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THE CHARACTER GRAPH OF A FINITE GROUP IS PERFECT

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Abstract

For a finite group G, let $\Delta(G)$ denote the character graph built on the set of degrees of the irreducible complex characters of G. A perfect graph is a graph Γ in which the chromatic number of every induced subgraph Δ of Γ equals the clique number of Δ. We show that the character graph $\Delta(G)$ of a finite group G is always a perfect graph. We also prove that the chromatic number of the complement of $\Delta(G)$ is at most three.

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1. Introduction

Let G be a finite group and let $\mathrm{cd}(G)$ be the set of all character degrees of G, that is, $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$, where $\mathrm{Irr}(G)$ is the set of all complex irreducible characters of G. The set of prime divisors of character degrees of G is denoted by $\rho(G)$. It is well known that the character degree set $\mathrm{cd}(G)$ may be used to provide information on the structure of the group G. For example, the Ito–Michler theorem [10] states that if a prime p divides no character degree of a finite group G, then G has a normal abelian Sylow p-subgroup. Another result due to Thompson [13] says that if a prime p divides every nonlinear character degree of a group G, then G has a normal p-complement.

A useful way to study the character degree set of a finite group G is to associate a graph to $\mathrm{cd}(G)$. One of these graphs is the character graph $\Delta(G)$ of G [9]. Its vertex set is $\rho(G)$ and two vertices p and q are joined by an edge if the product pq divides some character degree of G. We refer to the survey by Lewis [7] for results concerning this graph and related topics.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite simple graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A clique of Γ is a set of mutually adjacent vertices. The clique number of Γ, denoted by $\omega(\Gamma)$, is the maximum size of a clique of Γ. The chromatic number of Γ, denoted by $\chi(\Gamma)$, is the minimum number of colours needed to colour the vertices

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of the graph Γ so that any two adjacent vertices of Γ have different colours. Clearly $\omega(\Gamma) \leqslant \chi(\Gamma)$ for any graph Γ. The graph Γ is perfect if $\omega(\Delta) = \chi(\Delta)$ for every induced subgraph Δ of Γ.

The theory of perfect graphs relates the concept of graph colourings to the concept of cliques. Aside from having an interesting structure, perfect graphs are considered important for three reasons. First, several common classes of graphs are known to always be perfect. For instance, bipartite graphs, chordal graphs and comparability graphs are perfect. Second, a number of important algorithms only work on perfect graphs. Finally, perfect graphs can be used in a wide variety of applications, ranging from scheduling to order theory and communication theory.

One of the turning points for the investigation of the character degree graph is the 'three-vertex theorem' by Palfy [12]: the complement of $\Delta(G)$ does not contain any triangle whenever G is a finite solvable group. In a recent paper [1], this was extended by showing that, under the same solvability assumption, the complement of $\Delta(G)$ does not contain any cycle of odd length, which is equivalent to saying that the complement of $\Delta(G)$ is a bipartite graph. Thus, as the complement of a perfect graph is perfect [8], the character graph $\Delta(G)$ of a finite solvable group G is perfect. This argument motivates an interesting question: is the character graph of an arbitrary finite group perfect? In this paper, we wish to solve this problem.

THEOREM 1.1. *If G is a finite group, then $\Delta(G)$ is a perfect graph.*

Theorem 1.1 can be used to determine the chromatic number of the complement of $\Delta(G)$.

COROLLARY 1.2. *If G is a finite group, then the chromatic number of the complement of $\Delta(G)$ is at most three.*

2. Preliminaries

In this paper, all groups are assumed to be finite and all graphs are simple and finite. For a finite group G, the set of prime divisors of $|G|$ is denoted by $\pi(G)$ and, for an integer $n \geqslant 1$, the set of prime divisors of n is denoted by $\pi(n)$.

