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The motion of fibres in turbulent flow

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Equations of mean and fluctuating velocities in rotation and translation have been derived for rigid thin inertialess fibres moving in a turbulent fluid. The derived equations for mean motion are general to fluid velocities that vary nonlinearly along the length of the fibre. From the equations of fluctuating fibre velocity, rotational and translational dispersion coefficients were derived. The resulting dispersion coefficients were shown to decrease as the ratio of fibre length to Lagrangian integral length scale of the turbulence increased.

1. Introduction

The behaviour of fibres in a turbulent flow affects the transport, rheology and light scattering properties of suspensions that are of interest in many areas of science and industry. This includes the pulp and paper industry, where all fibre processing and papermaking is performed at high speeds in turbulent fluids. Particles suspended in a turbulent fluid undergo mean motion due to the mean fluid velocity and random motion due to the fluctuating component of the fluid velocity. For fibres, motion is a combination of translation and rotation.

The earliest investigation into the motion of non-spherical particles in a carrier fluid is that of Jeffery (1922). He calculated the total force and moment exerted on an ellipsoidal particle moving in a Stokes flow, assuming an inertialess particle and a constant fluid velocity gradient. The result is the classical expressions for the orbital motion of ellipsoidal particles in a simple shear flow. For rotation in a plane, the angular velocity is

$$\dot{\theta} = \frac{1}{\alpha^2+1}\left(\tfrac{1}{2}(\alpha^2-1)\sin(2\theta)\left(\frac{\partial U_2}{\partial \xi_2}-\frac{\partial U_1}{\partial \xi_1}\right)+(\sin^2(\theta)+\alpha^2\cos^2(\theta))\frac{\partial U_2}{\partial \xi_1}\right.$$
$$\left.-(\cos^2(\theta)+\alpha^2\sin^2(\theta))\frac{\partial U_1}{\partial \xi_2}\right) \quad (1.1)$$

where α is the aspect ratio of the two-dimensional ellipse, $\boldsymbol{U}$ is the fluid velocity and θ is the angle of the major axis of the ellipse with the ξ_1-axis. The centre of the particle follows the fluid streamlines.

Cox (1970) developed an insightful analytical series approximation for the force distribution along the length of a fiber in Stokes flow. As Reynolds number, based on fibre diameter, approaches zero (i.e. $\lim Re \to 0$) the force per unit length along the fibre, $\boldsymbol{f}$, is proportional to the relative velocity between the fluid and the fibre at that point, that is, a linear Stokes drag relation given by

$$\boldsymbol{f}(l) = \boldsymbol{D}[\boldsymbol{U}(l) - \boldsymbol{V}_f(l)] \quad (1.2)$$

where $\boldsymbol{D}$ is a constant drag tensor, independent of the position along the fibre, V_f is the velocity of the fibre and $\boldsymbol{U}$ is the velocity of the fluid as a function of position along the length of the fibre, l. The assumption of an inertialess particle and Stokesian drag has been used in almost all investigations of fibre motion in laminar flow (Riese, Spiegelberg & Ebeling 1969; Shanker, Gillespie & Guceri 1991; Pittman & Kasiri 1992; Tangsaghasaksri 1994).

The randomly fluctuating component of velocity due to turbulence gives rise to the dispersion of suspended particles in a manner similar to molecular diffusion. In gases, molecules randomly collide during travel along a mean free path. After repeated collisions, they become scattered. This scattering is called diffusion. In fluids, the random motion of fluid elements (eddies) imparts a continuous random motion on suspended particles. Repeated eddy interactions cause the particles to disperse over time.

Taylor (1921) provided one of the first investigations of turbulent dispersion. This classical paper relates the mean-square displacement of a diffusible property, originating from a point source, to the Lagrangian velocity correlation of the fluid. In this work, the diffusible property is assumed to follow the fluid exactly, that is, the Lagrangian particle velocity correlation is assumed to be the same as the fluid's. Although this assumption makes it easier to relate the dispersion of the particles to the turbulence of the fluid, the Lagrangian fluid velocity correlation is unknown. Much of the theoretical work following Taylor's analysis focused on methods of relating the statistical properties of turbulent flow in an Eulerian frame of reference to a Lagrangian one (Corrsin 1959; Saffman 1962; Kraichnan 1970; Lundgren & Pointin 1976).

Theoretical investigations of particle dispersion rely on a model of the force imposed on the particle by the fluid. A linear Stokes force has been used to successfully predict particle motion in turbulent flow by several investigators (Pismen & Nir 1978; Kallio & Reeks 1989; Shih & Lumley 1986; Underwood 1993). Drag relations based on empirical correlations have also been used (see Sommerfeld 1990; Call & Kennedy 1992; Yarin & Hetsroni 1994; Mei 1994), as well as a modified Stokes force that takes into account particle oscillations (for example, Lee & Durst 1982; Hinze 1975).

Calculation of particle dispersion also requires a model of fluid turbulence that relates the measurable Eulerian statistics to the Lagrangian statistics. Severals models have been proposed in the literature. In many studies of particle dispersion, the form of the Lagrangian correlation is assumed to be the same as the Eulerian correlation (e.g. Taylor 1921; Kaplan & Dinar 1988), and only the Lagrangian integral scales of the turbulence are related to the Eulerian statistics. For example, Shuen, Chen & Faeth (1983), Boysan, Ayers & Swithenbank (1982) and Gosman & Ionnides (1981) proposed direct relations between the Lagrangian integral time scale of the velocity correlation and the kinetic energy, k, and dissipation of kinetic energy, ϵ, of the turbulence. Such relations have been widely applied to random-walk models of particle dispersion, and to particle dispersion in non-homogeneous flows, including flows calculated using k–ϵ-numerical models of turbulent fluids. A summary of the proposed relations is given by MacInnes & Bracco (1992). Kraichnan (1970) suggested a turbulence model based on a series of random Fourier modes, chosen such that the imposed energy spectrum matches experimental results. Reeks (1977), using a linear Stokes force and Phythian's (1975) approximation of Kraichnan's turbulence model, looked at the effect of inertia and gravity on dispersion. He showed that, as particle inertia increased, the Lagrangian particle velocity correlation went from the Lagrangian fluid velocity correlation to the Eulerian fluid velocity correlation. Pismen

& Nir (1978) examined the effect of particle inertia using a linear drag function and the direct interaction approximation of Kraichnan's model turbulence. Their results were similar to Reeks'. Recently, direct numerical simulation of the Navier–Stokes equations has been used to study particle dispersion (McLaughlin 1994). Although this technique is potentially powerful, it is currently only applicable to low Reynolds number turbulence in simple flow conditions. Unfortunately, a universally applicable relationship between Lagrangian and Eulerian statistics remains elusive.

