J. Inst. Math. Jussieu (2021) **20**(2), 517–568
doi:10.1017/S1474748019000240 © Cambridge University Press 2019

ON STABILITY OF PHYSICALLY REASONABLE SOLUTIONS TO THE TWO-DIMENSIONAL NAVIER–STOKES EQUATIONS

YASUNORI MAEKAWA

Department of Mathematics, Kyoto University, Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan (maekawa@math.kyoto-u.ac.jp)

(Received 7 December 2017; revised 11 April 2019; accepted 11 April 2019; first published online 16 May 2019)

Abstract The flow past an obstacle is a fundamental object in fluid mechanics. In 1967 Finn and Smith proved the unique existence of stationary solutions, called the physically reasonable solutions, to the Navier–Stokes equations in a two-dimensional exterior domain modeling this type of flows when the Reynolds number is sufficiently small. The asymptotic behavior of their solution at spatial infinity has been studied in detail and well understood by now, while its stability has remained open due to the difficulty specific to the two-dimensionality. In this paper, we prove that the physically reasonable solutions constructed by Finn and Smith are asymptotically stable with respect to small and well-localized initial perturbations.

Keywords: Navier–Stokes equations; two-dimensional exterior flows; flows past an obstacle; Oseen equations

2010 *Mathematics subject classification*: Primary 76D05; 76D07
Secondary 35B35; 35Q30

1. Introduction

The two-dimensional motion of viscous incompressible fluid past an obstacle is a classical object of research in fluid mechanics. It is modeled by the two-dimensional Navier–Stokes system:

$$\begin{cases} \partial_t u - \dfrac{1}{Re}\Delta u + u\cdot\nabla u + \nabla p = 0, & \operatorname{div} u = 0, \quad t>0,\ x\in\Omega, \\ u|_{\partial\Omega} = 0, & \lim\limits_{|x|\to\infty} u = \mathbf{e}_1, \quad t>0, \\ u|_{t=0} = u_0, & x\in\Omega. \end{cases} \tag{1.1}$$

Here the fluid domain Ω is assumed to be an unbounded domain in $\mathbb{R}^2$ with a smooth and compact boundary, while $u = u(t,x) = (u_1(t,x), u_2(t,x))$ and $p = p(t,x)$, $x = (x_1,x_2)$ are the unknown velocity field and the pressure field of the fluid, respectively, $u_0 = (u_{0,1}(x), u_{0,2}(x))$ is a given initial velocity field, and $\mathbf{e}_1 = (1,0)$. The complement of the domain Ω represents the obstacle and is normalized in the following sense: $\operatorname{diam}(\mathbb{R}^2\setminus\Omega) = 1$ and the origin of the coordinates is located interior to $\mathbb{R}^2\setminus\Omega$ and $\operatorname{dist}(\{0\},\partial\Omega)\geqslant 1/4$. The positive parameter Re represents the Reynolds number. We use the standard notation for derivatives: $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $\Delta = \sum_{j=1}^2 \partial_j^2$, $\operatorname{div} u = \sum_{j=1}^2 \partial_j u_j$, $u\cdot\nabla u = \sum_{j=1}^2 u_j\partial_j u$.

By denoting $u - \mathbf{e}_1$ and $u_0 - \mathbf{e}_1$ by the same symbols u and u_0, respectively, and also by introducing the scaling of the time variable $t \mapsto \alpha t$ with $\alpha = Re$ for simplicity of notations, system (1.1) is equivalent with

$$\begin{cases} \partial_t u - \Delta u + \alpha \partial_1 u + \alpha u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad t > 0, \ x \in \Omega, \\ u|_{\partial\Omega} = -\mathbf{e}_1, \quad \lim_{|x| \to \infty} u = 0, \quad t > 0, \\ u|_{t=0} = u_0, \quad x \in \Omega. \end{cases} \tag{1.2}$$

Here αp is also denoted again by p. In this paper α is always assumed to be positive.

The important contribution to the study of (1.2) has been done by Finn and Smith [22], where the unique existence of stationary solutions $U = U(x; \alpha)$ to (1.2) is established when the Reynolds number α is sufficiently small, based on their analysis of the Oseen linearization in [21]. The stationary solutions in [22] decay in the order $O(|\alpha x|^{-\frac{1}{2}})$, which comes from the decay of the Oseen fundamental solution, and they are called the physically reasonable solutions. The asymptotic behavior of U as $|x| \to \infty$ is studied by Smith [49] and it is shown that the leading profile is described by the Oseen fundamental solution. The results of [21, 22, 49] have been extended in various aspects by Galdi [24, 25]. The reader is also refereed to the book by Galdi [26] for further details on the results for the stationary solutions to (1.2), and see also the work on Hillairet and Wittwer [34] for the stationary exterior problem in a half-space setting.

In spite of its importance, few is known so far about the stability of the physically reasonable solutions. This is contrasting with the three-dimensional case, where the unique existence of physically reasonable solutions for small Reynolds numbers are proved in the series of works by Finn [17–20], after a pioneering work of Leray [38] on the general existence result but with less information on the estimate at spatial infinity; see also the works by Fujita [23], Babenko [6], Farwig [16] about the stationary solutions to (1.1) in three dimensions. For the three-dimensional case the physically reasonable solutions decay in the order $O(|x|^{-1})$ and are known to be asymptotically stable when the Reynolds number is small enough. This asymptotic stability is proved by Heywood [30, 31] in the L^2 framework, and the L^2 stability is further studied and extended by many researches; see Masuda [43], Heywood [32], Miyakawa [44], Maremonti [41], Miyakawa and Sohr [45], and Borchers and Miyakawa [7]. In three dimensions another natural functional framework is the Lebesgue space L^3, for the physically reasonable solutions actually belong to L^3 as observed in the work of Shibata [47]. The L^3 stability is then established by Shibata [47], based on the analysis of the Oseen semigroup by Kobayashi and Shibata [37]; see also the work by Enomoto and Shibata [14, 15] for the L^p theory in higher dimensions. Finn's starting problem is also settled in three dimensions; see Galdi, Heywood, and Shibata [27].

In the two-dimensional case, recently the global well-posedness for nondecaying initial data is obtained by Abe [2] and Maremonti and Shimizu [42]. However, the stability of the physically reasonable solutions has been completely open, even when the Reynolds number is sufficiently small and the initial perturbations are small enough and compactly supported. The crucial difficulty comes from the slower spatial decay of U such as $O(|\alpha x|^{-\frac{1}{2}})$ as $|x| \to \infty$, which is only in the slow variable αx rather than the original variable x, while in the three-dimensional case it is at least in the order $O(|x|^{-1})$ already

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

in the original variable. In fact, even in the level of the Oseen semigroup the analysis becomes rather complicated due to the logarithmic singularity of the resolvent near the origin and also to the parabolic distribution of the continuous spectrum of the Oseen operator. Indeed, it is very recent that the Oseen semigroup in two dimensions is shown to define a bounded semigroup in the L^p space for $p \in (1, \infty)$, which is established by Hishida [35]. However, due to the slow decay of U, the L^p estimate for the Oseen semigroup in [35] does not yield either the linear or nonlinear stability of the physically reasonable solution itself, even when α is small enough. In fact, the stability of the physically reasonable solutions shares a common difficulty with the stability of the small scale-critical stationary solution (decaying in the order $O(|x|^{-1})$ for the Navier–Stokes equations ($\alpha = 0$) in a two-dimensional exterior domain, for which no general theory is established yet and only some partial results are known for specific examples (see Guillod [29], Y. M. [39], and Higaki [33] and references there in). A short discussion about this *scale-critical* nature of the problem is given in Remark 1.4 below.

In this paper we shall prove that, when α is sufficiently small, the physically reasonable solutions obtained by Finn and Smith are asymptotically stable for small and well-localized initial perturbations. As far as the author knows, this is the first stability result in the two-dimensional case. To be precise let $(U, \nabla P)$ be the stationary solution to (1.2):

$$\begin{cases} -\Delta U + \alpha \partial_1 U + \alpha U \cdot \nabla U + \nabla P = 0, \quad \operatorname{div} U = 0, \quad x \in \Omega, \\ \qquad\qquad U|_{\partial\Omega} = -\mathbf{e}_1, \quad \lim_{|x|\to\infty} U = 0. \end{cases} \tag{1.3}$$

We study the evolution of the perturbation $v = u - U$, which obeys the system:

$$\begin{cases} \partial_t v - \Delta v + \alpha \partial_1 v + \nabla q = -\alpha \nabla \cdot (U \otimes v + v \otimes U + v \otimes v), \quad t > 0, \ x \in \Omega, \\ \qquad \operatorname{div} v = 0, \quad t \geqslant 0, \ x \in \Omega, \\ \qquad v|_{\partial\Omega} = 0, \quad \lim_{|x|\to\infty} v = 0, \quad t > 0, \\ \qquad v|_{t=0} = v_0, \quad x \in \Omega. \end{cases} \tag{1.4}$$

Here $v_0 = (v_{0,1}, v_{0,2})$ is a given initial perturbation, and $f \otimes g = (f_i g_j)_{1 \leqslant i,j \leqslant 2}$. The decay structure of U clearly plays an important role for its stability. The key observation here is the well known *wake* structure of U, which gives the additional anisotropic spatial decay other than $O(|\alpha x|^{-\frac{1}{2}})$. In particular, the argument of [11, 22, 26, 49] gives the following decay estimate of U, which will be assumed in this paper.

Assumption. The following statements hold for any sufficiently small $\alpha > 0$.

(A1) The velocity $U = U(\alpha; \cdot)$ belongs to $L^\infty(\Omega)^2$ and satisfies

$$|U(x)| \leqslant \frac{C}{|\log \alpha|} \left(\frac{1}{|\alpha x|^{\frac{1}{2}} (1 + |\alpha x| - \alpha x_1)^{\frac{3}{4}}} + \frac{1}{1 + |\alpha x|} \right), \quad x \in \Omega$$

with a constant C independent of α.

(A2) $\nabla U \in L^4(\Omega)^{2\times 2} \cap L^\infty(\Omega)^{2\times 2}$ and $\|\nabla U\|_{L^4(\Omega)^{2\times 2} \cap L^\infty(\Omega)^{2\times 2}} \leqslant C$ with a constant C independent of α.

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

Since the pointwise estimate in **(A1)** is not written explicitly in the literature, for reader's convenience, we give a sketch of the proof of **(A1)** in Appendix C for the solutions constructed in [22].

Remark 1.1. (i) The decay of U is estimated in the rescaled variable $X=\alpha x$, which is compatible with (1.3). The factor $(1+|\alpha x|-\alpha x_1)$ in **(A1)** represents the wake structure behind the obstacle. The key point is that the slow decay $O(|\alpha x|^{-\frac{1}{2}})$ appears together with the wake structure, while the *nowake* part of U decays in the order $O(|\alpha x|^{-1})$. The factor $|\log\alpha|^{-1}$ gives the smallness of U in the rescaled variable when α is small enough and plays a key role in constructing the stationary solutions in [22]. This smallness factor is related to the Stokes paradox; cf. Chang and Finn [10].
(ii) The important property of the function $\frac{1}{|\alpha x|^{\frac{1}{2}}(1+|\alpha x|-\alpha x_1)^{\frac{3}{4}}}$ is that it still belongs to a scale-critical space but in the anisotropic sense. Indeed, as proved in Lemma A.1, we have

$$\frac{1}{|\alpha x|^{\frac{1}{2}}(1+|\alpha x|-\alpha x_1)^{\frac{1}{2}}}\leqslant\frac{C}{|\alpha x|^{\frac{1}{2}}+|\alpha x_2|},\qquad x\in\mathbb{R}^2\setminus\{0\},\tag{1.5}$$

and thus, the function $\frac{1}{|\alpha x|^{\frac{1}{2}}(1+|\alpha x|-\alpha x_1)^{\frac{1}{2}}}$ is bounded in the anisotropic norm $\|x_2 f\|_{L^\infty}$ which is invariant under the scaling $f_\lambda(x)=\lambda f(\lambda x)$; see Remark 1.4 for the discussion on the criticality. Moreover, in virtue of the factor $\frac{1}{(1+|\alpha x|-\alpha x_1)^{\frac{3}{4}}}$ rather than $\frac{1}{(1+|\alpha x|-\alpha x_1)^{\frac{1}{2}}}$, the function $\frac{1}{|\alpha x|^{\frac{1}{2}}(1+|\alpha x|-\alpha x_1)^{\frac{3}{4}}}$ even belongs to $L^\infty_{x_1}L^1_{x_2}$, i.e., it is integrable in x_2.

The main result of this paper is stated as follows. The precise definition of the function spaces is stated in § 2. Let us define the function $b(X)$ by the formula

$$\frac{1}{b(X)}=\frac{1}{|X|^{\frac{1}{2}}(1+|X|-X_1)^{\frac{3}{4}}}+\frac{1}{1+|X|}.\tag{1.6}$$

Theorem 1.2. *For any $\beta\in[0,1)$ there exists a constant $\bar\alpha=\bar\alpha(\beta)>0$ such that the following statement holds for any $\alpha\in(0,\bar\alpha]$. There exists a constant $\epsilon>0$ such that for any $v_0\in L^4_\sigma(\Omega)$ satisfying*

$$\|v_0\|_{L^\infty(\Omega)}+\|b(\alpha\cdot)v_0\|_{L^\infty(\Omega)}\leqslant(1-\beta)\epsilon,\tag{1.7}$$

system (1.4) admits a unique mild solution $v\in C([0,\infty);L^4_\sigma(\Omega))\cap C((0,\infty);W^{1,4}_0(\Omega)^2)$ satisfying $(\frac{t}{1+t})^{\frac{1}{2}}\nabla v(t)\in L^\infty(0,\infty;L^\infty(\Omega)^{2\times 2})$ and

$$\|v(t)\|_{L^\infty(\Omega)}\leqslant\frac{C|\log\alpha|}{1+(\alpha^2 t)^{\frac{\beta}{2}}}\|(1+b(\alpha\cdot))v_0\|_{L^\infty}.\tag{1.8}$$

Here ϵ and C are independent of α and β.

Remark 1.3. (i) One can check that any solenoidal vector field in Ω satisfying (1.7) belong to $L^4_\sigma(\Omega)$ (in virtue of the wake factor). The solutions treated in this paper are

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

mild solutions, i.e., the solutions to the integral equation associated with (1.4), which is well defined since the Oseen operator appearing in (1.4) generates the analytic semigroup in $L^4_\sigma(\Omega)$; see § 2 for details.
(ii) The condition (1.7) reflects the decay structure of U. Indeed, we are imposing the same decay on the perturbations as U. It should be emphasized that the positive number ϵ is small *but is independent of* α. Note that the estimate $\|v_0\|_{L^\infty(\Omega)} \leqslant \frac{C}{\alpha^{\frac{1}{2}}}\|b(\alpha\cdot)v_0\|_{L^\infty(\Omega)}$ holds under the normalized condition of the domain Ω stated above, but in order to make the smallness condition on ϵ uniformly in α, this estimate should not be used in describing the condition (1.7). Another important remark is that U itself does not satisfies the smallness condition of (1.7) in general. Indeed, for the physically reasonable solutions in [22] one can only claim that

$$\|U\|_{L^\infty(\Omega)} + |\log\alpha|\,\|b(\alpha\cdot)U\|_{L^\infty(\Omega)} \leqslant C$$

for some constant C which is uniform in small α, but this constant C is not necessarily small. In particular, $\|U\|_{L^\infty(\Omega)}$ is not necessarily small in general.
(iii) The weight function b can be slightly generalized. Indeed, Theorem 1.2 is valid even if the definition of b in (1.6) is replaced by

$$\frac{1}{b(X)} = d(X) + \frac{1}{1+|X|} \quad \text{with } (|X|^{\frac{1}{2}} + |X_2|)\,d(X) \in L^\infty(\mathbb{R}^2) \text{ and } d(X) \in L^\infty_{X_1}L^1_{X_2}. \tag{1.9}$$

In this case the number ϵ in Theorem 1.2 depends on $\|\rho d\|_{L^\infty_X} + \|d\|_{L^\infty_{X_1}L^1_{X_2}}$ with $\rho(X) = |X|^{\frac{1}{2}} + |X_2|$.
(iv) The number β represents the order of time decay and must be strictly less than 1 in our analysis. As indicated in (1.7), the decay order $O((\alpha^2 t)^{-\frac{1}{2}})$ exhibits a criticality in the estimate. In fact, if v_0 satisfies for some $\delta > 0$,

$$\text{ess.sup}_{x\in\Omega}(1 + |\alpha x|^{\frac{1}{2}} + |\alpha x_2|)^{1+\delta}|v_0(x)| \leqslant \epsilon(\delta) \ll 1, \tag{1.10}$$

then one can show $\|v(t)\|_{L^\infty(\Omega)} = o((\alpha^2 t)^{-\frac{1}{2}})$ as $\alpha^2 t \to \infty$, that is, v decays in time faster than the critical order in L^∞. However, to achieve this faster temporal decay (under stronger spatial decay of v_0 as in (1.10)) requires an even more technical computations. The discussion in this regime is stated in § 6.

Remark 1.4. It will be meaningful to point out here the scale-critical nature underlying this problem, though the phrase 'scale-critical' for (1.4) is somewhat confusing since there is no invariant rescaling for (1.4) even in the whole space setting, due to the presence of the term $\alpha\partial_1$. In fact, the criticality here is seen in the level of various estimates. One easy way to glimpse it is to recall the L^q-L^r estimate of the Oseen semigroup in $\mathbb{R}^2$, which is given by $\|e^{-t\mathbb{A}_\alpha} f\|_{L^q(\mathbb{R}^2)} \leqslant Ct^{-\frac{1}{r}+\frac{1}{q}}\|f\|_{L^r(\mathbb{R}^2)}$ for $1 \leqslant r \leqslant q \leqslant \infty$ with C uniformly in α (here $\mathbb{A}_\alpha = \mathbb{A}_{\alpha,\mathbb{R}^2} = -\Delta + \alpha\partial_1$ is the Oseen operator in $\mathbb{R}^2$; see § 2.2). Thus, in the level of the L^q-L^r estimate in $\mathbb{R}^2$, we do not see the effect of α, and then heuristically the borderline whether one can treat the linearized term $-\alpha\nabla\cdot(U\otimes v + v\otimes U)$ in (1.4) as a perturbation globally in time or not should be related to whether U is small in a scale-invariant norm for the standard Navier–Stokes equation ($\alpha = 0$), i.e., the norm which is invariant under

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

the scaling $f(x) \mapsto \lambda f(\lambda x)$. As stated in Remark 1.1, the physically reasonable solution U formally satisfies this requirement in an anisotropic way. Unfortunately, this view is of course too optimistic and naive in the actual proof, and we do not know whether U is stable or not in the simple framework of L^q spaces. Another way to see the criticality, which is indeed related to the actual proof, is to introduce the distance

$$\rho(X) = |X|^{\frac{1}{2}} + |X_2|, \quad X = \alpha x, \tag{1.11}$$

where the parabolic aspect in the low frequency of the operator $\alpha\partial_1 - \partial_2^2$ is taken into account. This distance is natural and useful in the analysis of (1.4). For example, as shown in Lemma A.1, U satisfying **(A1)** obeys the bound

$$|U(x)| \leqslant \frac{C}{|\log\alpha|\rho(X)}, \tag{1.12}$$

that reminds us the scale-critical decay $O(\frac{1}{|x|})$ for the standard Navier–Stokes equations. The criticality of the order $O(\rho(X)^{-1})$ will be seen in the analysis of the bilinear form in $\mathbb{R}^2$ in § 3.2. Roughly speaking, the result of § 3.2 is applied to the integral form of the linearized term $\Lambda_{\mathbb{A}_\alpha}[U, f](t, \cdot) := \alpha \int_0^t e^{-(t-s)\mathbb{A}_\alpha}\mathbb{P}\nabla\cdot(U \otimes f + f \otimes U)\, ds$, where $\mathbb{P}$ is the Helmholtz projection in $\mathbb{R}^2$, and after some computation using Lemma 2.3, we can show

$$|\Lambda_{\mathbb{A}_\alpha}[U, f](t, x)| \leqslant \frac{C}{|\log\alpha|}\int_{\mathbb{R}^2}\frac{1}{\rho(X-Y)^2\rho(Y)^{1+\gamma}}\, dY \sup_{s>0}\|\rho(\alpha\cdot)^\gamma f(s)\|_{L^\infty(\mathbb{R}^2)}, \tag{1.13}$$

where $\gamma > 0$ and $X = \alpha x$. Here (1.12) is used and the function $\frac{1}{\rho(X-Y)^2}$ comes from the time integral of the kernel $e^{-(t-s)\mathbb{A}_\alpha}\mathbb{P}\nabla\cdot$. Lemma 2.4 for the convolution of $\frac{1}{\rho}$ implies that

$$\int_{\mathbb{R}^2}\frac{1}{\rho(X-Y)^2\rho(Y)^{\gamma'+\gamma}}\, dY \leqslant \frac{C}{\rho(X)^{\gamma'-1+\gamma}}\left(\frac{1}{\gamma'-1+\gamma} + \frac{1}{3-\gamma-\gamma'}\right) \tag{1.14}$$

as long as $0 < \gamma' - 1 + \gamma < 2$, which yields by taking $\gamma' = 1$,

$$\sup_{t>0}\|\rho(\alpha\cdot)^\gamma\Lambda_{\mathbb{A}_\alpha}[U, f](t)\|_{L^\infty(\mathbb{R}^2)} \leqslant \frac{C}{|\log\alpha|}\left(\frac{1}{\gamma} + \frac{1}{2-\gamma}\right)\sup_{s>0}\|\rho(\alpha\cdot)^\gamma f(s)\|_{L^\infty(\mathbb{R}^2)}.$$

This estimate indicates how to close the estimate when $0 < \gamma < 2$ and α is small enough, by making use of the decay of U in a crucial way. Although we need more weighted norms (as stated below) and technical computations to achieve the nonlinear asymptotic stability, the proof of the linear estimate for the whole space problem proceeds along this basic idea. We observe from (1.14) that, if $|U(x)| = O(\rho(X)^{-\gamma'})$ with $\gamma' < 1$, then there is no hope to close the estimate in this approach: in this sense, the decay of U in (1.12) is just in the critical order.

The key step of the proof of Theorem 1.2 is to study the linearized problem

$$\begin{cases} \partial_t v - \Delta v + \alpha\partial_1 v + \nabla q = \alpha\nabla\cdot(f \otimes g + g \otimes f), \quad t > 0,\ x \in \Omega, \\ \operatorname{div} v = 0, \quad t \geqslant 0,\ x \in \Omega, \\ v|_{\partial\Omega} = 0, \quad \lim_{|x|\to\infty} v = 0, \quad t > 0, \\ v|_{t=0} = v_0, \quad x \in \Omega. \end{cases} \tag{1.15}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

The results for (1.15) are summarized in Propositions 4.3 and 4.4. We shall establish the weighted L^∞ estimate of the solution to (1.15), where the weight function is suitably chosen in view of the wake structure of U and the Oseen operator so that we can close the global estimate. The key idea is to introduce the weight function $\rho(X)$ in (1.11), which takes into account the anisotropic nature of the Oseen operator and the wake structure of the physically reasonable solution. Then we introduce the space–time norm $\|\cdot\|_{\alpha,\beta}$ with $\alpha \in (0, \frac{1}{2})$ and $\beta \in [0, 1]$ as

$$\begin{aligned}\|f\|_{\alpha,\beta} &= \sup_{t>0}(\|\rho(\alpha\cdot)f(t)\|_{L^\infty} + (1-\beta)(\alpha^2 t)^{\frac{\beta}{2}}\|\rho(\alpha\cdot)^{1-\beta}f(t)\|_{L^\infty})\\ &\quad + \sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{1}{|\log\alpha|}\|f(t)\|_{L^\infty} + \left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right).\end{aligned}$$

The first supremum in $\|\cdot\|_{\alpha,\beta}$ exactly controls the critical quantity (for example, let us recall (1.12) for $\|\rho(\alpha\cdot)f(t)\|_{L^\infty}$), while the second supremum is a subcritical quantity when $\beta < 1$. The introduction of the prefactors $1-\beta$ and $\frac{1}{|\log\alpha|}$ is aimed to achieve the result in a unified and sharp manner. The presence of $1-\beta$ is inevitable in the estimate of the bilinear form, already in the whole space problem. It describes the difficulty in achieving the critical temporal decay $O((\alpha^2 t)^{-\frac{1}{2}})$ for solutions to (1.15) when $f = U$ merely under the condition $\|g\|_{\alpha,1} < \infty$. The reason of the factor $\frac{1}{|\log\alpha|}$ is more specific to the exterior problem, for it is related with the large-time control of the flow produced near the boundary. Intuitively, the logarithmic order comes from the logarithmic singularity of the resolvent for the Stokes or Oseen operator when the resolvent parameter tends to zero.

