the second pair. Suppose that $(G_0, H_0) = (\mathrm{PSL}_3(r^t), [(r^t-1)^2/(3, r^t-1)].\mathrm{S}_3)$. Considering the structures of $\mathrm{Fit}(H_0)$ and $\mathbf{O}_p(H_0)$, either $(3, r^t-1) = 1$, $p = r^t - 1$ and $f \in \{1, 2\}$, or $f = 1$ and $p = r^t - 1 = 3$. The latter implies that $\mathrm{PSL}_2(q)$ is soluble, which is not the case. Assume that the former case holds. Then $|\mathrm{S}_3|$ is divisible by $r^t - 1 - 1$ or $(r^t-1)^2 - 1$. Then the only possibility is that $(p^f, r^t) = (7, 8)$. The other pairs can be determined in a similar way; the details are omitted here. Eventually, we conclude that (G_0, H_0, p, f) is one of $(\mathrm{PSL}_2(19), \mathrm{D}_{20}, 5, 1)$, $(\mathrm{PSL}_3(8), 7^2{:}\mathrm{S}_3, 7, 1)$ and $(\mathrm{Sz}(8), \mathbb{Z}_5{:}\mathbb{Z}_4, 5, 1)$. By the Atlas [3], neither $\mathrm{PSL}_3(8)$ nor $\mathrm{Sz}(8)$ has subgroup with a section $\mathrm{PSL}_2(p)$. Thus, in this case, G, X and $X_{\{u,v\}}$ are as given in Table 5.

Subcase 3.2. For the pairs (G_0, H_0) not appearing in Subcase 3.1, we check the finite number of H_0 one by one. We observe that either $p = 2$ or $H_0/\mathbf{O}_p(H_0)$ is a $\{2, 3\}$-group. Recall that $r \ne p$.

Suppose that $p = 2$. Recalling that $q = 2^f > 4$, we have $f \ge 3$. In particular, since $|H_0|$ is divisible by $2^f - 1$, H_0 is not a $\{2, 3\}$-group by Lemma 2.6. Then the only possibility is that $G_0 = {}^2\mathrm{F}_4(2)'$ and $H_0 = [2^9]{:}5{:}4$. Thus, $|\mathbf{O}_2(H_0)| = 2^9$; it follows from (a) that $f = 4$ or 9 and then G_0 has a section $\mathrm{PSL}_2(2^4)$ or $\mathrm{PSL}_2(2^9)$, which is impossible by checking the (maximal) subgroups of ${}^2\mathrm{F}_4(2)'$. Thus, $p > 2$ and $H_0/\mathbf{O}_p(H_0)$ is a $\{2, 3\}$-group; in particular, by (a), $\mathbf{O}_p(H_0) = \mathbb{Z}_p^{if}$ for some $i \in \{1, 2\}$.

Suppose that H_0 has a section A_4. Then H_0 has no normal Sylow 3-subgroup. Further, H_0 has no quotient A_4 as H_0 has a subgroup of index two. If $(3, (q-1)f) = 1$, then, by (b), we conclude that $p = 3$ and $\mathbf{O}_p(H_0)$ is the unique Sylow 3-subgroup of H_0, which is a contradiction. Thus, 3 is a divisor of $(q-1)f$. Check those H_0 in [17, Tables 16–20] which have a section A_4 and do not appear in Subcase 3.1. Recalling that $r \ne p > 2$ and $\mathbf{O}_p(H_0) = \mathbb{Z}_p^{if}$, it follows that either $\mathbf{O}_p(H_0) = \mathbb{Z}_3^2$ or $(G_0, H_0) = (\mathrm{F}_4(2).4, \mathbb{Z}_7^2{:}(3 \times \mathrm{SL}_2(3)).4)$. Since 3 is a divisor of $(q-1)f$, we get $G_0 = \mathrm{F}_4(2).4$ and $q = p^f = 7$ or 7^2. By (b), for $q = 7$ or 7^2, the order of H_0 should be a divisor of 72 or 192, respectively, which is impossible.

The above argument allows us to ignore many cases without further inspection. Inspecting carefully the remaining pairs, the possible candidates for $(X, X_{\{u,v\}})$ are as follows:

$(\mathrm{PGL}_2(9), \mathrm{D}_{20})$, $(\mathrm{M}_{10}, \mathbb{Z}_5{:}\mathbb{Z}_4)$, $(\mathrm{PGL}_2(11), \mathrm{D}_{20})$;
$(\mathrm{PSL}_3(r), 3^2{:}\mathrm{Q}_8)$, where $r \equiv 4, 7 \bmod 9$;
$(\mathrm{PSp}_4(4).4, \mathbb{Z}_{17}{:}\mathbb{Z}_{16})$, $(\mathrm{PSp}_4(4).4, 5^2{:}[2^5])$;
$(\mathrm{PSU}_3(r), 3^2{:}\mathrm{Q}_8)$, where $5 < r \equiv 2, 5 \bmod 9$;
$(\mathrm{PSU}_3(2^t), 3^2{:}\mathrm{Q}_8)$, where t is a prime no less than 5;
$({}^2\mathrm{F}_4(2), \mathbb{Z}_{13}{:}\mathbb{Z}_{12})$.

For the first three pairs, G, X and $X_{\{u,v\}}$ are easily determined and as given in Table 5. The pair $(\mathrm{PSp}_4(4).4, \mathbb{Z}_{17}{:}\mathbb{Z}_{16})$ is excluded as $\mathrm{PSp}_4(4).4$ has no subgroup $\mathrm{PSL}_2(17)$ and

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TABLE 6. Graphs for (3) of Lemma 3.3.

G	X	$X_{\{u,v\}}$	X_v	d	Remark
HS.2	HS.2	$[5^3]$:$[2^5]$	$\mathrm{PSU}_3(5)$:2	126	Γ bipartite
Ru	Ru	$[5^3]$:$[2^5]$	$\mathrm{PSU}_3(5)$:2	126	

the pair $({}^2\mathrm{F}_4(2), \mathbb{Z}_{13}{:}\mathbb{Z}_{12})$ is excluded as ${}^2\mathrm{F}_4(2)$ has no subgroup $\mathrm{PSL}_2(13)$. Suppose that $X \cong \mathrm{PSU}_3(2^t)$ and $X_{\{u,v\}} \cong 3^2{:}\mathrm{Q}_8$. Then we have $X_v \cong \mathrm{PSL}_2(9)$; however, by [1, Tables 8.3 and 8.4], $\mathrm{PSU}_3(2^t)$ has no subgroup $\mathrm{PSL}_2(9)$, which is a contradiction. Suppose that $(X, X_{\{u,v\}}) = (\mathrm{PSp}_4(4).4, 5^2{:}[2^5])$. Then X_v contains a Sylow 5-subgroup P of X and has a section $\mathrm{PSL}_2(5)$ or $\mathrm{PSL}_2(25)$. By the information for $\mathrm{PSp}_4(4).4$ given in the Atlas [3], we conclude that $X_v \le M \cong (\mathrm{A}_5 \times \mathrm{A}_5){:}2^2 < \mathrm{PSp}_4(4).2 < \mathrm{PSp}_4(4).4$. Note that $X_{uv} = 5^2{:}[2^4]$, which should be the normalizer of P in X_v. Using GAP [28], computation shows that $|\mathbf{N}_L(P)| \le 200$ for any maximal subgroup L of M with $P \le L$. It follows that $X_v = M \cong (\mathrm{A}_5 \times \mathrm{A}_5){:}2^2$, yielding $d = |X_v : X_{uv}| = 36 \ne q + 1$, which is a contradiction.

