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SPACES WHICH INVERT WEAK HOMOTOPY EQUIVALENCES

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Abstract It is well known that if X is a CW-complex, then for every weak homotopy equivalence $f: A \to B$, the map $f_*: [X, A] \to [X, B]$ induced in homotopy classes is a bijection. In fact, up to homotopy equivalence, only CW-complexes have that property. Now, for which spaces X is $f^*: [B, X] \to [A, X]$ a bijection for every weak equivalence f? This question was considered by J. Strom and T. Goodwillie. In this note we prove that a non-empty space inverts weak equivalences if and only if it is contractible.

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1. Introduction

A continuous map $f: A \to B$ is a weak homotopy equivalence if it induces a bijection $\pi_0(A) \to \pi_0(B)$ between the sets of path components and an isomorphism $\pi_n(A, a_0) \to \pi_n(B, f(a_0))$ for every base point $a_0 \in A$ and every $n \geq 1$. We say that an unpointed space X *inverts weak homotopy equivalences* if the functor $[-, X]$ inverts weak equivalences, that is, for every weak homotopy equivalence $f: A \to B$, the induced map $f^*: [B, X] \to [A, X]$ is a bijection. As usual, $[A, X]$ stands for the set of unpointed homotopy classes of maps from A to X. This property is clearly a homotopy invariant. In [**1**] Jeff Strom asked for the characterization of such spaces. Tom Goodwillie observed in [**1**] that if X inverts weak equivalences and is T_1 (i.e. its points are closed), then each path component is weakly contractible (has trivial homotopy groups) and then contractible. His idea was to use finite spaces weakly homotopy equivalent to spheres. A map from a connected finite space to a T_1-space has a connected and discrete image and is therefore constant. Goodwillie also proved that if a space inverts weak equivalences, then it must be connected. In this note we follow his ideas and give a further application of non-Hausdorff spaces to obtain the following characterization.

Theorem 1. *A non-empty space X inverts weak homotopy equivalences if and only if it is contractible.*

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We prove also a pointed version of the same result. A pointed space (X, x_0) *inverts pointed weak homotopy equivalences* if for every weak homotopy equivalence $f : A \to B$ and every base point $a_0 \in A$, $f^* : [(B, f(a_0)), (X, x_0)] \to [(A, a_0), (X, x_0)]$ is a bijection.

Theorem 2. *Let (X, x_0) be a pointed space which inverts pointed weak homotopy equivalences. Then X is contractible. Moreover, $\{x_0\}$ is a strong deformation retract of X.*

This second proof only involves locally compact metric spaces. Therefore we deduce the following.

Corollary 3. *Let $\mathcal{A}$ be one of the following categories of pointed spaces: all spaces, Hausdorff spaces, locally compact Hausdorff spaces or metric spaces. Then no non-trivial representable cohomology theory on $\mathcal{A}$ can satisfy Eilenberg and Steenrod's weak equivalence axiom.*

2. Proof of Theorem 1

Lemma 4 (Goodwillie). *Suppose that X inverts weak homotopy equivalences and is weakly contractible. Then it is contractible.*

Proof. Just take the weak homotopy equivalence $X \to *$. □

Lemma 5. *Let X be a space which inverts weak equivalences and let Y be a locally compact Hausdorff space. Then the mapping space X^Y, considered with the compact-open topology, also inverts weak equivalences.*

Proof. This follows from a direct application of the exponential law and the fact that a weak equivalence $f : A \to B$ induces a weak equivalence $f \times 1_Y : A \times Y \to B \times Y$. □

If A is a discrete space and B is a totally path-disconnected space (i.e. the path components are singletons) with the same cardinality as A, then any bijection $A \to B$ is a weak homotopy equivalence. This idea was used by Goodwillie, Strickland and Strom in [**1**]. We will use this remark in the following constructions. The first one is due to Goodwillie.

Construction 1. Let $A = \mathbb{N}_0$ be the set of non-negative integers with the discrete topology and $B = \{0\} \cup \{1/n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ with the usual subspace topology. The map $f : A \to B$, which maps 0 to 0 and n to $1/n$ for every n, is a weak homotopy equivalence.