We show the following estimate of the solution to (1.15):

$$\begin{aligned}\|v\|_{\alpha,\beta} &\leqslant C\|(1+b(\alpha\cdot))v_0\|_{L^\infty}\\ &\quad + \frac{C}{1-\beta}\|g\|_{\alpha,\beta}\sup_{t>0}\left(\|\rho(\alpha\cdot)f(t)\|_{L^\infty} + \alpha^{\frac{1}{2}}\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right). \qquad (1.16)\end{aligned}$$

Here C is independent of $\alpha \in (0, \frac{1}{2}]$ and $\beta \in [0, 1)$. This estimate is proved in Propositions 4.3 and 4.4. In (1.16) the norm for f allows us to handle the case when $f = U$ which is independent of t.

As in the previous works of [35, 37] for the Oseen semigroup in the L^p framework, our approach to (1.15) consists of two steps:

(i) analysis of (1.15) in $\mathbb{R}^2$;

(ii) local energy decay estimate of $e^{-t\mathbb{A}_{\alpha,\Omega}}$ in the exterior domain;

where in (ii) the resolvent problem is considered with a compactly supported sourcing term. In each step, however, obtaining the desired global bound is highly nontrivial, since we have to work with $f = U$ whose the spatial decay is not fast enough; it is just in a critical regime in an anisotropic form and only in the slow variable $X = \alpha x$.

In the step (i), to overcome the difficulty, we make full use of the anisotropic transport effect of the Oseen semigroup and our specific choice of the weight norm as above. More

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

precisely, the key is to apply Lemmas 2.3 and 2.4, which are compatible with the kernel of the Oseen semigroup. Lemma 2.3 describes how the wake structure appears from the time integral of the Oseen semigroup. Lemma 2.4 gives the convolution estimate of $\frac{1}{\rho(X)}$. It should be stressed that the use of the distance $\rho(X) = |X|^{\frac{1}{2}} + |X_2|$ enables us to obtain the sharp estimates in a unified manner. Another important lemma is Lemma A.1, which connects the wake decay $\frac{1}{|X|^{\frac{1}{2}}(1+|X|-X_1)^{\frac{1}{2}}}$ and $\frac{1}{\rho(X)}$.

The step (ii) is handled in [40] independently of this paper, and the result is stated in Theorem 4.1, which is a significant improvement of Hishida's result [35]. The estimate (1.16) in the exterior domain is obtained by combining the results from (i) and (ii) through a standard cut-off argument using the Bogovskii operator.

This paper is organized as follows. In § 2, the function spaces are introduced and some basic results on the Stokes semigroup and the Oseen semigroup are collected. In § 3, we study the linearized problem (1.15) in $\mathbb{R}^2$. Section 4 is then devoted to the analysis of (1.15) in the exterior domain. The nonlinear stability is proved in § 5 by applying a standard fixed point argument. Some key estimates used in the proof are collected in the appendix.

2. Preliminaries

In this section, we collect some notations frequently used in this paper and state some basic results on the Stokes semigroup that are standard and thus whose proofs are not given in details.

2.1. Function spaces of solenoidal vector fields

Let Ω be an exterior domain in $\mathbb{R}^2$ with smooth boundary. The class of smooth and compactly supported functions in Ω is denoted by $C_0^\infty(\Omega)$, and the class of test functions for solenoidal vector fields in Ω is defined by $C_{0,\sigma}^\infty(\Omega) = \{f \in C_0^\infty(\Omega)^2 \mid \operatorname{div} f = 0 \text{ in } \Omega\}$. We denote by $L^q(\Omega)$, $1 \leqslant q \leqslant \infty$, the usual Lebesgue space of all measurable functions whose L^q norm, $\|f\|_{L^q} = (\int_\Omega |f|^q dx)^{\frac{1}{q}}$ for $q < \infty$ and $\|f\|_{L^\infty} = \text{ess.sup}_{x\in\Omega}|f(x)|$ for $q = \infty$, is finite. For $p \in (1, \infty)$ and $q \in [1, \infty]$ the Lorentz space $L^{p,q}(\Omega)$ is defined to be the set of measurable functions f in Ω such that

$$\|f\|_{L^{p,q}(\Omega)} = \begin{cases} \left(\displaystyle\int_0^\infty (R|\{x \in \Omega \mid |f(x)| > R\}|^{\frac{1}{p}})^q \frac{dR}{R} \right)^{\frac{1}{q}}, & 1 \leqslant q < \infty, \\ \displaystyle\sup_{R>0} R|\{x \in \Omega \mid |f(x)| > R\}|^{\frac{1}{p}}, & q = \infty, \end{cases}$$

is finite. Here $|A|$ denotes the Lebesgue measure of the measurable set A. It is well known that $\|\cdot\|_{L^{p,q}(\Omega)}$ defines a quasi-norm and there exists a norm equivalent to it by which $L^{p,q}(\Omega)$ becomes a Banach space when $p > 1$; see [28, Ch. 1]. Moreover, by O'Neil [46] the generalized Hölder inequality and the Young inequality are known: if $1 < p, q, r < \infty$ and $1 \leqslant s_1, s_2 \leqslant \infty$ then

$$\|fg\|_{L^{r,s}(\Omega)} \leqslant C\|f\|_{L^{p,s_1}(\Omega)}\|g\|_{L^{q,s_2}(\Omega)}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \quad \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s}, \tag{2.1}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

and

$$\|f*g\|_{L^{r,s}(\mathbb{R}^2)} \leqslant C\|f\|_{L^{p,s_1}(\mathbb{R}^2)}\|g\|_{L^{q,s_2}(\mathbb{R}^2)}, \quad \frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1, \quad \frac{1}{s_1}+\frac{1}{s_2}=\frac{1}{s}, \tag{2.2}$$

where $(f*g)(x)=\int_{\mathbb{R}^2} f(x-y)g(y)\,dy$ is the usual convolution in $\mathbb{R}^2$. Moreover, we also have from [46, Theorem 3.6],

$$\begin{aligned} &\|f*g\|_{L^\infty(\mathbb{R}^2)} \leqslant C\|f\|_{L^{p,s_1}(\mathbb{R}^2)}\|g\|_{L^{q,s_2}(\mathbb{R}^2)}, \\ &\frac{1}{p}+\frac{1}{q}=\frac{1}{s_1}+\frac{1}{s_2}=1, \quad 1<p,\ q<\infty,\ 1\leqslant s_1,\ s_2\leqslant\infty. \end{aligned} \tag{2.3}$$

The space of all L^q solenoidal vector fields in Ω is denoted by $L^q_\sigma(\Omega)$, which is characterized as $L^q_\sigma(\Omega)=\overline{C^\infty_{0,\sigma}(\Omega)}^{\|f\|_{L^q}}=\{f\in L^q(\Omega)^2 \mid \operatorname{div} f=0 \text{ in } \Omega,\ f\cdot n=0 \text{ on } \partial\Omega\}$ for $q<\infty$. Here $n=n(x)$ is the exterior unit normal vector at $x\in\partial\Omega$. The L^q Sobolev space of order k in Ω is denoted by $W^{k,q}(\Omega)$ and we also introduce the space $W^{1,q}_0(\Omega)=\overline{C^\infty_0(\Omega)}^{\|f\|_{W^{1,q}}}$. As is well known, for $q\in(1,\infty)$, the space $L^q(\Omega)^2$ is written as the direct sum $L^q(\Omega)=L^q_\sigma(\Omega)\oplus G^q(\Omega)$, where $G^q(\Omega)=\{\nabla p\in L^q(\Omega)^2 \mid p\in L^q_{loc}(\overline{\Omega})\}$. Then the Helmholtz projection $\mathbb{P}_\Omega : L^q(\Omega)^2\to L^q_\sigma(\Omega)$ is well defined, which is an orthogonal projection when $q=2$; see Miyakawa [44] and Simader and Sohr [48] for details. For simplicity, the Helmholtz projection in $\mathbb{R}^2$ is denoted by $\mathbb{P}$, instead of $\mathbb{P}_{\mathbb{R}^2}$.

2.2. Oseen operator

For $q\in(1,\infty)$ the second order elliptic operator $A_{\alpha,\Omega}$ in $L^q(\Omega)$ is defined by

$$D_{L^q}(A_{\alpha,\Omega})=W^{2,q}(\Omega)\cap W^{1,q}_0(\Omega), \quad A_{\alpha,\Omega}f=-\Delta f+\alpha\partial_1 f, \quad f\in D_{L^q}(A_{\alpha,\Omega}).$$

Then the perturbed Stokes operators $\mathbb{A}_{\alpha,\Omega}$ in $L^q_\sigma(\Omega)$ is defined by

$$\begin{aligned} &D_{L^q}(\mathbb{A}_{\alpha,\Omega})=W^{2,q}(\Omega)^2\cap W^{1,q}_0(\Omega)^2\cap L^q_\sigma(\Omega), \\ &\mathbb{A}_{\alpha,\Omega}f=\mathbb{P}_\Omega A_{\alpha,\Omega}f, \quad f\in D_{L^q}(\mathbb{A}_{\alpha,\Omega}). \end{aligned}$$

The operator $\mathbb{A}_{\alpha,\Omega}$ is known as the Oseen operator. To simplify the notations the counterparts of these operators in $L^q_\sigma(\mathbb{R}^2)$ are written as $\mathbb{A}_\alpha$, instead of $\mathbb{A}_{\alpha,\mathbb{R}^2}$. As is well known, the Stokes operator $-\mathbb{A}_\Omega=-\mathbb{A}_{0,\Omega}$ generates a bounded C_0-analytic semigroup in $L^q_\sigma(\Omega)$; cf. Borchers and Varnhorn [9]. Since the terms $\alpha\mathbb{P}_\Omega\partial_1 f$ is lower order, it is not difficult to show that the Oseen operator $-\mathbb{A}_{\alpha,\Omega}$ also generates a C_0-analytic semigroup $e^{-t\mathbb{A}_{\alpha,\Omega}}$ in $L^q_\sigma(\Omega)$. Moreover, the following L^p-L^q estimates have been recently established by Hishida [35].

Proposition 2.1 (Hishida [35]). *Let $1<q\leqslant p<\infty$ and let $f\in L^q_\sigma(\Omega)$. Then for $\alpha\in(0,1]$,*

$$\|e^{-t\mathbb{A}_{\alpha,\Omega}}f\|_{L^p(\Omega)} \leqslant \frac{C}{\alpha^\kappa t^{\frac{1}{q}-\frac{1}{p}}}\|f\|_{L^q(\Omega)}, \quad t>0. \tag{2.4}$$

Here C depends only on Ω, p, and q, while $\kappa>1$ depends on α, p, and q.

Remark 2.2. In the approach of [35] the singularity $O(\alpha^{-\kappa})$ in (2.4) for small α is crucial. Estimate (2.4) is improved in [40], and in particular, it is shown in [40] that this singularity can be completely removed. This improvement (in the level of local energy decay estimate)

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

is essential in the proof of the stability of U, for the smallness of U in a weighted norm is only in a logarithmic order $O(\frac{1}{|\log\alpha|})$.

On the other hand, the following L^p-L^q estimates for the derivatives of the Stokes semigroup $e^{-t\mathbb{A}_\Omega}$ are classical:

$$\|\partial_t^k\nabla^j e^{-t\mathbb{A}_\Omega} f\|_{L^p(\Omega)} \leqslant C_T t^{-\frac{j}{2}-k-\frac{1}{q}+\frac{1}{p}}\|f\|_{L^q(\Omega)}, \quad 0<t\leqslant T<\infty, \tag{2.5}$$

for $j=0,1,2,\ k=0,1,\ 1<q\leqslant p<\infty$ or $1<q<p=\infty$, and $f\in L^q_\sigma(\Omega)$. Here C_T depends on Ω, p, q, and also on T in general. Recently, the L^∞ theory of the Stokes semigroup has been developed by Abe and Giga [3, 4], Abe, Giga, and Hieber [5], and Abe [1], and the following L^∞ estimates have been established by now:

$$\|\partial_t^k\nabla^j e^{-t\mathbb{A}_\Omega} f\|_{L^\infty(\Omega)} \leqslant C_T t^{-\frac{j}{2}-k}\|f\|_{L^\infty(\Omega)}, \quad 0<t\leqslant T<\infty, \tag{2.6}$$

$$\|e^{-t\mathbb{A}_\Omega}\mathbb{P}_\Omega\nabla\cdot F\|_{L^\infty(\Omega)} \leqslant C_T t^{-\frac{\sigma}{2}}\|F\|^\sigma_{L^\infty(\Omega)}\|\nabla F\|^{1-\sigma}_{L^\infty(\Omega)}, \quad 0<t\leqslant T<\infty, \tag{2.7}$$

for $j=0,1,2,\ k=0,1,\ \sigma\in(0,1]$, and $f\in L^\infty_\sigma(\Omega)=\{f\in L^\infty(\Omega)^2 \mid \operatorname{div} f=0 \text{ in } \Omega,\ f\cdot n=0 \text{ on } \partial\Omega\}$. Note that the semigroup property $e^{-t\mathbb{A}_\Omega}\mathbb{P}_\Omega\nabla\cdot F=e^{-\frac{t}{2}\mathbb{A}_\Omega}e^{-\frac{t}{2}\mathbb{A}_\Omega}\mathbb{P}_\Omega\nabla\cdot F$ together with (2.7) and (2.8) also give the estimate

$$\|\nabla e^{-t\mathbb{A}_\Omega}\mathbb{P}_\Omega\nabla\cdot F\|_{L^\infty(\Omega)} \leqslant C_T t^{-\frac{1+\sigma}{2}}\|F\|^\sigma_{L^\infty(\Omega)}\|\nabla F\|^{1-\sigma}_{L^\infty(\Omega)}, \quad 0<t\leqslant T<\infty, \tag{2.8}$$

Estimates (2.7) and (2.8) are useful in handling the inhomogeneous term of the form $\nabla\cdot(f\otimes g+g\otimes f)$. From (2.5), (2.6), and (2.7), it is not difficult to show that the following L^∞ estimates hold also for $e^{-t\mathbb{A}_{\alpha,\Omega}}$:

$$\|\nabla^j e^{-t\mathbb{A}_{\alpha,\Omega}} f\|_{L^\infty(\Omega)} \leqslant C_T t^{-\frac{j}{2}}\|f\|_{L^\infty(\Omega)}, \quad 0<t\leqslant T<\infty,\ j=0,1, \tag{2.9}$$

for all $f\in L^\infty_\sigma(\Omega)$. Indeed, estimate (2.9) is derived by solving the integral equation

$$v(t)=e^{-t\mathbb{A}_\Omega}f-\alpha\int_0^t e^{-(t-s)\mathbb{A}_\Omega}\mathbb{P}_\Omega\partial_1 v\,ds \tag{2.10}$$

in the function space

$$\begin{aligned} X_T &= \big\{f\in C_{w^*}([0,T];L^\infty_\sigma(\Omega)) \;\big|\\ &\qquad t^{\frac{1}{2}}\nabla f(t)\in L^\infty(0,T;L^\infty(\Omega)^{2\times 2}),\ f(t)=0 \text{ on } \partial\Omega, t\in(0,T]\big\}\\ \|f\|_{X_T} &= \sup_{0<t\leqslant T}(\|f(t)\|_{L^\infty(\Omega)}+t^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty(\Omega)}), \end{aligned}$$

where C_{w^*} denotes the $*$-weak topology. Precisely, the second term of the right-hand side of (2.10) is estimated by using (2.6), (2.7), (2.8) as

$$\begin{aligned} \left\|\nabla^j\alpha\int_0^t e^{-(t-s)\mathbb{A}_\Omega}\mathbb{P}_\Omega\partial_1 v\,ds\right\|_{L^\infty(\Omega)} &\leqslant C\alpha\int_0^t (t-s)^{-\frac{j+\sigma}{2}}\|v(s)\|^\sigma_{L^\infty(\Omega)}\|\nabla v(s)\|^{1-\sigma}_{L^\infty(\Omega)}\,ds\\ &\leqslant C\alpha\int_0^t (t-s)^{-\frac{j+\sigma}{2}}s^{-\frac{1-\sigma}{2}}\,ds\|v\|_{X_T}\\ &\leqslant C\alpha T^{\frac{1-j}{2}}\|v\|_{X_T}, \quad 0<t\leqslant T,\ |j|=0,1. \end{aligned}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

Here we have taken, for example, as $\sigma = \frac{1}{2}$. Thus, if $T \in (0, 2]$ is sufficiently small then there exists a unique solution $v \in X_T$ to (2.10). Repeating this argument, we obtain (2.9) for any $0 < T < \infty$ with a constant C_T depending on Ω, T, and α. In particular, if $T = 2$ and α is small enough then C_T depends only on Ω.

2.3. Key integral inequalities

Our approach is built upon the weighted L^∞ estimate of the Oseen semigroup which reflects the transport term $\alpha\partial_1$ and the wake decay structure of U. The following two integral inequalities play a crucial role, where the integrand is related to the kernel of the Oseen semigroup.

Lemma 2.3. *For any $M > 0$ there exists a positive constant $C = C(M)$ such that the following statements hold for any $\gamma_j \in \mathbb{R}$ with $|\gamma_j| \leqslant M$, $j = 1, 2$.*
(i) *Let $\gamma_1 > -1$. Then*

$$\int_0^\tau (\tau - s)^{-1} \left(1 + \frac{|X - Y - (\tau - s)\mathbf{e}_1|^2}{\tau - s}\right)^{-\frac{2+\gamma_1}{2}} ds \leqslant \begin{cases} -C \log |X - Y|, & \text{if } 0 < |X - Y| \leqslant \dfrac{1}{e}, \\ \dfrac{C}{(1+\gamma_1)|X - Y|^{\frac{1}{2}}(1 + |X - Y| - (X_1 - Y_1))^{\frac{1+\gamma_1}{2}}}, & \text{if } |X - Y| \geqslant \dfrac{1}{e}. \end{cases} \tag{2.11}$$

(ii) *Let $\gamma_1 > -1$, $\gamma_2 \neq 0$, and $\gamma_1 - |\gamma_2| > -2$. Then*

$$\int_0^\tau (\tau - s)^{-\frac{2+\gamma_2}{2}} \left(1 + \frac{|X - Y - (\tau - s)\mathbf{e}_1|^2}{\tau - s}\right)^{-\frac{2+\gamma_1}{2}} ds \leqslant \begin{cases} \dfrac{C}{|X - Y|^{\gamma_2}}\left(\dfrac{1}{\gamma_2} + \dfrac{1}{2 + \gamma_1 - \gamma_2}\right), \quad \text{if } 0 < |X - Y| \leqslant \dfrac{1}{e} \text{ and } \gamma_2 > 0, \\ C\left(\dfrac{1}{|\gamma_2|} + \dfrac{1}{2 + \gamma_1 - |\gamma_2|}\right), \quad \text{if } 0 < |X - Y| \leqslant \dfrac{1}{e} \text{ and } \gamma_2 < 0, \\ \dfrac{C}{|X - Y|^{\frac{1+\gamma_2}{2}}(1 + |X - Y| - (X_1 - Y_1))^{\frac{1+\gamma_1}{2}}}\left(\dfrac{1}{1 + \gamma_1} + \dfrac{1}{2 + \gamma_1 - |\gamma_2|}\right), \\ \qquad \text{if } |X - Y| \geqslant \dfrac{1}{e}. \end{cases} \tag{2.12}$$

Proof. Assume that $\gamma_1, \gamma_2 \in \mathbb{R}$ satisfy $\gamma_1 > -1$ and $\gamma_1 - |\gamma_2| > -2$. Then by changing the variable s as $\frac{|X-Y|}{\tau-s} = r$ and using $ds = \frac{|X-Y|}{r^2}\,dr$, we have

$$\int_0^\tau (\tau - s)^{-\frac{2+\gamma_2}{2}} \left(1 + \frac{|X - Y - (\tau - s)\mathbf{e}_1|^2}{\tau - s}\right)^{-\frac{2+\gamma_1}{2}} ds$$
$$= \int_{\frac{|X-Y|}{\tau}}^\infty \left(\frac{r}{|X - Y|}\right)^{\frac{2+\gamma_2}{2}} \left(1 + |X - Y|\left(r + \frac{1}{r}\right) - 2(X_1 - Y_1)\right)^{-\frac{2+\gamma_1}{2}} \frac{|X - Y|}{r^2}\,dr$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

$$\leqslant \frac{1}{|X-Y|^{\frac{\gamma_2}{2}}} \int_0^\infty \Big(1+2(|X-Y|-(X_1-Y_1))+\frac{|X-Y|}{r}(r-1)^2\Big)^{-\frac{2+\gamma_1}{2}} \frac{dr}{r^{1-\frac{\gamma_2}{2}}}$$
$$= \frac{1}{|X-Y|^{\frac{\gamma_2}{2}}} \Big(\int_0^{\frac12} + \int_{\frac12}^{\frac32} + \int_{\frac32}^\infty\Big) =: \frac{1}{|X-Y|^{\frac{\gamma_2}{2}}}(I+II+III).$$

(1) Estimate of I: We observe from $|X-Y|-(X_1-Y_1)\geqslant 0$ that

$$I \leqslant \int_0^{\frac12} \frac{r^{\frac{\gamma_1+\gamma_2}{2}}}{(r+|X-Y|(r-1)^2)^{\frac{2+\gamma_1}{2}}}\,dr \leqslant 2^{2+\gamma_1}\int_0^{\frac12} \frac{r^{\frac{\gamma_1+\gamma_2}{2}}}{(r+|X-Y|)^{\frac{2+\gamma_1}{2}}}\,dr.$$

If $|X-Y|\geqslant 1/e$ then it is easy to see that

$$\int_0^{\frac12} \frac{r^{\frac{\gamma_1+\gamma_2}{2}}}{(r+|X-Y|)^{\frac{2+\gamma_1}{2}}}\,dr \leqslant \frac{1}{|X-Y|^{\frac{2+\gamma_1}{2}}}\int_0^{\frac12} r^{\frac{\gamma_1+\gamma_2}{2}}\,dr \leqslant \frac{2}{(2+\gamma_1+\gamma_2)|X-Y|^{\frac{2+\gamma_1}{2}}}.$$

Next we consider the case $0<|X-Y|\leqslant 1/e$. In this case we see

$$\begin{aligned}\int_0^{\frac12} \frac{r^{\frac{\gamma_1+\gamma_2}{2}}}{(r+|X-Y|)^{\frac{2+\gamma_1}{2}}}\,dr &= \int_0^{|X-Y|} + \int_{|X-Y|}^{\frac12}\\ &\leqslant \frac{1}{|X-Y|^{\frac{2+\gamma_1}{2}}}\int_0^{|X-Y|} r^{\frac{\gamma_1+\gamma_2}{2}}\,dr + \int_{|X-Y|}^{\frac12} r^{-1+\frac{\gamma_2}{2}}\,dr\\ &\leqslant \frac{2}{2+\gamma_1+\gamma_2}|X-Y|^{\frac{\gamma_2}{2}} + \begin{cases} -\log|X-Y| & \text{if } \gamma_2=0\\ \dfrac{2}{\gamma_2} & \text{if } \gamma_2>0\\ \dfrac{2}{|\gamma_2|}|X-Y|^{\frac{\gamma_2}{2}} & \text{if } \gamma_2<0.\end{cases}\end{aligned}$$

The estimate of I is done.

