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NON-STANDARD, NORMAL SUBGROUPS AND NON-NORMAL, STANDARD SUBGROUPS OF THE MODULAR GROUP

BY

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ABSTRACT. Let R be a commutative ring with identity. A subgroup S of $GL_n(R)$, where $n \geqq 2$, is said to be *standard* if and only if S contains all the $\mathbf{q}$-elementary matrices and all conjugates of those matrices by products of elementary matrices, where $\mathbf{q}$ is the ideal in R generated by $x_{ij}, x_{ii} - x_{jj} (i \neq j)$, for all $(x_{ij}) \in S$. It is known that, when $n \geqq 3$, the standard subgroups of $GL_n(R)$ are precisely those normalized by the elementary matrices. To demonstrate how completely this result can break down for $n = 2$ we prove that $GL_2(\mathbf{Z})$, where $\mathbf{Z}$ is the ring of rational integers, has uncountably many non-normal, standard subgroups and uncountably many non-standard, normal subgroups.

1. **Introduction**. Let R be a commutative ring with identity and let $\mathbf{q}$ be an ideal in R. For each $n \geqq 2$ let $E_n(R)$ be the subgroup of $GL_n(R)$ generated by the elementary matrices and let $E_n(R, \mathbf{q})$ be the normal subgroup of $E_n(R)$ generated by the $\mathbf{q}$-elementary matrices. The *order* of a subgroup S of $GL_n(R)$, denoted by $o(S)$, is the ideal in R generated by all x_{ij}, $x_{ii} - x_{jj}$ $(i \neq j)$, where $(x_{ij}) \in S$. The subgroup S is called *standard* if and only if $E_n(R, \mathbf{q}_o) \leqq S$, where $\mathbf{q}_o = o(S)$. We say that S has *level zero* if and only if $E_n(R, \mathbf{q}) \leqq S$, only when $\mathbf{q} = \{0\}$. Let $\mathcal{S}(n, R)$ be the set of standard subgroups of $GL_n(R)$ and $\mathcal{E}(n, R)$ be the set of subgroups of $GL_n(R)$ normalized by $E_n(R)$. Vaserstein [15] has proved that $\mathcal{S}(n, R) = \mathcal{E}(n, R)$, for all $n \geqq 3$. (He has extended this result [15], [16], [17] to subgroups of $Gl_n(S)$, for particular classes of non-commutative rings S, where $n \geqq 3$.)

For Vaserstein's result (or an even weaker version of his result) to carry over to the case where $n = 2$ it appears that R has to contain "sufficiently many units". We recall that R is said to be an SR_m-ring if and only if R satisfies Bass's stable range condition "$(7.2)_m$", for some integer $m \geqq 2$. (See [2] p. 106.) Let A be an SR_2-ring. (Semi-local rings, for example, are SR_2-rings, by [2] Theorem 7.4.). If $N \in \mathcal{S}(2, R)$ and $\mathbf{q} = o(N)$, then the commutator subgroup $[Gl_2(A), N]$ is contained in $E_2(A, \mathbf{q})$, by [2] Theorems 7.4, 7.5(b). It follows that $\mathcal{S}(2, A) \subseteq \mathcal{E}(2, A)$. Costa and Keller [4] Theorem 2.6 have shown that $\mathcal{S}(2, A) = \mathcal{E}(2, A)$, when $\frac{1}{6} \in R$. On the other hand it is also known that (in general) $\mathcal{S}(2, A) \neq \mathcal{E}(2, A)$ when $\frac{i}{2} \notin A$ or $\frac{1}{3} \notin A$. (See [9] Theorems 2.4, 4.1).

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Suppose now that B is a Dedekind ring of arithmetic type [2] p. 83 with infinitely many units. (Every Dedekind ring is an SR_3-ring by [2] Theorem 7.4) It is known [10] Corollary 1.3 that $\mathcal{S}(2,B) \subseteq \mathcal{E}(2,B)$. It is also known [10] Theorem 2.2 that $\mathcal{S}(2,B) = \mathcal{E}(2,B)$ when $\frac{1}{6} \in B$. Although in general $\mathcal{S}(2,B) \neq \mathcal{E}(2,B)$ (see [10] Examples 2.3, 2.4) Serre [14] Proposition 2, p. 492 has proved that every member of $\mathcal{E}(2,B)$ is "almost standard". More precisely he proves that if $N \in \mathcal{E}(2,B)$ and $o(N) \neq \{0\}$ then, $E_2(B, \mathbf{q}_o) \leqq N$, for some *non-zero* $\mathbf{q}_o$. (We note that the only subgroups of $Gl_n(R)$ of order zero are central subgroups.)