Let $(X, X_{\{u,v\}}) = (\mathrm{PSL}_3(r), 3^2{:}\mathrm{Q}_8)$. Then $X_{uv} \cong 3^2{:}4$. It is easily shown that $p = 3$ and $X_v \cong \mathrm{PSL}_2(9)$. Since $r \equiv 4, 7 \bmod 9$, we know that $\mathrm{PSL}_3(r)$ has a Sylow 3-subgroup $\mathbb{Z}_3^2$. By [1, Tables 8.3 and 8.4], $\mathrm{PSL}_3(r)$ has a subgroup $\mathrm{PSL}_2(9)$ if and only if $r \equiv 1, 4 \bmod 15$. Thus, in this case, we have $r \equiv 11, 14, 29, 41 \bmod 45$. For a subgroup $\mathrm{PSL}_2(9)$ of $\mathrm{PSL}_3(r)$, taking a Sylow 3-subgroup Q of $\mathrm{PSL}_2(9)$, the normalizers of Q in $\mathrm{PSL}_2(9)$ and $\mathrm{PSL}_3(r)$ are (isomorphic to) $3^2{:}4$ and $3^2{:}\mathrm{Q}_8$, respectively. Then these two normalizers of Q can serve as the roles of X_{uv} and $X_{\{u,v\}}$, respectively. Thus, X and $X_{\{u,v\}}$ are as given in Table 5. Noting that $G = XG_{\{u,v\}}$, we have $G_{\{u,v\}}/X_{\{u,v\}} \cong G/X \lesssim \mathsf{Out}(\mathrm{PSL}_3(r)) \cong \mathrm{S}_3$ and so $G = X.[m]$ and $G_{\{u,v\}} = X_{\{u,v\}}.[m]$, where m is a divisor of 6. Thus, $|G_{uv}{:}X_{uv}| = m$; since $|G_v{:}G_{uv}| = 10 = |X_v{:}X_{uv}|$, we have $|G_v{:}X_v| = m$. By [1, Table 8.4], $\mathbf{N}_{\mathsf{Aut}(\mathrm{PSL}_3(r))}(X_v) = X_v.2$. Since $X_v \trianglelefteq G_v$, it follows that $m \le 2$. Thus, $G = X$ or $X.2$ and, if $G = X.2$, then $G_v = X_v.2 \cong \mathrm{PGL}_2(9)$ and $G_{\{u,v\}} \cong 3^2{:}\mathrm{Q}_8.2$. The pair $(\mathrm{PSU}_3(r), 3^2{:}\mathrm{Q}_8)$ is similarly dealt with; the details are omitted. This completes the proof. □

LEMMA 4.4. *If (3) of Lemma 3.3 holds, then G, X, $X_{\{u,v\}}$ and X_v are as listed in Table 6.*

PROOF. Let $X_v^{\Gamma(v)} = \mathrm{PSU}_3(q).[o]$ and $q = p^f > 2$, where p is a prime and $o \mid 2(3, q{+}1)f$. Then $(X_v^{\Gamma(v)})_u = p^{f+2f}{:}((q^2-1)/(3, q{+}1)).[o]$ and $X_{uv}^{[1]} = 1$, by Theorem 2.4. Thus, $|\mathbf{O}_p(X_{\{u,v\}})| = p^{3f}.a$, $p^{4f}.a$, $p^{5f}.a$ or $p^{6f}.a$, where a is a divisor of $(2, p)$. Moreover, $\mathbf{O}_p(X_{\{u,v\}})$ is nonabelian and $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ has a subgroup $\mathbb{Z}_{(q^2-1)/(3,q+1)}$. We next determine which pair (G_0, H_0) in [17, Tables 14–20] is a possible candidate for $(X, X_{\{u,v\}})$. Note that we may ignore those H_0 that either have no subgroup of index two or have an abelian maximal normal p-subgroup. In particular, $\mathrm{soc}(X)$ is not an alternating group.

Case 1. Let (G_0, H_0) be a pair with H_0 included in some infinite families given in [17, Tables 16–20]. Since $\mathbf{O}_p(X_{\{u,v\}})$ is nonabelian, we conclude that $(X, \mathbf{O}_p(X_{\{u,v\}}))$ is one of the following pairs:

$$(\mathrm{PSL}_3(p^t).2, [p^{3t}]),\ (\mathrm{PGL}_3(p^t).2, [p^{3t}])\ (\text{with } p = 2),$$
$$(\mathrm{PSU}_3(p^t), [p^{3t}]),\ (\mathrm{PSp}_4(p^t).\mathbb{Z}_{2^{l+1}}, [p^{4t}])\ (\text{with } p = 2),$$
$$(\mathrm{Sz}(p^t), [p^{2t}]),\ (\mathrm{Ree}(p^t), [p^{3t}]) \text{ and } (\mathrm{G}_2(p^t).\mathbb{Z}_{2^{l+1}}, [p^{6t}]),$$

where 2^l is the 2-part of t. Check the maximal subgroups of $\mathrm{PSp}_4(p^t).\mathbb{Z}_{2^{l+1}}$, $\mathrm{Sz}(p^t)$ and $\mathrm{Ree}(p^t)$; refer to [1, Table 8.14], [27, Theorem 9] and [15, Theorem C], respectively. We conclude that none of $\mathrm{PSp}_4(t^f).\mathbb{Z}_{2^{l+1}}$, $\mathrm{Sz}(p^t)$ and $\mathrm{Ree}(p^t)$ has maximal subgroups with a simple section $\mathrm{PSU}_3(q)$ and they are excluded. For the first three and the last pairs, $|X/\mathbf{O}_p(X_{\{u,v\}})|$ is a divisor of $2(p^t-1)^2$ and $\mathbf{O}_p(X_{\{u,v\}}) = [p^{3t}]$ or $[p^{6t}]$. Clearly, $t \le 2f$.

Suppose that $t = 2f$. Then $\mathrm{soc}(X) = \mathrm{PSL}_3(q^2)$ or $\mathrm{PSU}_3(q^2)$, and $\mathbf{O}_p(X_{\{u,v\}}) = [q^6]$. It follows that $\mathbf{O}_p(X_v^{[1]}) = [q^3]$. Thus, $\mathbf{O}_p(X_v) \ne 1$ and X_v has an almost simple quotient $\mathrm{PSU}_3(q).[o]$. Checking Tables 8.3 and 8.5 given in [1], we conclude that X has no maximal subgroup containing X_v, which is a contradiction. If $t = f$, then we have $(X, \mathbf{O}_p(X_{\{u,v\}})) = (\mathrm{G}_2(p^t).\mathbb{Z}_{2^{l+1}}, [q^6])$ and we get a similar contradiction by checking the maximal subgroups of $\mathrm{G}_2(p^t).\mathbb{Z}_{2^{l+1}}$.

Suppose that $f \ne t < 2f$. Then $f > 1$. Recalling that $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ has a subgroup $\mathbb{Z}_{(q^2-1)/(3,q+1)}$, we know that $p^{2f}-1$ is a divisor of $2(3,q+1)(p^t-1)^2$. If $p^{2f}-1$ has a primitive prime divisor, say s, then $s \ge 2f+1 \ge 5$, and s is not a divisor of $2(3,q+1)(p^t-1)^2$, which is a contradiction. It follows from Zsigmondy's theorem that $2f = 6$ and $p = 2$ and so $t = 1$ or 2. Then 7 is a divisor of $p^{2f}-1$ but not a divisor of $2(3,q+1)(p^t-1)^2$, which is a contradiction.

Case 2. Let (G_0, H_0) be one of the pairs in [17, Tables 15–20] that is not considered in Case 1. Assume that $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ is a $\{2,3\}$-group. Then $p^{2f}-1$ has no prime divisor other than 2 and 3. It follows that $f = 1$ and so $p = q > 2$. Calculation shows that $p \in \{3,5,7\}$. For $q = p = 3$, it is easily shown that $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ is a 2-group. These observations yield that either $q = p = 3$ and $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ is a 2-group, or $X_{\{u,v\}}$ is not a $\{2,3\}$-group.

Recall that $X_{\{u,v\}}/\mathbf{O}_p(X_{\{u,v\}})$ has a subgroup $\mathbb{Z}_{(q^2-1)/(3,q+1)}$ and $\mathbf{O}_p(X_{\{u,v\}})$ has order $p^{if}.a$, where $3 \le i \le 6$. It follows that $(X, X_{\{u,v\}})$ is one of the following pairs:

$$(\mathrm{HS}.2, [5^3]{:}[2^5]),\ (\mathrm{Ru}, [5^3]{:}[2^5]),\ (\mathrm{McL}, [5^3]{:}3{:}8),\ (\mathrm{Co}_2, [5^3]{:}4\mathrm{S}_4),$$
$$(\mathrm{Th}, [5^3]{:}4\mathrm{S}_4),\ (\mathrm{J}_4, [11^3]{:}(5 \times 2\mathrm{S}_4)).$$

Then $q = p \in \{5, 11\}$ and $X_v^{[1]} = 1$. In particular, $\mathrm{soc}(X_v) = \mathrm{PSU}_3(p)$, and $X_{\{u,v\}}$ is the normalizer $\mathbf{N}_X(P)$ of some Sylow p-subgroup P of X. Thus, $X_{uv} = X_v \cap X_{\{u,v\}} \le \mathbf{N}_{X_v}(P)$. For the pairs $(\mathrm{HS}.2, [5^3]{:}[2^5])$ and $(\mathrm{Ru}, [5^3]{:}[2^5])$, by the Atlas [3], $X_{\{u,v\}}$ is a normalizer of some Sylow 5-subgroup that intersects a maximal subgroup $\mathrm{PSU}_3(5){:}2$ of $\mathrm{soc}(X)$ at $[5^3]{:}8{:}2$; thus G, X and $X_{\{u,v\}}$ are as listed in Table 6. The other pairs are excluded as follows.