LEMMA 2.1 [6, Corollary 11.29]. *Let $N \lhd G$ and $\chi \in \mathrm{Irr(G)}$. Let $\varphi \in \mathrm{Irr(N)}$ be a constituent of χ_N. Then $\chi(1)/\varphi(1)$ divides $[G : N]$.*

Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A cycle on n vertices $v_1, \ldots, v_n$, $n \geq 3$, is a graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence and are nonadjacent otherwise. A cycle with n vertices is said to be of length n and is denoted by C_n, that is, $C_n : v_1, \ldots, v_n, v_1$. The join $\Gamma * \Delta$ of graphs Γ and Δ is the graph $\Gamma \cup \Delta$ together with all edges joining $V(\Gamma)$ and $V(\Delta)$. An independent set is a set of vertices of a graph Γ such that no two of them are adjacent. A maximum independent set is an independent set of largest possible size. This size is called the independence number of Γ and is denoted by $\alpha(\Gamma)$. The complement of Γ and the induced subgraph

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of Γ on $X \subseteq V(\Gamma)$ are denoted by Γ^c and $\Gamma[X]$, respectively. For more details, we refer to basic textbooks such as [3]. Now we present some properties of perfect graphs.

LEMMA 2.2 [8]. *A graph Γ is perfect if and only if the complement of Γ is perfect.*

LEMMA 2.3 [4]. *A graph Γ is perfect if and only if it has no induced subgraph isomorphic either to a cycle of odd order at least five or to the complement of such a cycle.*

We next state some results on character graphs needed in the next section.

LEMMA 2.4. *Let G be a group and let $\pi \subseteq \rho(G)$.*

(a) (Palfy's condition [12]) *If G is solvable and $|\pi| \geqslant 3$, then there exist two distinct primes u, v in π and $\chi \in \mathrm{Irr(G)}$ such that $uv \mid \chi(1)$.*
(b) (Moreto–Tiep's condition [11]) *If $|\pi| \geqslant 4$, then there exists $\chi \in \mathrm{Irr(G)}$ such that $\chi(1)$ is divisible by two distinct primes in π.*

LEMMA 2.5 [14]. *Let $G \cong \mathrm{PSL_2(q)}$, where $q \geqslant 4$ is a power of a prime p.*

(a) *If q is even, then $\Delta(G)$ has three connected components, $\{2\}$, $\pi(q-1)$ and $\pi(q+1)$, and each component is a complete graph.*
(b) *If $q > 5$ is odd, then $\Delta(G)$ has two connected components, $\{p\}$ and $\pi((q-1)(q+1))$.*
    (i) *The connected component $\pi((q-1)(q+1))$ is a complete graph if and only if $q-1$ or $q+1$ is a power of two.*
    (ii) *If neither of $q-1$ nor $q+1$ is a power of two, then $\pi((q-1)(q+1))$ can be partitioned as $\{2\} \cup M \cup P$, where the subsets $M = \pi(q-1) - \{2\}$ and $P = \pi(q+1) - \{2\}$ are both nonempty sets. The subgraph of $\Delta(G)$ corresponding to each of the subsets M, P is complete, all primes are adjacent to 2 and no prime in M is adjacent to any prime in P.*

LEMMA 2.6 [2]. *Let G be a finite group and let π be a subset of the vertex set of $\Delta(G)$ such that $|\pi|$ is an odd number larger than one. Then π is the set of vertices of a cycle in $\Delta(G)^c$ if and only if $O^{\pi'}(G) = S \times A$, where A is abelian, $S \cong \mathrm{SL_2(u^\alpha)}$ or $S \cong \mathrm{PSL_2(u^\alpha)}$ for a prime $u \in \pi$ and a positive integer α, and the primes in $\pi - \{u\}$ are alternately odd divisors of $u^\alpha + 1$ and $u^\alpha - 1$.*

LEMMA 2.7 [5]. *If G and H are two nonabelian groups that satisfy $\rho(G) \cap \rho(H) = F$ with $|F| = n$, then $\Delta(G \times H) = K_n * \Delta(G)[\rho(G) - F] * \Delta(H)[\rho(H) - F]$, where K_n is a complete graph with vertex set F.*

3. Proof of the main results

PROOF OF THEOREM 1.1. On the contrary, assume that $\Delta(G)$ is not perfect. Then, by Lemma 2.3, there exists a subset $\pi \subseteq \rho(G)$ such that $|\pi| = 2n+1$ for some integer $n \geqslant 2$, and $\Delta(G)^c[\pi]$ or $\Delta(G)[\pi]$ is a cycle. Now one of the following cases occurs.