Although there is a large body of scientific literature on particle motion in turbulent flow, there are few studies directly relevant to highly elongated, orientable, particles (fibres) in turbulent flow. One of the first on fibre motion in turbulent flow was by Cho, Iribane & Richard (1980) who investigated the effect of atmospheric turbulence on the preferred settling orientation of high aspect ratio ice crystals (fibres), in cumulonimbus clouds. After estimating the time for a fibre to become oriented and the time between eddy interactions, they concluded that atmospheric turbulence does not greatly affect the mean particle orientation. Other studies of turbulent fibre motion include glass fibre deposition onto human lung trachea (Asgharian 1988, 1989).

Kagermann & Köhler (1984) modelled turbulent translational and orientational dispersion assuming Stokes drag, Jeffery's rotation equation, and Kraichnan's turbulence model. Inherent in these simplifications is the assumption that the particle is smaller than the smallest scales of turbulence. The orientation distribution function of small fibres in turbulent flow has also been theoretically calculated by Krushkal & Gallily (1988) using the Fokker–Planck equation. A key parameter in this equation was the rotational Péclet number, $\mathscr{P}_e$, which is the ratio of a typical velocity gradient, $\mathrm{d}U/\mathrm{d}\xi$, to the rotational dispersion coefficient, D_p, i.e. $\mathscr{P}_e$ is the ratio of orienting to randomizing effects, and is given by

$$\mathscr{P}_e = \frac{\mathrm{d}U/\mathrm{d}\xi}{D_p}. \tag{1.3}$$

The dispersion coefficient was related to the turbulent dissipation, ϵ, and the kinematic viscosity, v, by dimensional analysis and the Kolmogorov local isotropy hypothesis for small eddies, as

$$D_p = (\epsilon/v)^{1/2}. \tag{1.4}$$

This was related to the fluctuating component of the fluids velocity, u_i, through the relationship

$$\epsilon/v = \sum_{ij} \overline{\left(\frac{\mathrm{d}u_i}{\mathrm{d}\xi_j}\right)^2}. \tag{1.5}$$

Recently, the axial and lateral orientation of glass fibres in turbulent and laminar pipe flow has been directly measured (Bernstein & Shapiro 1994). The flow was characterized by its Reynolds number and the rotational Péclet number as defined above. It was found that near the pipe centre, for low Reynolds number laminar flow, the axial fibre orientation distribution is broad, with no preferred orientation. As Reynolds number increased within the laminar regime, consequently increasing the Péclet number, the axial orientation distribution became sharper. In high Reynolds number, low Péclet number, turbulent flow the randomizing component of the microturbulence created an almost uniform orientation distribution.

The principal application of a model of turbulent translational dispersion is to calculate the effect of turbulence on particle concentration. There are two approaches to solving such a problem: the Eulerian approach and the Lagrangian approach.

In the Eulerian approach, the dispersed particles are treated as a continuum and turbulent dispersion is described by Fick's diffusion equation, analogous to molecular diffusion. This method has been used to calculate particle concentration for several flow conditions, assuming an appropriate dispersion coefficient, for example Hinze (1975); Hishida, Ando & Maeda (1992). In the Lagrangian approach, the trajectory of a single particle is calculated by solving the particles equations of motion, assuming a known turbulent fluid velocity field. Particle concentration is estimated from the statistics of a large number of trajectories. This method has been used by many investigators, for example Lu, Fontaine & Aubertain (1993); MacInnes & Bracco (1992); Call & Kennedy (1992). A comparison of the advantages and disadvantages of the two approaches is given by Durst, Milojevic & Schonung (1984). Both approaches have been utilized to calculate particle concentration throughout the suspension for several cases of simple and complex flows. For fibres, calculating the fibre orientation distribution is similar to calculating the particle concentration. Krushkal & Gallily (1988) calculated the fibre orientation distribution, using the Fokker–Planck equation and a turbulent orientation dispersion coefficient, for two flow conditions: the free circular turbulent jet and the lower atmospheric boundary layer.

Dispersed particles in turn may affect the turbulent properties of the carrier fluid. Such particle–turbulence interaction is a highly complex phenomenon that is governed by many factors, including: the size distribution of the particles, the particle concentration, the nature of the fluid flow, and the length scales of the turbulence. From experimental studies (Gore & Crowe 1989), small particles decrease the turbulent intensity of the fluid, probably by increasing the apparent viscosity of the fluid. Larger particles, greater than some critical particle Reynolds number, increase the intensity of the turbulence, perhaps due to vortex shedding. Both mechanisms are strongly affected by particle concentration. However, for the case of a single particle in suspension in our study, we assume that the turbulence of the fluid is not significantly affected.

In this study, we derive the equations of motion of a single fibre suspended in a turbulent fluid, neglecting Brownian effects, in terms of the Lagrangian particle velocity correlation, similar to Taylor (1921). The mean and fluctuating components of motion are given. Translational and orientational dispersion is characterized by a dispersion coefficient, and the dependence on fibre length is discussed.

2. Equations of motion

The derivation of the equations of fibre motion in a turbulent flow requires a model of the force imposed on the fibre by the fluid. Unfortunately, there is no slender body theory that is strictly valid for the high Reynolds number turbulent flow of interest here. As a necessary simplification, the form of the force on the fibre under creeping flow conditions, given by (1.2), is assumed to be retained for higher Reynolds number flows. However, (1.2) was derived for a small Reynolds number flow where Reynolds number is based on fibre length, L. Therefore, (1.2) is only strictly valid for infinitely thin fibres with L less than the Kolmogorov length scale, η, of the turbulence. Equation (1.2) is applied to longer fibres suspended in turbulent flow, by imposing the free-draining approximation used to model flexible fibre motion (Ross & Klingenberg 1997), and to model polymer dynamics (Doi & Chen 1989; Doi & Edwards 1988). Hence, the fibre is considered to be composed of a series of elements Δ_l long, where $\Delta_l < \eta$, and each element is assumed to be hydrodynamically independent. In this model of a fibre, each element meets the necessary conditions for

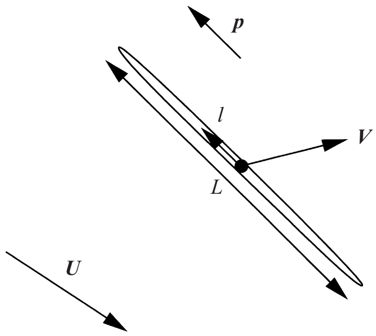

FIGURE 1. A straight rigid fibre of length L, moving with an average velocity $\boldsymbol{V}$, in a turbulent fluid with velocity $\boldsymbol{U}$. The fibre is pointing in the direction given by unit vector $\boldsymbol{p}_f$.

(1.2) to be valid, and assuming hydrodynamic independence of each element allows (1.2) to be applied to all elements, thus the entire fibre.