Construction 2. Let α be a cardinal which is greater than or equal to c, the cardinality $\#\mathbb{R}$ of the continuum. Let B be a set with cardinality $\#B > \alpha$. Consider the following topology in B: a proper subset $F \subseteq B$ is closed if and only if $\#F \leq \alpha$. Note that B is totally path-disconnected, as if $\gamma : I \to B$ is a path, then its image has cardinality at most $c \leq \alpha$, so it is connected and discrete and then a point. Let A be the discretization of B, i.e. the same set with the discrete topology. Then the function $i : A \to B$, which is the identity on underlying sets, is a weak homotopy equivalence.

Goodwillie used Construction 1 to prove the following.

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Proposition 6 (Goodwillie). *Let X be a space which inverts weak homotopy equivalences. Then it is connected.*

Proof. We can assume that X is non-empty. Suppose that X_0 and X_1 are two path components of X. Let $x_0 \in X_0$ and $x_1 \in X_1$. Let $f : A \to B$ be the weak homotopy equivalence considered in Construction 1. Take $g : A \to X$ defined by $g(0) = x_0$ and $g(n) = x_1$ for every $n \geq 1$. By hypothesis there exists a map $h : B \to X$ such that $h(0) \in X_0$ and $h(1/n) \in X_1$ for every $n \geq 1$. Since $1/n \to 0$, X_0 intersects the closure of X_1. Thus X_0 and X_1 are contained in the same connected component of X. □

We use Construction 2 to prove something stronger.

Lemma 7. *Let X be a space which inverts weak homotopy equivalences. Then X is path-connected.*

Proof. We can assume that X is non-empty. Let X_0 and X_1 be path components of X. For $\alpha = \max\{\#X, c\}$, let B, A and $i : A \to B$ be as in Construction 2. Let b_0 and b_1 be two different points of B. Define $g : A \to X$ in such a way that $g(b_0) \in X_0$ and $g(b_1) \in X_1$ (define g arbitrarily in the remaining points of A). Then g is continuous. Since $i^* : [B, X] \to [A, X]$ is surjective, there exists a map $h : B \to X$ such that $hi \simeq g$. In particular, $h(b_0) \in X_0$ and $h(b_1) \in X_1$. Since $\#B > \alpha \geq \#X$ and $B = \bigcup_{x \in X} h^{-1}(x)$, there exists $x \in X$ such that $\#h^{-1}(x) > \alpha$. Let $U \subseteq X$ be an open neighbourhood of $h(b_0)$. Then $h^{-1}(U^c) \subseteq B$ is a proper closed subset, so $\#h^{-1}(U^c) \leq \alpha$. Thus, $h^{-1}(x)$ is not contained in $h^{-1}(U^c)$ and then $x \in U$. Since every open neighbourhood of $h(b_0)$ contains x, there is a continuous path from x to $h(b_0)$, namely $t \mapsto x$ for $t < 1$ and $1 \mapsto h(b_0)$. In particular, $x \in X_0$. Symmetrically, $x \in X_1$. Therefore, $X_0 = X_1$. □

Proof of Theorem 1. It is clear that a contractible space inverts weak equivalences. Suppose that $X \neq \emptyset$ is a space which inverts weak equivalences. By Lemma 5, X^{S^n} inverts weak equivalences for every $n \geq 0$ and then it is path-connected. Therefore, $\pi_n(X)$ is trivial for every $n \geq 0$ and, by Lemma 4, X is contractible. □

3. Proof of Theorem 2

In the proof of Lemma 7, non-Hausdorff spaces played a central part. If we change our hypothesis that X inverts weak equivalences for the pointed version, then it can be proved that X is path-connected using only Hausdorff spaces.

Lemma 8. *Let (X, x_0) be a pointed space which inverts pointed weak homotopy equivalences between Hausdorff spaces. Then X is path-connected.*

Proof. Let X_0 be the path component of x_0. Let X_1 be any path component of X and let $x_1 \in X_1$. We use once again Construction 1. Let A, B, f, g be as in Proposition 6. Let $a_0 = 0 \in A$. By hypothesis there exists $h : (B, 0) \to (X, x_0)$ such that $hf \simeq g$ rel $\{0\}$. In particular, $h(1) \in X_1$. Define $h' : B \to X$ by $h'(0) = x_0$ and $h'(1/n) = h(1/(n+1))$ for $n \geq 1$. The continuity of h' follows from that of h. Since $h'(1/n) = h(1/(n+1)) \in X_1$ and $h(1/n) \in X_1$ for every $n \geq 1$, there exists a homotopy $H : A \times I \to X$ from $h'f$