To see how completely Vaserstein's result breaks down for $n = 2$ when R has a "small number of units", we consider now the case where R is a Dedekind ring of arithmetic type with only finitely many units. It follows then that either (i) $R = \mathbf{Z}$, the ring of rational integers, (ii) R is the ring of integers of an imaginary quadratic number field or (iii) R is the coordinate ring of an affine curve obtained by removing a point from a projective curve over a finite field. In this note we prove that $GL_2(\mathbf{Z})$ has uncountably many normal subgroups of level zero contained in the modular group, $SL_2(\mathbf{Z}) = E_2(\mathbf{Z})$. We also prove that, for all but finitely many $\mathbf{q}$, there exist uncountably many non-normal subgroups of $SL_2(\mathbf{Z})$ which are standard subgroups of order q. It follows that both $\mathcal{E}(2,\mathbf{Z})\backslash\mathcal{S}(2,\mathbf{Z})$ and $\mathcal{S}(2,\mathbf{Z})\backslash\mathcal{E}(2,\mathbf{Z})$ are uncountable. (We note that $GL_2(\mathbf{Z})$ is countable.

The author [8] §3 has proved similar results for the case where R is of type (iii). Serre [14, Corollaire 2, p. 519] has provided (non-trivial) examples of normal subgroups of level zero of $GL_2(R)$, for all but finitely many R of type (ii).

2. **Results** We simplify our notation. Let the ideal $\mathbf{q}$ in $\mathbf{Z}$ have non-negative generator m. In our terminology we replace "order $\mathbf{q}$" with "order m." We put $G = GL_2(\mathbf{Z})$, $\Gamma = SL_2(\mathbf{Z})$, $\Delta(m) = E_2(\mathbf{Z}, \mathbf{q})$ and $\Gamma(m) = \ker(SL_2(\mathbf{Z}) \longrightarrow SL_2(\mathbf{Z}/\mathbf{q}))$. (By definition we have $\Gamma(1) = \Delta(1) = \Gamma$.) We denote $PSL_2(\mathbf{Z})$ by $\hat{\Gamma}$ and the image of any subgroup S of Γ in $\hat{\Gamma}$ by $\hat{S}$. We note that $S \cong \hat{S}$, when $-I_2 \notin S$.

Let H, K be subgroups of a group L. As usual $[H, K]$ denotes the subgroup generated by all the commutators $[h,k] = h^{-1}k^{-1}hk$, where $h \in H, k \in K$.

We recall [11] that a group L is said to be *SQ-universal* if and only if every countable group is embeddable in some factor group of L. (The author wishes to thank Dr. S.J. Pride for referring him to [11].)

THEOREM 1. *There exist $2^{\aleph_0}$ normal subgroups of G which have level zero.*

PROOF. Choose $m > 2$. Then $\Gamma(m) \cong \hat{\Gamma}(m)$ and so $\Gamma(m)$ is free, non-cyclic by [13] Theorem VIII. 7, p. 144. It is well known that every such group is SQ-universal and so G is SQ-universal by a result of Neumann, [11] Lemma. It follows from a remark of Neumann [11] p. 4 that G has $2^{\aleph_0}$ normal subgroups.

Now we put

$$N_1 = \{N : N \lhd G\} \text{ and } N_2 = \{N \in N_1 : N \leqq \Gamma(m)\}.$$

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Consider the surjective map p: $N_1 \longrightarrow N_2$, defined by

$$p(N) = \Gamma(m) \cap N.$$

Let $M \in N_2$. If $N \in p^{-1}(M)$ then $|N : M| \leqq |G : \Gamma(m)|$. It follows that $p^{-1}(M)$ is at most countably infinite and hence that

$$\text{card } N_2 = 2^{\aleph_0}.$$

Let $N_3 = \{N' = [N, N] : N \in N_2\}$. Then each element of N_3 is a normal subgroup of G which has level zero by [6] Corollary 8. By a theorem of Auslander and Lyndon [1] it follows that

$$N_1' = N_2' \leftrightarrow N_1 = N_2,$$

where N_1, $N_2 \in N_3$. We conclude that card $N_3 = 2^{\aleph_0}$ □

The set $\{\Gamma(m)' : m > 1\}$ is a countably infinite set of normal subgroups of G of level zero.

THEOREM 2. *Every standard subgroup of G of order m, where $m \leqq 5$, is normal in G.*

PROOF. Let S be a subgroup of G of order m. Then $[G, S] \leqq \Gamma(m)$. The result follows since $\Gamma(m) = \Delta(m)$ when $m \leqq 5$, by [3] Lemmas 8, 9 □

THEOREM 3. *Let $\mathcal{S}(m)$ be the set of non-normal subgroups of Γ which are standard subgroups (of G) of order m. Then*

(a) card $\mathcal{S}(6) = 2^{\aleph_0}$.
(b) card $\mathcal{S}(m) = 2^{\aleph_0}$, when $m > 6$.