Assuming a straight rigid fibre, the velocity at any point along the fibre, $\boldsymbol{V}_f$, will be the sum of the fibre translational velocity, $\boldsymbol{V}$, and rotational velocity, $l\dot{\boldsymbol{p}}_f$, where $\dot{\boldsymbol{p}}_f$ is the time derivative of a unit vector parallel to the fibre's major axis, and l is the distance from the fibre centre (see figure 1). The net external force, $\boldsymbol{F}$, on the particle is obtained by integrating (1.2) along the length of the fibre, given by

$$\boldsymbol{F} = \int_{-L/2}^{L/2} \mathbf{D}[\boldsymbol{U}(l) - (\boldsymbol{V} + l\dot{\boldsymbol{p}}_f)]\mathrm{d}l. \tag{2.1}$$

The net external moment acting on the fibre is similarly given by

$$\boldsymbol{M} = \int_{-L/2}^{L/2} l\boldsymbol{p}_f \times \mathbf{D}[\boldsymbol{U}(l) - (\boldsymbol{V} + l\dot{\boldsymbol{p}}_f)]\mathrm{d}l. \tag{2.2}$$

Assuming the fibre is neutrally buoyant, and inertial forces are negligible in generating relative velocities between the particle and fluid results in $\boldsymbol{F} = 0$ and $\boldsymbol{M} = 0$. For this condition,

$$\boldsymbol{V} = \frac{1}{L}\int_{-L/2}^{L/2} \boldsymbol{U}(l)\mathrm{d}l \tag{2.3}$$

and

$$\dot{\boldsymbol{p}}_f = \frac{12}{L^3}\int_{-L/2}^{L/2} l\boldsymbol{U}(l)\mathrm{d}l. \tag{2.4}$$

2.1. *The equations of mean motion*

In a turbulent flow field, the fluid velocity consists of a time-averaged or mean component, $\bar{\boldsymbol{U}}$, and a fluctuating component, $\boldsymbol{u}$. The total fluid velocity is given by

$$\boldsymbol{U} = \bar{\boldsymbol{U}} + \boldsymbol{u}. \tag{2.5}$$

The mean motion of the fibres is obtained by substituting (2.5) into (2.3) and (2.4), and averaging. The equations of mean fibre motion are given by

$$\bar{\boldsymbol{V}} = \frac{1}{L}\int_{-L/2}^{L/2} \bar{\boldsymbol{U}}(l)\mathrm{d}l \tag{2.6}$$

and

$$\dot{\bar{\boldsymbol{p}}} = \frac{12}{L^3}\int_{-L/2}^{L/2} l\bar{\boldsymbol{U}}(l)\mathrm{d}l. \tag{2.7}$$

Similarly, the fibre's translational velocity and rotational velocity will have mean and fluctuating components, i.e. $\boldsymbol{V} = \bar{\boldsymbol{V}} + \boldsymbol{v}$ and $\boldsymbol{p}_f = \bar{\boldsymbol{p}} + \boldsymbol{p}$.

The mean fibre velocity is the average of the mean fluid velocity field, averaged over the length of the fibre. The fibre's mean rotational velocity is the first moment of the mean fluid velocity, taken over the length of the fibre. These equation have been derived for laminar flow by several investigators (Riese 1969; Shanker *et al.* 1991; Pittman & Kasiri 1992; Tangsaghasaksri 1994; Olson 1996).

2.2. *The equations of turbulent motion*

The turbulent motion of the fluid causes the fibre velocity and orientation to fluctuate in a random manner, leading to dispersion. The fibre translational and orientational dispersion, based on the equations of motion, are derived below.

2.2.1. *Translational dispersion*

Substituting (2.5) into the equation of translational fibre motion (2.3), and changing the limits of integration, the fluctuating component of the fibre's translational velocity, v_i, is given by

$$v_i = \frac{1}{L}\int_0^L u_i(l)\mathrm{d}l. \tag{2.8}$$

Following a single particle as it moves through the carrier fluid (i.e. using a Lagrangian particle frame of reference) the position, x_i, of the particle as a function of time is given by

$$x_i = \int_0^t v_i \mathrm{d}t = \frac{1}{L}\int_0^t \int_0^L u_i(l,t')\mathrm{d}l\mathrm{d}t'. \tag{2.9}$$

The magnitude of the particle's mean-square displacement is calculated from (2.9) as

$$\overline{x_i x_j} = \overline{\frac{1}{L}\int_0^t \int_0^L u_i(l,t')\mathrm{d}l\mathrm{d}t' \frac{1}{L}\int_0^t \int_0^L u_j(l',t'')\mathrm{d}l'\mathrm{d}t''}. \tag{2.10}$$

The above equation is related to the turbulent velocity correlation following the analysis of Taylor (1921) for a tracer particle or fluid element, but extended to account for fibres with lengths that extend through the fluid. Simplifying the above yields

$$\overline{x_i x_j} = \frac{1}{L^2}\int_0^t \int_0^t \int_0^L \int_0^L \overline{u_i(l,t')u_j(l',t'')}\mathrm{d}l\mathrm{d}l'\mathrm{d}t'\mathrm{d}t''. \tag{2.11}$$

This equation is further simplified by first considering the spatial integrals and realizing that for the same random function (i.e. $i = j$) the integrand is symmetric over the area of integration defined by the square $[0, L]$ and $[0, L]$. Therefore, integration over the square is equivalent to twice the integration over a triangular region, depicted by

$$\int_0^L \int_0^L \mathrm{d}l\mathrm{d}l' = 2\int_0^L \int_0^l \mathrm{d}l'\mathrm{d}l. \tag{2.12}$$

Doing this for both time and space results in

$$\overline{x_i^2} = \frac{4}{L^2}\int_0^t \int_0^{t'} \int_0^L \int_0^l \overline{u_i(l,t')u_i(l',t'')}\mathrm{d}l'\mathrm{d}l\mathrm{d}t''\mathrm{d}t'. \tag{2.13}$$

Here the summation notation for repeated indices is not used. Introducing a change of variables $l' = l + \delta$ and $t'' = t' + \tau$ the above becomes

$$\overline{x_i^2} = \frac{4}{L^2}\int_0^t \int_{-t'}^{0} \int_0^L \int_{-l}^0 \overline{u_i(l,t')u_i(l+\delta,t'+\tau)}\mathrm{d}\delta\mathrm{d}l\mathrm{d}\tau\mathrm{d}t'. \tag{2.14}$$

The integrand equals the particle's Lagrangian velocity correlation function, $\mathscr{R}$, multiplied by the square of the fluctuating component of the fluid. When averaged over l and t' for steady, homogeneous turbulence, $\mathscr{R}$ is a function of τ and δ, and is independent of l and t', that is

$$\frac{\overline{u_i(l,t')u_i(l+\delta,t'+\tau)}}{\overline{u_iu_i}} = \mathscr{R}_{ii}(\delta,\tau). \tag{2.15}$$

In general, the spatial $\mathscr{R}_{ii}$ is a function of the spatial vector separating the points of velocity measurement, that is $\mathscr{R} = \mathscr{R}_{ii}(l\boldsymbol{p})$. Assuming the fibre is randomly oriented, a simplifying assumption is introduced that the fluid velocity correlation in the direction of the fibre axis (longitudinal) and perpendicular to the fibre axis (transverse) is identical. This is not strictly true, with classic theoretical relationships existing between the two correlations (for example, Hinze 1975).