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to hf. Moreover, we can take H to be stationary on $0 \in A$. Since $f^* : [(B, 0), (X, x_0)] \to [(A, 0), (X, x_0)]$ is injective, there exists a homotopy $F : B \times I \to X$, $F : h' \simeq h$ rel $\{0\}$. The map F gives a collection of paths from $h(1/(n+1))$ to $h(1/n)$. We glue all these paths to form a path from x_0 to $h(1)$. That is, define $\gamma : I \to X$ by $\gamma(0) = x_0$ and $\gamma(t) = F(1/n, (1/n - 1/(n+1))^{-1}(t - 1/(n+1)))$ if $t \in [1/(n+1), 1/n]$. Note that γ is continuous at $t = 0$, as if $U \subseteq X$ is a neighbourhood of x_0, then $\{0\} \times I \subseteq F^{-1}(U)$, and by the tube lemma there exists $n_0 \geq 1$ such that $\{1/n\} \times I \subseteq F^{-1}(U)$ for every $n \geq n_0$. Then $[0, 1/(n_0)] \subseteq \gamma^{-1}(U)$. Hence, x_0 and $h(1)$ lie in the same path component, so $X_0 = X_1$. □

We will refine the proof of the previous result to obtain a proof for Theorem 2.

Construction 3. Let $k \geq 0$. Define $A = \{0\} \amalg (S^k \times \mathbb{N})$, where $\mathbb{N}$ has the discrete topology. For $n \in \mathbb{N}$, denote by $1/nS^k$ the subspace of $\mathbb{R}^{k+1}$ of points with norm $1/n$. Let $B = \{0\} \cup \bigcup_{n \in \mathbb{N}} 1/nS^k \subseteq \mathbb{R}^{k+1}$ with the usual subspace topology. The map $f : A \to B$, which maps 0 to 0 and $(s, n) \in S^k \times \mathbb{N}$ to $s/n \in 1/nS^k$, is a homeomorphism in each path component. Therefore, f is a weak homotopy equivalence.

Proof of Theorem 2. Let $k \geq 0$ and let $f : (A, 0) \to (B, 0)$ be the weak homotopy equivalence of Construction 3. Let $\omega : S^k \to X$ be any continuous map. Define $g : (A, 0) \to (X, x_0)$ by $g(0) = x_0$ and $g(s, n) = \omega(s)$ for every $s \in S^k$, $n \in \mathbb{N}$. Then g is continuous and by hypothesis there exists $h : (B, 0) \to (X, x_0)$ such that $hf \simeq g$ rel $\{0\}$. In particular, $hf|_{S^k \times \{n\}} : S^k \to X$ is homotopic to ω. Define $h' : (B, 0) \to (X, x_0)$ by $h'(0) = x_0$ and $h'(s/n) = h(s/(n+1))$ for every $s \in S^k$ and $n \in \mathbb{N}$. The continuity of h' follows from that of h. For each $s \in S^k$ and $n \in \mathbb{N}$, $h'f(s, n) = hf(s, n+1)$, so $h'f|_{S^k \times \{n\}} = hf|_{S^k \times \{n+1\}} \simeq \omega \simeq hf|_{S^k \times \{n\}}$. Then there exists a homotopy $H : A \times I \to S^k$, $H : h'f \simeq hf$ rel $\{0\}$. By hypothesis $h' \simeq h$ rel $\{0\}$. Let $F : B \times I \to X$ be a homotopy from h' to h which is stationary on $\{0\}$.

Define $G : S^k \times I \to X$ by $G(s, 0) = x_0$ for every $s \in S^k$ and $G(s, t) = F(s/n, (1/n - 1/(n+1))^{-1}(t - 1/(n+1)))$ if $t \in [1/(n+1), 1/n]$ (see Figure 1). It is clear then that G is well defined and $G|_{S^k \times (0,1]}$ is continuous. We prove that G is also continuous at those points of the form $(s, 0)$. If $U \subseteq X$ is a neighbourhood of x_0, then $\{0\} \times I \subseteq F^{-1}(U) \subseteq \mathbb{R}^{k+1} \times I$. By the tube lemma there exists $n_0 \geq 1$ such that $\{1/n\} \times I \subseteq F^{-1}(U)$ for every $n \geq n_0$. Then $S^k \times [0, 1/(n_0)]$ is a neighbourhood of $(s, 0)$ contained in $G^{-1}(U)$. Therefore, G is a (free) homotopy between the constant map x_0 and $G(-, 1)$. Since $G(s, 1) = F(s, 1) = h(s) = hf(s, 1)$, $G(-, 1) \simeq \omega$. This proves that every map $S^k \to X$ is freely homotopic to a constant. Since this holds for every $k \geq 0$, X is weakly contractible.