(2) Estimate of II: We have

$$II \leqslant C\int_{\frac12}^{\frac32} (1+|X-Y|-(X_1-Y_1)+|X-Y|(r-1)^2)^{-\frac{2+\gamma_1}{2}}\,dr.$$

Here C is uniform in $\gamma_j\in\mathbb{R}$ with $|\gamma_j|\leqslant M$, $j=1,2$. Then, when $0<|X-Y|\leqslant 1/e$ it is easy to see that $II\leqslant C$. Thus it suffices to consider the case $|X-Y|\geqslant 1/e$. In this case we have

$$\begin{aligned}&\int_{\frac12}^{\frac32} (1+|X-Y|-(X_1-Y_1)+|X-Y|(r-1)^2)^{-\frac{2+\gamma_1}{2}}\,dr\\ &\leqslant \int_{\mathbb{R}} (1+|X-Y|-(X_1-Y_1)+|X-Y|(r-1)^2)^{-\frac{2+\gamma_1}{2}}\,dr\\ &= \frac{2}{|X-Y|^{\frac12}}\int_0^\infty (1+|X-Y|-(X_1-Y_1)+\eta^2)^{-\frac{2+\gamma_1}{2}}\,d\eta\\ &\leqslant \frac{C}{(1+\gamma_1)|X-Y|^{\frac12}(1+|X-Y|-(X_1-Y_1))^{\frac{1+\gamma_1}{2}}}.\end{aligned}$$

The estimate of II is done.

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

(3) Estimate of *III*: We have

$$III \leqslant C \int_{\frac{3}{2}}^{\infty} \frac{1}{(1+r|X-Y|)^{\frac{2+\gamma_1}{2}}} \frac{dr}{r^{1-\frac{\gamma_2}{2}}}.$$

Here C is uniform in $\gamma_j \in \mathbb{R}$ with $|\gamma_j| \leqslant M,\ j=1,2$. If $|X-Y| \geqslant 1/e$ then it is easy to see that

$$\int_{\frac{3}{2}}^{\infty} \frac{1}{(1+r|X-Y|)^{\frac{2+\gamma_1}{2}}} \frac{dr}{r^{1-\frac{\gamma_2}{2}}} \leqslant \frac{1}{|X-Y|^{\frac{2+\gamma_1}{2}}} \int_{\frac{3}{2}}^{\infty} \frac{dr}{r^{2+\frac{\gamma_1-\gamma_2}{2}}}$$
$$\leqslant \frac{2}{(2+\gamma_1-\gamma_2)|X-Y|^{\frac{2+\gamma_1}{2}}}.$$

When $0 < |X-Y| \leqslant 1/e$ we have

$$\int_{\frac{3}{2}}^{\infty} \frac{1}{(1+r|X-Y|)^{\frac{2+\gamma_1}{2}}} \frac{dr}{r^{1-\frac{\gamma_2}{2}}}$$
$$= \frac{1}{|X-Y|^{\frac{\gamma_2}{2}}} \int_{\frac{3}{2}|X-Y|}^{\infty} \frac{1}{(1+s)^{\frac{2+\gamma_1}{2}} s^{1-\frac{\gamma_2}{2}}}\, ds$$
$$\leqslant \frac{C}{|X-Y|^{\frac{\gamma_2}{2}}} \begin{cases} -\log|X-Y| + \dfrac{1}{2+\gamma_1} & \text{if } \gamma_2 = 0 \\ \dfrac{1}{\gamma_2} + \dfrac{1}{2+\gamma_1-\gamma_2} & \text{if } \gamma_2 > 0 \\ \dfrac{|X-Y|^{\gamma_2/2}}{|\gamma_2|} + \dfrac{1}{2+\gamma_1-\gamma_2} & \text{if } \gamma_2 < 0. \end{cases}$$

Finally we note that for $|X-Y| \geqslant 1/e$,

$$\frac{1}{|X-Y|^{1+\frac{\gamma_1+\gamma_2}{2}}} \leqslant \frac{C}{|X-Y|^{\frac{1+\gamma_2}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1+\gamma_1}{2}}}$$

since $1+\gamma_1 > 0$. Thus, collecting the estimates of I, *II*, and *III*, we obtain the desired estimates. The proof is complete. □

Next we set

$$\rho(X) = |X|^{\frac{1}{2}} + |X_2|. \tag{2.13}$$

Note that ρ defines a distance in $\mathbb{R}^2$.

Lemma 2.4. *Let* $-1 < \gamma_3, \gamma_4 < 2$ *and* $\gamma_3 + \gamma_4 > 1$. *Then for* $X \neq \mathbb{R}^2$,

$$\int_{\mathbb{R}^2} \frac{1}{\rho(X-Y)^{1+\gamma_3}\rho(Y)^{1+\gamma_4}}\, dY \leqslant \frac{C}{\rho(X)^{\gamma_3+\gamma_4-1}} \left(\frac{1}{2-\gamma_3} + \frac{1}{2-\gamma_4} + \frac{1}{\gamma_3+\gamma_4-1}\right). \tag{2.14}$$

Here C *is independent of* γ_3 *and* γ_4.

Proof. We decompose the left-hand side of (2.14) into three parts:

$$\text{L.H.S of (2.14)} = \int_{2\rho(Y)\leqslant\rho(X)} + \int_{\rho(X)\leqslant 2\rho(Y)\leqslant 4\rho(X)} + \int_{2\rho(X)\leqslant\rho(Y)} = I + II + III.$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

From $\rho(X-Y) \geqslant \rho(X)/2$ when $2\rho(Y) \leqslant \rho(X)$, by $1+\gamma_3 > 0$, I is estimated as

$$
\begin{aligned}
I &\leqslant \frac{C}{\rho(X)^{1+\gamma_3}} \int_{2\rho(Y)\leqslant\rho(X)} \frac{dY}{\rho(Y)^{1+\gamma_4}} \\
&\leqslant \frac{C}{\rho(X)^{1+\gamma_3}} \int_{2|Y_1|^{\frac{1}{2}}\leqslant\rho(X),2|Y_2|\leqslant\rho(X)} \frac{dY}{(|Y_1|^{\frac{1}{2}}+|Y_2|)^{1+\gamma_4}} \\
&\leqslant \frac{C}{\rho(X)^{1+\gamma_3}} \begin{cases} \dfrac{C}{\gamma_4} \displaystyle\int_{2|Y_1|^{\frac{1}{2}}\leqslant\rho(X)} \frac{dY_1}{|Y_1|^{\frac{\gamma_4}{2}}}, & \dfrac{1}{2} \leqslant \gamma_4 < 2, \\ C\rho(X) \displaystyle\int_{2|Y_1|^{\frac{1}{2}}\leqslant\rho(X)} \frac{dY_1}{|Y_1|^{\frac{1+\gamma_4}{2}}}, & -1 < \gamma_4 \leqslant \dfrac{1}{2} \end{cases} \\
&\leqslant \frac{C}{(2-\gamma_4)\rho(X)^{\gamma_3+\gamma_4-1}}.
\end{aligned}
$$

Here C is taken independently of γ_3, γ_4. As for II, we have

$$
\begin{aligned}
II &\leqslant \frac{C}{\rho(X)^{1+\gamma_4}} \int_{\rho(Y)\leqslant 2\rho(X)} \frac{dY}{\rho(X-Y)^{1+\gamma_3}} \\
&\leqslant \frac{C}{\rho(X)^{1+\gamma_4}} \int_{|X_1-Y_1|^{\frac{1}{2}}\leqslant 3\rho(X),|X_2-Y_2|\leqslant 3\rho(X)} \frac{dY}{(|X_1-Y_1|^{\frac{1}{2}}+|X_2-Y_2|)^{1+\gamma_3}} \\
&\leqslant \frac{C}{\rho(X)^{1+\gamma_4}} \begin{cases} \dfrac{C}{\gamma_3} \displaystyle\int_{|X_1-Y_1|^{\frac{1}{2}}\leqslant 3\rho(X)} \frac{dY_1}{|X_1-Y_1|^{\frac{\gamma_3}{2}}}, & \dfrac{1}{2} \leqslant \gamma_3 < 2, \\ C\rho(X) \displaystyle\int_{|X_1-Y_1|^{\frac{1}{2}}\leqslant 3\rho(X)} \frac{dY_1}{|X_1-Y_1|^{\frac{1+\gamma_3}{2}}}, & -1 < \gamma_3 \leqslant \dfrac{1}{2} \end{cases} \\
&\leqslant \frac{C}{(2-\gamma_3)\rho(X)^{\gamma_3+\gamma_4-1}}.
\end{aligned}
$$

Finally let us estimate III. We use $\rho(X-Y) \geqslant \rho(Y)/2$, which gives

$$
\begin{aligned}
III &\leqslant C \int_{2\rho(X)\leqslant\rho(Y)} \frac{dY}{\rho(Y)^{2+\gamma_3+\gamma_4}} \\
&= C\left(\int_{2\rho(X)\leqslant\rho(Y),|Y|^{\frac{1}{2}}\leqslant|Y_2|} \frac{dY}{\rho(Y)^{2+\gamma_3+\gamma_4}} + \int_{2\rho(X)\leqslant\rho(Y),|Y_2|\leqslant|Y|^{\frac{1}{2}}} \frac{dY}{\rho(Y)^{2+\gamma_3+\gamma_4}} \right) \\
&=: III_1 + III_2.
\end{aligned}
$$

The term III_1 is estimated from $\rho(Y) \leqslant 2|Y_2|$ for $|Y|^{\frac{1}{2}} \leqslant |Y_2|$ as

$$
\begin{aligned}
III_1 \leqslant C \int_{\rho(X)\leqslant|Y_2|} \frac{dY}{(|Y_1|+Y_2^2)^{\frac{2+\gamma_3+\gamma_4}{2}}} &\leqslant C \int_{\rho(X)\leqslant|Y_2|} \frac{dY_2}{|Y_2|^{\gamma_3+\gamma_4}} \\
&\leqslant \frac{C}{(\gamma_3+\gamma_4-1)\rho(X)^{\gamma_3+\gamma_4-1}}.
\end{aligned}
$$

The term III_2 is estimated as, by using the fact that $2\rho(X) \leqslant \rho(Y)$ with $|Y_2| \leqslant |Y|^{\frac{1}{2}}$ and $|Y_2| \geqslant 8$ implies $2\rho(X) \leqslant 8|Y_1|^{\frac{1}{2}}$,

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

$$
\begin{aligned}
III_2 &\leqslant C \int_{2\rho(X)\leqslant\rho(Y),|Y_2|\leqslant|Y|^{\frac{1}{2}},|Y_2|\leqslant 8} \frac{dY}{\rho(Y)^{2+\gamma_3+\gamma_4}} \\
&\quad + C \int_{2\rho(X)\leqslant\rho(Y),|Y_2|\leqslant|Y|^{\frac{1}{2}},|Y_2|\geqslant 8} \frac{dY}{\rho(Y)^{2+\gamma_3+\gamma_4}} \\
&\leqslant \frac{C}{\rho(X)^{\gamma_3+\gamma_4-1}} \int_{|Y_2|\leqslant 8} \frac{dY}{\rho(Y)^3} + C \int_{\rho(X)\leqslant 4|Y_1|^{\frac{1}{2}}} \frac{dY}{\rho(Y)^{2+\gamma_3+\gamma_4}} \\
&\leqslant \frac{C}{\rho(X)^{\gamma_3+\gamma_4-1}} \int_{|Y_2|\leqslant 8} \frac{dY}{|Y|^{\frac{3}{2}}} + C \int_{\rho(X)\leqslant 4|Y_1|^{\frac{1}{2}}} \frac{dY_1}{|Y_1|^{\frac{1+\gamma_3+\gamma_4}{2}}} \\
&\leqslant \frac{C}{\rho(X)^{\gamma_3+\gamma_4-1}} + \frac{C}{(\gamma_3+\gamma_4-1)\rho(X)^{\gamma_3+\gamma_4-1}}.
\end{aligned}
$$

Collecting these above, we obtain (2.14). The proof is complete. □

3. Estimate of Oseen semigroup in $\mathbb{R}^2$

In this section we establish weighted L^∞ estimates of the Oseen semigroup in $\mathbb{R}^2$, written as $e^{-t\mathbb{A}_\alpha}$, and the Duhamel term $\int_0^t e^{-(t-s)\mathbb{A}_\alpha}\mathbb{P}F(s)\,ds$. Here the operator $e^{-t\mathbb{A}_\alpha}\mathbb{P}$ is realized as a convolution operator $e^{-t\mathbb{A}_\alpha}\mathbb{P}f = \Phi_\alpha(t) * f$, and the kernel $\Phi_\alpha(t,x)$ is given by

$$\Phi_\alpha(t,x) = \Phi(t, x - \alpha t\mathbf{e}_1), \tag{3.1}$$

where $\mathbf{e}_1 = (1,0)$ and Φ is defined in terms of the Fourier transform

$$\Phi(\tau, X) = \mathcal{F}^{-1}\left[e^{-\tau|\xi|^2}\left(\mathbb{I} - \frac{\xi\otimes\xi}{|\xi|^2}\right)\right](X). \tag{3.2}$$

Here $\mathbb{I} = (\delta_{jk})_{1\leqslant j,k\leqslant 2}$ and $\xi\otimes\xi = (\xi_j\xi_k)_{1\leqslant j,k\leqslant 2}$. Note that Φ is nothing but the kernel of $e^{\tau\Delta}\mathbb{P}$. By its definition the pointwise estimate of Φ_α is available from that of Φ, which is well known by now as follows.

Proposition 3.1. *Let $k = 0, 1, \ldots,$ and let $j = (j_1, j_2)$ be a multi-index. Then*

$$|\partial_\tau^k \nabla_X^j \Phi(\tau, X)| \leqslant C\tau^{-1-\frac{|j|}{2}-k}\left(1 + \frac{|X|^2}{\tau}\right)^{-1-\frac{|j|}{2}-k}. \tag{3.3}$$

Here C depends only on k and j.

3.1. Estimate of semigroup in $\mathbb{R}^2$

We start from the estimates of $e^{-t\mathbb{A}_\alpha}f$ in the weighted L^∞ norm. Let us recall that $b(X)$ is defined as (1.6) and $\rho(X) = |X|^{\frac{1}{2}} + |X_2|$.

Proposition 3.2. *Let $\beta \in [0,1]$. Let $b(\alpha\cdot)f \in L^\infty(\mathbb{R}^2)^2$ and* $\operatorname{div} f = 0$ *in $\mathbb{R}^2$. Then*

$$\sup_{t>0}(1 + (\alpha^2 t)^{\frac{\beta}{2}})\|\rho(\alpha\cdot)^{1-\beta}e^{-t\mathbb{A}_\alpha}f\|_{L^\infty} \leqslant C\|(1 + b(\alpha\cdot))f\|_{L^\infty}, \tag{3.4}$$

and

$$\sup_{t>0} t^{\frac{1}{2}}\|\nabla e^{-t\mathbb{A}_\alpha}f\|_{L^\infty} \leqslant C\|f\|_{L^\infty}. \tag{3.5}$$

Here C is independent of α and β.

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

Proof. Let $G(t,x)=\frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}$ be the Gauss kernel in $\mathbb{R}^2$. Then, in virtue of the condition $\operatorname{div} f=0$ in $\mathbb{R}^2$, the following representation holds:

$$(e^{-t\mathbb{A}_\alpha}f)(x)=\int_{\mathbb{R}^2}G(t,x-y-\alpha t\mathbf{e}_1)f(y)\,dy. \tag{3.6}$$

Then it is straightforward to see

$$\|\nabla^j e^{-t\mathbb{A}_\alpha}f\|_{L^\infty}\leqslant Ct^{-\frac{j}{2}}\|f\|_{L^\infty},\quad t>0,\ j=0,1. \tag{3.7}$$

Next we observe that

$$|(e^{-t\mathbb{A}_\alpha}f)(x)|\leqslant\int_{\mathbb{R}^2}G(t,x-y-\alpha t\mathbf{e}_1)\left(\frac{1}{|\alpha y|^{\frac12}(1+|\alpha y|-\alpha y_1)^{\frac34}}+\frac{1}{1+|\alpha y|}\right)dy \times\|b(\alpha\cdot)f\|_{L^\infty}.$$

Let us recall from Lemma A.1 that $\frac{1}{|Y|^{\frac12}(1+|Y|-Y_1)^{\frac34}}\in L^\infty_{Y_1}L^1_{Y_2}$, and we also note $\frac{1}{1+|Y|}\in L^{2,\infty}_Y(\mathbb{R}^2)$. Thus we conclude from $\|G(t,x-\cdot-\alpha t\mathbf{e}_1)\|_{L^1_{y_1}L^\infty_{y_2}}+\|G(t,x-\cdot-\alpha t\mathbf{e}_1)\|_{L^{2,1}_y}\leqslant Ct^{-\frac12}$ that

$$\|e^{-t\mathbb{A}_\alpha}f\|_{L^\infty}\leqslant\frac{C}{\alpha t^{\frac12}}\|b(\alpha\cdot)f\|_{L^\infty},\quad t>0. \tag{3.8}$$

Estimates (3.7) and (3.8) yield (3.4) with $\beta=1$. Next, since ρ defines a distance in $\mathbb{R}^2$, we have

$$\begin{aligned}
&\rho(\alpha x)|(e^{-t\mathbb{A}_\alpha}f)(x)|\\
&\leqslant\int_{\mathbb{R}^2}(\rho(\alpha x-\alpha y-\alpha^2t\mathbf{e}_1)+\rho(\alpha y)+\rho(\alpha^2t\mathbf{e}_1))G(t,x-y-\alpha t\mathbf{e}_1)|f(y)|\,dy\\
&\leqslant\int_{\mathbb{R}^2}(\alpha^{\frac12}|x-y-\alpha t\mathbf{e}_1|^{\frac12}+\alpha|x_2-y_2|)G(t,x-y-\alpha t\mathbf{e}_1)|f(y)|\,dy\\
&\quad+\|\rho(\alpha\cdot)f\|_{L^\infty}+\alpha t^{\frac12}\int_{\mathbb{R}^2}G(t,x-y-\alpha t\mathbf{e}_1)|f(y)|\,dy\\
&\leqslant C\alpha^{\frac12}t^{\frac14}\int_{\mathbb{R}^2}G(2t,x-y-\alpha t\mathbf{e}_1)\frac{1}{\rho(\alpha y)}\,dy\|\rho(\alpha\cdot)f\|_{L^\infty}\\
&\quad+C\alpha t^{\frac12}\int_{\mathbb{R}^2}G(2t,x-y-\alpha t\mathbf{e}_1)|f(y)|\,dy\\
&\quad+\|\rho(\alpha\cdot)f\|_{L^\infty}+\alpha t^{\frac12}\int_{\mathbb{R}^2}G(t,x-y-\alpha t\mathbf{e}_1)|f(y)|\,dy.
\end{aligned}$$

Then, by using the fact that $\frac{1}{\rho(\alpha y)}\leqslant\frac{1}{|\alpha y|^{1/2}}$ with $|y|^{-\frac12}\in L^{4,\infty}(\mathbb{R}^2)$ and also by arguing as in the derivation of (3.8) (for the second and the forth terms), we obtain

$$\rho(\alpha x)|(e^{-t\mathbb{A}_\alpha}f)(x)|\leqslant C(\|\rho(\alpha\cdot)f\|_{L^\infty}+\|b(\alpha\cdot)f\|_{L^\infty}).$$

Thus the inequality $\|\rho(\alpha\cdot)f\|_{L^\infty}\leqslant C\|b(\alpha\cdot)f\|_{L^\infty}$ (by Lemma A.1) yields

$$\|\rho(\alpha\cdot)e^{-t\mathbb{A}_\alpha}f\|_{L^\infty}\leqslant C\|b(\alpha\cdot)f\|_{L^\infty},\quad t>0. \tag{3.9}$$

Here C is independent of α. Estimates (3.9) and (3.7) prove (3.4) with $\beta=1$. Then the case $\beta\in(0,1)$ of (3.4) follows from the interpolation of the endpoint cases $\beta=0,1$. The proof is complete. □

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

3.2. Estimate of bilinear form in $\mathbb{R}^2$

We set the bilinear form $\Lambda_{\mathbb{A}_\alpha}[f,g]$ as

$$\Lambda_{\mathbb{A}_\alpha}[f,g](t) = \alpha \int_0^t e^{-(t-s)\mathbb{A}_\alpha} \mathbb{P}\nabla \cdot (f \otimes g + g \otimes f)\, ds. \tag{3.10}$$

Our aim is to establish the weighted L^∞ estimate of $\Lambda_{\mathbb{A}_\alpha}[f,g]$ globally in time by using the scale-critical anisotropic norm.

Proposition 3.3. *Let $\alpha \in (0, \frac{1}{2}]$ and $\beta \in [0,1)$. Then the following estimates hold for any $f, g \in L^\infty(0,\infty; W^{1,\infty}(\mathbb{R}^2)^2)$ such that $\rho(\alpha\cdot)f, \rho(\alpha\cdot)g \in L^\infty(0,\infty; L^\infty(\mathbb{R}^2)^2)$.*

$$\begin{aligned} &\sup_{t>0} (\alpha^2 t)^{\frac{\beta}{2}} \|\rho(\alpha\cdot)^{1-\beta} \Lambda_{\mathbb{A}_\alpha}[f,g](t)\|_{L^\infty} \\ &\quad \leqslant \frac{C}{1-\beta} \sup_{t>0} \|\rho(\alpha\cdot) f(t)\|_{L^\infty} \sup_{t>0} ((\alpha^2 t)^{\frac{\beta}{2}} \|\rho(\alpha\cdot)^{1-\beta} g(t)\|_{L^\infty} + \|\rho(\alpha\cdot) g(t)\|_{L^\infty}), \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\sup_{t>0} (1 + (\alpha^2 t)^{\frac{\beta}{2}}) \|\Lambda_{\mathbb{A}_\alpha}[f,g](t)\|_{L^\infty} \\ &\quad \leqslant \frac{C}{1-\beta} \sup_{t>0} \|\rho(\alpha\cdot) f(t)\|_{L^\infty} \\ &\qquad \times \sup_{t>0} ((\alpha^2 t)^{\frac{\beta}{2}} \|\rho(\alpha\cdot)^{1-\beta} g(t)\|_{L^\infty} + \|\rho(\alpha\cdot) g(t)\|_{L^\infty} + (1 + (\alpha^2 t)^{\frac{\beta}{2}}) \|g(t)\|_{L^\infty}), \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\sup_{t>0} (1 + (\alpha^2 t)^{\frac{\beta}{2}}) \left(\frac{t}{1+t}\right)^{\frac{1}{2}} \|\nabla \Lambda_{\mathbb{A}_\alpha}[f,g](t)\|_{L^\infty} \\ &\quad \leqslant C\alpha^{\frac{1}{2}} \sup_{t>0} \left(\|\rho(\alpha\cdot) f(t)\|_{L^\infty} + \alpha^{\frac{1}{2}} \left(\frac{t}{1+t}\right)^{\frac{1}{2}} \|\nabla f(t)\|_{L^\infty} \right) \\ &\qquad \times \sup_{t>0} (1 + (\alpha^2 t)^{\frac{\beta}{2}}) \left(\|g(t)\|_{L^\infty} + \left(\frac{t}{1+t}\right)^{\frac{1}{2}} \|\nabla g(t)\|_{L^\infty} \right). \end{aligned} \tag{3.13}$$

Here C is independent of α and β.