Noting that $\mathscr{R}_{ii}(\delta,\tau)$ is symmetric about the origin, (2.14) becomes

$$\overline{x_i^2} = \overline{u_i^2}\frac{4}{L^2}\int_0^t \int_0^{t'} \int_0^L \int_0^l \mathscr{R}_{ii}(\delta,\tau)\mathrm{d}\delta\mathrm{d}l\mathrm{d}\tau\mathrm{d}t'. \tag{2.16}$$

The spatial and temporal integrals, in the above equation, are further simplified by integrating each pair by parts. First the spatial integrals are simplified:

$$\overline{x_i^2} = \overline{u_i^2}\frac{4}{L^2}\int_0^t \int_0^{t'} \left[l\int_0^l \mathscr{R}_{ii}(\delta,\tau)\mathrm{d}\delta \Big|_0^L - \int_0^L l\mathscr{R}_{ii}(l,\tau)\mathrm{d}l \right]\mathrm{d}\tau\mathrm{d}t'. \tag{2.17}$$

Changing variables in the second part of the integration, from l to δ, and rearranging slightly yields

$$\overline{x_i^2} = \overline{u_i^2}\frac{4}{L^2}\int_0^t \int_0^{t'} \int_0^L (L-\delta)\mathscr{R}_{ii}(\delta,\tau)\mathrm{d}\delta\mathrm{d}\tau\mathrm{d}t'. \tag{2.18}$$

The last three steps are repeated for the temporal integrals to arrive at our final result of

$$\overline{x_i^2} = \overline{u_i^2}\frac{4t}{L}\int_0^t \int_0^L (1-\tau/t)(1-\delta/L)\mathscr{R}_{ii}(\delta,\tau)\mathrm{d}\delta\mathrm{d}\tau. \tag{2.19}$$

Following the above procedure, it is relatively straightforward to show that the magnitude of the velocity fluctuations of the fibre is given by

$$\overline{v_i^2} = \overline{u_i^2}\frac{2}{L}\int_0^L (1-\delta/L)\mathscr{R}_{ii}(\delta)\mathrm{d}\delta. \tag{2.20}$$

This equation demonstrates that the magnitude of the fluctuations of the fibre will be less than the fluctuations of the fluid, and will diminish with increasing fibre length.

2.2.2. *Rotational dispersion*

The rotational dispersion of the fibre is calculated following the same procedure as the translational dispersion. From (2.7) the fluctuating component of the rotational velocity, $\dot{p}_i$, is given by

$$\dot{p}_i = \frac{12}{L^3}\int_{-L/2}^{L/2} l u_i(l)\mathrm{d}l. \tag{2.21}$$

The mean square of the rotational velocity is

$$\overline{\dot{p}_i\dot{p}_j} = \overline{\frac{12}{L^3}\int_{-L/2}^{L/2} l u_i(l)\mathrm{d}l \frac{12}{L^3}\int_{-L/2}^{L/2} l u_j(l)\mathrm{d}l} \tag{2.22}$$

or

$$\overline{\dot{p}_i\dot{p}_j} = \frac{144}{L^6}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} l l' \overline{u_i(l)u_j(l')}\mathrm{d}l'\mathrm{d}l. \tag{2.23}$$

As before, the integrand is symmetric over the area of integration defined by the square $[-L/2, L/2]$ and $[-L/2, L/2]$. Therefore, the integration over the square is equivalent to twice the integration over a triangle, i.e.

$$\overline{\dot{p}_i{}^2} = \frac{288}{L^6}\int_{-L/2}^{L/2}\int_{-L/2}^{l} l l' \overline{u_i(l)u_i(l')}\mathrm{d}l'\mathrm{d}l. \tag{2.24}$$

The same change of variables, $l' = l + \delta$, is introduced and

$$\overline{\dot{p}_i{}^2} = \frac{288}{L^6}\int_{-L/2}^{L/2}\int_{-(l+L/2)}^{0} l(l+\delta)\overline{u_i(l)u_i(l+\delta)}\mathrm{d}\delta\mathrm{d}l. \tag{2.25}$$

The velocity correlation, $\mathscr{R}$, is again defined as

$$\frac{\overline{u_i(l)u_i(l+\delta)}}{\overline{u_i u_i}} = \mathscr{R}_{ii}(\delta) \tag{2.26}$$

and substituting for the correlation function results in

$$\overline{\dot{p}_i{}^2} = \overline{u_i^2}\frac{288}{L^6}\int_{-L/2}^{L/2}\int_{-(l+L/2)}^{0} l(l+\delta)\mathscr{R}_{ii}(\delta)\mathrm{d}\delta\mathrm{d}l. \tag{2.27}$$

Integrating and simplifying results in the following expression for the mean-square fluctuating angular velocity:

$$\overline{\dot{p}_i{}^2} = \overline{u_i^2}\frac{24}{L^3}\int_0^L \left(1 - 3\frac{\delta}{L} + 2\left(\frac{\delta}{L}\right)^3\right)\mathscr{R}_{ii}(\delta)\mathrm{d}\delta. \tag{2.28}$$

As with the translational dispersion, the angular velocity decreases with increasing length. Integrating with respect to time results in the following expression for the orientation dispersion:

$$\overline{p_i{}^2} = \overline{u_i^2}\frac{24}{L^3}\int_0^t\int_0^L 2(t-\tau)\left(1 - 3\frac{\delta}{L} + 2\left(\frac{\delta}{L}\right)^3\right)\mathscr{R}_{ii}(\delta,\tau)\mathrm{d}\delta\mathrm{d}\tau. \tag{2.29}$$

Of course, this expression is only strictly valid for small orientational dispersion, i.e. $\overline{p_i{}^2} \ll 1$. A more general expression for orientation dispersion will be discussed later.

2.2.3. *Dispersion coefficient*

Particle transport by turbulent dispersion is often approximated as a gradient-driven process with the mean flux of a fluctuating quantity related to the gradient of the mean quantity by a dispersion coefficient. The dispersion coefficient for both fibre translation and orientation is related to the mean-square displacements, derived in the previous section, following the analysis of Hinze (1975).

Assuming the concentration of fibres, C, has a mean, $\bar{C}$, and fluctuating component, c, that is, $C = \bar{C} + c$, and the particles have a fluctuating velocity, v, the concentration flux, $\mathscr{F}$, due to the turbulent fluctuations is given by

$$\mathscr{F} = \overline{vc}. \tag{2.30}$$

The fluctuating component of concentration is approximately related to the mean gradient of the concentration by

$$c = -x\frac{\mathrm{d}\bar{C}}{\mathrm{d}x} \tag{2.31}$$

where x is the displacement of the particles during a fluctuation. The velocity of the particle is then the time derivative of the displacement, i.e.