First, the group Th is excluded as it has no maximal subgroup with a simple section $\mathrm{PSU}_3(5)$; refer to [36, Table 5.8]. For the pair $(\mathrm{McL}, [5^3]{:}3{:}8)$, by the Atlas [3], we have

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$X_v = \mathrm{PSU}_3(5)$ and so $X_{uv} \le \mathbf{N}_{\mathrm{PSU}_3(5)}(P) = [5^3]{:}8$, which contradicts that $|X_{\{u,v\}} : X_{uv}| = 2$. For the pair $(\mathrm{J}_4, [11^3]{:}(5 \times 2\mathrm{S}_4))$, by [36, Table 5.8], $X_v = \mathrm{PSU}_3(11).2$, yielding $X_{uv} \le \mathbf{N}_{X_v}(P) = [11^3]{:}(5 \times 8{:}2)$ and we get a similar contradiction. For the pair $(X, X_{\{u,v\}}) = (\mathrm{Co}_2, [5^3]{:}4\mathrm{S}_4)$, by the Atlas [3], $X_v < \mathrm{HS}.2 < \mathrm{Co}_2$. Checking the maximal subgroups of HS.2, we have $X_v = \mathrm{PSU}_3(5)$ or $X_v = \mathrm{PSU}_3(5){:}2$. It follows that $X_{uv} \le \mathbf{N}_{X_v}(P) = [5^3]{:}8$ or $[5^3]{:}[2^5]$ and then $|X_{\{u,v\}} : X_{uv}| \ne 2$, which is a contradiction. This completes the proof. □

5. Graphs with soluble vertex-stabilizers

Let G, T, X and $\Gamma = (V, E)$ be as in Hypothesis 3.1. The following lemma says that if Γ is a complete bipartite graph, then $\Gamma \cong \mathsf{K}_{6,6}$ and $G_v^{\Gamma(v)}$ is insoluble.

LEMMA 5.1. *Assume that $\Gamma \cong \mathsf{K}_{d,d}$. Then $T \cong \mathrm{A}_6$, $d = 6$, $T_v = \mathrm{PSL}_2(5)$ and $T_{uv} \cong \mathrm{D}_{10}$. In particular, X_{uv} is nonabelian.*

PROOF. Let G^+ be the subgroup of G fixing the bipartition of Γ. Then $G_v \le G^+$ and G_v is 2-transitive on the partite set that does not contain v. Thus, G^+ acts 2-transitively on each partite set and these two actions are not equivalent. Check the almost simple 2-transitive groups; refer to [2, Table 7.4]. We conclude that $T \cong \mathrm{A}_6$ or M_{12}, $T_v \cong \mathrm{A}_5$ or M_{11} and $T_{uv} \cong \mathrm{D}_{10}$ or $\mathrm{PSL}_2(11)$, respectively. Since T_{uv} is soluble, the lemma follows. □

Assume that G_v is soluble and let $\mathrm{soc}(G_v^{\Gamma(v)}) = \mathbb{Z}_p^f$, where p is a prime. By Lemma 5.1, since G_v is soluble, Γ is not a complete bipartite graph. Then we have the following result by [21, Theorem 3.3].

LEMMA 5.2. *Assume that X_{uv} is abelian. Then one of the following holds:*

(1) $T \cong \mathrm{PSL}_2(p^f)$, $T_{\{u,v\}} \cong \mathrm{D}_{2(p^f-1)/(2,p-1)}$, $T_v \cong \mathbb{Z}_p^f{:}\mathbb{Z}_{(p^f-1)/(2,p-1)}$ *and* $\Gamma \cong \mathsf{K}_{p^f+1}$;
(2) $T = \mathrm{Sz}(2^f)$, $T_{\{u,v\}} \cong \mathrm{D}_{2(2^f-1)}$, $T_v \cong \mathbb{Z}_2^f{:}\mathbb{Z}_{2^f-1}$ *and Γ is $(T, 2)$-arc-transitive, where $f \ge 3$ is odd.*

REMARK 5.3. In Lemma 5.2, $T_{\{u,v\}}$ is soluble and maximal in T and thus $X = T$ by the choice of X. For part (1), since Γ is $(G, 2)$-arc-transitive, G is a 3-transitive group of degree $p^f + 1$ and thus $X \ne G$ if p is odd. The graphs satisfying part (2) are determined by [5, Construction 5.4 and Proposition 5.5]; in particular, for any given odd $f \ge 3$, there is a unique $(\mathrm{Sz}(2^f), 2)$-arc-transitive graph of valency 2^f that has automorphism group $\mathsf{Aut}(\mathrm{Sz}(2^f))$.

LEMMA 5.4. *Assume that (1) or (2) of Lemma 3.5 holds and X_{uv} is nonabelian. Then one of the following holds:*

(1) $G = X$ *or* $X.2$, $X = \mathrm{M}_{10}$, $X_{\{u,v\}} \cong \mathbb{Z}_8{:}\mathbb{Z}_2$, $X_v \cong 3^2{:}\mathrm{Q}_8$ *and* $\Gamma \cong \mathsf{K}_{10}$;
(2) $G = X = \mathrm{PSL}_3(3).2$, $X_{\{u,v\}} \cong \mathrm{GL}_2(3){:}2$, $X_v \cong 3^2{:}\mathrm{GL}_2(3)$ *and Γ is the point–line nonincidence graph of* $\mathrm{PG}(2, 3)$.

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PROOF.

Case 1. Assume that Lemma 3.5(1) holds. Suppose first that $(X_v^{\Gamma(v)})_u = \mathrm{Q}_8$. Then $X_{uv} \lesssim \mathrm{Q}_8{\times}\mathrm{Q}_8$. This implies that $|X_{\{u,v\}}|$ is a divisor of 2^7 and divisible by 2^4. Checking Tables 14–20 in [17], we have $X \cong \mathrm{PSL}_2(9).2 = \mathrm{M}_{10}$ and $X_{\{u,v\}} \cong \mathbb{Z}_8{:}\mathbb{Z}_2$. In this case, $X_v \cong 3^2{:}\mathrm{Q}_8$ and $d = 9$. Since Γ has valency nine and order $|X : X_v| = 10$, we have $\Gamma \cong \mathsf{K}_{10}$, desired as in part (1).

Suppose that $(X_v^{\Gamma(v)})_u \neq \mathrm{Q}_8$. If $p = 3$ and $(G_v^{\Gamma(v)})_u = \mathrm{Q}_8$, then $(X_v^{\Gamma(v)})_u$ is abelian; it follows that X_{uv} is abelian, which is a contradiction. Thus, we have $\mathrm{SL}_2(3) \trianglelefteq (G_v^{\Gamma(v)})_u \leq \mathrm{GL}_2(p)$ and $p \in \{3, 5, 7, 11, 23\}$. Then $(G_v^{\Gamma(v)})_u \leq \mathbf{N}_{\mathrm{GL}_2(p)}(\mathrm{SL}_2(3)) = \mathbb{Z}_{p-1} \circ \mathrm{GL}_2(3)$. Since $(X_v^{\Gamma(v)})_u$ is nonabelian and normal in $(G_v^{\Gamma(v)})_u$, we have $\mathrm{Q}_8 \trianglelefteq (X_v^{\Gamma(v)})_u$ and hence $\mathrm{SL}_2(3) \trianglelefteq (X_v^{\Gamma(v)})_u$. Moreover, $|X_{\{u,v\}}|$ is a divisor of $2^7 \cdot 3^2 \cdot (p-1)^2$ and divisible by 2^4. Let M be an arbitrary normal abelian subgroup of $X_{\{u,v\}}$. Then $M \cap X_{uv}$ has index at most 2 in M, and $(M \cap X_{uv})X_v^{[1]}/X_v^{[1]}$ is isomorphic to a normal subgroup of $(X_v^{\Gamma(v)})_u$. Thus, $(M \cap X_{uv})X_v^{[1]}/X_v^{[1]} \lesssim \mathbb{Z}_{p-1}$. Since $M \cap X_v^{[1]} \trianglelefteq X_v^{[1]}$ and $X_v^{[1]}$ is isomorphic to a normal subgroup of $(X_v^{\Gamma(v)})_u$, we have $M \cap X_v^{[1]} \lesssim \mathbb{Z}_{p-1}$. Noting that $(M \cap X_{uv})X_v^{[1]}/X_v^{[1]} \cong M \cap X_{uv}/(M \cap X_v^{[1]})$, it follows that $|M \cap X_{uv}|$ is a divisor of $(p-1)^2$. Thus, $|M|$ is a divisor of $2(p-1)^2$.