Remark 3.4. (i) In Proposition 3.3 the function f does not need to decay in time, which is important to handle the linear term $\nabla \cdot (U \otimes u + u \otimes U)$ with the physically reasonable solution U.
(ii) Estimate (3.13) is easier, for it is subcritical estimate in view of scaling. Estimate (3.12) is still subcritical, but the proof requires a detailed computation in order to derive the prefactor $\frac{1}{1-\beta}$, which could be optimal. Estimate (3.11) is just critical in view of scaling and is the most nontrivial estimate in Proposition 3.3. Its proof consists of the pointwise estimate of the kernel given by (3.1) and Proposition 3.1, and the integral inequalities obtained in Lemmas 2.3, 2.4.

Proof of Proposition 3.3. (i) Proof of (3.11) and (3.12): We observe that

$$\Lambda_{\mathbb{A}_\alpha}[f,g](t,x) = \Lambda_{\mathbb{A}_1}[\tilde{f},\tilde{g}](\tau, X), \quad \tau = \alpha^2 t,\ \ X = \alpha x$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

by the change of variables in the integral, where $\tilde{f}(\tau, X) = f(t, x)$ and $\tilde{g}(\tau, X) = g(t, x)$ with $\tau = \alpha^2 t$ and $X = \alpha x$. Then (3.11) and (3.12) are respectively equivalent with the weighted estimate

$$\sup_{\tau>0} \tau^{\frac{\beta}{2}} \|\rho^{1-\beta} \Lambda_{\mathbb{A}_1}[\tilde{f}, \tilde{g}](\tau)\|_{L^\infty} \leqslant \frac{C}{1-\beta} \sup_{\tau>0} \|\rho \tilde{f}(\tau)\|_{L^\infty} \sup_{\tau>0} (\tau^{\frac{\beta}{2}} \|\rho^{1-\beta} \tilde{g}(\tau)\|_{L^\infty} + \|\rho \tilde{g}(\tau)\|_{L^\infty}), \tag{3.14}$$

and the L^∞ estimate

$$\sup_{\tau>0} (1+\tau^{\frac{\beta}{2}}) \|\Lambda_{\mathbb{A}_1}[\tilde{f}, \tilde{g}](\tau)\|_{L^\infty} \leqslant \frac{C}{1-\beta} \sup_{\tau>0} \|\rho \tilde{f}(\tau)\|_{L^\infty} \sup_{\tau>0} (\tau^{\frac{\beta}{2}} \|\rho^{1-\beta} \tilde{g}(\tau)\|_{L^\infty} + \|\rho \tilde{g}(\tau)\|_{L^\infty} + (1+\tau^{\frac{\beta}{2}}) \|\tilde{g}(\tau)\|_{L^\infty}). \tag{3.15}$$

Let us first consider the L^∞ estimate for a short time. To simplify the notation we set

$$K(\tau, X-Y) = \left(1 + \frac{|X - Y - \tau \mathbf{e}_1|^2}{\tau}\right)^{-1}. \tag{3.16}$$

Then, from (3.1)–(3.3), the kernel of $e^{-\tau \mathbb{A}_1} \mathbb{P} \nabla \cdot$ is pointwise bounded from above by $C\tau^{-\frac{3}{2}} K(\tau, X-Y)^{\frac{3}{2}}$ for all $\tau > 0, X, Y \in \mathbb{R}^2$. Thus we have $\|e^{-\tau \mathbb{A}_1} \mathbb{P} \nabla \cdot F\|_{L^\infty} \leqslant C\tau^{-\frac{3}{2}} \|K(\tau)\|_{L^{\frac{4}{3},1}} \|F\|_{L^{4,\infty}} \leqslant C\tau^{-\frac{3}{4}} \|F\|_{L^{4,\infty}}$ by (2.3) and the real interpolation. Then the definition of $\Lambda_{\mathbb{A}_1}[\tilde{f}, \tilde{g}]$ implies

$$\begin{aligned} \|\Lambda_{\mathbb{A}_1}[\tilde{f}, \tilde{g}](\tau)\|_{L^\infty} &\leqslant C \int_0^\tau (\tau - s)^{-\frac{3}{4}} \|\tilde{f} \otimes \tilde{g}(s) + \tilde{g} \otimes \tilde{f}(s)\|_{L^{4,\infty}} \, ds \\ &\leqslant C\tau^{\frac{1}{4}} \sup_{s>0} \|\tilde{g}(s)\|_{L^\infty} \sup_{s>0} \|\rho \tilde{f}(s)\|_{L^\infty}, \quad 0 < \tau \leqslant 2. \end{aligned} \tag{3.17}$$

Here we have used $\rho(X)^{-1} \leqslant |X|^{-1/2} \in L^{4,\infty}(\mathbb{R}^2)$, and hence, $\|\tilde{f}(s)\|_{L^{4,\infty}} \leqslant C\|\rho \tilde{f}(s)\|_{L^\infty}$. Next we prove the weighted (3.14), and the L^∞ estimate (3.15) for the case $\tau \geqslant 2$ is discussed later. To this end we recall that

$$|\Lambda_{\mathbb{A}_1}[\tilde{f}, \tilde{g}](\tau, X)| \leqslant C \int_0^\tau \int_{\mathbb{R}^2} (\tau - s)^{-\frac{3}{2}} K(\tau - s, X - Y)^{\frac{3}{2}} |\tilde{f} \otimes \tilde{g} + \tilde{g} \otimes \tilde{f}(\tau, Y)| \, dY \, ds, \tag{3.18}$$

and then, by splitting the time integral $\int_0^\tau$ into $\int_0^{\frac{\tau}{2}}$ and $\int_{\frac{\tau}{2}}^\tau$,

$$\begin{aligned} &|\Lambda_{\mathbb{A}_1}[\tilde{f}, \tilde{g}](\tau, X)| \\ &\leqslant C\tau^{-\frac{\beta}{2}} \int_0^{\frac{\tau}{2}} \int_{\mathbb{R}^2} (\tau - s)^{-\frac{3-\beta}{2}} K(\tau - s, X - Y)^{\frac{3}{2}} \frac{1}{\rho(Y)^2} \, dY \, ds \\ &\quad \times \sup_{\tau>0} \|\rho \tilde{g}(\tau)\|_{L^\infty} \sup_{\tau>0} \|\rho \tilde{f}(\tau)\|_{L^\infty} \end{aligned}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

$$+C\tau^{-\frac{\beta}{2}}\int_{\frac{\tau}{2}}^{\tau}\int_{\mathbb{R}^2}(\tau-s)^{-\frac{3}{2}}K(\tau-s,X-Y)^{\frac{3}{2}}\frac{1}{\rho(Y)^{2-\beta}}\,dY\,ds$$
$$\times\sup_{\tau>0}\tau^{\frac{\beta}{2}}\|\rho^{1-\beta}\tilde{g}(\tau)\|_{L^\infty}\sup_{\tau>0}\|\rho\tilde{f}(\tau)\|_{L^\infty}. \tag{3.19}$$

Now we apply Lemma 2.3 with $\gamma_1 = 1$ and $\gamma_2 = 1-\beta, 1$, which is the key in our argument. Then it follows that

$$\begin{aligned}
&|\Lambda_{\mathbb{A}_1}[\tilde{f},\tilde{g}](\tau,X)|\\
&\leqslant C\tau^{-\frac{\beta}{2}}\left\{\frac{1}{1-\beta}\int_{\{|X-Y|\leqslant\frac{1}{e}\}}\frac{1}{|X-Y|^{1-\beta}}\frac{1}{\rho(Y)^2}\,dY\right.\\
&\quad\left.+\int_{\{|X-Y|\geqslant\frac{1}{e}\}}\frac{1}{|X-Y|^{\frac{2-\beta}{2}}(1+|X-Y|-(X_1-Y_1))}\frac{1}{\rho(Y)^2}\,dY\right\}\\
&\quad\times\sup_{\tau>0}\|\rho\tilde{g}(\tau)\|_{L^\infty}\sup_{\tau>0}\|\rho\tilde{f}(\tau)\|_{L^\infty}\\
&\quad+C\tau^{-\frac{\beta}{2}}\int_{\mathbb{R}^2}\frac{1}{|X-Y|(1+|X-Y|-(X_1-Y_1))}\frac{1}{\rho(Y)^{2-\beta}}\,dY\\
&\quad\times\sup_{\tau>0}\tau^{\frac{\beta}{2}}\|\rho^{1-\beta}\tilde{g}(\tau)\|_{L^\infty}\sup_{\tau>0}\|\rho\tilde{f}(\tau)\|_{L^\infty}.
\end{aligned}$$

As for the first term, by decomposing as $\int_{\{|X-Y|\leqslant\frac{1}{e}\}}=\int_{\{|X-Y|\leqslant\frac{1}{e},\ |Y|\leqslant 1\}}+\int_{\{|X-Y|\leqslant\frac{1}{e},|Y|\geqslant 1\}}$ and by using $\frac{1}{\rho(Y)^2}\leqslant\frac{1}{|Y|}$ and the triangle inequality $\rho(X)^{1-\beta}\leqslant\rho(X-Y)^{1-\beta}+\rho(Y)^{1-\beta}$, it is not difficult to see that $\int_{\{|X-Y|\leqslant\frac{1}{e}\}}\frac{1}{|X-Y|^{1-\beta}\rho(Y)^2}\,dY\leqslant\frac{C}{\rho(X)^{1-\beta}}$ with C independent of β. As for the second and the third terms, we recall the inequality

$$\frac{1}{|X-Y|^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}}\leqslant\frac{C}{\rho(X-Y)},$$

which follows from Lemma A.1 (the square root of (A 3)). Then we arrive at

$$\begin{aligned}
&|\Lambda_{\mathbb{A}_1}[\tilde{f},\tilde{g}](\tau,X)|\\
&\leqslant C\tau^{-\frac{\beta}{2}}\left\{\frac{1}{(1-\beta)\rho(X)^{1-\beta}}+\int_{\mathbb{R}^2}\frac{1}{\rho(X-Y)^{2-\beta}\rho(Y)^2}\,dY\right\}\\
&\quad\times\sup_{\tau>0}\|\rho\tilde{g}(\tau)\|_{L^\infty}\sup_{\tau>0}\|\rho\tilde{f}(\tau)\|_{L^\infty}\\
&\quad+C\tau^{-\frac{\beta}{2}}\int_{\mathbb{R}^2}\frac{1}{\rho(X-Y)^2\rho(Y)^{2-\beta}}\,dY\sup_{\tau>0}\tau^{\frac{\beta}{2}}\|\rho^{1-\beta}\tilde{g}(\tau)\|_{L^\infty}\sup_{\tau>0}\|\rho\tilde{f}(\tau)\|_{L^\infty}.
\end{aligned}\tag{3.20}$$

Note that $\int_{\mathbb{R}^2}\frac{1}{\rho(X-Y)^2\rho(Y)^{2-\beta}}\,dY=\int_{\mathbb{R}^2}\frac{1}{\rho(X-Y)^{2-\beta}\rho(Y)^2}\,dY$. Then Lemma 2.4 with $\gamma_3 = 1-\beta$ and $\gamma_4 = 1$ gives

$$\begin{aligned}
&|\Lambda_{\mathbb{A}_1}[\tilde{f},\tilde{g}](\tau,X)|\\
&\leqslant\frac{C}{(1-\beta)\tau^{\frac{\beta}{2}}\rho(X)^{1-\beta}}\sup_{\tau>0}(\|\rho\tilde{g}(\tau)\|_{L^\infty}+\tau^{\frac{\beta}{2}}\|\rho^{1-\beta}\tilde{g}(\tau)\|_{L^\infty})\sup_{\tau>0}\|\rho\tilde{f}(\tau)\|_{L^\infty},
\end{aligned}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

that is,

$$\sup_{\tau>0} \tau^{\frac{\beta}{2}} \|\rho^{1-\beta} \Lambda_{\mathbb{A}_1}[\tilde{f}, \tilde{g}](\tau)\|_{L^\infty} \leqslant \frac{C}{1-\beta} \sup_{\tau>0} (\|\rho \tilde{g}(\tau)\|_{L^\infty} + \tau^{\frac{\beta}{2}} \|\rho^{1-\beta} \tilde{g}(\tau)\|_{L^\infty}) \sup_{\tau>0} \|\rho \tilde{f}(\tau)\|_{L^\infty}. \tag{3.21}$$

This proves the weighted estimate (3.14). The L^∞ estimate (3.15) is proved in the similar manner. Indeed, instead of (3.19), we compute as

$$\begin{aligned} &|\Lambda_{\mathbb{A}_1}[\tilde{f}, \tilde{g}](\tau, X)| \\ &\leqslant C\tau^{-\frac{\beta}{2}} \int_0^{\frac{\tau}{2}} \int_{\mathbb{R}^2} (\tau - s)^{-\frac{3-\beta}{2}} K(\tau - s, X - Y)^{\frac{3}{2}} \frac{1}{\rho(Y)(1+\rho(Y))} \, dY \, ds \\ &\quad \times \sup_{\tau>0} \|(1+\rho)\tilde{g}(\tau)\|_{L^\infty} \sup_{\tau>0} \|\rho \tilde{f}(\tau)\|_{L^\infty} \\ &\quad + C\tau^{-\frac{\beta}{2}} \int_{\frac{\tau}{2}}^{\tau} \int_{\mathbb{R}^2} (\tau - s)^{-\frac{3}{2}} K(\tau - s, X - Y)^{\frac{3}{2}} \frac{1}{\rho(Y)(1+\rho(Y)^{1-\beta})} \, dY \, ds \\ &\quad \times \sup_{\tau>0} \tau^{\frac{\beta}{2}} \|(1+\rho^{1-\beta})\tilde{g}(\tau)\|_{L^\infty} \sup_{\tau>0} \|\rho \tilde{f}(\tau)\|_{L^\infty}. \end{aligned} \tag{3.22}$$

Then the above argument using Lemma 2.3 yields

$$\begin{aligned} &|\Lambda_{\mathbb{A}_1}[\tilde{f}, \tilde{g}](\tau, X)| \\ &\leqslant C\tau^{-\frac{\beta}{2}} \left\{ \frac{1}{1-\beta} \int_{\{|X-Y| \leqslant \frac{1}{e}\}} \frac{1}{|X-Y|^{1-\beta}} \frac{1}{\rho(Y)(1+\rho(Y))} \, dY \right. \\ &\quad \left. + \int_{\{|X-Y| \geqslant \frac{1}{e}\}} \frac{1}{\rho(X-Y)^{2-\beta} \rho(Y)(1+\rho(Y))} \, dY \right\} \sup_{\tau>0} \|(1+\rho)\tilde{g}(\tau)\|_{L^\infty} \sup_{\tau>0} \|\rho \tilde{f}(\tau)\|_{L^\infty} \\ &\quad + C\tau^{-\frac{\beta}{2}} \int_{\mathbb{R}^2} \frac{1}{\rho(X-Y)^2 \rho(Y)(1+\rho(Y))^{1-\beta}} \, dY \\ &\quad \times \sup_{\tau>0} \tau^{\frac{\beta}{2}} \|(1+\rho^{1-\beta})\tilde{g}(\tau)\|_{L^\infty} \sup_{\tau>0} \|\rho \tilde{f}(\tau)\|_{L^\infty}. \end{aligned} \tag{3.23}$$

The difference from (3.20) is that the singularity around $Y = 0$ is relaxed thanks to the factor $\frac{1}{1+\rho(Y)}$ rather than $\frac{1}{\rho(Y)}$. Then it is not difficult to show that for $|X| \leqslant 1$,

$$\begin{aligned} &\int_{\{|X-Y| \leqslant \frac{1}{e}\}} \frac{1}{|X-Y|^{1-\beta}} \frac{1}{\rho(Y)(1+\rho(Y))} \, dY \leqslant C, \\ &\int_{\{|X-Y| \geqslant \frac{1}{e}\}} \frac{1}{\rho(X-Y)^{2-\beta} \rho(Y)(1+\rho(Y))} \, dY \leqslant \frac{C}{1-\beta}, \\ &\int_{\mathbb{R}^2} \frac{1}{\rho(X-Y)^2 \rho(Y)(1+\rho(Y))^{1-\beta}} \, dY \leqslant \frac{C}{1-\beta}. \end{aligned}$$

On the other hand, we already know from Lemma 2.4 that

$$\int_{\{|X-Y| \leqslant \frac{1}{e}\}} \frac{1}{|X-Y|^{1-\beta}} \frac{1}{\rho(Y)(1+\rho(Y))} \, dY \leqslant \frac{C}{\rho(X)^{1-\beta}},$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

$$\int_{\{|X-Y|\geqslant \frac{1}{e}\}} \frac{1}{\rho(X-Y)^{2-\beta}\rho(Y)(1+\rho(Y))}\, dY \leqslant \frac{C}{(1-\beta)\rho(X)^{1-\beta}},$$
$$\int_{\mathbb{R}^2} \frac{1}{\rho(X-Y)^2\rho(Y)(1+\rho(Y))^{1-\beta}}\, dY \leqslant \frac{C}{(1-\beta)\rho(X)^{1-\beta}}.$$

These three bounds are applied for $|X| \geqslant 1$. Thus we have shown for $\tau \geqslant 2$,

$$\tau^{\frac{\beta}{2}}\|\Lambda_{\mathbb{A}_1}[\tilde{f},\tilde{g}](\tau)\|_{L^\infty} \leqslant \frac{C}{1-\beta}\sup_{\tau>0}\|(1+\rho)\tilde{g}(\tau)\|_{L^\infty}\sup_{\tau>0}\|\rho\tilde{f}(\tau)\|_{L^\infty} + \frac{C}{1-\beta}\sup_{\tau>0}\tau^{\frac{\beta}{2}}\|(1+\rho^{1-\beta})\tilde{g}(\tau)\|_{L^\infty}\sup_{\tau>0}\|\rho\tilde{f}(\tau)\|_{L^\infty}. \tag{3.24}$$

Combining (3.17) and (3.24), we obtain (3.15).
(ii) Proof of (3.13): We estimate in the original variables t and x. We denote by χ_B the characteristic function of the set B. Let us recall the estimate $\|\nabla e^{-t\mathbb{A}_\alpha}\mathbb{P}h\|_{L^\infty} \leqslant Ct^{-\frac{1}{2}}\|h\|_{L^\infty}$, $\|\nabla e^{-t\mathbb{A}_\alpha}\mathbb{P}h\|_{L^\infty} \leqslant Ct^{-\frac{3}{4}}\|h\|_{L^{4,\infty}}$, and $\|\nabla e^{-t\mathbb{A}_\alpha}\mathbb{P}\nabla\cdot F\|_{L^\infty} \leqslant Ct^{-\frac{5}{4}}\|F\|_{L^{4,\infty}}$, which yield by using $\|f\nabla^j g\|_{L^{4,\infty}} \leqslant C\|f\|_{L^{4,\infty}}\|\nabla^j g\|_{L^\infty}$ and $\|f\|_{L^{4,\infty}} \leqslant C\alpha^{-\frac{1}{2}}\|\rho(\alpha\cdot)f\|_{L^\infty}$,

$$\begin{aligned}
&\|\nabla\Lambda_{\mathbb{A}_\alpha}[f,g](t)\|_{L^\infty}\\
&\quad\leqslant C\alpha\int_0^t \chi_{\{t-s\leqslant 1\}}(t-s)^{-\frac{1}{2}}\|g\nabla f(s)\|_{L^\infty}\, ds\\
&\qquad + C\alpha\int_0^t \chi_{\{t-s\leqslant 1\}}(t-s)^{-\frac{3}{4}}\|f\nabla g(s)\|_{L^{4,\infty}}\, ds\\
&\qquad + C\alpha\int_0^t \chi_{\{t-s\geqslant 1\}}(t-s)^{-\frac{5}{4}}\|fg(s)\|_{L^{4,\infty}}\, ds\\
&\quad\leqslant C\alpha\int_0^t \chi_{\{t-s\leqslant 1\}}(t-s)^{-\frac{1}{2}}(1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1}(\frac{1+s}{s})^{\frac{1}{2}}\, ds\\
&\qquad \times\sup_{s>0}(1+(\alpha^2 s)^{\frac{\beta}{2}})\|g(s)\|_{L^\infty}\sup_{s>0}\left(\frac{s}{1+s}\right)^{\frac{1}{2}}\|\nabla f(s)\|_{L^\infty}\\
&\qquad + C\alpha^{\frac{1}{2}}\int_0^t \chi_{\{t-s\leqslant 1\}}(t-s)^{-\frac{3}{4}}(1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1}\left(\frac{1+s}{s}\right)^{\frac{1}{2}}\, ds\\
&\qquad \times\sup_{s>0}\|\rho(\alpha\cdot)f(s)\|_{L^\infty}\sup_{s>0}(1+(\alpha^2 s)^{\frac{\beta}{2}})\left(\frac{s}{1+s}\right)^{\frac{1}{2}}\|\nabla g(s)\|_{L^\infty}\\
&\qquad + C\alpha^{\frac{1}{2}}\int_0^t \chi_{\{t-s\geqslant 1\}}(t-s)^{-\frac{5}{4}}(1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1}\, ds\\
&\qquad \times\sup_{s>0}\|\rho(\alpha\cdot)f(s)\|_{L^\infty}\sup_{s>0}(1+(\alpha^2 s)^{\frac{\beta}{2}})\|g(s)\|_{L^\infty}.
\end{aligned}\tag{3.25}$$

Thus (3.13) follows. The proof is complete. □

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

3.3. Estimate for compactly supported data

In this subsection we consider the estimate of the term

$$\Upsilon_{\mathbb{A}_\alpha}[R](t) = \int_0^t e^{-(t-s)\mathbb{A}_\alpha}\mathbb{P}R(s)\,ds, \tag{3.26}$$

when $R(s)$ is compactly supported for each s. The term of this kind naturally appears in the cut-off procedure near the boundary in the exterior problem. We note that estimate (3.28) in Proposition 3.5 below is valid for $0 \leqslant \beta \leqslant 1$ (including $\beta = 1$), but with the logarithmic singularity $|\log\alpha|$ in the coefficient. This logarithmic singularity is inevitable in general, for the kernel of $\int_0^t e^{-(t-s)\mathbb{A}_\alpha}\mathbb{P}\,ds$ has a similar behavior as the one of $\mathbb{A}_\alpha^{-1}$ for $t \gg 1$. Indeed, the kernel of $\mathbb{A}_\alpha^{-1}$ is uniformly bounded in the regime $|\alpha x| \geqslant O(1)$ in virtue of the transport term $\alpha\partial_1$, while, as indicated by (2.11), it contains the term $C\log|\alpha x|$ in the regime $|\alpha x| \leqslant O(1)$ and this is responsible for the singularity $|\log\alpha|$ in (3.28).