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}. \tag{2.32}$$

Therefore, the average particle flux, $\mathscr{F}$, is given as

$$\mathscr{F} = -\overline{x\frac{\mathrm{d}\bar{C}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}} \tag{2.33}$$

which is equal to

$$\mathscr{F} = -\frac{1}{2}\frac{\mathrm{d}\overline{x^2}}{\mathrm{d}t}\frac{\mathrm{d}\bar{C}}{\mathrm{d}x}. \tag{2.34}$$

By definition, the flux is proportional to the concentration gradient multiplied by the dispersion coefficient, i.e.

$$\mathscr{F} = -D_t\frac{\mathrm{d}\bar{C}}{\mathrm{d}x}. \tag{2.35}$$

Thus,

$$D_t = \frac{1}{2}\frac{\mathrm{d}\overline{x^2}}{\mathrm{d}t}. \tag{2.36}$$

Although the dispersion coefficient, D_t, is properly defined as a tensor, here isotropy is assumed and it is defined as a scalar property. Substituting (2.19) into (2.36), results in

$$D_t = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{u^2}\frac{4}{L^2}\int_0^t\int_0^L (t-\tau)(L-\delta)\mathscr{R}(\delta,\tau)\mathrm{d}\delta\mathrm{d}\tau\right). \tag{2.37}$$

By replacing concentration with the orientation distribution function, and the velocity of the fibre with the rotational velocity, the orientation dispersion coefficient, D_p, is similarly derived, and is given by

$$D_p = \frac{1}{2}\frac{\mathrm{d}\overline{p^2}}{\mathrm{d}t}. \tag{2.38}$$

Substituting (2.29) into the above yields

$$D_p = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{u^2}\frac{24}{L^3}\int_0^t\int_0^L 2(t-\tau)\left(1-3\frac{\delta}{L}+2\left(\frac{\delta}{L}\right)^3\right)\mathscr{R}(\delta,\tau)\mathrm{d}\delta\mathrm{d}\tau\right). \tag{2.39}$$

3. Discussion

The mean and fluctuating equations of turbulent fibre motion are examined separately and compared with previous work.

3.1. *Mean motion*

In the limit as fibre length decreases to zero ($L \to 0$), the equation of fibre translation (2.6) is identical to that of the fluid, i.e.

$$\bar{V}_i = \bar{U}_i. \tag{3.1}$$

Thus, short, inertialess fibres act as tracer particles and follow the fluid streamlines. Also, centres of fibres of any length will follow the fluid streamlines in a linear shear field.

In linear shear, the equation of rotation, given by (2.7), is identical to Jeffery's equation of rotation in the limit as aspect ratio goes to infinity. This can be easily shown in two dimensions. A general flow field with linear velocity gradients is given by

$$U_i = U_i|_0 + \left.\frac{\partial U_i}{\partial \xi_j}\right|_0 \xi_j. \tag{3.2}$$

In a plane, $\xi_1 = l\cos\theta$ and $\xi_2 = l\sin\theta$. Substituting this into (2.7) and integrating yields

$$\dot{\theta} = \frac{1}{2}\sin(2\theta)\left(\frac{\partial U_2}{\partial \xi_2} - \frac{\partial U_1}{\partial \xi_1}\right) + \cos^2(\theta)\frac{\partial U_2}{\partial \xi_1} - \sin^2(\theta)\frac{\partial U_1}{\partial \xi_2}. \tag{3.3}$$

By inspection we see that this equation is identical to Jeffery's equation of rotation given by (1.1), in the limit as $\alpha \to \infty$.

The difference between the two theoretical expressions for rotational velocity, (1.1) and (3.3), for a finite value of α, is that Cox's (1970) first-order approximation neglects the moment over the fibre width. Cox states that third-order terms in his expansion are required before any moment from the fibre width is retained; however, these third-order terms are not given in his analysis. The result of using Cox's first-order approximation is that fibres no longer rotate in classical orbits, but rotate until the fibre is parallel to the fluid streamlines, at which point the rotational velocity becomes zero. The advantage of (2.7) over the corresponding equation of Jeffrey is that the former is generalized to velocity fields that are nonlinear over the length of the fibre.

3.2. *Turbulent motion*

To illustrate the theoretically-predicted effect of length on fibre dispersion in (2.19) and (2.29), the particle's Lagrangian velocity correlation function must be modelled. This is discussed below, followed by a discussion of dispersion of short fibres and of fibres with lengths comparable to the integral length scale of turbulence.

3.2.1. *Lagrangian particle velocity correlation*

As with the problem of Taylor's (1921) closure assumptions, our derivation is in terms of the unknown Lagrangian particle velocity correlation. It is well known, experimentally (Snyder & Lumley 1971) and theoretically (Reeks 1977), that particle inertia and external forces affect the Lagrangian particle velocity correlation function.

The Lagrangian particle velocity correlation will be a function of fibre length. In the limit as fibre length approaches zero, the fibre follows the fluid exactly, the equation of translational dispersion is identical to that proposed by Taylor (1921), and the Lagrangian particle velocity correlation is identical to the Lagrangian fluid velocity correlation. As fibre length increases, the fibre extends further through the fluid and the fluctuations along the length of the fibre begin to cancel each other. This results in the fibre's fluctuations becoming increasingly damped. In the limit of an infinitely long fibre, all the fibre fluctuations are damped out and fibre motion is governed by the mean fluid motion. Therefore, the Lagrangian particle velocity correlation is the same as the fluid's Eulerian velocity correlation. For the intermediate case, the Lagrangian particle velocity correlation will be between the Lagrangian and Eulerian correlations of the fluid. This is similar to the transition from Lagrangian to Eulerian correlations for small particles as a function of increasing inertia (Reeks 1977; Pismen & Nir 1978).

In this work, we relate fibre dispersion to the integral temporal and spatial scales of the turbulence. To do so, we refer to the Lagrangian particle integral time scale, $\mathscr{T}$, realizing it is bracketed between the Lagrangian fluid integral time scale for short fibres, and the Eulerian integral time scale in the limit of long fibres. If the turbulence is ergodic, and the fibres are randomly oriented and positioned in the fluid, then averaging the velocity along the length of a fibre, over an ensemble of fibres, results in the fluid's spatial Eulerian velocity correlation. Therefore, the velocity correlation along the length of the fibre is approximated as the fluid Eulerian spatial velocity correlation, with the corresponding Eulerian integral length scale.