The above observations allow us to consider only the pairs (G_0, H_0) in [17, Tables 14–20] that satisfy the following conditions:

(c1) $|H_0|$ is a divisor of $2^7 \cdot 3^2 \cdot (p-1)^2$ and divisible by 2^4; H_0 has a factor (a quotient of some subnormal subgroup) Q_8; and H_0 has no element of order 3^2, 5^2 or 11^2;

(c2) if M is a normal abelian subgroup of H_0, then $|M|$ is a divisor of $2(p-1)^2$; if $p \in \{7, 11, 23\}$, the order of $\mathbf{O}_{(p-1)/2}(H_0)$ is a divisor of $(p-1)^2/4$.

Checking those H_0 that satisfy conditions (c1) and (c2), we conclude that the possible pairs $(X, X_{\{u,v\}})$ are listed as follows:

$(\mathrm{M}_{11}, 3^2{:}\mathrm{Q}_8.2)$, $(\mathrm{M}_{11}, 2\mathrm{S}_4)$, $(\mathrm{M}_{12}, [2^5].\mathrm{S}_3)$, $(\mathrm{M}_{12}, 3^2{:}2\mathrm{S}_4)$,
$(\mathrm{J}_2, [2^6]{:}(3{\times}\mathrm{S}_3))$, $(\mathrm{J}_3, [2^6]{:}(3{\times}\mathrm{S}_3))$, $(\mathrm{Co}_3, [2^9].3^2.\mathrm{S}_3))$,
$(\mathrm{He}.2, [2^8]{:}3^2.\mathrm{D}_8))$, $(\mathrm{McL}.2, [2^6]{:}\mathrm{S}_3^2)$,
$(\mathrm{PSL}_3(3), 3^2{:}2\mathrm{S}_4)$, $(\mathrm{PSL}_3(3).2, 2\mathrm{S}_4{:}2)$, $(\mathrm{PSL}_3(4).2, 2^{2+4}.3.2)$,
$(\mathrm{PGL}_3(4).2, [2^6].3.\mathrm{S}_3)$, $(\mathrm{PSL}_4(3).2, 2.\mathrm{S}_4^2.2)$, $(\mathrm{PSL}_5(2).2, [2^8].\mathrm{S}_3^2.2)$,
$(\mathrm{PSp}_4(4).4, [2^8]{:}3.12)$, $(\mathrm{PSp}_4(4).4, 5^2{:}[2^5])$, $(\mathrm{PSp}_6(2), [2^7]{:}\mathrm{S}_3^2)$,
$(\mathrm{PSp}_6(3), [2^8]{:}3^3.S_3)$, $(\mathrm{PSU}_3(3), 4.\mathrm{S}_4)$, $(\mathrm{PSU}_4(2), 2.\mathrm{A}_4^2.2)$,
$(\mathrm{PSU}_4(3), 2.\mathrm{A}_4^2.4)$, $(\mathrm{PSU}_4(3).2, [2^5].\mathrm{S}_4)$, $(\mathrm{P}\Omega_8^+(3).\mathrm{A}_4, 10^2{:}4\mathrm{A}_4)$,
$(\mathrm{G}_2(2)', 4.\mathrm{S}_4)$, $(\mathrm{G}_2(3), \mathrm{SL}_2(3) \circ \mathrm{SL}_2(3){:}2)$, $({}^2\mathrm{F}_4(2)', 5^2{:}4\mathrm{A}_4)$.

Note that these groups X are included in the Atlas [3]. Inspecting the subgroups of X, only the pair $(\mathrm{PSL}_3(3).2, 2\mathrm{S}_4{:}2)$ gives a desired $X_v \cong 3^2{:}\mathrm{GL}_2(3)$ and then the desired graph Γ has valency $d = 9$. In this case, the socle $\mathrm{PSL}_3(3)$ of X has two orbits on the vertex set of Γ; each of them has size 13 and can be viewed as the point set or the line set of the projective plane $\mathrm{PG}(2, 3)$. This forces that Γ is (isomorphic to) one of the following graphs: $\mathsf{K}_{13,13} - 13\mathsf{K}_2$, the point–line incidence graph and the point–line nonincidence graph of $\mathrm{PG}(2,3)$. Since Γ has valency 9, the graph Γ is the point–line nonincidence graph of $\mathrm{PG}(2,3)$. Then part (2) of this lemma follows.

Case 2. Let $2_+^{1+4}{:}\mathbb{Z}_5 \le (G_v^{\Gamma(v)})_u \le 2_+^{1+4}.(\mathbb{Z}_5{:}\mathbb{Z}_4)$. Then $2_+^{1+4} \trianglelefteq (X_v^{\Gamma(v)})_u$ and so $|X_{\{u,v\}}|$ is a divisor of $2^{15}\cdot 5^2$ and divisible by 2^6. Further, if M is a normal abelian subgroup of $X_{\{u,v\}}$, then a similar argument as in *Case 1* yields that $|M|$ is a divisor of 2^5. It is easily shown that $\mathbf{O}_2(X_{uv}) \ne 1$ and hence $\mathbf{O}_2(X_{\{u,v\}}) \ne 1$. Checking the pairs (G_0, H_0) in [17, Tables 14–20], either $\mathbf{O}_2(H_0) = 1$ or $|H_0|$ has an odd prime divisor other than 5. Thus, in this case, no desired pair $(X, X_{\{u,v\}})$ exists. This completes the proof. □

We assume next that Lemma 3.5(3) occurs. Thus, $(G_v^{\Gamma(v)})_u \not\le \mathrm{GL}_1(p^f)$ and $(G_v^{\Gamma(v)})_u \le \Gamma\mathrm{L}_1(p^f)$. Then $f > 1$ and $(G_v^{\Gamma(v)})_u \lesssim \mathbb{Z}_{p^f-1}{:}\mathbb{Z}_f$. Recalling that $X_{uv} \lesssim (X_u^{\Gamma(u)})_v \times (X_v^{\Gamma(v)})_u \le (G_u^{\Gamma(u)})_v \times (G_v^{\Gamma(v)})_u$, we have the following simple fact.

LEMMA 5.5. *If (3) of Lemma 3.5 occurs, then $X_{\{u,v\}}$ has no section $\mathbb{Z}_t^3$, $\mathbb{Z}_r^5$ or $\mathbb{Z}_2^6$, where t is a primitive prime divisor of $p^f - 1$ and r is an arbitrary odd prime.*

LEMMA 5.6. *Assume that X_{uv} is nonabelian and (3) of Lemma 3.5 occurs. Then $p^f \ne 2^6$.*

PROOF. Suppose that $p^f = 2^6$. Then X has order divisible by 2^6, $X_{uv} \lesssim \mathbb{Z}_{63}{:}\mathbb{Z}_6 \times \mathbb{Z}_{63}{:}\mathbb{Z}_6$ and thus $X_{\{u,v\}}$ has a normal Hall $2'$-subgroup and $|X_{\{u,v\}}|$ is indivisible by 2^4. Checking Tables 14–20 given in [17], $(X, X_{\{u,v\}})$ is one of the following pairs:

$(\mathrm{S}_7, \mathbb{Z}_7{:}\mathbb{Z}_6)$, $(\mathrm{M}_{12}.2, 3_+^{1+2}{:}\mathrm{D}_8)$, $(\mathrm{PSL}_2(2^6), \mathrm{D}_{126})$, $(\mathrm{PSL}_2(5^3), \mathrm{D}_{126})$,
$(\mathrm{PSL}_2(7937), \mathrm{D}_{7938})$, $(\mathrm{PSL}_3(8), 7^2{:}\mathrm{S}_3)$, $(\mathrm{Sz}(8), \mathrm{D}_{14})$, $(\mathrm{G}_2(3).2, [3^6]{:}\mathrm{D}_8)$.