Proposition 3.5. *Let* $\alpha \in (0, \frac{1}{2}]$ *and* $\beta \in [0, 1]$. *Suppose that* $R(t) \in L^4(\mathbb{R}^2)^2$ *and* $\operatorname{supp} R(t) \subset \{|x| \leqslant 6\}$ *for all* $t > 0$. *Then*

$$\sup_{t>0}(\alpha^2 t)^{\frac{\beta}{2}}\|\rho(\alpha\cdot)^{1-\beta}\Upsilon_{\mathbb{A}_\alpha}[R](t)\|_{L^\infty} \leqslant \frac{C}{1-\beta}\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{t}{1+t}\right)^{\frac{3}{4}}\|R(t)\|_{L^4}, \tag{3.27}$$

$$\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\|\Upsilon_{\mathbb{A}_\alpha}[R](t)\|_{L^\infty} \leqslant C|\log\alpha|\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{t}{1+t}\right)^{\frac{3}{4}}\|R(t)\|_{L^4}, \tag{3.28}$$

$$\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla\Upsilon_{\mathbb{A}_\alpha}[R](t)\|_{L^\infty} \leqslant C\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{t}{1+t}\right)^{\frac{3}{4}}\|R(t)\|_{L^4}. \tag{3.29}$$

Here C *is independent of* α *and* β.

Proof. (i) Proof of (3.27) and (3.28): Let $0 < t \leqslant 2$. Then from the pointwise estimate of the kernel stated in (3.1) and Proposition 3.1, it is straightforward to show for $h \in L^4(\mathbb{R}^2)^2$ with $\operatorname{supp} h \subset \{|x| \leqslant 6\}$,

$$\|(1+\rho(\alpha\cdot)^{1-\beta})e^{-t\mathbb{A}_\alpha}\mathbb{P}h\|_{L^\infty} \leqslant Ct^{-\frac{1}{4}}\|h\|_{L^4}, \quad 0 < t \leqslant 2,\ \beta \in [0, 1]. \tag{3.30}$$

The details are omitted here. Thus we have for $t \in (0, 2]$,

$$\begin{aligned}\|(1+\rho(\alpha\cdot)^{1-\beta})\Upsilon_{\mathbb{A}_\alpha}[R](t)\|_{L^\infty} &\leqslant C\int_0^t (t-s)^{-\frac{1}{4}}\|R(s)\|_{L^4}\,ds \\ &\leqslant C\sup_{s>0}\left(\frac{s}{1+s}\right)^{\frac{3}{4}}\|R(s)\|_{L^4}.\end{aligned} \tag{3.31}$$

Next we consider the case $t \geqslant 2$. In this case we can write

$$\Upsilon_{\mathbb{A}_\alpha}[R](t) = \int_{t-1}^t e^{-(t-s)\mathbb{A}_\alpha}\mathbb{P}R(s)\,ds + \int_0^{t-1} e^{-(t-s)\mathbb{A}_\alpha}\mathbb{P}R(s)\,ds =: I(t) + II(t). \tag{3.32}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

The term $I(t)$ is handled as in the proof of (3.31), and we have for $\beta \in [0, 1]$,

$$\begin{aligned}
&\|(1+\rho(\alpha\cdot)^{1-\beta})I(t)\|_{L^\infty} \\
&\leqslant C\int_{t-1}^{t}(t-s)^{-\frac{1}{4}}(1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1}\,ds \sup_{s>0}(1+(\alpha^2 s)^{\frac{\beta}{2}})\left(\frac{s}{1+s}\right)^{\frac{3}{4}}\|R(s)\|_{L^4} \\
&\leqslant C(1+(\alpha^2 t)^{\frac{\beta}{2}})^{-1}\sup_{s>0}(1+(\alpha^2 s)^{\frac{\beta}{2}})\left(\frac{s}{1+s}\right)^{\frac{3}{4}}\|R(s)\|_{L^4}. \qquad (3.33)
\end{aligned}$$

To estimate $II(t)$ let us recall that $e^{-t\mathbb{A}_\alpha} = e^{-tA_\alpha}$ (on the space of solenoidal vector fields) and $\mathbb{P}$ are convolution operator in the spatial variables and hence we have the identity $e^{-t\mathbb{A}_\alpha}\mathbb{P}h = e^{-tA_\alpha}\mathbb{P}h = e^{-(t-1)A_\alpha}\mathbb{P}e^{-A_\alpha}h$. When $h \in L^4(\mathbb{R}^2)^2$ with $\operatorname{supp} h \subset \{|x| \leqslant 6\}$, due to the exponential decay of the Gaussian, we have

$$\|(1+|x|^2)^4 e^{-A_\alpha}h\|_{L^\infty} \leqslant C\|h\|_{L^1} \leqslant C\|h\|_{L^4}. \qquad (3.34)$$

Here we have used $\operatorname{supp} h \subset \{|x| \leqslant 6\}$ in the last line. Set $\tilde{R}(s) = e^{-A_\alpha}R(s)$. Then (3.34) and the pointwise estimate of the kernel for $e^{-(t-s-1)\mathbb{A}_\alpha}\mathbb{P}$ stated in (3.1) and Lemma 3.1 imply

$$\begin{aligned}
|II(t,x)| &= \left|\int_0^{t-1} e^{-(t-s-1)A_\alpha}\mathbb{P}\tilde{R}(s)\,ds(x)\right| \\
&\leqslant C\int_0^{t-1}\int_{\mathbb{R}^2}(t-s-1)^{-1}\left(1+\frac{|x-y-\alpha(t-s-1)\mathbf{e}_1|^2}{t-s-1}\right)^{-1}\frac{dy}{1+|y|^8}\|R(s)\|_{L^4}\,ds \\
&\leqslant C\int_0^{t-1}\int_{\mathbb{R}^2}(t-s-1)^{-1}\left(1+\frac{|x-y-\alpha(t-s-1)\mathbf{e}_1|^2}{t-s-1}\right)^{-1}\frac{dy}{1+|y|^8} \\
&\quad\times(1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1}\left(\frac{1+s}{s}\right)^{\frac{3}{4}}ds \\
&\quad\times\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{t}{1+t}\right)^{\frac{3}{4}}\|R(t)\|_{L^4}.
\end{aligned}$$

To estimate the integral in the last term we set

$$\begin{aligned}
II_1 &= \int_0^{\frac{t-1}{2}}\int_{\mathbb{R}^2}(t-s-1)^{-1}\left(1+\frac{|x-y-\alpha(t-s-1)\mathbf{e}_1|^2}{t-s-1}\right)^{-1}\frac{dy}{1+|y|^8} \\
&\qquad\times(1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1}\left(\frac{1+s}{s}\right)^{\frac{3}{4}}ds, \\
II_2 &= \int_{\frac{t-1}{2}}^{t-1}\int_{\mathbb{R}^2}(t-s-1)^{-1}\left(1+\frac{|x-y-\alpha(t-s-1)\mathbf{e}_1|^2}{t-s-1}\right)^{-1}\frac{dy}{1+|y|^8} \\
&\qquad\times(1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1}\left(\frac{1+s}{s}\right)^{\frac{3}{4}}ds.
\end{aligned}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

Let us estimate II_1 for $t \geqslant 2$. When $\alpha t^{1/2} \geqslant \rho(\alpha x)$ we compute as

$$
\begin{aligned}
II_1 \leqslant\ & \frac{C}{t^{\frac{1}{2}}} \int_{\mathbb{R}^2} \int_{\frac{1}{4}}^{t} (t-s)^{-\frac{1}{2}} \left(1 + \frac{|x-y-\alpha(t-s)\mathbf{e}_1|^2}{t-s}\right)^{-1} ds \frac{dy}{1+|y|^8} \\
& + \frac{C}{t} \int_{\mathbb{R}^2} \int_0^{\frac{1}{4}} \left(1 + \frac{|x-y-\alpha(t-s)\mathbf{e}_1|^2}{t-s}\right)^{-1} s^{-\frac{3}{4}}\, ds \frac{dy}{1+|y|^8},
\end{aligned} \tag{3.35}
$$

while when $\rho(\alpha x) \geqslant \alpha t^{1/2}$ we use the bound

$$
\begin{aligned}
II_1 \leqslant\ & C \int_{\mathbb{R}^2} \int_{\frac{1}{4}}^{t} (t-s)^{-1} \left(1 + \frac{|x-y-\alpha(t-s)\mathbf{e}_1|^2}{t-s}\right)^{-1} ds \frac{dy}{1+|y|^8} \\
& + \frac{C}{t} \int_{\mathbb{R}^2} \int_0^{\frac{1}{4}} \left(1 + \frac{|x-y-\alpha(t-s)\mathbf{e}_1|^2}{t-s}\right)^{-1} s^{-\frac{3}{4}}\, ds \frac{dy}{1+|y|^8}.
\end{aligned} \tag{3.36}
$$

In each case the second term (the time integral over $[0, \frac{1}{4}]$) is not a leading term in large time and we may focus on the estimate of the first term in the right-hand side of (3.35) and (3.36). Let $\kappa \in [0, 1]$. We observe that from the change of the variable $\tilde{s} = \alpha^2 s$ and Lemma 2.3 with $\gamma_1 = 0$ and $\gamma_2 = -\kappa$,

$$
\begin{aligned}
& \int_0^t (t-s)^{-1+\frac{\kappa}{2}} \left(1 + \frac{|x-y-\alpha(t-s)\mathbf{e}_1|^2}{t-s}\right)^{-1} ds \\
& = \frac{1}{\alpha^\kappa} \int_0^\tau (\tau-\tilde{s})^{-1+\frac{\kappa}{2}} \left(1 + \frac{|X-Y-(\tau-\tilde{s})\mathbf{e}_1|^2}{\tau-\tilde{s}}\right)^{-1} d\tilde{s}, \quad \tau = \alpha^2 t \\
& \leqslant \frac{C}{\alpha^\kappa} \begin{cases} -\chi_{\{|X-Y|\leqslant 1/e\}} \log|X-Y| + \dfrac{\chi_{\{|X-Y|\geqslant 1/e\}}}{|X-Y|^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}}, & \kappa = 0, \\ \dfrac{\chi_{\{|X-Y|\leqslant 1/e\}}}{\kappa} + \dfrac{\chi_{\{|X-Y|\geqslant 1/e\}}}{|X-Y|^{\frac{1-\kappa}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}}, & \kappa \in (0, 1]. \end{cases}
\end{aligned} \tag{3.37}
$$

Here χ_B denotes the characteristic function of the set B. Thus, when $\alpha t^{1/2} \geqslant \rho(\alpha x)$ we have from (3.35) and (3.37) with $\kappa = 1$ that

$$
\rho(\alpha x)^{1-\beta} II_1(t, x) \leqslant (\alpha^2 t)^{\frac{1-\beta}{2}} II_1(t, x) \leqslant \frac{C}{(\alpha^2 t)^{\frac{\beta}{2}}}, \tag{3.38}
$$

and when $\alpha t^{1/2} \leqslant \rho(\alpha x)$ we have from (3.36) and (3.37) with $\kappa = 0$, by writing $X = \alpha x$ and $Y = \alpha y$,

$$
\begin{aligned}
\rho(\alpha x)^{1-\beta} II_1(t, x) \leqslant\ & \frac{1}{(\alpha^2 t)^{\frac{\beta}{2}}} \rho(\alpha x) II_1(t, x) \\
\leqslant\ & \frac{C}{(\alpha^2 t)^{\frac{\beta}{2}}} \int_{\mathbb{R}^2} \frac{\rho(X-Y) + \rho(Y)}{|X-Y|^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}} \frac{dy}{1+|y|^8}
\end{aligned}
$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

$$\leqslant \frac{C}{(\alpha^2 t)^{\frac{\beta}{2}}} \int_{\mathbb{R}^2} \left(1 + \frac{1}{|x-y|^{\frac{1}{2}}}\right) \frac{dy}{1+|y|^7}$$
$$\leqslant \frac{C}{(\alpha^2 t)^{\frac{\beta}{2}}}. \tag{3.39}$$

Here we have used

$$\frac{\rho(X-Y)}{|X-Y|^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}} \leqslant C,$$

which follows from Lemma A.1, and

$$\frac{\rho(Y)}{|X-Y|^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}} \leqslant \frac{|\alpha y|^{\frac{1}{2}}+|\alpha y|}{|X-Y|^{\frac{1}{2}}} \leqslant \frac{C(1+|y|)}{|x-y|^{\frac{1}{2}}}.$$

Hence (3.38) and (3.39) shows

$$\|\rho(\alpha\cdot)^{1-\beta} II_1(t)\|_{L^\infty} \leqslant \frac{C}{(\alpha^2 t)^{\frac{\beta}{2}}}. \tag{3.40}$$

Next we estimate II_2 for $t \geqslant 2$. Since

$$II_2 \leqslant \frac{C}{1+(\alpha^2 t)^{\frac{\beta}{2}}} \int_{\mathbb{R}^2} \int_0^t (t-s)^{-1} \left(1 + \frac{|x-y-\alpha(t-s)\mathbf{e}_1|^2}{t-s}\right)^{-1} ds \frac{dy}{1+|y|^8}, \tag{3.41}$$

the computation as above and (3.37) with $\kappa = 0$ imply

$$\rho(\alpha x)^{1-\beta} II_2(t,x)$$
$$\leqslant \frac{C}{1+(\alpha^2 t)^{\frac{\beta}{2}}} \int_{\mathbb{R}^2} (\rho(X-Y)^{1-\beta} + \rho(Y)^{1-\beta})$$
$$\times \left(-\chi_{\{|X-Y|\leqslant 1/e\}} \log|X-Y| + \frac{\chi_{\{|X-Y|\geqslant 1/e\}}}{|X-Y|^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}}\right) \frac{dy}{1+|y|^8}.$$

By using the inequality of the form $\sup_{a\in(0,1]} a^{1-\beta}(-\log a) \leqslant \frac{C}{1-\beta}$ one can check that

$$\rho(X-Y)^{1-\beta} \left(-\chi_{\{|X-Y|\leqslant 1/e\}} \log|X-Y| + \frac{\chi_{\{|X-Y|\geqslant 1/e\}}}{|X-Y|^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}}\right)$$
$$\leqslant \frac{C}{1-\beta},$$

while from $\rho(Y)^{1-\beta} \leqslant C\alpha^{\frac{1-\beta}{2}}(1+|y|)^{1-\beta}$ and

$$-\chi_{\{|X-Y|\leqslant 1/e\}} \log|X-Y| \leqslant C(|\log\alpha| + \chi_{\{|x-y|\leqslant 1\}} |\log|x-y||),$$

we have

$$\rho(Y)^{1-\beta} \left(-\chi_{\{|X-Y|\leqslant 1/e\}} \log|X-Y| + \frac{\chi_{\{|X-Y|\geqslant 1/e\}}}{|X-Y|^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}}\right)$$
$$\leqslant \frac{C}{1-\beta}(1+|y|)(1+\chi_{\{|x-y|\leqslant 1\}}|\log|x-y||).$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

Hence we have arrived at

$$\|\rho(\alpha\cdot)^{1-\beta} II_2(t)\|_{L^\infty} \leqslant \frac{C}{(1-\beta)(1+(\alpha^2 t)^{\frac{\beta}{2}})}. \tag{3.42}$$

Estimates (3.40) and (3.42) give

$$\|\rho(\alpha\cdot)^{1-\beta} II(t)\|_{L^\infty} \leqslant \frac{C}{(1-\beta)(\alpha^2 t)^{\frac{\beta}{2}}} \sup_{s>0}(1+(\alpha^2 s)^{\frac{\beta}{2}}) \left(\frac{s}{1+s}\right)^{\frac{3}{4}} \|R(s)\|_{L^4}, \quad t \geqslant 2. \tag{3.43}$$

Then, (3.32), (3.33), and (3.43) for $t \geqslant 2$ and (3.31) for $t \in (0,2]$ prove (3.27). The proof of (3.28) is similar to the one of (3.27). Indeed, in virtue of (3.31) for $t \in (0,2]$ and of (3.32) with (3.33) for $t \geqslant 2$, it suffices to consider the estimate of $\|II(t)\|_{L^\infty}$ for $t \geqslant 2$. Let us recall that $II(t)$ is bounded from above in term of II_1 and II_2. Then II_1 is estimated as

$$\begin{aligned} II_1(t,x) &\leqslant \frac{C}{t-1}\int_0^{\frac{t-1}{2}} \int_{\mathbb{R}^2} \frac{dy}{1+|y|^8}(1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1}\left(\frac{1+s}{s}\right)^{\frac{3}{4}} ds \\ &\leqslant \frac{C}{1+(\alpha^2 t)^{\frac{\beta}{2}}}, \quad t \geqslant 2. \end{aligned} \tag{3.44}$$

As for II_2, we have from (3.41) and (3.37) with $\kappa = 0$,

$$\begin{aligned} II_2(t,x) &\leqslant \frac{C}{1+(\alpha^2 t)^{\frac{\beta}{2}}} \\ &\quad \times \int_{\mathbb{R}^2}\left(-\chi_{\{|X-Y|\leqslant 1/e\}}\log|X-Y| + \frac{\chi_{\{|X-Y|\geqslant 1/e\}}}{|X-Y|^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}}\right)\frac{dy}{1+|y|^8} \\ &\leqslant \frac{C}{1+(\alpha^2 t)^{\frac{\beta}{2}}}\int_{\mathbb{R}^2}(|\log\alpha| + \chi_{\{|x-y|\leqslant 1\}}|\log|x-y||)\frac{dy}{1+|y|^8} \\ &\leqslant \frac{C|\log\alpha|}{1+(\alpha^2 t)^{\frac{\beta}{2}}}. \end{aligned} \tag{3.45}$$

Thus, (3.44) and (3.45) give the desired estimate of $II(t)$ for $t \geqslant 2$. The proof of (3.28) is complete.

(ii) Proof of (3.29): from the estimate $\|\nabla e^{-t\mathbb{A}_\alpha}\mathbb{P}h\|_{L^\infty} \leqslant Ct^{-\frac{1}{2}-\frac{1}{q}}\|h\|_{L^q}$, $t > 0$, $1 \leqslant q \leqslant \infty$, we have

$$\begin{aligned} &\|\nabla \Upsilon_{\mathbb{A}_\alpha}[R](t)\|_{L^\infty} \\ &\quad \leqslant C\int_0^t (\chi_{\{t-s\leqslant 1\}}(t-s)^{-\frac{3}{4}}\|R(s)\|_{L^4} + \chi_{\{t-s\geqslant 1\}}(t-s)^{-\frac{3}{2}}\|R(s)\|_{L^1})\,ds \\ &\quad \leqslant C\int_0^t (\chi_{\{t-s\leqslant 1\}}(t-s)^{-\frac{3}{4}} + \chi_{\{t-s\geqslant 1\}}(t-s)^{-\frac{3}{2}})(1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1}\left(\frac{1+s}{s}\right)^{\frac{3}{4}} ds \\ &\qquad \times \sup_{s>0}(1+(\alpha^2 s)^{\frac{\beta}{2}})\left(\frac{s}{1+s}\right)^{\frac{3}{4}}\|R(s)\|_{L^4}. \end{aligned}$$

Here we have used $\operatorname{supp} R(s) \subset \{|x| \leqslant 6\}$. Hence (3.29) easily follows. The proof is complete. □

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

4. Estimate of Oseen semigroup in exterior domain

In this section we establish the stability estimates for the Oseen semigroup $\{e^{-\mathbb{A}_{\alpha,\Omega}}\}_{t\geqslant 0}$, which extends the estimates in the whole space case in the previous sections. Let us recall that Ω is normalized so that $\mathrm{diam}\,(\mathbb{R}^2\setminus\Omega)=1$ and the origin is in the interior of the obstacle $\mathbb{R}^2\setminus\Omega$ and $\mathrm{dist}\,(\{0\},\partial\Omega)\geqslant\frac{1}{4}$.

In establishing the estimate of $\{e^{-t\mathbb{A}_{\alpha,\Omega}}\}_{t\geqslant 0}$ in large time, it is important to obtain the local (in space) estimate taking into account the flow produced from the boundary. This estimate is called the local energy decay estimate in the literature for the exterior problem, and is a key step for the global L^p-L^q estimates of the Stokes semigroup in exterior domains (Iwashita [36] for 3D case, Dan and Shibata [12, 13] for 2D case) or of the Oseen semigroup $\{e^{-t\mathbb{A}_{\alpha,\Omega}}\}_{t\geqslant 0}$ (Kobayashi and Shibata [37] for 3D case, Hishida [35] for 2D case). Let us consider the Stokes system

$$\begin{cases}\partial_t u-\Delta u+\alpha\partial_1 u+\nabla p_u=0, & t>0,\ x\in\Omega,\\ \operatorname{div} u=0, & t\geqslant 0,\ x\in\Omega,\\ u=0, & t>0,\ x\in\partial\Omega,\\ u|_{t=0}=\mathbb{P}_\Omega f, & x\in\Omega.\end{cases}\tag{4.1}$$

Our analysis relies on the following theorem, which gives the temporal decay estimate of the solution $u(t)=e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f$ and the associated pressure $p_u(t)=p[\mathbb{P}_\Omega f](t)$ near the boundary when the given data $f\in L^q(\Omega)^2$, $1<q<\infty$, is compactly supported.

Theorem 4.1 (Local energy decay estimate [40, Theorem 1.2]). *Set* $\Omega_4=\Omega\cap\{|x|\leqslant 4\}$. *Let* $\alpha\in(0,\frac{1}{2}]$ *and* $1<q<\infty$. *Assume that* $f\in L^q(\Omega)^2$ *and* $\mathrm{supp}\, f\subset\{|x|\leqslant 5\}$. *Then for* $j=0,1,2$,

$$\|\nabla^j e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f\|_{L^q(\Omega_4)}\leqslant\begin{cases}\dfrac{C}{t^{\frac{j}{2}}}\|f\|_{L^q(\Omega)}, & 0<t\leqslant 3,\\ \left(\dfrac{C}{t(\log t)^2}+\dfrac{C\alpha^2}{|\log t|}\right)\|f\|_{L^q(\Omega)}, & 2\leqslant t\leqslant\alpha^{-2},\\ \dfrac{C}{t^2\alpha^2|\log\alpha|}\|f\|_{L^q(\Omega)}, & t\geqslant\alpha^{-2},\end{cases}\tag{4.2}$$

and the associated pressure field $p[\mathbb{P}_\Omega f](t)=p_u(t)$, $\int_{\Omega_4}p[\mathbb{P}_\Omega f](t)\,dx=0$, *satisfies*

$$\|p[\mathbb{P}_\Omega f](t)\|_{L^q(\Omega_4)}\leqslant\begin{cases}\dfrac{C}{t^{\frac{1}{2}(1+\frac{1}{q})}}\|f\|_{L^q(\Omega)}, & 0<t\leqslant 3,\\ \left(\dfrac{C}{t(\log t)^2}+\dfrac{C\alpha^2}{|\log t|}\right)\|f\|_{L^q(\Omega)}, & 2\leqslant t\leqslant\alpha^{-2},\\ \dfrac{C}{t^2\alpha^2|\log\alpha|}\|f\|_{L^q(\Omega)}, & t\geqslant\alpha^{-2},\end{cases}\tag{4.3}$$

Here the constant C *depend only on* q *and* Ω.

Remark 4.2. (i) The constant C in Theorem 4.1 is independent of $\alpha\in(0,\frac{1}{2}]$. Hence, by taking the limit $\alpha\to 0$, we recover the local energy decay estimate of the Stokes

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

semigroup $e^{-t\mathbb{A}_\Omega}$ established by Dan and Shibata [12]. Note that Theorem 4.1 is a significant improvement of [35], where the algebraic singularity in α^{-1} is present.

(ii) The key point of (4.2)–(4.3) is that the quantities $\int_1^\infty \|\nabla^j e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f\|_{L^q(\Omega_4)}\,dt$ and $\int_1^\infty \|p[\mathbb{P}_\Omega f](t)\|_{L^q(\Omega_4)}\,dt$ are estimated uniformly in small α. Such bounds are essential to obtain the estimate in the exterior domain as in the whole space case and to handle the term $\alpha\nabla\cdot(U\otimes v+v\otimes U)$ with U whose scale-critical (anisotropic) norm is small only in the logarithmic order.