3.2.2. *Short fibre dispersion*

The Lagrangian particle velocity correlation function is approximated assuming that the spatial correlation and the temporal correlation are statistically independent. This enables the combined spatial and temporal correlation to be expressed as

$$\mathscr{R}(\delta,\tau) = \mathscr{R}_\delta(\delta)\mathscr{R}_\tau(\tau) \tag{3.4}$$

where $\mathscr{R}_\delta(\delta)$ is approximately the same as the fluid's Eulerian velocity correlation and $\mathscr{R}_\tau(\tau)$ is the particle's Lagrangian temporal velocity correlation. Using this simplification, we estimate the translational and orientational dispersion of fibres with lengths less than the scale of turbulence. Taking the limit as $L \to 0$ of (2.19) yields

$$\overline{x^2} = 2\overline{u^2}\int_0^t (t-\tau)\mathscr{R}(\tau)\,\mathrm{d}\tau \tag{3.5}$$

which is identical to the expression given by Taylor for dispersion of a passive scalar. For infinitely small particles, it is also assumed that the particle's temporal Lagrangian velocity correlation is the same as the fluids. As t gets large with respect to the Lagrangian time scale (i.e. $t \to \infty$), the translational dispersion becomes

$$\overline{x^2} = 2\overline{u^2}\mathscr{T}t. \tag{3.6}$$

Substituting the above into the equation for the translational dispersion coefficient, (2.36), results in the classical Taylor dispersion coefficient

$$D_t = \overline{u^2}\mathscr{T}. \tag{3.7}$$

The orientation dispersion in the limit as L goes to zero is calculated by first taking a Taylor series expansion of the correlation coefficient, with respect to its spatial variable, i.e.

$$\mathscr{R}(\delta) = 1 + \frac{1}{2}\frac{\mathrm{d}^2\mathscr{R}(\delta)}{\mathrm{d}^2\delta}\bigg|_{\delta=0}\delta^2 + \ldots \tag{3.8}$$

where

$$-\frac{1}{2}\frac{\mathrm{d}^2\mathscr{R}(\delta)}{\mathrm{d}^2\delta} = \frac{1}{\lambda^2} \tag{3.9}$$

and λ is the Taylor micro-length scale, or dissipation scale, of the turbulence, which represents the average dimension of the smallest eddies (Hinze 1975). This series expansion is then substituted into (2.29) and taking the limit as $L \to 0$ results in the following expression for the dispersion of small fibres as a function of time:

$$\overline{p^2} = 2\frac{\overline{u^2}}{\lambda^2}\int_0^t 2(t-\tau)\mathscr{R}(\tau)\mathrm{d}\tau. \tag{3.10}$$

The long time dispersion ($t \to \infty$) is then

$$\overline{p^2} = 4\frac{\overline{u^2}}{\lambda^2}\mathscr{T}t. \tag{3.11}$$

Substituting the above into the equation for the orientation dispersion coefficient, (2.38), results in the following expression:

$$D_p = 2\frac{\overline{u^2}}{\lambda^2}\mathscr{T}. \tag{3.12}$$

This expression is compared with the rotational dispersion coefficient of Krushkal & Gallily (1988), derived by dimensional analysis (equation (1.4)). The total kinetic energy dissipation is given by

$$\epsilon = 15\nu\frac{\overline{u^2}}{\lambda^2} \tag{3.13}$$

or

$$\frac{\epsilon}{\nu} \propto \frac{\overline{u^2}}{\lambda^2} \tag{3.14}$$

where ν is the kinematic viscosity. Substituting the above equation and (1.4) into (3.12) results in

$$\mathscr{T} \propto \frac{\lambda}{(\overline{u^2})^{1/2}}. \tag{3.15}$$

That is, the dispersion coefficient of Krushkal & Gallily makes the approximation that the Lagrangian integral time scale is proportional to the Taylor micro-length scale.

3.2.3. *Long fibre dispersion*

As fibre length increases, the velocity fluctuations and dispersion are diminished. To illustrate the effect of fibre length on fibre dispersion an approximate equation for the velocity correlation, $\mathscr{R}(\delta, \tau)$, is substituted into (2.19) and (2.29) and integrated to obtain dispersion as a function of time.

The temporal correlation is approximated by a negative exponential function which has been shown to reasonably reflect experimental data (MacInnes & Bracco 1992; Hinze 1975). The spatial correlation is approximated by a Gaussian function, also

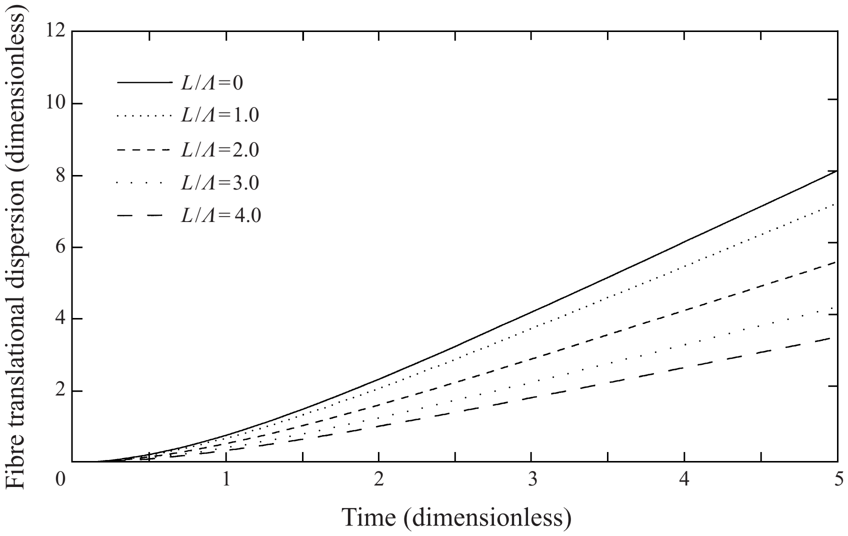

FIGURE 2. The dimensionless dispersion, $\overline{x^2}/\overline{u^2}\mathscr{T}^2$, plotted against dimensionless time, $t/\mathscr{T}$, for values of dimensionless fibre length L/Λ equal to 0, 1.0, 2.0, 3.0 and 4.0.

used as a model velocity correlation (Hinze 1975). The Gaussian function is chosen as a model over the exponential function, for the spatial correlation, because it is symmetric, with a zero first derivative at $\delta = 0$. A correlation function with a non-zero first derivative at the origin results in an infinite orientation dispersion in the limit as fibre length goes to zero.