The pair $(\mathrm{PSL}_2(2^6), \mathrm{D}_{126})$ yields that $X_v \cong 2^6{:}\mathbb{Z}_{63}$ and thus X_{uv} is abelian; this is not the case. The other pairs are easily excluded as none of them gives a desired X_v. This completes the proof. □

LEMMA 5.7. *Assume that X_{uv} is nonabelian and (3) of Lemma 3.5 occurs. Suppose that X_{uv} has a normal abelian Hall $2'$-subgroup. Then $G = X$ or $X.2$, $X = \mathrm{M}_{10}$, $X_{\{u,v\}} \cong \mathbb{Z}_8{:}\mathbb{Z}_2$, $X_v \cong 3^2{:}\mathrm{Q}_8$ and $\Gamma \cong \mathsf{K}_{10}$.*

PROOF. Note that $X_{\{u,v\}} = X_{uv}.2$. The unique Hall $2'$-subgroup of X_{uv} is also the Hall $2'$-subgroup of $X_{\{u,v\}}$. Checking Tables 14–20 given in [17], we know that $(X, X_{\{u,v\}})$ is one of the following pairs:

(i) $(\mathrm{PGL}_2(7), \mathrm{D}_{16})$, $(\mathrm{PSL}_3(2).2, \mathrm{D}_{16})$, $(\mathrm{PGL}_2(9), \mathrm{D}_{16})$, $(\mathrm{M}_{10}, \mathbb{Z}_8{:}\mathbb{Z}_2)$,
$(\mathrm{A}_5, \mathrm{D}_{10})$, $(\mathrm{A}_6, 3^2{:}\mathbb{Z}_4)$, $(\mathrm{M}_{11}, 3^2{:}\mathrm{Q}_8.2)$, $(\mathrm{J}_1, \mathrm{D}_6\times\mathrm{D}_{10})$,
$(\mathrm{PGL}_2(7), \mathrm{D}_{12})$, $(\mathrm{PGL}_2(9), \mathrm{D}_{20})$, $(\mathrm{M}_{10}, \mathbb{Z}_5{:}\mathbb{Z}_4)$, $(\mathrm{PGL}_2(11), \mathrm{D}_{20})$,
$(\mathrm{PSL}_2(t^a), \mathrm{D}_{2(t^a\pm1)/(2,t-1)})$, $(\mathrm{PSp}_4(4).4, \mathbb{Z}_{17}{:}\mathbb{Z}_{16})$;

(ii) $(\mathrm{PSL}_2(t^a), \mathbb{Z}_t^a{:}\mathbb{Z}_{(t^a-1)/2})$, t is a prime, $a \le 4$ and $t^a - 1$ is a power of 2;
$(\mathrm{PSL}_3(t), \mathbb{Z}_3^2{:}\mathrm{Q}_8)$, t is a prime with $t \equiv 4, 7 \bmod 9$;
$(\mathrm{PSU}_3(t), \mathbb{Z}_3^2{:}\mathrm{Q}_8)$, t is a prime with $t \equiv 2, 5 \bmod 9$;

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$(\mathrm{PSU}_3(2^a), \mathbb{Z}_3^2{:}\mathrm{Q}_8)$ with a prime $a > 3$;
$(\mathrm{PSp}_4(2^a).\mathbb{Z}_{2^{b+1}}, \mathrm{D}^2_{2(q\pm1)}{:}2.\mathbb{Z}_{2^{b+1}})$, $(\mathrm{PSp}_4(2^a).\mathbb{Z}_{2^{b+1}}, \mathbb{Z}_{2^{2a}+1}.4.\mathbb{Z}_{2^{b+1}})$, 2^b is the 2-part of a;
$(\mathrm{Sz}(2^{2a+1}), \mathrm{D}_{2(2^{2a+1}-1)})$, $(\mathrm{Sz}(2^{2a+1}), \mathbb{Z}_{2^{2a+1}\pm2^{a+1}+1}{:}\mathbb{Z}_4)$;
$({}^3\mathrm{D}_4(t^a), \mathbb{Z}_{t^{4a}-t^{2a}+1}{:}\mathbb{Z}_4)$, t is a prime.

The pair $(\mathrm{M}_{10}, \mathbb{Z}_8{:}\mathbb{Z}_2)$ yields that $X_v \cong 3^2{:}\mathrm{Q}_8$ and $d = 9$. The third pair in (i) implies that $X_v \cong \mathbb{Z}_3^2{:}\mathbb{Z}_8$; however, then X_{uv} is abelian, which is not the case. For $(\mathrm{PSL}_2(t^a), \mathrm{D}_{2(t^a\pm1)/(2,t^a-1)})$, checking the subgroups of $\mathrm{PSL}_2(t^a)$, we have $t^a = p^f$ and $X_v \cong \mathbb{Z}_p^f{:}\mathbb{Z}_{(p^f-1)/(2,p-1)}$ and then X_{uv} is abelian, which a contradiction. The other pairs in (i) are also excluded as $|X|$ is indivisible by p^f. (Note that $f > 1$.)

Now we deal with the pairs in (ii). Note that, for an odd prime r, the edge-stabilizer $X_{\{u,v\}}$ has a unique Sylow r-subgroup $\mathbf{O}_r(X_{\{u,v\}})$. Then $\mathbf{O}_r(X_{\{u,v\}})$ is a Sylow subgroup of X by Lemma 2.7. This implies that the unique Hall $2'$-subgroup of $X_{\{u,v\}}$, say K, is a Hall subgroup of X. Since $X_{\{u,v\}} = X_{uv}.2$, we have $K \le X_{uv}$. Note that $|X_v{:}X_{uv}| = d = p^f$ and X_v is contained in a maximal subgroup of X. We now check the maximal subgroups of X that contain K; refer to [12, Hauptsatz II.8.27], [1, Tables 8.3–8.6, 8.14 and 8.15] and [13, 14, 27]. Then one of the following occurs:

(iii) $X = \mathrm{Sz}(2^{2a+1})$ and $X_v \cong \mathbb{Z}_2^{2a+1}{:}\mathbb{Z}_{2^{2a+1}-1}$;
(iv) $X = \mathrm{PSp}_4(2^a).\mathbb{Z}_{2^{b+1}}$ and $X_v \lesssim \mathrm{Sp}_2(2^{2a}){:}2.\mathbb{Z}_{2^b}$;
(v) $X = \mathrm{PSp}_4(2^a).\mathbb{Z}_{2^{b+1}}$ and $X_v \lesssim \mathrm{Sp}_2(2^a) \wr \mathrm{S}_2.\mathbb{Z}_{2^b}$.

Item (iii) yields that X_{uv} is abelian, which is not the case. Item (iv) gives $X_{uv} = X_v$, which is a contradiction. Suppose that (v) occurs; we have $X_v \cong (\mathbb{Z}_2^a{:}\mathbb{Z}_{2^a-1})^2{:}2.\mathbb{Z}_{2^b}$. Then $1 \ne \mathbf{O}_2(X_v) \le \mathbf{O}_2(G_v)$ and hence $d = |\mathbf{O}_2(G_v)|$, by Lemma 2.5. Since X_v is transitive on $\Gamma(v)$, it follows that $p^f = d = 2^{2a}$. Thus, $|X_{uv}| = (2^a-1)^2 2^{b+1}$ and so $|X_{\{u,v\}}{:}X_{uv}| = 8 > 2$, which is a contradiction. □

COROLLARY 5.8. *Assume that X_{uv} is nonabelian and (3) of Lemma 3.5 occurs. If $f = 2$, then $G = X$ or $X.2$, $X = \mathrm{M}_{10}$, $X_{\{u,v\}} \cong \mathbb{Z}_8{:}\mathbb{Z}_2$, $X_v \cong 3^2{:}\mathrm{Q}_8$ and $\Gamma \cong \mathsf{K}_{10}$.*

PROOF. Let $f = 2$. Then $(X_v^{\Gamma(v)})_u \lesssim \mathbb{Z}_{p^2-1}.\mathbb{Z}_2$. Note that $X_{\{u,v\}} = X_{uv}.2$ and $X_{uv} \lesssim \mathbb{Z}_{p^2-1}.\mathbb{Z}_2{\times}\mathbb{Z}_{p^2-1}.\mathbb{Z}_2$. Then Lemma 5.7 is applicable and the result follows. □

Let $\pi_0(p^f-1)$ be the set of primitive primes of p^f-1. By Zsigmondy's theorem, if $\pi_0(p^f-1) = \emptyset$ and $f > 1$, then $p^f = 2^6$, or $f = 2$ and $p = 2^t - 1$, where t is a prime. Thus, in view of Lemma 5.6 and Corollary 5.8, we assume next that $\pi_0(p^f-1) \ne \emptyset$.