4.1. Estimate of semigroup in exterior domain

In this subsection we establish the weighted L^∞ estimate of the Oseen semigroup $e^{-t\mathbb{A}_{\alpha,\Omega}}$. Let us recall that the function $b(X)$ is defined as (1.6).

Proposition 4.3. *Let $\alpha\in(0,\frac12]$ and $\beta\in[0,1)$. Let $f\in L^4_\sigma(\Omega)$ satisfy $b(\alpha\cdot)f\in L^\infty(\Omega)^2$. Then*

$$\sup_{t>0}(\alpha^2 t)^{\frac{\beta}{2}}\|\rho(\alpha\cdot)^{1-\beta}e^{-t\mathbb{A}_{\alpha,\Omega}}f\|_{L^\infty}\leqslant\frac{C}{1-\beta}\|(1+b(\alpha\cdot))f\|_{L^\infty},\tag{4.4}$$

$$\sup_{t>0}(1+(\alpha^2 t)^{\frac12})\|e^{-t\mathbb{A}_{\alpha,\Omega}}f\|_{L^\infty}\leqslant C|\log\alpha|\,\|(1+b(\alpha\cdot))f\|_{L^\infty},\tag{4.5}$$

$$\sup_{t>0}(1+(\alpha^2 t)^{\frac12})\left(\frac{t}{1+t}\right)^{\frac12}\|\nabla e^{-t\mathbb{A}_{\alpha,\Omega}}f\|_{L^\infty}\leqslant C\|(1+b(\alpha\cdot))f\|_{L^\infty}.\tag{4.6}$$

Here C is independent of α and β.

Proof. Let us introduce a cut-off function $\chi\in C_0^\infty(\mathbb{R}^2)$, $0\leqslant\chi\leqslant1$, such that $\chi(x)=1$ for $|x|\leqslant1$ and $\chi(x)=0$ for $|x|\geqslant2$. Then we set

$$v(t)=(1-\chi)e^{-t\mathbb{A}_{\alpha,\Omega}}f+\mathbb{B}[\nabla\chi\cdot e^{-t\mathbb{A}_{\alpha,\Omega}}f].$$

Here $\mathbb{B}$ is the Bogovskii operator in the annulus $D_0=\{x\in\mathbb{R}^2\mid 1<|x|<2\}$, that is, $\mathbb{B}[g]$ satisfies

$$\operatorname{div}\mathbb{B}[g]=g\quad\text{in } D_0,\qquad \mathbb{B}[g]=0\quad\text{on }\partial D_0$$

for a given function $g\in C_0^\infty(D_0)$ with $\int_{D_0}g\,dx=0$. As is well known (see, e.g., Borchers and Sohr [8]), the Bogovskii operator $\mathbb{B}$ is extended to a bounded operator from $W_0^{k,q}(D_0)$ to $W_0^{k+1,q}(D_0)^2$ for any $1<q<\infty$ and $k=0,1,\ldots,$ together with the estimate

$$\|\nabla^{k+1}\mathbb{B}[g]\|_{L^q(D_0)}\leqslant C\|\nabla^k g\|_{L^q(D_0)},\qquad 1<q<\infty,\ k=0,1,\ldots.\tag{4.7}$$

Then $v(t)$ is regarded as a solenoidal vector field in $\mathbb{R}^2$, and satisfies the equations

$$\partial_t v+A_\alpha v+\nabla p_v=R,\quad \operatorname{div}v=0,\quad t>0,\quad x\in\mathbb{R}^2\tag{4.8}$$

and $v|_{t=0}=(1-\chi)f+\mathbb{B}[\nabla\chi\cdot f]$, with a suitable pressure p_v and

$$\begin{aligned}R(t)&=(\Delta\chi)e^{-t\mathbb{A}_{\alpha,\Omega}}f+2\nabla\chi\cdot\nabla e^{-t\mathbb{A}_{\alpha,\Omega}}f-\alpha(\partial_1\chi)e^{-t\mathbb{A}_{\alpha,\Omega}}f\\&\quad-(\nabla\chi)p_\Omega(t)+(\partial_t+A_\alpha)\mathbb{B}[\nabla\chi\cdot e^{-t\mathbb{A}_{\alpha,\Omega}}f].\end{aligned}\tag{4.9}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

Here p_Ω is the pressure associated with $e^{-t\mathbb{A}_{\alpha,\Omega}}f$, while $R(t)$ is supported in $\overline{D_0}$. Thus v satisfies the integral equation

$$\begin{aligned} v(t) &= e^{-t\mathbb{A}_\alpha}v_0 + \Upsilon_{\mathbb{A}_\alpha}[R](t), \quad \Upsilon_{\mathbb{A}_\alpha}[R](t) = \int_0^t e^{-(t-s)\mathbb{A}_\alpha}\mathbb{P}R(s)\,ds, \\ v_0 &= (1-\chi)f + \mathbb{B}[\nabla\chi\cdot f]. \end{aligned} \tag{4.10}$$

By Proposition 3.2 we have

$$\begin{aligned} &\sup_{t>0}\big(\|\rho(\alpha\cdot)e^{-t\mathbb{A}_\alpha}v_0\|_{L^\infty} + (1+(\alpha^2t)^{\frac{\beta}{2}})\|\rho(\alpha\cdot)^{1-\beta}e^{-t\mathbb{A}_\alpha}v_0\|_{L^\infty} \\ &\qquad + (1+(\alpha^2t)^{\frac12})\|e^{-t\mathbb{A}_\alpha}v_0\|_{L^\infty} + t^{\frac12}\|\nabla e^{-t\mathbb{A}_\alpha}v_0\|_{L^\infty}\big) \\ &\quad \leqslant C\|(1+b(\alpha\cdot))v_0\|_{L^\infty} \\ &\quad \leqslant C\|(1+b(\alpha\cdot))f\|_{L^\infty}. \end{aligned} \tag{4.11}$$

Here we have used in the last line that $\|(1+b(\alpha\cdot))\mathbb{B}[\nabla\chi\cdot f]\|_{L^\infty} \leqslant C\|\mathbb{B}[\nabla\chi\cdot f]\|_{L^\infty} \leqslant C\|\mathbb{B}[\nabla\chi\cdot f]\|_{W^{1,3}} \leqslant C\|\nabla\chi\cdot f\|_{L^3} \leqslant C\|f\|_{L^\infty}$. On the other hand, Proposition 3.5 implies

$$\begin{aligned} &\sup_{t>0}\Bigg(\|\rho(\alpha\cdot)\Upsilon_{\mathbb{A}_\alpha}[R](t)\|_{L^\infty} + (1-\beta)(\alpha^2t)^{\frac{\beta}{2}}\|\rho(\alpha\cdot)^{1-\beta}\Upsilon_{\mathbb{A}_\alpha}[R](t)\|_{L^\infty} \\ &\qquad + \frac{(1+(\alpha^2t)^{\frac12})}{|\log\alpha|}\|\Upsilon_{\mathbb{A}_\alpha}[R](t)\|_{L^\infty} + (1+(\alpha^2t)^{\frac12})\left(\frac{t}{1+t}\right)^{\frac12}\|\nabla\Upsilon_{\mathbb{A}_\alpha}[R](t)\|_{L^\infty}\Bigg) \\ &\leqslant C\sup_{t>0}(1+(\alpha^2t)^{\frac12})\left(\frac{t}{1+t}\right)^{\frac34}\|R(t)\|_{L^4}, \end{aligned} \tag{4.12}$$

since $\operatorname{supp} R(t) \subset \{|x| \leqslant 2\}$. For the estimate of $R(t)$ we have

$$\|R(t)\|_{L^4} \leqslant C(\|e^{-t\mathbb{A}_{\alpha,\Omega}}f\|_{W^{1,4}(D_0)} + \|p_\Omega(t)\|_{L^4(D_0)}). \tag{4.13}$$

Estimate (4.13) is apparently nontrivial, due to the presence of the time derivative in the definition of R. This difficulty is overcome by applying the estimate for the Bogovskii operator in Sobolev spaces with a negative order, which is obtained by [8]. The details of the proof of (4.13) are postponed to the appendix. Admitting (4.13), let us focus on the estimate near the boundary. To this end we take a smooth cut-off denoted again by χ such that $\chi = 1$ for $|x| \leqslant 2$ and $\chi = 0$ for $|x| \geqslant 3$, and set

$$u(t) = (1-\chi)e^{-t\mathbb{A}_\alpha}f + \mathbb{B}[\nabla\chi\cdot e^{-t\mathbb{A}_\alpha}f], \tag{4.14}$$

where f is extended to $\mathbb{R}^2$ by zero in $\mathbb{R}^2\setminus\Omega$. The operator $\mathbb{B}$ denotes the Bogovskii operator but in the annulus $D_1 = \{x\in\mathbb{R}^2 \mid 2<|x|<3\}$ in this case. Since $\operatorname{div} f = 0$ in $\mathbb{R}^2$, we have $e^{-t\mathbb{A}_\alpha}f = e^{-tA_\alpha}f$, and in particular, the associated pressure is zero. Then the direct calculation shows that $w(t) = e^{-t\mathbb{A}_{\alpha,\Omega}}f - u(t)$ satisfies

$$\begin{cases} \partial_t w + A_\alpha w + \nabla p_w = -\tilde{R}, & t>0,\ x\in\Omega, \\ \operatorname{div} w = 0, & t\geqslant 0,\ x\in\Omega, \\ w = 0, & t>0,\ x\in\partial\Omega, \\ w|_{t=0} = w_0, & x\in\Omega, \end{cases}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

with $p_w = p_\Omega$, $w_0 = \chi f - \mathbb{B}[\nabla\chi\cdot f]$, and

$$\tilde{R}(t) = (\Delta\chi)e^{-t\mathbb{A}_\alpha} f + 2\nabla\chi\cdot\nabla e^{-t\mathbb{A}_\alpha} f - \alpha(\partial_1\chi)e^{-t\mathbb{A}_\alpha} f + (\partial_t + A_\alpha)\mathbb{B}[\nabla\chi\cdot e^{-t\mathbb{A}_\alpha} f].$$

Thus w is written as

$$\begin{aligned} w(t) &= e^{-t\mathbb{A}_{\alpha,\Omega}} w_0 - \int_0^t e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega \tilde{R}(s)\,ds \\ &= w^{(1)}(t) + w^{(2)}(t). \end{aligned} \tag{4.15}$$

Then the associated pressure p_w is written in the form $p_w = p_{w^{(1)}} + p_{w^{(2)}}$, where

$$p_{w^{(1)}}(t) = p[w_0](t), \quad p_{w^{(2)}}(t) = -\int_0^t p[\mathbb{P}_\Omega\tilde{R}(s)](t-s)\,ds \tag{4.16}$$

by following the notation used in Theorem 4.1. Since $e^{-t\mathbb{A}_{\alpha,\Omega}} f = w(t)$ for $|x|\leqslant 2$, it suffices to estimate $w(t)$.

(i) Estimate of $w^{(1)}$: from the L^∞ estimate of the Oseen semigroup, see (2.9), we have

$$\|\nabla^j w^{(1)}(t)\|_{L^\infty} \leqslant \frac{C}{t^{\frac{j}{2}}}\|w_0\|_{L^\infty} \leqslant \frac{C}{t^{\frac{j}{2}}}\|f\|_{L^\infty}, \quad 0<t\leqslant 3,\ \ j=0,1, \tag{4.17}$$

while Theorem 4.1 and the Sobolev embedding inequality yield

$$\|w^{(1)}(t)\|_{W^{1,\infty}(\Omega\cap\{|x|\leqslant 3\})} \leqslant \frac{C}{t}\|w_0\|_{L^4} \leqslant \frac{C}{t}\|f\|_{L^\infty}, \quad t\geqslant 3. \tag{4.18}$$

Here C is independent of α.

(ii) Estimate of $w^{(2)}$: From the estimate of the Oseen semigroup it is easy to see that

$$\begin{aligned} \|\nabla^j w^{(2)}(t)\|_{L^\infty} &\leqslant C\int_0^t (t-s)^{-\frac{j}{2}-\frac{1}{4}}\|\mathbb{P}_\Omega\tilde{R}(s)\|_{L^4}\,ds \\ &\leqslant C\int_0^t (t-s)^{-\frac{j}{2}-\frac{1}{4}}\|\tilde{R}(s)\|_{L^4}\,ds, \quad 0<t\leqslant 3,\ \ j=0,1. \end{aligned} \tag{4.19}$$

When $t\geqslant 3$ we have also from Theorem 4.1,

$$\begin{aligned} &\|w^{(2)}(t)\|_{W^{1,\infty}(\Omega\cap\{|x|\leqslant 3\})} \\ &\leqslant \int_{t-2}^t \|e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega\tilde{R}(s)\|_{W^{1,\infty}}\,ds \\ &\quad + C\int_0^{t-2}\|e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega\tilde{R}(s)\|_{W^{2,4}(\Omega\cap\{|x|\leqslant 3\})}\,ds \\ &\leqslant C\int_{t-2}^t (t-s)^{-\frac{3}{4}}\|\tilde{R}(s)\|_{L^4}\,ds \\ &\quad + C\int_0^{t-2}\left(\frac{1}{(t-s)(\log(t-s))^2} + \frac{\alpha^2\chi_{\{t-s\leqslant\alpha^{-2}\}}}{|\log(t-s)|} + \frac{\chi_{\{t-s\geqslant\alpha^{-2}\}}}{(t-s)^2\alpha^2|\log\alpha|}\right)\|\tilde{R}(s)\|_{L^4}\,ds. \end{aligned} \tag{4.20}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

Note that the function $\frac{1}{t(\log t)^2} + \frac{\alpha^2 \chi_{\{t \leqslant \alpha^{-2}\}}}{|\log t|} + \frac{\chi_{\{t \geqslant \alpha^{-2}\}}}{t^2\alpha^2|\log \alpha|}$ is integrable over $(2, \infty)$ and its L^1 norm is uniformly bounded in $\alpha \in (0, \frac{1}{2}]$. Next, as in the estimate of R, by recalling that the associated pressure of $e^{-t\mathbb{A}_\alpha} f$ is zero, we have

$$\|\tilde{R}(t)\|_{L^4} \leqslant C\|e^{-t\mathbb{A}_\alpha} f\|_{W^{1,\infty}(\{|x|\leqslant 3\})} \leqslant \frac{C}{1+(\alpha^2 t)^{\frac{1}{2}}}(1+t^{-\frac{1}{2}})\|(1+b(\alpha\cdot))f\|_{L^\infty}. \tag{4.21}$$

Here we have also used Proposition 3.2 in the last line. Thus, (4.19) and (4.21) give

$$\|w^{(2)}(t)\|_{L^\infty} + t^{\frac{1}{2}}\|\nabla w^{(2)}(t)\|_{L^\infty} \leqslant C\|(1+b(\alpha\cdot))f\|_{L^\infty}, \quad 0 < t \leqslant 3, \tag{4.22}$$

while it is not difficult to see from (4.20) and (4.21) that

$$\|w^{(2)}(t)\|_{W^{1,\infty}(\Omega\cap\{|x|\leqslant 3\})} \leqslant \frac{C}{1+(\alpha^2 t)^{\frac{1}{2}}}\|(1+b(\alpha\cdot))f\|_{L^\infty}, \quad t \geqslant 3. \tag{4.23}$$

(iii) Estimate of $p_{w^{(1)}}$: By Theorem 4.1,

$$\|p_{w^{(1)}}(t)\|_{L^4(\Omega\cap\{|x|\leqslant 3\})} \leqslant \frac{C}{t^{\frac{3}{4}}}\|w_0\|_{L^4} \leqslant \frac{C}{t^{\frac{3}{4}}}\|f\|_{L^\infty}, \quad t > 0,$$

which implies

$$(1+(\alpha^2 t)^{\frac{1}{2}})\left(\frac{t}{1+t}\right)^{\frac{3}{4}}\|p_{w^{(1)}}(t)\|_{L^4(\Omega\cap\{|x|\leqslant 3\})} \leqslant C\|f\|_{L^\infty}, \quad t > 0. \tag{4.24}$$

(iv) Estimate of $p_{w^{(2)}}$: By Theorem 4.1 we have

$$\begin{aligned}\|p_{w^{(2)}}(t)\|_{L^4(\Omega\cap\{|x|\leqslant 3\})} \leqslant C\int_0^t \Bigg(&\frac{\chi_{\{t-s\leqslant 2\}}}{(t-s)^{\frac{3}{4}}} + \frac{\chi_{\{2\leqslant t-s\leqslant \alpha^{-2}\}}}{(t-s)(\log(t-s))^2} \\ &+ \frac{\alpha^2\chi_{\{2\leqslant t-s\leqslant\alpha^{-2}\}}}{|\log(t-s)|} + \frac{\chi_{\{t-s\geqslant\alpha^{-2}\}}}{(t-s)^2\alpha^2|\log\alpha|}\Bigg)\|\tilde{R}(s)\|_{L^4}\,ds,\end{aligned}$$

and then, combining with (4.21),

$$(1+(\alpha^2 t)^{\frac{1}{2}})\left(\frac{t}{1+t}\right)^{\frac{3}{4}}\|p_{w^{(2)}}(t)\|_{L^4(\Omega\cap\{|x|\leqslant 3\})} \leqslant C\|(1+b(\alpha\cdot)f\|_{L^\infty}, \quad t > 0. \tag{4.25}$$

Let us recall that $e^{-t\mathbb{A}_{\alpha,\Omega}} f = w^{(1)}(t) + w^{(2)}(t)$ and $p_\Omega = p_{w^{(1)}} + p_{w^{(2)}}$ for $|x| \leqslant 2$. Then, from estimate (4.13) for $\|R(t)\|_{L^4}$ and (4.17), (4.18), (4.22), (4.23), (4.24), and (4.25), we have

$$\|R(t)\|_{L^4} \leqslant C\left(\frac{1}{t^{\frac{3}{4}}} + \frac{1}{1+(\alpha^2 t)^{\frac{1}{2}}}\right)\|(1+b(\alpha\cdot))f\|_{L^\infty}.$$

Therefore, the right-hand side of (4.12) is bounded from above by $C\|(1+b(\alpha\cdot))f\|_{L^\infty}$ with C independent of α and β, as desired. Then, by recalling (4.10) and (4.11), we obtain (4.4), (4.5), and (4.6) but with L^∞ norm in the left-hand side replaced by $L^\infty(\{|x| \geqslant 2\})$ norm (since $v(t) = e^{-t\mathbb{A}_{\alpha,\Omega}} f$ for $|x| \geqslant 2$). The estimates in $\{|x| \leqslant 2\}$ follow from (4.17), (4.18), (4.22), and (4.23). The proof is complete. □

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

For $\alpha \in (0, \frac{1}{2}]$ and $\beta \in [0, 1]$ let us introduce the space/time norm $\|\cdot\|_{\alpha,\beta}$ as

$$\|f\|_{\alpha,\beta} = \sup_{t>0}(\|\rho(\alpha\cdot)f(t)\|_{L^\infty} + (1-\beta)(\alpha^2 t)^{\frac{\beta}{2}}\|\rho(\alpha\cdot)^{1-\beta}f(t)\|_{L^\infty}) + \sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{1}{|\log\alpha|}\|f(t)\|_{L^\infty} + \left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right). \quad (4.26)$$

Note that the factors $(1-\beta)$ and $\frac{1}{|\log\alpha|}$ in (4.26) are compatible with the estimates in Proposition 4.3 for $e^{-t\mathbb{A}_{\alpha,\Omega}}$. The choice of the weight $(\alpha^2 t)^{\frac{\beta}{2}}$ rather than $(\alpha^2 t)^{\frac{1}{2}}$ in front of $\|f(t)\|_{L^\infty}$ and $\|\nabla f(t)\|_{L^\infty}$ even in the case $\beta < 1$ is due to the bilinear estimate in the next subsection, although we have the critical decay $O((\alpha^2 t)^{-\frac{1}{2}})$ in (4.5) for the Oseen semigroup. In this sense the norm

$$\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{1}{|\log\alpha|}\|f(t)\|_{L^\infty} + \left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right)$$

is subcritical when $\beta < 1$. Proposition 4.3 implies

$$\|e^{-\cdot\mathbb{A}_{\alpha,\Omega}}f\|_{\alpha,\beta} \leqslant C\|(1+b(\alpha\cdot))f\|_{L^\infty}. \quad (4.27)$$

Here C is independent of α and β.

4.2. Estimate of bilinear form in exterior domain

In this subsection we establish the estimate of the bilinear form

$$\Lambda_{\mathbb{A}_{\alpha,\Omega}}[f, g](t) = \alpha\int_0^t e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega\nabla\cdot(f\otimes g + g\otimes f)\,ds. \quad (4.28)$$

Note that the norm $\|\cdot\|_{\alpha,\beta}$ is defined as (4.26).

Proposition 4.4. *Let $\alpha \in (0, \frac{1}{2}]$ and $\beta \in [0, 1)$. Let $b(\alpha\cdot)f, b(\alpha\cdot)g \in L^\infty(0, \infty; L^\infty(\Omega)^2)$, and $\nabla f, \nabla g \in L^\infty_{loc}((0, \infty); L^\infty(\Omega))$. Then we have*

$$\|\Lambda_{\mathbb{A}_{\alpha,\Omega}}[f, g]\|_{\alpha,\beta} \leqslant \frac{C}{1-\beta}\|g\|_{\alpha,\beta}\sup_{t>0}\left(\|\rho(\alpha\cdot)f(t)\|_{L^\infty} + \alpha^{\frac{1}{2}}\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right). \quad (4.29)$$

Here C is independent of α and β.