The spatial and temporal correlations are given as functions of their integral length and time scales, Λ and $\mathscr{T}$, respectively, by

$$\mathscr{R}_\delta(\delta) = \exp\left(\frac{-\pi\delta^2}{4\Lambda^2}\right) \tag{3.16}$$

and

$$\mathscr{R}_\tau(\tau) = \exp\left(\frac{-\tau}{\mathscr{T}}\right). \tag{3.17}$$

Using these model equations the translational and orientation dispersion as a function of length and time are given as

$$\overline{x^2} = 4\overline{u^2}\mathscr{T}^2\frac{\Lambda}{L}\left(\mathrm{erf}\left(\frac{\pi^{1/2}L}{2\Lambda}\right) + \frac{2}{\pi}\frac{\Lambda}{L}\left(\mathrm{e}^{-\pi L^2/4\Lambda^2} - 1\right)\right)\left(\frac{t}{\mathscr{T}} + \mathrm{e}^{-t/\mathscr{T}} - 1\right) \tag{3.18}$$

and

$$\overline{p^2} = 48\overline{u^2}\frac{\mathscr{T}^2}{L^2}\frac{\Lambda}{L}\left(\mathrm{erf}\left(\frac{\pi^{1/2}L}{2\Lambda}\right) + \frac{16}{\pi^2}\left(\frac{\Lambda}{L}\right)^3\left(1 - \mathrm{e}^{-\pi L^2/4\Lambda^2}\right) + \frac{2}{\pi}\frac{\Lambda}{L}\left(\mathrm{e}^{-\pi L^2/4\Lambda^2} - 3\right)\right) \times\left(\frac{t}{\mathscr{T}} + \mathrm{e}^{-t/\mathscr{T}} - 1\right). \tag{3.19}$$

The expression for the translational dispersion (3.18) is plotted in figure 2 for values of L/Λ equal to 0, 1.0, 2.0, 3.0 and 4.0. From this plot, it is evident that the effect of fibre length on dispersion can be considerable.

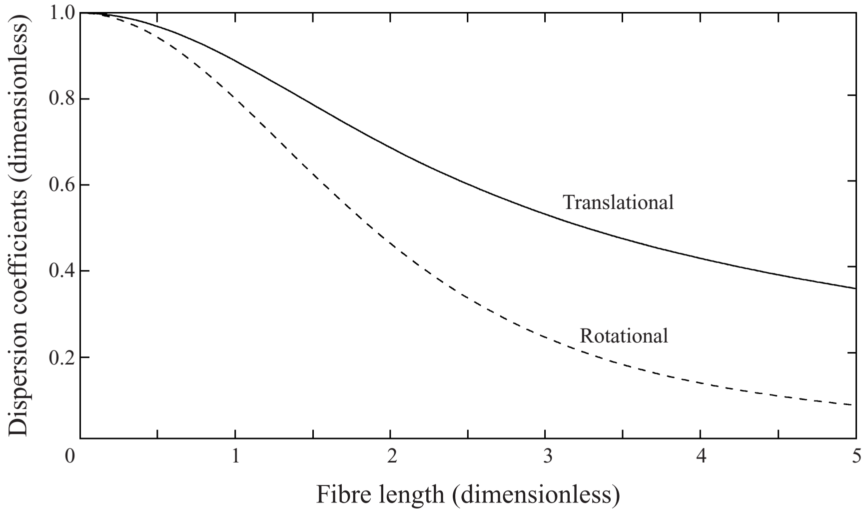

FIGURE 3. The dimensionless dispersion coefficients, $D(L)/D(0)$, as a function of fibre length normalized by the integral length scale, L/Λ.

The length dependence on dispersion is best illustrated by long-time translation and orientation dispersion coefficients, given by

$$D_t = 2\overline{u^2}\mathscr{T}\frac{\Lambda}{L}\left(\operatorname{erf}\left(\frac{\pi^{1/2}L}{2\Lambda}\right) + \frac{2}{\pi}\frac{\Lambda}{L}\left(\mathrm{e}^{-\pi L^2/4\Lambda^2} - 1\right)\right) \tag{3.20}$$

and

$$D_p = 24\overline{u^2}\frac{\mathscr{T}}{L^2}\frac{\Lambda}{L}\left(\operatorname{erf}\left(\frac{\pi^{1/2}L}{2\Lambda}\right) + \frac{16}{\pi^2}\left(\frac{\Lambda}{L}\right)^3\left(1 - \mathrm{e}^{-\pi L^2/4\Lambda^2}\right) + \frac{2}{\pi}\frac{\Lambda}{L}\left(\mathrm{e}^{-\pi L^2/4\Lambda^2} - 3\right)\right). \tag{3.21}$$

The dispersion coefficients are normalized by the dispersion of a zero length fibre, and plotted as a function of L/Λ in figure 3. From this figure, it is evident that fibre dispersion is dramatically damped as fibre length increases beyond the integral length scale of the turbulence. The effect of length is more pronounced for rotational dispersion, with the long-time dispersion coefficient nearly zero at $L = 5\Lambda$.

The dissipation scale, λ, of the model spatial correlation function used to calculate the long-time dispersion coefficients is related to the integral length scale as $\lambda = 2\Lambda/\pi^{1/2}$. This is unusual, in that the dissipation scale is greater than the integral length scale. This is not the case and is an artifact of the model correlation function. A smaller dissipation scale with respect to the integral length scale would result in a greater dampening of the orientation dispersion coefficient relative to the translational dispersion coefficient than is indicated in figure 3.

4. Application

An important application of the equations of mean and fluctuating fibre motion is the calculation of the spatial and orientational distribution of fibres in a dilute turbulent suspension, especially in the industrial papermaking process, where the spatial and orientational distribution of fibres greatly affects the physical properties of the fibre suspension and final paper product. These distributions can be determined by considering either a Lagrangian problem, where fibre trajectories are calculated by integrating the equations of motion directly, or by considering an Eulerian problem where the fibre concentration changes at each point in the fluid is considered. Both rely on the equations of motion for a fibre.

In this section, we demonstrate the application of the equations of motion derived previously, by considering an Eulerian approach. Following Doi & Edwards (1989), the convection-dispersion equation for fibre position and orientation is given and solved for a few highly simplified cases. The approach is general and more complicated applications could be considered; however, the solutions would most likely require numerical treatment.

The number of fibres having orientation $\boldsymbol{p}$ and position $\boldsymbol{r}$ at time t is denoted as $\Psi(\boldsymbol{r}, \boldsymbol{p}, t)$. The convection-dispersion that governs the evolution of Ψ is given by

$$\frac{\partial \Psi}{\partial t} = D_r \nabla_r^2 \Psi - \nabla_r \cdot (\boldsymbol{\omega} \Psi) + D_t \nabla^2 \Psi - \nabla \cdot (\bar{\boldsymbol{V}} \Psi) \tag{4.1}$$

where ∇_r is sometimes referred to as the rotational operator, and is expressed as

$$\nabla_r = \boldsymbol{p} \times \frac{\partial}{\partial \boldsymbol{p}}. \tag{4.2}$$

The fibre's angular velocity, $\boldsymbol{\omega}$, is related to the fibre's rotational vector by

$$\boldsymbol{\omega} = \boldsymbol{p} \times \dot{\boldsymbol{p}} \tag{4.3}$$

where $\dot{\boldsymbol{p}}$ is dependent on the fluid velocity surrounding the fibres, as defined in (2.7). Similarly, the average velocity of the fibre, $\bar{\boldsymbol{V}}$, is given in (2.6). The translational and rotational dispersion coefficients are assumed to be independent of fibre orientation and position, and are given by (2.36) and (2.38), respectively. Substituting (2.6), (2.7), (2.36) and (2.38) into (4.1) results in (4.1) expressed in terms of the turbulent velocity field.