LEMMA 5.9. *Assume that $\pi{:} = \pi_0(p^f-1) \ne \emptyset$, X_{uv} is nonabelian and (3) of Lemma 3.5 occurs. Then $f \ge 3$ and:*

(1) $\pi \ne \pi(|X_{\{u,v\}}|) \setminus \{2\}$, $\min(\pi) \ge \max\{5, f{+}1\}$;
(2) $p \not\equiv \pm1 \bmod r$ *and* $\boldsymbol{O}_r(X_{\{u,v\}}) \ne 1$ *for each* $r \in \pi$;
(3) $X_{\{u,v\}}$ *has a unique (nontrivial) Hall π-subgroup that is either cyclic or a direct product of two cyclic subgroups.*

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PROOF. By the assumptions in this lemma and Lemma 3.6, we have that $(X_v^{\Gamma(v)})_u \cong \mathbb{Z}_{m'}.\mathbb{Z}_{f/e'}$ and $\emptyset \neq \pi = \pi_0(p^f - 1) \subseteq \pi(m')$. For $r \in \pi$, since $p^{r-1} \equiv 1 \bmod r$, we have $f \leq r-1$ and so $r \geq f+1$. In particular, $r \geq 5$ and $p \not\equiv \pm 1 \bmod r$. Recall that $X_{\{u,v\}} = X_{uv}.2$ and $X_{uv} \lesssim \mathbb{Z}_{m'}.\mathbb{Z}_{f/e'} \times \mathbb{Z}_{m'}.\mathbb{Z}_{f/e'}$. It follows that $\mathbf{O}_r(X_{\{u,v\}}) \neq 1$ and $\mathbf{O}_r(X_{\{u,v\}})$ is the unique Sylow r-subgroup of $X_{\{u,v\}}$. Clearly, $\mathbf{O}_r(X_{\{u,v\}})$ is either cyclic or a direct product of two cyclic subgroups. Then $X_{\{u,v\}}$ has a unique Hall π-subgroup F that is either cyclic or a direct product of two cyclic subgroups. Clearly, $F \neq 1$ and, by Lemma 5.7, $X_{\{u,v\}}$ has no normal abelian Hall $2'$-subgroup. Then $\pi \neq \pi(|X_{\{u,v\}}|) \setminus \{2\}$ and the lemma follows. □

Recall that $X_{\{u,v\}}$ has no section $\mathbb{Z}_2^6$ or $\mathbb{Z}_3^5$; see Lemma 5.5. Combining with Lemma 5.9, we next check the pairs (G_0, H_0) listed in [17, Tables 14–20].

LEMMA 5.10. *Assume that $\pi_0(p^f - 1) \neq \emptyset$, X_{uv} is nonabelian and (3) of Lemma 3.5 occurs. Then $T = \mathrm{soc}(X)$ is not a simple group of Lie type.*

PROOF. Suppose that T is a simple group of Lie type over a finite field of order $q' = t^a$, where t is a prime. Since $T \trianglelefteq G$, we know that T is transitive on the edge set of Γ. Then $T_v^{\Gamma(v)} \neq 1$. Noting that $T_v^{\Gamma(v)} \trianglelefteq G_v^{\Gamma(v)}$, we have $\mathrm{soc}(G_v^{\Gamma(v)}) \leq T_v^{\Gamma(v)}$. In particular, T_v is transitive on $\Gamma(v)$ and so $|T_v| = p^f|T_{uv}|$. In view of this, noting that $T_v = T \cap X_v = T \cap G_v$ and $T_{\{u,v\}} = T \cap X_{\{u,v\}} = T \cap G_{\{u,v\}}$, we sometimes work on the triple $(T, T_v, T_{\{u,v\}})$ instead of $(X, X_v, X_{\{u,v\}})$.

By Lemmas 5.7 and 5.9, $X_{\{u,v\}}$ is not a $\{2,3\}$-group and has no normal abelian Hall $2'$-subgroup. Assume that $t \in \pi_0(p^f - 1)$. By Lemmas 5.5 and 5.9, $t \geq 5$, $X_{\{u,v\}}$ has no section $\mathbb{Z}_t^3$ and $\mathbf{O}_t(X_{\{u,v\}}) \neq 1$ is abelian. Checking the pairs (G_0, H_0) listed in [17, Tables 16–20], we have $X = \mathrm{PSL}_2(t^2)$ and $X_{\{u,v\}} \cong \mathbb{Z}_t^2{:}\mathbb{Z}_{(t^2-1)/2}$. For this case, checking the subgroups of $\mathrm{PSL}_2(t^2)$, no desired X_v arises, which is a contradiction. Therefore, $t \notin \pi_0(p^f - 1)$.

By Lemma 5.9, $\mathbf{O}_r(X_{\{u,v\}}) \neq 1$ for each $r \in \pi_0(p^f - 1)$. Recall that $X_{\{u,v\}}$ is not a $\{2,3\}$-group and has a subgroup of index two. Checking the pairs (G_0, H_0) listed in [17, Tables 16–20], we conclude that $\mathbf{O}_t(X_{\{u,v\}}) = 1$. Further, we observe that a desired $X_{\{u,v\}}$ if it exists has the form of $N.K$, where N is an abelian subgroup of T and either K is a $\{2,3\}$-group or $(X, K) = (\mathrm{E}_8(q'), \mathbb{Z}_{30})$. For the case where $K \not\cong \mathbb{Z}_{30}$, by Lemma 3.6, $\pi_0(p^f - 1) \subseteq \pi(|N|)$ and thus, by Lemma 5.5, N has no subgroup $\mathbb{Z}_r^3$ for $r \in \pi_0(p^f - 1)$. With these restrictions, only one of the following Cases 1–4 occurs.

Case 1. Either $X = \mathrm{PSL}_3(q')$ and $X_{\{u,v\}} \cong (1/(3, q'-1))\mathbb{Z}_{q'-1}^2.\mathrm{S}_3$ with $q' \neq 2, 4$, or $X = \mathrm{PSU}_3(q')$ and $X_{\{u,v\}} \cong (1/(3, q'+1))\mathbb{Z}_{q'+1}^2.\mathrm{S}_3$. Then $|X_v| = (3/(3, q' \mp 1))p^f(q' \mp 1)^2$. Checking Tables 8.3–8.6 given in [1], we have $X = \mathrm{PSL}_3(q')$ and $X_v \lesssim [q'^3]{:}(1/(3, q'-1))\mathbb{Z}_{q'-1}^2$. It follows that $p = t = 3$ and $|\mathbf{O}_3(X_v)| = 3^{f+1} = 3d$, which contradicts Lemma 2.5.

Case 2. $T = \mathrm{soc}(X) = \mathrm{P}\Omega_8^+(q')$ and $T_{\{u,v\}} \cong \mathrm{D}^2_{2(q'^2+1)/(2,q'-1)}.[2^2]$. In this case, noting that $|T_{\{u,v\}}{:}T_{uv}| \leq 2$, we have $|T_v| = 2^4 p^f((q'^2+1)^2/(2, q'-1)^2)$ or $2^3 p^f((q'^2+1)^2/(2, q'-1)^2)$. Let M be a maximal subgroup of T with $T_v \leq M$. By [13], since $|M|$ is divisible by

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TABLE 7. Edge-stabilizers in exceptional groups.

T	N	$\lvert T_{\{u,v\}}:N\rvert$	Remark
$\mathrm{Ree}(3^a)$	$\mathbb{Z}_{3^a\pm3^{(a+1)/2}+1}$	6	odd $a \geq 3$
	$\mathbb{Z}_2\times\mathbb{Z}_{(3^a+1)/2}$	6	& $X = T$
$\mathrm{G}_2(3^a)$	$\mathbb{Z}^2_{3^a\pm1}$	12	odd $a \geq 2$
	$\mathbb{Z}_{3^{2a}\pm3^a+1}$	6	
	$\mathbb{Z}^2_{2^a+1}$	48	odd $a \geq 3$
${}^2\mathrm{F}_4(2^a)$	$\mathbb{Z}^2_{2^a\pm2^{(a+1)/2}+1}$	96	& $X = T$
	$\mathbb{Z}_{2^{2a}\pm2^{(3a+1)/2}+2^a\pm2^{(a+1)/2}+1}$	12	& $2^a\pm2^{(a+1)/2}+1 > 5$
	$\mathbb{Z}^2_{2^{2a}\pm2^a+1}$	72	
$\mathrm{F}_4(2^a)$	$\mathbb{Z}^2_{2^{2a}+1}$	96	$a \geq 2$
	$\mathbb{Z}_{2^{4a}-2^{2a}+1}$	12	
$\mathrm{E}_8(q')$	$\mathbb{Z}^2_{q'^4-q'^2+1}$	288	$X = T$
	$\mathbb{Z}_{q'^8\pm q'^7\mp q'^5-q'^4\mp q'^3\pm q'+1}$	30	

$(q'^2+1)^2$, we have $M \cong \mathrm{PSL}_2(q'^2)^2.2^2$. It is easily shown that $\mathrm{PSL}_2(q'^2)^2.2^2$ does not have subgroups of order $2^4p^f((q'^2+1)^2/(2, q'-1)^2)$ or $2^3p^f((q'^2+1)^2/(2, q'-1)^2)$, which is a contradiction.