Proof. The proof is similar to the proof of Proposition 4.3. We first take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^2)$, $0 \leqslant \chi \leqslant 1$, such that $\chi(x) = 1$ for $|x| \leqslant 1$ and $\chi(x) = 0$ for $|x| \geqslant 2$. Then we set

$$v(t) = (1-\chi)\Lambda_{\mathbb{A}_{\alpha,\Omega}}[f, g](t) + \mathbb{B}[\nabla\chi\cdot\Lambda_{\mathbb{A}_{\alpha,\Omega}}[f, g](t)].$$

Here $\mathbb{B}$ is the Bogovskii operator in the annulus $D_0 = \{x \in \mathbb{R}^2 \mid 1 < |x| < 2\}$. Then $v(t)$ is regarded as a solenoidal vector field in $\mathbb{R}^2$, and satisfies the equations

$$\begin{cases} \partial_t v + A_\alpha v + \nabla p_v = \alpha\nabla\cdot((1-\chi)f\otimes g + g\otimes(1-\chi)f) + R, & t > 0,\ x \in \mathbb{R}^2, \\ \operatorname{div} v = 0, & t \geqslant 0,\ x \in \mathbb{R}^2, \\ v|_{t=0} = 0, & x \in \mathbb{R}^2, \end{cases}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

with a suitable pressure p_v and

$$\begin{aligned} R(t) = {} & (\Delta\chi)\Lambda_{\mathbb{A}_{\alpha,\Omega}}[f,g](t) + 2\nabla\chi\cdot\nabla\Lambda_{\mathbb{A}_{\alpha,\Omega}}[f,g](t) - \alpha(\partial_1\chi)\Lambda_{\mathbb{A}_{\alpha,\Omega}}[f,g](t) \\ & - (\nabla\chi)p_\Omega(t) + (\partial_t + A_\alpha)\mathbb{B}[\nabla\chi\cdot\Lambda_{\mathbb{A}_{\alpha,\Omega}}[f,g](t)] \\ & + \alpha(\nabla\chi)\cdot(f\otimes g + g\otimes f). \end{aligned} \tag{4.30}$$

Here p_Ω is the pressure associated with $\Lambda_{\mathbb{A}_{\alpha,\Omega}}[f,g]$, while $R(t)$ is supported in $\overline{D_0}$. Thus v satisfies the integral equation

$$\begin{aligned} v(t) &= \Lambda_{\mathbb{A}_\alpha}[(1-\chi)f,g](t) + \Upsilon_{\mathbb{A}_\alpha}[R](t), \\ \Lambda_{\mathbb{A}_\alpha}[f,g](t) &= \alpha\int_0^t e^{-(t-s)\mathbb{A}_\alpha}\mathbb{P}\nabla\cdot(f\otimes g + g\otimes f)\,ds, \\ \Upsilon_{\mathbb{A}_\alpha}[R](t) &= \int_0^t e^{-(t-s)\mathbb{A}_\alpha}\mathbb{P}R(s)\,ds. \end{aligned} \tag{4.31}$$

By Proposition 3.3 we have

$$\begin{aligned} & \|\Lambda_{\mathbb{A}_\alpha}[(1-\chi)f,g]\|_{\alpha,\beta} \\ & \leqslant \frac{C}{1-\beta}\|g\|_{\alpha,\beta}\sup_{t>0}\left(\|\rho(\alpha\cdot)(1-\chi)f\|_{L^\infty} + \alpha^{\frac{1}{2}}\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla(1-\chi)f\|_{L^\infty}\right) \\ & \leqslant \frac{C}{1-\beta}\|g\|_{\alpha,\beta}\sup_{t>0}\left(\|\rho(\alpha\cdot)f\|_{L^\infty} + \alpha^{\frac{1}{2}}\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f\|_{L^\infty}\right). \end{aligned} \tag{4.32}$$

Here we have also used in the last line that $\alpha^{\frac{1}{2}}\|(\nabla\chi)f\|_{L^\infty} \leqslant C\|\rho(\alpha\cdot)f\|_{L^\infty}$. On the other hand, Proposition 3.5 implies

$$\|\Upsilon_{\mathbb{A}_\alpha}[R]\|_{\alpha,\beta} \leqslant C\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{t}{1+t}\right)^{\frac{3}{4}}\|R(t)\|_{L^4}, \tag{4.33}$$

since $\operatorname{supp} R(t) \subset \{|x| \leqslant 2\}$. For the estimate of $R(t)$ we have from (4.30),

$$\|R(t)\|_{L^4} \leqslant C(\|\Lambda_{\mathbb{A}_{\alpha,\Omega}}[f,g](t)\|_{W^{1,4}(D_0)} + \|p_\Omega(t)\|_{L^4(D_0)} + \alpha\|fg(t)\|_{L^4(D_0)}). \tag{4.34}$$

The proof of (4.34) is exactly the same as the proof of (4.13). As in the proof of Proposition 4.3, next we derive the estimate near the boundary. To this end we take a smooth cut-off denoted again by χ such that $\chi = 1$ for $|x| \leqslant 2$ and $\chi = 0$ for $|x| \geqslant 3$, and set

$$u(t) = (1-\chi)\Lambda_{\mathbb{A}_\alpha}[(1-\chi)f,g](t) + \mathbb{B}[\nabla\chi\cdot\Lambda_{\mathbb{A}_\alpha}[(1-\chi)f,g](t)], \tag{4.35}$$

where f,g are extended to $\mathbb{R}^2$ by zero in $\mathbb{R}^2\setminus\Omega$. The operator $\mathbb{B}$ denotes the Bogovskii operator but in the annulus $D_1 = \{x \in \mathbb{R}^2 \mid 2 < |x| < 3\}$ in this case. Then the direct calculation shows that $w(t) = \Lambda_{\mathbb{A}_{\alpha,\Omega}}[f,g](t) - u(t)$ satisfies

$$\begin{cases} \partial_t w + A_\alpha w + \nabla p_w = -\tilde{R}, & t > 0,\ x \in \Omega, \\ \operatorname{div} w = 0, & t \geqslant 0,\ x \in \Omega, \\ w = 0, & t > 0,\ x \in \partial\Omega, \\ w|_{t=0} = 0, & x \in \Omega, \end{cases}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

with $p_w = p_\Omega - (1-\chi)p_{\mathbb{R}^2}$, where $p_{\mathbb{R}^2}$ is the pressure field associated with $\Lambda_{\mathbb{A}_\alpha}[(1-\chi) f, g]$, and

$$\begin{aligned}\tilde{R}(t) &= (\Delta\chi)\Lambda_{\mathbb{A}_\alpha}[(1-\chi)f,g](t) + 2\nabla\chi\cdot\nabla\Lambda_{\mathbb{A}_\alpha}[(1-\chi)f,g](t)\\ &\quad -\alpha(\partial_1\chi)\Lambda_{\mathbb{A}_\alpha}[(1-\chi)f,g](t) + (\partial_t + A_\alpha)\mathbb{B}[\nabla\chi\cdot\Lambda_{\mathbb{A}_\alpha}[(1-\chi)f,g](t)]\\ &\quad -(\nabla\chi)p_{\mathbb{R}^2} - \alpha\nabla\cdot(\chi f\otimes g + g\otimes\chi f) + \alpha\chi\nabla\cdot((1-\chi)f\otimes g + g\otimes(1-\chi)f).\end{aligned}$$

Thus w is written as

$$w(t) = -\int_0^t e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega\tilde{R}(s)\,ds, \tag{4.36}$$

while the associated pressure p_w is written in the form

$$p_w(t) = -\int_0^t p[\mathbb{P}_\Omega\tilde{R}(s)](t-s)\,ds \tag{4.37}$$

by following the notation used in Theorem 4.1. Since $\Lambda_{\mathbb{A}_{\alpha,\Omega}}[f,g](t) = w(t)$ for $|x|\leqslant 2$, it suffices to estimate $w(t)$.

(i) Estimate of w: from the estimate of the Oseen semigroup, see (2.9), it is easy to see that

$$\begin{aligned}\|\nabla^j w(t)\|_{L^\infty} &\leqslant C\int_0^t (t-s)^{-\frac{j}{2}-\frac{1}{4}}\|\mathbb{P}_\Omega\tilde{R}(s)\|_{L^4}\,ds\\ &\leqslant C\int_0^t (t-s)^{-\frac{j}{2}-\frac{1}{4}}\|\tilde{R}(s)\|_{L^4}\,ds, \quad 0<t\leqslant 3,\ \ j=0,1.\end{aligned} \tag{4.38}$$

When $t\geqslant 3$ we have also from Theorem 4.1,

$$\begin{aligned}&\|w(t)\|_{W^{1,\infty}(\Omega\cap\{|x|\leqslant 3\})}\\ &\quad\leqslant \int_{t-2}^t \|e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega\tilde{R}(s)\|_{W^{1,\infty}}\,ds\\ &\qquad + C\int_0^{t-2}\|e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega\tilde{R}(s)\|_{W^{2,4}(\Omega\cap\{|x|\leqslant 3\})}\,ds\\ &\quad\leqslant C\int_{t-2}^t (t-s)^{-\frac{3}{4}}\|\tilde{R}(s)\|_{L^4}\,ds\\ &\qquad + C\int_0^{t-2}\left(\frac{1}{(t-s)(\log(t-s))^2} + \frac{\alpha^2\chi_{\{t-s\leqslant\alpha^{-2}\}}}{|\log(t-s)|} + \frac{\chi_{\{t-s\geqslant\alpha^{-2}\}}}{(t-s)^2\alpha^2|\log\alpha|}\right)\|\tilde{R}(s)\|_{L^4}\,ds.\end{aligned} \tag{4.39}$$

Next, Proposition 3.3 for the bilinear form $\Lambda_{\mathbb{A}_\alpha}[(1-\chi)f,g]$ yields

$$\begin{aligned}\|\tilde{R}(t)\|_{L^4} &\leqslant C(\|\Lambda_{\mathbb{A}_\alpha}[(1-\chi)f,g](t)\|_{W^{1,\infty}(\{|x|\leqslant 3\})} + \|p_{\mathbb{R}^2}(t)\|_{|L^4(\{|x|\leqslant 3\})}\\ &\quad + \alpha\|\chi\nabla(fg)\|_{L^4} + \alpha\|fg\nabla\chi\|_{L^4})\\ &\leqslant \frac{C(1+t^{-\frac{1}{2}})}{(1-\beta)(1+(\alpha^2 t)^{\frac{\beta}{2}})}\|g\|_{\alpha,\beta}\sup_{t>0}\left(\|\rho(\alpha\cdot)f(t)\|_{L^\infty} + \alpha^{\frac{1}{2}}\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right)\\ &\quad + C\|p_{\mathbb{R}^2}(t)\|_{L^4}.\end{aligned} \tag{4.40}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

To estimate $p_{\mathbb{R}^2}$ we recall that

$$\partial_t \Lambda_{\mathbb{A}_\alpha}[f,g] + A_\alpha \Lambda_{\mathbb{A}_\alpha}[f,g] + \nabla p_{\mathbb{R}^2} = \alpha \nabla\cdot((1-\chi) f\otimes g + g\otimes (1-\chi) f), \quad t>0,\ x\in\mathbb{R}^2,$$

and thus, from $\operatorname{div}\Lambda_{\mathbb{A}_\alpha}[f,g]=0$, $p_{\mathbb{R}^2}$ is given by the formula

$$p_{\mathbb{R}^2} = -\alpha(-\Delta_{\mathbb{R}^2})^{-1}\nabla\cdot\nabla\cdot((1-\chi) f\otimes g + g\otimes(1-\chi) f). \tag{4.41}$$

Since $(-\Delta_{\mathbb{R}^2})^{-1}\nabla\cdot\nabla\cdot$ defines a singular integral operator, we have from the Calderon–Zygmund inequality

$$\begin{aligned}\|p_{\mathbb{R}^2}(t)\|_{L^8(\mathbb{R}^2)} &\leqslant C\alpha \|f\otimes g(t) + g\otimes f(t)\|_{L^8(\mathbb{R}^2)}\\ &= C\alpha^{\frac{3}{4}}\|f\otimes g(t)+g\otimes f(t)\|_{L^8_X}, \quad X=\alpha x\end{aligned}$$

and then, using

$$\begin{aligned}\|f\otimes g(t)+g\otimes f(t)\|_{L^8_X} &\leqslant C\|f(t)\|_{L^8_X}\|g(t)\|_{L^\infty}\\ &\leqslant C\left\|\frac{1}{\rho}\right\|_{L^8_X(\{|X|\geqslant\alpha/4\})}\|\rho(\alpha,\cdot) f(t)\|_{L^\infty}\|\,\|g(t)\|_{L^\infty}\end{aligned}$$

due to $f=0$ for $|x|\leqslant 1/4$ (since f is zero extension to $\mathbb{R}^2$), by using $\|\frac{1}{\rho}\|_{L^8_X(\{|X|\geqslant\alpha/4\})}\leqslant C\alpha^{-\frac{1}{4}}$, we end up with the estimate

$$\begin{aligned}\|p_{\mathbb{R}^2}(t)\|_{L^4(D_0)} &\leqslant C\alpha^{\frac{1}{2}}\|\rho(\alpha\cdot) f(t)\|_{L^\infty}\|g(t)\|_{L^\infty}\\ &\leqslant \frac{C\alpha^{\frac{1}{2}}|\log\alpha|}{1+(\alpha^2 t)^{\frac{\beta}{2}}}\|g\|_{\alpha,\beta}\sup_{t>0}\|\rho(\alpha\cdot) f(t)\|_{L^\infty}.\end{aligned} \tag{4.42}$$

Collecting (4.40) and (4.42), we obtain

$$\|\tilde{R}(t)\|_{L^4} \leqslant \frac{C(1+t^{-\frac{1}{2}})}{(1-\beta)(1+(\alpha^2 t)^{\frac{\beta}{2}})}\|g\|_{\alpha,\beta}\sup_{t>0}\left(\|\rho(\alpha\cdot) f(t)\|_{L^\infty} + \alpha^{\frac{1}{2}}\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right). \tag{4.43}$$

Hence, combining (4.38) and (4.39) with (4.43), we arrive at

$$\begin{aligned}&\|w(t)\|_{L^\infty(\Omega\cap\{|x|\leqslant 3\})} + \left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla w(t)\|_{L^\infty(\Omega\cap\{|x|\leqslant 3\})}\\ &\quad\leqslant \frac{C}{(1-\beta)(1+(\alpha^2 t)^{\frac{\beta}{2}})}\|g\|_{\alpha,\beta}\sup_{t>0}\left(\|\rho(\alpha\cdot) f(t)\|_{L^\infty} + \alpha^{\frac{1}{2}}\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right).\end{aligned} \tag{4.44}$$

(ii) Estimate of p_Ω in D_0: Recall that $p_\Omega = p_w + (1-\chi)p_{\mathbb{R}^2}$. From (4.37) and Theorem 4.1 we have

$$\begin{aligned}\|p_w\|_{L^4(\Omega\cap\{|x|\leqslant 3\})} \leqslant C\int_0^t &\left(\frac{\chi_{\{t-s\leqslant 2\}}}{(t-s)^{\frac{3}{4}}} + \frac{\chi_{\{2\leqslant t-s\leqslant\alpha^{-2}\}}}{(t-s)|\log(t-s)|^2}\right.\\ &\left. + \frac{\alpha^2\chi_{\{2\leqslant t-s\leqslant\alpha^{-2}\}}}{|\log(t-s)|} + \frac{\chi_{\{t-s\geqslant\alpha^{-2}\}}}{(t-s)^2\alpha^2|\log\alpha|}\right)\|\tilde{R}(s)\|_{L^4}\,ds,\end{aligned}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

and then, combining with (4.43), we obtain

$$(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{t}{1+t}\right)^{\frac{3}{4}}\|p_w(t)\|_{L^4(\Omega\cap\{|x|\leqslant 3\})} \leqslant \frac{C}{1-\beta}\|g\|_{\alpha,\beta}\sup_{t>0}\left(\|\rho(\alpha\cdot)f(t)\|_{L^\infty}+\alpha^{\frac{1}{2}}\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right). \tag{4.45}$$

Thus, (4.42) and (4.45) yield

$$(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{t}{1+t}\right)^{\frac{3}{4}}\|p_\Omega(t)\|_{L^4(D_0)} \leqslant \frac{C}{1-\beta}\|g\|_{\alpha,\beta}\sup_{t>0}\left(\|\rho(\alpha\cdot)f(t)\|_{L^\infty}+\alpha^{\frac{1}{2}}\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right). \tag{4.46}$$

Estimates (4.34) with $\alpha\|fg\|_{L^4(D_0)} \leqslant \frac{C\alpha^{\frac{1}{2}}|\log\alpha|}{1+(\alpha^2 t)^{\frac{\beta}{2}}}\|g\|_{\alpha,\beta}\sup_{t>0}\|\rho(\alpha\cdot)f(t)\|_{L^\infty}$, (4.44), and (4.46) give

$$(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{t}{1+t}\right)^{\frac{3}{4}}\|R(t)\|_{L^4} \leqslant \frac{C}{1-\beta}\|g\|_{\alpha,\beta}\sup_{t>0}\left(\|\rho(\alpha\cdot)f(t)\|_{L^\infty}+\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right). \tag{4.47}$$

Since $\Lambda_{\alpha,\Omega}[f,g]=v$ for $|x|\geqslant 2$, estimates (4.32), (4.33), and (4.47) yield (4.29) away from the boundary ($|x|\geqslant 2$), while (4.44) yields (4.29) near the boundary ($|x|\leqslant 2$). The proof is complete. □

5. Nonlinear stability

Based on the linear analysis in the previous sections we solve the integral equation of the Navier–Stokes equations

$$\begin{aligned} v(t) &= e^{-t\mathbb{A}_{\alpha,\Omega}}v_0-\alpha\int_0^t e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega\nabla\cdot(U\otimes v+v\otimes U)\,ds \\ &\quad -\alpha\int_0^t e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega\nabla\cdot(v\otimes v)\,ds \\ &= e^{-t\mathbb{A}_{\alpha,\Omega}}v_0-\Lambda_{\alpha,\Omega}[U,v](t)-\frac{1}{2}\Lambda_{\alpha,\Omega}[v,v](t). \end{aligned} \tag{5.1}$$

Proof of Theorem 1.2. Let $\alpha\in(0,\frac{1}{2}]$ and $\beta\in[0,1)$, and let $\|\cdot\|_{\alpha,\beta}$ be the norm defined as (4.26). Let X be the Banach space defined by

$$X=\{f\in C([0,\infty);L^4_\sigma(\Omega))\cap C((0,\infty);W^{1,4}_0(\Omega)^2)\mid \|f\|_{\alpha,\beta}<\infty\}.$$

Then we define the nonlinear map $\mathcal{N}$ on X as

$$\mathcal{N}[f](t)=e^{-t\mathbb{A}_{\alpha,\Omega}}v_0-\Lambda_{\alpha,\Omega}[U,f](t)-\tfrac{1}{2}\Lambda_{\alpha,\Omega}[f,f](t), \tag{5.2}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

and our aim is to show that $\mathcal{N}$ defines a contraction map in a small closed ball of X. Set $w_0(t) = e^{-t\mathbb{A}_{\alpha,\Omega}}v_0$, which belongs to $C([0,\infty); L^4_\sigma(\Omega)) \cap C((0,\infty); W^{1,4}_0(\Omega)^2)$ since $v_0 \in L^4_\sigma(\Omega)$. We observe from (4.27) that

$$\|w_0\|_{\alpha,\beta} \leqslant C_1\|(1+b(\alpha\cdot))v_0\|_{L^\infty}. \tag{5.3}$$

Here C_1 is independent of α and β. On the other hand, we have from Proposition 4.4,

$$\begin{aligned}\|\Lambda_{\alpha,\Omega}[U,f]\|_{\alpha,\beta} &\leqslant \frac{C}{1-\beta}\|f\|_{\alpha,\beta}(\|\rho(\alpha\cdot)U\|_{L^\infty} + \alpha^{\frac{1}{2}}\|\nabla U\|_{L^\infty})\\ &\leqslant \frac{C}{1-\beta}\left(\frac{1}{|\log\alpha|} + \alpha^{\frac{1}{2}}\right)\|f\|_{\alpha,\beta}\\ &\leqslant \frac{1}{8}\|f\|_{\alpha,\beta},\end{aligned} \tag{5.4}$$

where we have used the assumptions **(A1)**–**(A2)** for U, and we have taken α small enough depending on $\beta \in [0,1)$. We also have

$$\begin{aligned}\|\Lambda_{\alpha,\Omega}[f,g]\|_{\alpha,\beta} &\leqslant \frac{C}{1-\beta}\|g\|_{\alpha,\beta}\sup_{t>0}\left(\|\rho(\alpha\cdot)f(t)\|_{L^\infty} + \alpha^{\frac{1}{2}}\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right)\\ &\leqslant \frac{C_2}{1-\beta}\|g\|_{\alpha,\beta}\|f\|_{\alpha,\beta}.\end{aligned} \tag{5.5}$$

Here C_2 is independent of α and β. We note that $\nabla\cdot(f\otimes g) \in L^4(\Omega)$ when $\|\rho(\alpha\cdot)f\|_{L^\infty} + \|\rho(\alpha\cdot)g\|_{L^\infty} + \|\nabla f\|_{L^\infty} + \|\nabla g\|_{L^\infty} < \infty$. Therefore, we have for $0 < t < T$ with any $T > 0$ and $j = 0, 1$,

$$\begin{aligned}&\left\|\nabla^j \int_0^t e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega\nabla\cdot(f\otimes g)\,ds\right\|_{L^4}\\ &\quad\leqslant C_T\int_0^t (t-s)^{-\frac{j}{2}}\|\mathbb{P}_\Omega\nabla\cdot(f\otimes g)\|_{L^4}\,ds\\ &\quad\leqslant C_T\int_0^t (t-s)^{-\frac{j}{2}}(\|f\nabla g(s)\|_{L^4} + \|g\nabla f(s)\|_{L^4})\,ds\\ &\quad\leqslant C_{T,\alpha}\int_0^t (t-s)^{-\frac{j}{2}}(\|\rho(\alpha\cdot)f(s)\|_{L^\infty}\|\nabla g(s)\|_{L^\infty} + \|\rho(\alpha\cdot)g\|_{L^\infty}\|\nabla f(s)\|_{L^\infty})\,ds.\end{aligned} \tag{5.6}$$

Here the constant $C_{T,\alpha}$ depends on T and α. From the estimate of the form (5.6) it is not difficult to show that $\Lambda_{\alpha,\Omega}[U,f]$ and $\Lambda_{\alpha,\Omega}[f,g]$ belong to $C((0,\infty); W^{1,4}_0(\Omega)^2)$ and $\lim_{t\to 0}(\|\Lambda_{\alpha,\Omega}[U,f](t)\|_{L^4} + \|\Lambda_{\alpha,\Omega}[f,g](t)\|_{L^4}) = 0$ when $f, g \in X$. The details are omitted here. Thus we have proved that the nonlinear map $\mathcal{N}$ is a map from X into X.

Estimates (5.3), (5.4), and (5.5) imply that $\mathcal{N}$ is a contraction map in the ball

$$\left\{f \in X \mid \|f\|_{\alpha,\beta} \leqslant \frac{1-\beta}{8C_2}\right\}$$

if $C_1\|(1+b(\alpha\cdot))v_0\|_{L^\infty} \leqslant \frac{1-\beta}{16C_2}$. Hence, if this smallness condition is satisfied, by the Banach fixed point theorem there exists a unique fixed point v of $\mathcal{N}$ in this ball, which is the solution to (5.1). Note that the proof of the Banach fixed point theorem also implies

$$\|v\|_{\alpha,\beta} \leqslant 2\|w_0\|_{\alpha,\beta} \leqslant 2C_1\|(1+b(\alpha\cdot))v_0\|_{L^\infty}. \tag{5.7}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

By the definition of $\|\cdot\|_{\alpha,\beta}$ we have

$$\|v(t)\|_{L^\infty} \leqslant \frac{|\log\alpha|}{(1+(\alpha^2 t)^{\frac{\beta}{2}})}\|v\|_{\alpha,\beta} \leqslant \frac{2C_1|\log\alpha|}{(1+(\alpha^2 t)^{\frac{\beta}{2}})}\|(1+b(\alpha\cdot))v_0\|_{L^\infty}.$$

Thus (1.8) follows. The proof is complete. □

6. Note on the class of perturbations and temporal decay

In Theorem 1.2 and the key linear estimate (1.16) the solution decays in time with the rate $O((\alpha^2)^{-\frac{\beta}{2}})$, $\beta < 1$, in L^∞_x. In this section we discuss how to obtain the critical or supercritical rate $\beta \geqslant 1$ under the additional condition on the initial data. In fact, in this regime we need more complicated weighted norm. Let $\alpha \in (0, \frac{1}{2}]$, $\delta \in (0,1)$, and $\beta \in [1, 1+\delta)$. Then we set $\|f\|_{\alpha,\beta,\delta}$ as

$$\begin{aligned}\|f\|_{\alpha,\beta,\delta} &= \sup_{t>0}\left(\|\rho(\alpha\cdot)^{1+\delta} f(t)\|_{L^\infty} + \frac{\delta(1+\delta-\beta)}{1+2\delta-\beta}(\alpha^2 t)^{\frac{\beta}{2}}\|\rho(\alpha\cdot)^{1+\delta-\beta} f(t)\|_{L^\infty}\right)\\ &\quad + \sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{2-\beta}{(2-\beta)|\log\alpha|+1}\|f(t)\|_{L^\infty} + \left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right).\end{aligned} \tag{6.1}$$

As in $\|f\|_{\alpha,\beta}$, the first two terms in the right-hand side lie in the same scale, while the last two terms are in the subscale compared with the first ones (in view of time and spatial decay). Then, instead of the linear estimate (1.16), we have for $\alpha \in (0, \frac{1}{2}]$, $\delta \in (0,1)$, $\beta \in [1, 1+\delta)$ with $\beta \geqslant 2\delta$, and for the solution v to (1.15),

$$\begin{aligned}\|v\|_{\alpha,\beta,\delta} &\leqslant \frac{C}{\delta(1-\delta)}\|(1+\rho(\alpha\cdot))^{1+2\delta} v_0\|_{L^\infty}\\ &\quad + C\left(\frac{1}{1+\delta-\beta}+\frac{1}{\delta}\right)\|g\|_{\alpha,\beta,\delta}\sup_{t>0}\left(\|\rho(\alpha\cdot) f(t)\|_{L^\infty} + \alpha^{\frac{1}{2}}\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right).\end{aligned} \tag{6.2}$$

Here C is independent of α, β, δ. The main difference with (1.16) is that (i) there are additional singularity on δ near $\delta = 0$ and $\delta = 1$, and (ii) the norm $\|(1+\rho(\alpha\cdot))^{1+2\delta} v_0\|_{L^\infty}$ is needed, rather than $\|(1+\rho(\alpha\cdot))^{1+\delta} v_0\|_{L^\infty}$. The proof of (6.2) proceeds in the same manner as in the proof of (1.16): (Step 1) obtain the estimates in the whole space problem, (Step 2) invoke the local energy decay estimate in Theorem 4.1, (Step 3) obtain the estimates in the exterior problem by combining the results of Step 1 and Step 2 through a cut-off argument. Step 2 and Step 3 are exactly the same as discussed in § 4, and the modifications are needed only for Step 1, i.e., the estimates in the whole space problem. Thus we state here the counterparts of Propositions 3.2, 3.3, and 6.3, but without details of the proof.