Previously, the fibre orientation correlation was derived for small changes in orientation (2.29). Using (4.1), a more general expression for the fibre orientation correlation can be derived. From (4.1), the fibre orientation distribution in isotropic turbulence with no mean shear is governed by

$$\frac{\partial \Psi}{\partial t} = D_r \nabla_r^2 \Psi. \tag{4.4}$$

The conditional probability of a fibre having orientation $\boldsymbol{p}$, given that it had orientation $\boldsymbol{p}_0$ at time $t = 0$, corresponds to the particular solution of (4.4) with initial condition

$$\Psi(\boldsymbol{p}, t = 0) = \delta(\boldsymbol{p} - \boldsymbol{p}_0). \tag{4.5}$$

The solution of (4.4) and (4.5) is the fundamental solution, or Green's function, and is denoted by $G(\boldsymbol{p}, \boldsymbol{p}_0, t)$. From this solution, we calculate $\overline{\boldsymbol{p}(t) \cdot \boldsymbol{p}(0)}$, which is a measure of the orientation correlation, as

$$\overline{\boldsymbol{p}(t) \cdot \boldsymbol{p}(0)} = \frac{1}{4\pi} \int \boldsymbol{p}(t) \cdot \boldsymbol{p}(0) G(\boldsymbol{p}, \boldsymbol{p}_0, t) \mathrm{d}\boldsymbol{p} \mathrm{d}\boldsymbol{p}_0. \tag{4.6}$$

Although, $G(\boldsymbol{p}, \boldsymbol{p}_0, t)$ can be solved analytically (Berne & Pecora 1976), it is not always necessary. Again, following Doi & Edwards (1988), we solve the above by considering the time derivative of (4.6), to arrive at a simple ordinary differential equation, i.e.

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\boldsymbol{p}(t)\cdot\boldsymbol{p}(0)} &= \frac{1}{4\pi}\int \boldsymbol{p}(t)\cdot\boldsymbol{p}(0)\frac{\partial G(\boldsymbol{p},\boldsymbol{p}_0,t)}{\partial t}\mathrm{d}\boldsymbol{p}\mathrm{d}\boldsymbol{p}_0\\ &= \frac{D_r}{4\pi}\int \boldsymbol{p}(t)\cdot\boldsymbol{p}(0)\nabla_r^2 G(\boldsymbol{p},\boldsymbol{p}_0,t)\mathrm{d}\boldsymbol{p}\mathrm{d}\boldsymbol{p}_0\\ &= \frac{D_r}{4\pi}\int \nabla_r^2(\boldsymbol{p}(t)\cdot\boldsymbol{p}(0))G(\boldsymbol{p},\boldsymbol{p}_0,t)\mathrm{d}\boldsymbol{p}\mathrm{d}\boldsymbol{p}_0\\ &= \frac{-2D_r}{4\pi}\int \boldsymbol{p}(t)\cdot\boldsymbol{p}(0)G(\boldsymbol{p},\boldsymbol{p}_0,t)\mathrm{d}\boldsymbol{p}\mathrm{d}\boldsymbol{p}_0\\ &= -2D_r\overline{\boldsymbol{p}(t)\cdot\boldsymbol{p}(0)}.\end{aligned} \tag{4.7}$$

The solution of this equation is given by

$$\overline{\boldsymbol{p}(t)\cdot\boldsymbol{p}(0)} = \exp(-2D_r t). \tag{4.8}$$

This solution is general to large orientation dispersion, i.e. large $\overline{\boldsymbol{p}(t)\cdot\boldsymbol{p}(0)}$, and demonstrates how the fibre orientation correlation decreases exponentially with time. It also shows that fibres with a small orientation dispersion coefficient, i.e. fibres longer than the length scale of the turbulence, disperse slowly. This analysis included the simplification that D_r is constant with respect to time, and is approximated by (3.21). However, the solution for the time-dependent dispersion coefficient could be solved numerically.

As a further application of (4.1), we calculate the steady-state fibre orientation distribution for a suspension undergoing extensional flow. Extensional flow is considered since an analytical solution exists and is readily found (Doi & Edwards 1988). The mean fluid velocity for elongational flow in the $\boldsymbol{e}_3$ direction with strength $\dot{\epsilon}$, can be expressed as

$$\bar{\boldsymbol{U}}(\boldsymbol{r}) = \frac{\dot{\epsilon}}{2}(3\boldsymbol{e}_3\boldsymbol{e}_3 - \boldsymbol{I})\cdot\boldsymbol{r}. \tag{4.9}$$

The angular velocity, $\boldsymbol{\omega}$, is calculated from (2.7) and (4.3), and substituted into (4.1), to arrive at the differential equation

$$\frac{\partial\Psi}{\partial t} = D_r\nabla_r\cdot\left(\nabla_r\Psi - \frac{3\dot{\epsilon}}{2D_r}\boldsymbol{p}\times\boldsymbol{e}_3(\boldsymbol{p}\cdot\boldsymbol{e}_3)\Psi\right) \tag{4.10}$$

which has a steady-state solution of

$$\Psi = \text{Const.}\times\exp\left(\frac{3\dot{\epsilon}}{4D_r}p_3^2\right). \tag{4.11}$$

Since, we consider only the steady-state solution the long-time dispersion coefficient (3.21) is used. Equation (4.11) demonstrates that fibres in elongational flow are uniformly distributed around the $\boldsymbol{e}_3$ axis, and orientation is dependent on the ratio of elongation rate $\dot{\epsilon}$ to the rotational dispersion coefficient. A high $\dot{\epsilon}/D_r$ results in a narrow orientation distribution, while a low $\dot{\epsilon}/D_r$ results in a broader orientation distribution. From (3.21), the rotational dispersion coefficient decreases as fibre length increases, resulting in long fibres having a narrower orientation distribution than short fibres.

5. Conclusion

A theoretical analysis of fibre motion in turbulent flow has been presented, assuming a single straight thin rigid inertialess fibre acted on by a linear drag force. The analysis provides equations for the mean and fluctuating components of fibre translation and rotation, as well as expressions for the resulting dispersion coefficients. For short fibres, the equation for mean rotation was shown to equivalent to Jeffery's (1922) equation of rotation, for infinite aspect ratio particles. The translational dispersion coefficient is the same as Taylor (1921), and the rotational dispersion coefficient is given by $D_p = 2\overline{u^2}\mathscr{T}/\lambda^2$. For longer fibres, the mean fibre velocity is the average of the mean fluid velocity field, taken over the length of the fibre (equation (2.6)). The fibre mean rotational velocity is the first moment of the mean fluid velocity (2.7). The equations of translational and orientational dispersion are given in terms of the fibre's Lagrangian velocity correlation (2.20) and (2.28). The resulting expressions demonstrate that translational and orientational dispersion are significantly diminished as the ratio of fibre length to integral length scale of the velocity correlation increases.

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