Finally, since X is weakly contractible, the constant map x_0, $c : (X, x_0) \to (X, x_0)$ is a weak homotopy equivalence. Then $c^* : [(X, x_0), (X, x_0)] \to [(X, x_0), (X, x_0)]$ is bijective. In particular, the class of the identity $1_X : (X, x_0) \to (X, x_0)$ is in the image of c^*, which means that $1_X \simeq c$ rel x_0. In other words, $\{x_0\}$ is a strong deformation retract of X. □

Of course, the converse of Theorem 2 is also true: if $\{x_0\}$ is a strong deformation retract of X, (X, x_0) inverts pointed weak homotopy equivalences.

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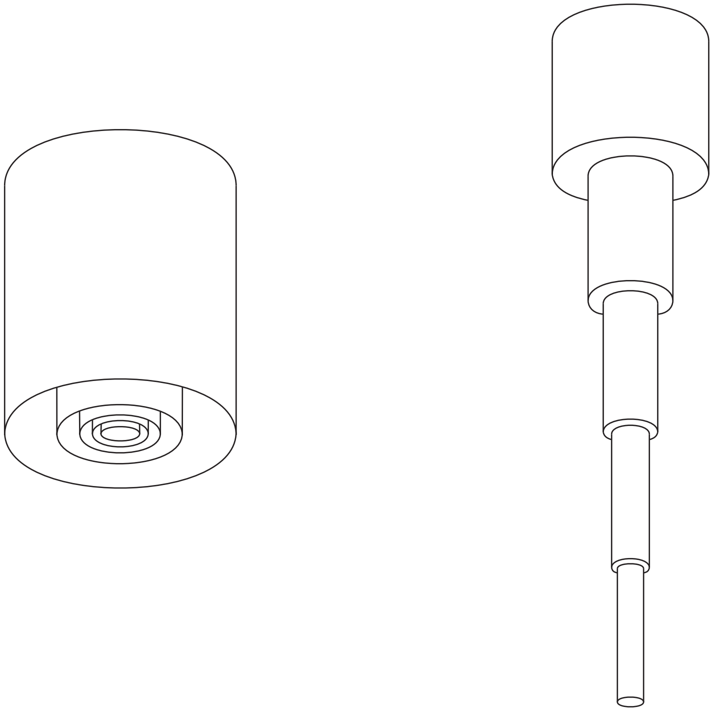

Figure 1. The homotopy F is defined in the cylinder $B \times I$ pictured on the left, which looks like a shut retractable telescope for $k = 1$. The values of F at the bottom of the cylinder $1/nS^k \times I$ coincide with the values of F at the top of the next smaller cylinder $1/(n+1)S^k \times I$. The homotopy G is constructed by pulling the telescope to its full length.

Since the spaces A and B of Construction 3 are locally compact, Hausdorff and metric, the proof of Theorem 2 implies Corollary 3.

4. A final comment

The characterization mentioned in the abstract of those unpointed spaces X for which $[X, -]$ inverts weak equivalences is easy to obtain.

Proposition 9. *Let X be a space. Then the following are equivalent:*

(i) $f_* : [X, A] \to [X, B]$ *is a bijection for every weak equivalence* $f : A \to B$*;*

(ii) X *has the homotopy type of a CW-complex.*

Proof. It is well known that any CW-complex satisfies the first property (see [**2**, Proposition 4.22], for instance). Now, suppose X is such that $[X, -]$ inverts weak equivalences. Let $f : Y \to X$ be a CW-approximation. Then $f_* : [X, Y] \to [X, X]$ is a bijection.

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In particular, there exists a map $g : X \to Y$ such that $fg \simeq 1_X$. Then g is a weak equivalence. Now, since $[Y, -]$ inverts weak equivalences, $g_* : [Y, X] \to [Y, Y]$ is a bijection, so there exists $h : Y \to X$ with $gh \simeq 1_Y$. Then g is a homotopy equivalence. □

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