Case 3. $(X, X_{\{u,v\}})$ is one of $({}^2\mathrm{F}_4(2)', 5^2{:}4\mathrm{A}_4)$ and $({}^2\mathrm{F}_4(2), 13{:}12)$. For the first pair, we have $\pi_0(p^f-1) = \{5\}$ and, since p^f is a divisor of $\lvert{}^2\mathrm{F}_4(2)'\rvert$, we conclude that $p^f = 2^4$ or 3^4. The second pair implies that $\pi_0(p^f-1) = \{13\}$ and then $p^f = 2^{12}$ or 3^3. By the Atlas [3], X has no maximal subgroup containing X_{uv} as a subgroup of index divisible by p^f, which is a contradiction.

Case 4. $T_{\{u,v\}}$ has a normal abelian subgroup N listed in Table 7. Let M be a maximal subgroup of T with $T_v \leq M$. Then $\lvert M\rvert$ is divisible by $p^f\lvert N\rvert$. Check the maximal subgroups of T of order divisible by $\lvert N\rvert$; refer to [15, 20, 22]. Then we may deduce a contradiction. First, by [15, Theorem C], we conclude that $\mathrm{Ree}(3^a)$ has no maximal subgroup of order divisible by $p^f\lvert N\rvert$. Similarly, by [22], the group ${}^2\mathrm{F}_4(2^a)$ is excluded. We next deal with the remaining cases.

Suppose that $T = \mathrm{G}_2(3^a)$. For $\lvert N\rvert = 3^{2a}\pm3^a+1$, since $\lvert M\rvert$ is divisible by $3^{2a}\pm3^a+1$, we have $M \cong \mathrm{SL}_3(3^a){:}2$ or $\mathrm{SU}_3(3^a){:}2$ by [15, Theorems A and B]. By [1, Tables 8.3–8.6], we conclude that $T_v \lesssim \mathbb{Z}_{3^{2a}\pm3^a+1}{:}[6]$, which is impossible. Similarly, for $\lvert N\rvert = (3^a\pm1)^2$, we have that $T_v \lesssim (\mathrm{SL}_2(3^a)\circ\mathrm{SL}_2(3^a)).2$, $\mathrm{SL}_3(3^a){:}2$ or $\mathrm{SU}_3(3^a){:}2$. Since $\lvert T_v\rvert$ is divisible by $\frac{1}{2}\lvert T_{\{u,v\}}\rvert p^f = 6p^f(3^a\pm1)^2$, checking the maximal subgroups of $\mathrm{SL}_2(3^a)$, $\mathrm{SL}_3(3^a)$ and $\mathrm{SU}_3(3^a)$, we have $p = 3$ and $T_v \lesssim [3^{ba}]{:}\mathbb{Z}^2_{3^a-1}.2$ for $b = 2$ or 3. Since T_{uv} has order divisible by 3, it follows that $\mathbf{O}_3(T_{uv}) \neq 1$, which contradicts Lemma 2.5.

Suppose that $T = \mathrm{F}_4(2^a)$. By [19, 20], noting that $\lvert M\rvert$ is divisible by $p^f\lvert N\rvert$, we conclude that $M \cong \mathrm{Sp}_8(2^a)$ or $\mathrm{P}\Omega^+_8(2^a).\mathrm{S}_3$ with $\lvert N\rvert = (2^{2a}+1)^2$, or $M \cong c.\mathrm{PSL}_3(2^a)^2.c.2$ or $c.\mathrm{PSU}_3(2^a)^2.c.2$ with $\lvert N\rvert = (2^{2a}\pm2^a+1)^2$, where $c = (3, 2^a \pm 1)$. Then a contradiction

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TABLE 8. Primitive prime divisors.

X	J_1	J_2	J_4	Co_1	$O'N.2$	He	B
$\lvert X_{\{u,v\}}\rvert$	$2{\cdot}3{\cdot}7$	$2^2{\cdot}3{\cdot}5^2$	$2{\cdot}7{\cdot}43$	$2^3{\cdot}3^2{\cdot}7^2$	$2{\cdot}3{\cdot}5{\cdot}31$	$2^4{\cdot}3{\cdot}5^2$	$2^5{\cdot}3^4{\cdot}13$
r	7	5	43	7	31	5	13
p^f	2^3	2^4	2^{14}	2^3, 3^6	2^5	2^4	$3^3, 5^4, 2^{12}$
$p^f - 1 \mid \lvert G_{uv}\rvert$	✓	✓	×	✓, ×	✓	✓	✓, ✓, ×
X	B	B	M	M	M	M	M
$\lvert X_{\{u,v\}}\rvert$	$2^2{\cdot}3^2{\cdot}19$	$2{\cdot}11{\cdot}23$	$2^3{\cdot}3{\cdot}11{\cdot}23$	$2^2{\cdot}3{\cdot}7{\cdot}29$	$2{\cdot}3^2{\cdot}5{\cdot}31$	$2^3{\cdot}5{\cdot}41$	$2{\cdot}23{\cdot}47$
r	19	23	23	29	31	41	47
p^f	2^{18}	$2^{11}, 3^{11}$	$2^{11}, 3^{11}$	2^{28}	$2^5, 5^3$	$2^{20}, 3^8$	2^{23}
$p^f-1 \mid \lvert G_{uv}\rvert$	×	×, ×	×, ×	×	✓, ×	×, ×	×

follows from checking the maximal subgroups of $\mathrm{Sp}_8(2^a)$, $\mathrm{P\Omega}_8^+(2^a)$, $\mathrm{PSL}_3(2^a)$ and $\mathrm{PSU}_3(2^a)$; refer to [1, Tables 8.3–8.6 and 8.48–8.50].

Finally, suppose that $T = \mathrm{E}_8(q')$. Then $|N| = (q'^4 - q'^2+1)^2$ and $M \cong \mathrm{PSU}_3(q'^2)^2.8$. For this case, checking the maximal subgroups $\mathrm{PSU}_3(q'^2)$, we get a contradiction. This completes the proof. □

LEMMA 5.11. *Assume that $\pi_0(p^f - 1) \neq \emptyset$, X_{uv} is not abelian and (3) of Lemma 3.5 occurs. Then $G = X = \mathrm{J}_1$, $X_{\{u,v\}} \cong \mathbb{Z}_7{:}\mathbb{Z}_6$, $X_v \cong \mathbb{Z}_2^3{:}\mathbb{Z}_7{:}\mathbb{Z}_3$ and $d = 8$.*

PROOF. By Lemma 5.10, $T = \mathrm{soc}(X)$ is either an alternating group or a sporadic simple group. Note that $X_{\{u,v\}}$ is not a $\{2,3\}$-group and has no normal abelian Hall $2'$-subgroup.

Assume that T is an alternating group. Then, by [17, Table 14], either $X = \mathrm{A}_r$ and $X_{\{u,v\}} \cong \mathbb{Z}_r{:}\mathbb{Z}_{(r-1)/2}$ for $r \notin \{7, 11, 17, 23\}$, or $X = \mathrm{S}_r$ and $X_{\{u,v\}} \cong \mathbb{Z}_r{:}\mathbb{Z}_{r-1}$ for $r \in \{7, 11, 17, 23\}$. For these two cases, X_v is a transitive subgroup of S_r in the natural action of S_r. Then either X_v is almost simple or $X_v \lesssim \mathbb{Z}_r{:}\mathbb{Z}_{r-1}$ (refer to [4, page 99, Corollary 3.5B]), which is a contradiction.

Assume that T is a sporadic simple group and let $r \in \pi_0(p^f - 1)$. Then $(X, X_{\{u,v\}}, r)$ is one of the following triples:

$(\mathrm{J}_1, \mathbb{Z}_7{:}\mathbb{Z}_6, 7)$, $(\mathrm{J}_1, \mathbb{Z}_{11}{:}\mathbb{Z}_{10}, 11)$, $(\mathrm{J}_1, \mathbb{Z}_{19}{:}\mathbb{Z}_6, 19)$, $(\mathrm{J}_2, \mathbb{Z}_5^2{:}\mathrm{D}_{12}, 5)$,
$(\mathrm{J}_3.2, \mathbb{Z}_{19}{:}\mathbb{Z}_{18}, 19)$, $(\mathrm{J}_4, \mathbb{Z}_{29}{:}\mathbb{Z}_{28}, 29)$, $(\mathrm{J}_4, \mathbb{Z}_{37}{:}\mathbb{Z}_{12}, 37)$, $(\mathrm{J}_4, \mathbb{Z}_{43}{:}\mathbb{Z}_{14}, 43)$,
$(\mathrm{O'N}.2, \mathbb{Z}_{31}{:}\mathbb{Z}_{30}, 31)$, $(\mathrm{He}, \mathbb{Z}_5^2{:}4\mathrm{A}_4, 5)$, $(\mathrm{Co}_1, \mathbb{Z}_7^2{:}(3{\times}2\mathrm{A}_4), 7)$,
$(\mathrm{Ly}, \mathbb{Z}_{37}{:}\mathbb{Z}_{18}, 37)$, $(\mathrm{Ly}, \mathbb{Z}_{67}{:}\mathbb{Z}_{22}, 67)$, $(\mathrm{Fi}_{24}', \mathbb{Z}_{29}{:}\mathbb{Z}_{14}, 29)$,
$(\mathrm{B}, \mathbb{Z}_{13}{:}\mathbb{Z}_{12}{\times}\mathrm{S}_4, 13)$, $(\mathrm{B}, \mathbb{Z}_{19}{:}\mathbb{Z}_{18}{\times}\mathbb{Z}_2, 19)$, $(\mathrm{B}, \mathbb{Z}_{23}{:}\mathbb{Z}_{11}{\times}2, 23)$,
$(\mathrm{M}, \mathbb{Z}_{23}{:}\mathbb{Z}_{11}{\times}\mathrm{S}_4, 23)$, $(\mathrm{M}, (\mathbb{Z}_{29}{:}\mathbb{Z}_{14}{\times}3).2, 29)$, $(\mathrm{M}, \mathbb{Z}_{31}{:}\mathbb{Z}_{15}{\times}\mathrm{S}_3, 31)$,
$(\mathrm{M}, \mathbb{Z}_{41}{:}\mathbb{Z}_{40}, 41)$, $(\mathrm{M}, \mathbb{Z}_{47}{:}\mathbb{Z}_{23}{\times}2, 47)$.

Recall that p^f is a divisor of $|X|$ and r is a primitive prime divisor of $p^f - 1$. Searching all possible pairs (p^f, r), we get Table 8.

Recalling that $G_{\{u,v\}} = X_{\{u,v\}}.(G/X)$, we have $2|G_{uv}| = |G_{\{u,v\}}| = |X_{\{u,v\}}||G{:}X| = 2|X_{uv}||G{:}X|$ and so $|G_{uv}| = |X_{uv}||G{:}X|$. Since G_v is 2-transitive on $\Gamma(v)$, we know that $(p^f - 1)$ is a divisor of $|G_{uv}| = |X_{uv}||G{:}X|$. It follows that $(X, X_{\{u,v\}}, r, p^f)$ is one of the

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following quadruples:

$$(\mathrm{J}_1, \mathbb{Z}_7{:}\mathbb{Z}_6, 7, 2^3),\ (\mathrm{J}_2, \mathbb{Z}_5^2{:}\mathrm{D}_{12}, 5, 2^4),\ (\mathrm{Co}_1, \mathbb{Z}_7^2{:}(3{\times}2\mathrm{A}_4), 7, 2^3),$$
$$(\mathrm{O'N}.2, \mathbb{Z}_{31}{:}\mathbb{Z}_{30}, 31, 2^5),\ (\mathrm{He}, \mathbb{Z}_5^2{:}4\mathrm{A}_4, 5, 2^4),\ (\mathrm{B}, \mathbb{Z}_{13}{:}\mathbb{Z}_{12}{\times}\mathrm{S}_4, 13, 3^3),$$
$$(\mathrm{B}, \mathbb{Z}_{13}{:}\mathbb{Z}_{12}{\times}\mathrm{S}_4, 13, 5^4),\ (\mathrm{M}, \mathbb{Z}_{31}{:}\mathbb{Z}_{15}{\times}\mathrm{S}_3, 31, 2^5).$$

For $(\mathrm{Co}_1, \mathbb{Z}_7^2{:}(3{\times}2\mathrm{A}_4), 7, 2^3)$, we have $X_{uv} \lesssim \Gamma\mathrm{L}_1(2^3){\times}\Gamma\mathrm{L}_1(2^3)$, yielding that $|X_{uv}|$ is odd, which is a contradiction. Similarly, for $(\mathrm{B}, \mathbb{Z}_{13}{:}\mathbb{Z}_{12}{\times}\mathrm{S}_4, 13, 3^3)$, the order of X_{uv} is indivisible by 2^4, which is a contradiction; for $(\mathrm{M}, \mathbb{Z}_{31}{:}\mathbb{Z}_{15}{\times}\mathrm{S}_3, 31, 2^5)$, the order of X_{uv} is indivisible by 3, which is a contradiction. For $(\mathrm{He}, \mathbb{Z}_5^2{:}4\mathrm{A}_4, 5, 2^4)$, the order of X_{uv} is divisible by $2^3 \cdot 3 \cdot 5^2$ and, since $p^f = 2^4$, the order of X_u is divisible by $2^7 \cdot 3 \cdot 5^2$; however, He has no soluble subgroup of order divisible by $2^7 \cdot 3 \cdot 5^2$, which is a contradiction. Similarly, $(\mathrm{O'N}.2, \mathbb{Z}_{31}{:}\mathbb{Z}_{30}, 31, 2^5)$ is excluded as O′N.2 has no soluble subgroup with order divisible by $2^5 \cdot 31$. (Note that G_v is soluble.) By the Atlas [3], J_2 has no subgroup with order divisible by $2^4 \cdot 5^2$ and then $(\mathrm{J}_2, \mathbb{Z}_5^2{:}\mathrm{D}_{12}, 5, 2^4)$ is excluded. By the Atlas [3] and [35, Theorem 2.1], B has no subgroup with order divisible by $3^2 \cdot 5^4 \cdot 13$ and so $(\mathrm{B}, \mathbb{Z}_{13}{:}\mathbb{Z}_{12}{\times}\mathrm{S}_4, 13, 5^4)$ is excluded. Then only $(\mathrm{J}_1, \mathbb{Z}_7{:}\mathbb{Z}_6, 7, 2^3)$ is left, which gives $X_v \cong \mathbb{Z}_2^3{:}\mathbb{Z}_7{:}\mathbb{Z}_3$, $d = p^f = 8$ and $G = X$. This completes the proof. □

Finally, we summarize the argument for proving Theorem 1.1 as follows.

Proof of Theorem 1.1. Clearly, each $(G, G_v, G_{\{u,v\}})$ in Table 1 gives a G-edge-primitive graph $\mathsf{Cos}(G, G_v, G_{\{u,v\}})$. It is not difficult to check the 2-arc-transitivity of G acting on $\mathsf{Cos}(G, G_v, G_{\{u,v\}})$; we omit the details.

Now let G and $\Gamma = (V, E)$ satisfy the assumptions in Theorem 1.1. Let $T = \mathrm{soc}(G)$ and $\{u, v\} \in E$. Choose a minimal X among the normal subgroups of G that act primitively on E. Then $\mathrm{soc}(X) = T$. Since $G_{\{u,v\}}$ is soluble, $X_{\{u,v\}}$ is soluble. Then $(X, X_{\{u,v\}})$ is one of the pairs (G_0, H_0) listed in [17, Tables 14–20]. Thus, Γ, G, $G_{\{u,v\}}$, X and $X_{\{u,v\}}$ satisfy Hypothesis 3.1 and then Lemmas 3.3 and 3.5 work here. If $G_v^{\Gamma(v)}$ is an almost simple 2-transitive group, then, by Lemma 3.3 and Lemmas 4.1–4.4, the triple $(G, G_v, G_{\{u,v\}})$ is as listed in Table 1. Assume next that $G_v^{\Gamma(v)}$ is a soluble 2-transitive group of degree $d = p^f$, where p is a prime.

If X_{uv} is abelian, then the triple $(G, G_v, G_{\{u,v\}})$ is as desired in Table 1, by Lemma 5.2. Thus, assume further that X_{uv} is nonabelian. Then G_{uv} is nonabelian. By Lemma 3.5, either $G_v^{\Gamma(v)} \nleq \mathrm{GL}_1(p^f)$ and $G_v^{\Gamma(v)} \leq \Gamma\mathrm{L}_1(p^f)$, or $G_v^{\Gamma(v)}$ has a normal subgroup $\mathrm{SL}_2(3)$ or 2_+^{1+4}. For the latter case, the triple $(G, G_v, G_{\{u,v\}})$ is known by Lemma 5.4. Let $G_v^{\Gamma(v)} \leq \Gamma\mathrm{L}_1(p^f)$ and consider the primitive prime divisors of $p^f - 1$. If $p^f - 1$ has no primitive prime divisor, then, by Lemma 5.6 and Corollary 5.8, $(G, G_v, G_{\{u,v\}})$ is as listed in Table 1. If $p^f - 1$ has primitive prime divisors, then $(G, G_v, G_{\{u,v\}})$ is known by Lemma 5.11. This completes the proof. □

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