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Case 1. $\Delta(G)^c[\pi]$ is a cycle. Then there exist primes $p_0, p_1, \ldots, p_{2n} \in \rho(G)$ such that $\Delta(G)^c[\pi]$ is the cycle $C : p_0, p_1, \ldots, p_{2n}, p_0$. Using Lemma 2.6, $N := O^{\pi'}(G) = S \times A$, where A is abelian, $S \cong \mathrm{SL}_2(p^m)$ or $S \cong \mathrm{PSL}_2(p^m)$ for a prime $p \in \pi$ and a positive integer m, and the primes in $\pi - \{p\}$ are alternately odd divisors of $p^m + 1$ and $p^m - 1$. Without loss of generality, we can assume that $p = p_0$. Since $\Delta(G)^c[\pi]$ is a cycle of length at least five, p_1 is not adjacent to both vertices p_3 and p_4 in $\Delta(G)^c$. Also, for some $\epsilon \in \{\pm 1\}$, $p_3 \in \pi(p^m - \epsilon) - \{2\}$ and $p_4 \in \pi(p^m + \epsilon) - \{2\}$. Note that p_1 is an element of either $\pi(p^m - \epsilon) - \{2\}$ or $\pi(p^m + \epsilon) - \{2\}$. Without loss of generality, we can assume that $p_1 \in \pi(p^m - \epsilon) - \{2\}$. Since p_1 and p_4 are adjacent vertices in $\Delta(G)$, for some $\chi \in \mathrm{Irr(G)}$, $p_1 p_4 \mid \chi(1)$. Now let $\varphi \in \mathrm{Irr(N)}$ be a constituent of χ_N. Then, by Lemma 2.1, $\chi(1)/\varphi(1)$ divides $[G : N]$. Therefore, as G/N is a π'-group, $p_1 p_4 \mid \varphi(1)$. This is a contradiction since, by Lemma 2.5, p_1 and p_4 are nonadjacent vertices in $\Delta(N)$.

Case 2. $\Delta(G)[\pi]$ is a cycle. If $\Delta(G)[\pi] \cong C_5$, then $\Delta(G)^c[\pi] \cong C_5$, which is in contradiction with Case 1. Thus, $|\pi| \geqslant 7$ and there exist distinct primes $p_1, p_2, p_3, q_1, q_2, q_3$ in π such that the induced subgraphs of $\Delta(G)^c$ on the sets $\pi_1 := \{p_1, p_2, p_3\}$ and $\pi_2 := \{q_1, q_2, q_3\}$ are cycles of length three. Hence, applying Lemma 2.6, for $i = 1, 2$, $N_i := O^{\pi_i'}(G) = S_i \times A_i$, where A_i is abelian, $S_i \cong \mathrm{SL}_2(u_i^{\alpha_i})$ or $S_i \cong \mathrm{PSL}_2(u_i^{\alpha_i})$ for a prime $u_i \in \pi_i$ and a positive integer α_i, and the primes in $\pi_i - \{u_i\}$ are alternately odd divisors of $u_i^{\alpha_i} + 1$ and $u_i^{\alpha_i} - 1$. Let $N := S_1 S_2$. Now it is easy to see that $N/Z(N) \cong \mathrm{PSL}_2(u_1^{\alpha_1}) \times \mathrm{PSL}_2(u_2^{\alpha_2})$. Therefore, $\Delta(N/Z(N))[\pi_1 \cup \pi_2] \subseteq \Delta(G)$ is a complete bipartite graph with parts π_1 and π_2, by Lemmas 2.5 and 2.7. But this is a contradiction as $\Delta(G)[\pi]$ is a cycle of $\Delta(G)$. □

PROOF OF COROLLARY 1.2. Using Theorem 1.1, $\Delta(G)$ is a perfect graph. Thus, by Lemma 2.2, so is $\Delta(G)^c$. Hence, $\chi(\Delta(G)^c) = \omega(\Delta(G)^c)$. It is clear that $\omega(\Delta(G)^c) = \alpha(\Delta(G))$. By Lemma 2.4, $\alpha(\Delta(G)) \leqslant 3$. Hence, $\chi(\Delta(G)^c) \leqslant 3$, which completes the proof. □

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