PROOF. For each $m \geqq 6$ we have

$$\Gamma(m)/\Delta(m) \cong \hat{\Gamma}(m)/\hat{\Delta}(m) \cong \Phi_g,$$

where

$$\Phi_g = \langle a_1, b_1, \ldots, a_g, b_g : \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle,$$

with

$$g = 1 + \mu(m - 6)/12m,$$

and $\mu = |\hat{\Gamma} : \hat{\Gamma}(m)|$. (See, for example, [13] §22, p. 156, and [18] p. 532.) Clearly every subgroup of G lying between $\Gamma(m)$ and $\Delta(m)$ is a standard subgroup of order m.

(*a*) When $m = 6$, $g = 1$ and so card $\mathcal{S}(6) \leqq 2^{\aleph_0}$. Now Newman [12] has classified all the normal subgroups of $\hat{\Gamma}$ lying between $\hat{\Gamma}'$ and $\hat{\Delta}(6)$, which include all those lying between $\hat{\Gamma}(6)$ and $\hat{\Delta}(6)$. From his classification it is clear that card $\mathcal{S}(6) \geqq 2^{\aleph_0}$. Hence result.

(*b*) When $m > 6, g \geqq 2$. Let X be the normal subgroup of Φ_g generated by b_1, b_2 and a_1, b_i, where $i > 2$. Then ϕ_g/X is the free group on 2 generators, F_2. (See [5] p. 257 for the case $g = 2$.) In the proof of [8] Theorem 3.2 it is shown that F_2 has $2^{\aleph_0}$ non-normal subgroups. The result follows □

We note that every standard subgroup of G of level (6) is contained in the subgroup $\langle -I_2, \Gamma(6)\rangle$. It follows that G has countably many non- normal standard subgroups of level 6, by Theorem 3(a).

3. Remarks. Let $\mathcal{N}(n,R)$ be the set of normal subgroups of $GL_n(R)$. The only general result relating $\mathcal{N}(n,R)$ and $\mathcal{S}(n,R)$ (or $\mathcal{E}(n,r)$) is the trivial observation that $\mathcal{N}(n,R) \subseteq \mathcal{E}(n,R)$. We mention some known results.

(*a*) Suppose that R is an SR_2-ring. Then (as in the introduction) it is known [2] Theorems 7.4, 7.5(*b*) that $\mathcal{N}(n,R) = \mathcal{S}(n,R)(= \mathcal{E}(n,R))$, for all $n \geqq 3$, and that $\mathcal{S}(2,R) \subseteq \mathcal{N}(2,R)$. It is also known [4] Theorem 2.6 the $\mathcal{S}(2,R) = \mathcal{E}(2,R) = \mathcal{N}(2,R)$, when $\frac{1}{6} \in R$. By [9] Theorem 2.4 it is possible to have $\mathcal{N}(2,R) \neq \mathcal{S}(2,R)$ and $\mathcal{N}(2,R) \neq \mathcal{E}(2,R)$.

(*b*) Suppose that R is a Dedekind ring of arithmetic type with infinitely many units. Then by [10] Theorem 3.1, Example 3.4 it is possible to have ($\mathcal{S}(n,R) = \mathcal{E}(n,R) = \mathcal{N}(n,R)$, for all $n > 2$) or ($\mathcal{N}(n,R) \neq \mathcal{E}(n,R)$, for infinitely many n, and, simultaneously, $\mathcal{S}(n,R) = \mathcal{E}(n,R)$, for all $n > 2$).

(*c*) Suppose finally that R is a Dedekind ring of arithmetic type with only finitely many units. (See introduction.) It is known [7] Theorem 8 that, if, $R = \mathbf{Z}$ or R is a type (iii), then $\mathcal{N}(n,R) = \mathcal{S}(n,R)(= \mathcal{E}(n,R))$, for all $n \geqq 3$. It is also known [6] §4 that $\mathcal{E}(2,\mathbf{Z})\backslash\mathcal{N}(2,\mathbf{Z})$ is infinite. By Theorem 1 of this note it follows that $\mathcal{N}(2,\mathbf{Z})\backslash\mathcal{S}(2,\mathbf{Z})$ is uncountable. For R of type (ii) it is possible [7] Theorems 15, 16 to have $\mathcal{N}(n,R) \neq \mathcal{E}(n,R)$, for infinitely many $n \geqq 3$, and Serre [14] Corollaire 2, p. 519 has proved that (in general) $\mathcal{N}(2,R) \neq \mathcal{S}(2,R)$.

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