Proposition 6.1. *Let* $\delta \in (0,1)$ *and* $\beta \in [0, 1+\delta]$. *Let* $\rho(\alpha\cdot)^{1+2\delta} f \in L^\infty(\mathbb{R}^2)^2$ *and* $\operatorname{div} f = 0$ *in* $\mathbb{R}^2$. *Then*

$$\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\|\rho(\alpha\cdot)^{1+\delta-\beta} e^{-t\mathbb{A}_\alpha} f\|_{L^\infty} \leqslant \frac{C}{\delta(1-\delta)}\|(1+\rho(\alpha\cdot))^{1+2\delta} f\|_{L^\infty}, \tag{6.3}$$

and

$$\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})t^{\frac{1}{2}}\|\nabla e^{-t\mathbb{A}_\alpha} f\|_{L^\infty} \leqslant C\|(1+\rho(\alpha\cdot))^{1+2\delta} f\|_{L^\infty}. \tag{6.4}$$

Here C is independent of α, β, and δ.

Proof. We give a proof only for (6.3) with $\beta = 1+\delta$. Since $\operatorname{div} f = 0$ ensures that the kernel of $e^{-t\mathbb{A}_\alpha}$ is given by the heat kernel, we have

$$|(e^{-t\mathbb{A}_\alpha} f)(x)| \leqslant \int_{\mathbb{R}^2} G(t, x-y-\alpha t\mathbf{e}_1)\frac{dy}{(1+\rho(\alpha y))^{1+2\delta}}\|(1+\rho(\alpha\cdot))^{1+2\delta} f\|_{L^\infty}.$$

From the definition $\rho(Y) = |Y|^{\frac{1}{2}} + |Y_2|$, we have $\int_{\mathbb{R}} \frac{dy_2}{(1+\rho(\alpha y))^{1+2\delta}} \leqslant \frac{C}{\delta|\alpha y_1|^\delta}$, which yields

$$\begin{aligned}|(e^{-t\mathbb{A}_\alpha} f)(x)| &\leqslant \frac{C}{\alpha t^{\frac{1}{2}}\delta}\int_{\mathbb{R}} e^{-\frac{|x_1-y_1-\alpha t|^2}{4t}}\frac{dy_1}{|\alpha y_1|^\delta}\|(1+\rho(\alpha\cdot))^{1+2\delta} f\|_{L^\infty}\\ &\leqslant \frac{C}{\delta(1-\delta)(\alpha^2 t)^{\frac{1+\delta}{2}}}\|(1+\rho(\alpha\cdot))^{1+2\delta} f\|_{L^\infty}.\end{aligned}$$

Note that we need the norm $\|(1+\rho(\alpha\cdot))^{1+2\delta} f\|_{L^\infty}$ rather than $\|(1+\rho(\alpha\cdot))^{1+\delta} f\|_{L^\infty}$. The proof is complete. □

Proposition 6.2. *Let $\alpha \in (0, \frac{1}{2}]$, $\delta \in (0,1)$, and $\beta \in [1, 1+\delta)$. Then the following estimates hold for any $f, g \in L^\infty(0,\infty; W^{1,\infty}(\mathbb{R}^2)^2)$ such that $\rho(\alpha\cdot) f, \rho(\alpha\cdot)^{1+\delta} g \in L^\infty(0,\infty; L^\infty(\mathbb{R}^2)^2)$.*

$$\begin{aligned}&\sup_{t>0}(\alpha^2 t)^{\frac{\beta}{2}}\|\rho(\alpha\cdot)^{1+\delta-\beta}\Lambda_{\mathbb{A}_\alpha}[f,g](t)\|_{L^\infty}\\ &\quad\leqslant C\left(\frac{1}{1+\delta-\beta}+\frac{1}{1-\delta}+\frac{1}{\delta}\right)\sup_{t>0}\|\rho(\alpha\cdot) f(t)\|_{L^\infty}\\ &\qquad\times\sup_{t>0}((\alpha^2 t)^{\frac{\beta}{2}}\|\rho(\alpha\cdot)^{1+\delta-\beta} g(t)\|_{L^\infty} + \|\rho(\alpha\cdot)^{1+\delta} g(t)\|_{L^\infty}),\end{aligned} \tag{6.5}$$

$$\begin{aligned}&\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\|\Lambda_{\mathbb{A}_\alpha}[f,g](t)\|_{L^\infty}\\ &\quad\leqslant C\left(\frac{1}{1+\delta-\beta}+\frac{1}{1-\delta}+\frac{1}{\delta}\right)\sup_{t>0}\|\rho(\alpha\cdot) f(t)\|_{L^\infty}\\ &\qquad\times\sup_{t>0}((\alpha^2 t)^{\frac{\beta}{2}}\|\rho(\alpha\cdot)^{1+\delta-\beta} g(t)\|_{L^\infty} + \|\rho(\alpha\cdot)^{1+\delta} g(t)\|_{L^\infty} + (1+(\alpha^2 t)^{\frac{\beta}{2}})\|g(t)\|_{L^\infty}),\end{aligned} \tag{6.6}$$

$$\begin{aligned}&\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla\Lambda_{\mathbb{A}_\alpha}[f,g](t)\|_{L^\infty}\\ &\quad\leqslant C\alpha^{\frac{1}{2}}\sup_{t>0}\left(\|\rho(\alpha\cdot) f(t)\|_{L^\infty} + \alpha^{\frac{1}{2}}\left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla f(t)\|_{L^\infty}\right)\\ &\qquad\times\sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}})\left(\|g(t)\|_{L^\infty} + \left(\frac{t}{1+t}\right)^{\frac{1}{2}}\|\nabla g(t)\|_{L^\infty}\right).\end{aligned} \tag{6.7}$$

Here C is independent of α, β, and δ.

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

Note that $1-\delta \geqslant 1+\delta-\beta$ when $\beta \geqslant 2\delta$. The proof of Proposition 6.2 is very parallel to the proof of Proposition 3.3, which is based on Lemmas 2.3 and 2.4. The singularity $\frac{1}{\delta}$ is related to the case when β is close to 1, and if $\beta > 1$ then this factor can be replaced by $\frac{1}{\beta-1}$. Indeed, these singularities are originated from the computation of the form (3.19), which now reads as

$$\begin{aligned}|\Lambda_{\mathbb{A}_1}[\tilde{f}, \tilde{g}](\tau, X)| \leqslant\ & C\tau^{-\frac{\eta}{2}} \int_0^{\frac{\tau}{2}} \int_{\mathbb{R}^2} (\tau-s)^{-\frac{3-\eta}{2}} K(\tau-s, X-Y)^{\frac{3}{2}} \frac{1}{\rho(Y)^{2+\delta}}\, dY\, ds \\ & \times \sup_{\tau>0} \|\rho^{1+\delta}\tilde{g}(\tau)\|_{L^\infty} \sup_{\tau>0} \|\rho \tilde{f}(\tau)\|_{L^\infty} \\ & + C\tau^{-\frac{\beta}{2}} \int_{\frac{\tau}{2}}^{\tau} \int_{\mathbb{R}^2} (\tau-s)^{-\frac{3}{2}} K(\tau-s, X-Y)^{\frac{3}{2}} \frac{1}{\rho(Y)^{2+\delta-\beta}}\, dY\, ds \\ & \times \sup_{\tau>0} \tau^{\frac{\beta}{2}} \|\rho^{1+\delta-\beta}\tilde{g}(\tau)\|_{L^\infty} \sup_{\tau>0} \|\rho \tilde{f}(\tau)\|_{L^\infty}. \end{aligned} \tag{6.8}$$

Here the number η must be taken suitably to meet our purpose. If $\beta \geqslant 1$ then we may take $\eta = \beta$, which yields the singularity $\frac{1}{\beta-1}$ when $\beta > 1$ by the application of Lemma 2.3 for the kernel estimate in the regime $|X-Y| \leqslant \frac{1}{e}$. When $\beta > 1$, in order to replace the factor $\frac{1}{\beta-1}$ by $\frac{1}{\delta}$ we may assume $\beta \leqslant 1+\frac{\delta}{2}$ (otherwise the replacement to $\frac{1}{\delta}$ is trivial) and then consider two cases $\tau^{\frac{1}{2}} \geqslant \rho(X)$ and $\tau^{\frac{1}{2}} \leqslant \rho(X)$. When $\tau^{\frac{1}{2}} \geqslant \rho(X)$ we take $\eta = 1+\frac{\delta}{2}$ and then apply Lemma 2.3. By using $\tau^{-\frac{\eta}{2}} \leqslant \tau^{-\frac{\beta}{2}}\rho(X)^{-\eta+\beta}$ we finally obtain the desired estimate, after the similar computation as in the proof of Proposition 3.3. If $\tau^{\frac{1}{2}} \leqslant \rho(X)$ then we take $\eta = 0$, which yields the decay $\frac{1}{\rho(X)^{1+\delta}}$ from the first term in the right-hand side of (6.8) and thus we obtain the decay $\tau^{-\frac{\beta}{2}}\rho(X)^{-1-\delta+\beta}$. One can check that the singularity $\frac{1}{\delta}$ appears through the applications of Lemmas 2.3 and 2.4 in this argument. Since these are technical issues, we omit the details.

The nontrivial modification is required in obtaining the estimate of the inhomogeneous term with a compactly supported data, as stated below.

Proposition 6.3. *Let $\alpha \in (0, \frac{1}{2}]$, $\delta \in (0,1)$, $\beta \in [1, 1+\delta)$, and $\beta \geqslant 2\delta$. Suppose that $R(t) \in L^4(\mathbb{R}^2)^2$ and $\operatorname{supp} R(t) \subset \{|x| \leqslant 6\}$ for all $t > 0$. Then*

$$\sup_{t>0} (\alpha^2 t)^{\frac{\beta}{2}} \|\rho(\alpha\cdot)^{1+\delta-\beta} \Upsilon_{\mathbb{A}_\alpha}[R](t)\|_{L^\infty} \leqslant \frac{C}{1+\delta-\beta} \sup_{t>0} (1+(\alpha^2 t)^{\frac{\beta}{2}}) \left(\frac{t}{1+t}\right)^{\frac{3}{4}} \|R(t)\|_{L^4}, \tag{6.9}$$

$$\sup_{t>0} (1+(\alpha^2 t)^{\frac{\beta}{2}}) \|\Upsilon_{\mathbb{A}_\alpha}[R](t)\|_{L^\infty} \leqslant C\left(|\log\alpha| + \frac{1}{2-\beta}\right) \sup_{t>0} (1+(\alpha^2 t)^{\frac{\beta}{2}}) \left(\frac{t}{1+t}\right)^{\frac{3}{4}} \|R(t)\|_{L^4}, \tag{6.10}$$

$$\sup_{t>0} (1+(\alpha^2 t)^{\frac{\beta}{2}}) \left(\frac{t}{1+t}\right)^{\frac{1}{2}} \|\nabla\Upsilon_{\mathbb{A}_\alpha}[R](t)\|_{L^\infty} \leqslant C \sup_{t>0} (1+(\alpha^2 t)^{\frac{\beta}{2}}) \left(\frac{t}{1+t}\right)^{\frac{3}{4}} \|R(t)\|_{L^4}. \tag{6.11}$$

Here C is independent of α, β, and δ.

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

Proof. It suffices to focus on the estimate for $t \geqslant 2$; the estimate for $t \in (0, 2]$ is proved in the same way as in the proof of Proposition 3.5 without difficulty. We decompose $\Upsilon_{\mathbb{A}_\alpha}[R](t)$ into $I(t)+II(t)$ as in (3.32), and the term $I(t)$ is handled in parallel to (3.33). As for the term $II(t)$, we argue as in the proof of Proposition 3.5, which gives

$$|II(t,x)| \leqslant C \int_0^{t-1} \int_{\mathbb{R}^2} (t-s-1)^{-1} \left(1+\frac{|x-y-\alpha(t-s-1)\mathbf{e}_1|^2}{t-s-1}\right)^{-1} \frac{dy}{1+|y|^8} \times (1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1} \left(\frac{1+s}{s}\right)^{\frac{3}{4}} ds \times \sup_{t>0}(1+(\alpha^2 t)^{\frac{\beta}{2}}) \left(\frac{t}{1+t}\right)^{\frac{3}{4}} \|R(t)\|_{L^4}.$$

Thus, the proof is reduced to the computation of

$$II_1 = \int_0^{\frac{t-1}{2}} \int_{\mathbb{R}^2} (t-s-1)^{-1} \left(1+\frac{|x-y-\alpha(t-s-1)\mathbf{e}_1|^2}{t-s-1}\right)^{-1} \frac{dy}{1+|y|^8} \times (1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1} \left(\frac{1+s}{s}\right)^{\frac{3}{4}} ds$$

$$II_2 = \int_{\frac{t-1}{2}}^{t-1} \int_{\mathbb{R}^2} (t-s-1)^{-1} \left(1+\frac{|x-y-\alpha(t-s-1)\mathbf{e}_1|^2}{t-s-1}\right)^{-1} \frac{dy}{1+|y|^8} \times (1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1} \left(\frac{1+s}{s}\right)^{\frac{3}{4}} ds.$$

The estimate of II_2 is obtained in the same way as in the proof of (3.42), and we have

$$\|\rho(\alpha\cdot)^{1+\delta-\beta} II_2(t)\|_{L^\infty} \leqslant \frac{C}{(1+\delta-\beta)(1+(\alpha^2 t)^{\frac{\beta}{2}})}.$$

We omit the details and focus on the estimate of II_1. When $\alpha t^{\frac{1}{2}} \leqslant 4$ we have from (3.37) with $\kappa = 0$ that

$$\begin{aligned} II_1 &\leqslant C \int_0^{\frac{1}{4}} \int_{\mathbb{R}^2} (t-s-1)^{-1} \left(1+\frac{|x-y-\alpha(t-s)\mathbf{e}_1|^2}{t-s}\right)^{-1} \frac{dy}{1+|y|^8} s^{-\frac{3}{4}}\, ds \\ &\quad + C \int_{\frac{1}{4}}^{t} \int_{\mathbb{R}^2} (t-s)^{-1} \left(1+\frac{|x-y-\alpha(t-s)\mathbf{e}_1|^2}{t-s}\right)^{-1} \frac{dy}{1+|y|^8}\, ds \\ &\leqslant C \int_0^{\frac{1}{4}} \int_{\mathbb{R}^2} (t-s-1)^{-1} \left(1+\frac{|x-y-\alpha(t-s)\mathbf{e}_1|^2}{t-s}\right)^{-1} \frac{dy}{1+|y|^8} s^{-\frac{3}{4}}\, ds \\ &\quad + C \int_{\mathbb{R}^2} \left(-\chi_{\{|X-Y|\leqslant 1/e\}} \log|X-Y| + \frac{\chi_{\{|X-Y|\geqslant 1/e\}}}{|X-Y|^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}}\right) \\ &\quad \times \frac{dy}{1+|y|^8}, \end{aligned}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

which leads to the bound for $\alpha t^{\frac{1}{2}} \leqslant 4$ as

$$(1+(\alpha^2 t)^{\frac{\beta}{2}})\|\rho(\alpha\cdot)^{1+\delta-\beta} II_1(t)\|_{L^\infty} \leqslant \frac{C}{1+\delta-\beta}.$$

Thus we consider the case $\alpha t^{\frac{1}{2}} \geqslant 4$. In this case we further decompose II_1 into $\int_0^{t^{\frac{1}{2}}/4\alpha} + \int_{t^{\frac{1}{2}}/4\alpha}^{\frac{t-1}{2}} = II_{1,1} + II_{1,2}$, and then it is not difficult to see

$$\begin{aligned}\rho(\alpha x)^{1+\delta-\beta} II_{1,1}(t,x) &\leqslant \frac{C}{t}(\alpha^2 t)^{\frac{1+\delta-\beta}{2}} \int_0^{\frac{t^{\frac{1}{2}}}{4\alpha}} (1+(\alpha^2 s)^{\frac{\beta}{2}})^{-1} \left(\frac{1+s}{s}\right)^{\frac{3}{4}} ds \\ &\leqslant \frac{C}{(2-\beta)(\alpha^2 t)^{\frac{\beta}{2}+\beta-\delta}} \leqslant \frac{C}{(2-\beta)(\alpha^2 t)^{\frac{\beta}{2}}}.\end{aligned}$$

It remains to estimate $II_{1,2}$. We have from (3.37) with $\kappa = \frac{\beta}{2}$,

$$\begin{aligned}II_{1,2} &\leqslant \frac{C}{(1+(\alpha^2 t)^{\frac{\beta}{4}})} t^{-\frac{\beta}{4}} \int_{\mathbb{R}^2} \int_0^t (t-s)^{-1+\frac{\beta}{4}} \left(1 + \frac{|x-y-\alpha(t-s)\mathbf{e}_1|^2}{t-s}\right)^{-1} ds \frac{dy}{1+|y|^8} \\ &\leqslant \frac{C}{(1+(\alpha^2 t)^{\frac{\beta}{4}})} t^{-\frac{\beta}{4}} \alpha^{-\frac{\beta}{2}} \\ &\quad \times \int_{\mathbb{R}^2} \left(\chi_{\{|X-Y|\leqslant 1/e\}} + \frac{\chi_{\{|X-Y|\geqslant 1/e\}}}{|X-Y|^{\frac{1-\frac{\beta}{2}}{2}} (1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}}\right) \frac{dy}{1+|y|^8}.\end{aligned}$$

Then, under the condition $2\delta \leqslant \beta < 2$ we have $1+\delta-\beta \leqslant 1-\frac{\beta}{2}$ and $1-\frac{\beta}{2} > 0$, and thus,

$$\frac{\rho(X-Y)^{1+\delta-\beta}\, \chi_{\{|X-Y|\geqslant 1/e\}}}{|X-Y|^{\frac{1-\frac{\beta}{2}}{2}} (1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}} \leqslant C < \infty$$

holds with a universal constant C. Hence, by also recalling $\alpha t^{\frac{1}{2}} \geqslant 4$,

$$\|\rho(\alpha\cdot)^{1+\delta-\beta} II_{1,2}(t)\|_{L^\infty} \leqslant \frac{C}{1+(\alpha^2 t)^{\frac{\beta}{2}}}.$$

The proof of (6.9) is complete, for $2-\beta > 1-\delta \geqslant 1+\delta-\beta$ under the condition on the parameters. The proof of (6.10) and (6.11) is similar to the proof of (3.28) and (3.29). Note that the factor $\frac{1}{2-\beta}$ in (6.10) comes from the time integral in (3.44). The proof of Proposition 6.3 is complete. □

Remark 6.4. Once the linear estimate (6.2) is obtained, the application to the nonlinear stability is a routine work. If the initial data is localized enough so that $(1+\rho(\alpha\cdot)^{1+2\delta})v_0 \in L^\infty(\Omega)^2$ with $\delta > 0$ and if this weighted L^∞ norm is small enough (depending also on δ), then the solution $v(t)$ decays with the rate $o((\alpha^2 t)^{\frac{1}{2}})$ in L^∞_x as $t \to \infty$.

Acknowledgements. The author is partially supported by JSPS grant 17K05320 and 25220702.

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

Appendix A. Key inequality

Lemma A.1. *There exists* $\kappa \in (0, 1)$ *such that*

$$|X| - X_1 \geqslant \begin{cases} C|X|, & \text{if } |X_2| \geqslant \kappa|X_1|, \\ \dfrac{X_2^2}{4|X_1|}, & \text{if } |X_2| \leqslant \kappa|X_1|. \end{cases} \tag{A 1}$$

Here C *is independent of* X. *As a consequence, there exists* $C' > 0$ *such that*

$$\frac{1}{|X|^{\frac{1}{2}}(1+|X|-X_1)^{\frac{3}{4}}} \leqslant \frac{C'}{|X|^{\frac{1}{2}}+|X_2|}\left(\frac{1}{1+|X|^{\frac{1}{4}}}+\min\left\{1, \frac{|X_1|^{\frac{1}{4}}}{|X_2|^{\frac{1}{2}}}\right\}\right), \quad X \neq 0, \tag{A 2}$$

$$\frac{1}{|X|(1+|X|-X_1)} \leqslant \frac{C'}{|X|+X_2^2}, \quad X \neq 0. \tag{A 3}$$

In particular, the function $\frac{1}{|X|^{\frac{1}{2}}(1+|X|-X_1)^{\frac{3}{4}}}$ *belongs to* $L^\infty_{X_1} L^1_{X_2}$.

Proof. Fix sufficiently small $\kappa > 0$. If $|X_2| \geqslant \kappa|X_1|$ we have for any $\epsilon > 0$,

$$|X| - X_1 = \epsilon|X| + (1-\epsilon)|X| - X_1 \geqslant \epsilon|X| + (1-\epsilon)\sqrt{(1+\kappa^2)}|X_1| - X_1.$$

Thus, by taking $\epsilon > 0$ sufficiently small but depending only on κ, we obtain $|X| - X_1 \geqslant \epsilon|X|$ in this case. Next we consider the case $|X_2| \leqslant \kappa|X_1|$. In this case we have from the smallness of $|\frac{X_2}{X_1}|$,

$$\begin{aligned} |X| - X_1 &= |X_1|\left(1+\left(\frac{X_2}{X_1}\right)^2\right)^{\frac{1}{2}} - X_1 \\ &\geqslant |X_1|\left(1+\frac{1}{4}\left(\frac{X_2}{X_1}\right)^2\right) - X_1 \geqslant \frac{X_2^2}{4|X_1|}. \end{aligned}$$

This proves (A 1). To show (A 2) we set $g(X) = \dfrac{1}{|X|^{\frac{1}{2}}(1+|X|-X_1)^{\frac{3}{4}}}$. It is obvious that $|X|^{\frac{1}{2}} g(X) \in L^\infty(\mathbb{R}^2)$, and moreover, if $|X_2| \geqslant \kappa|X_1|$ then (A 1) gives the bound

$$g(X) \leqslant \frac{C}{|X|^{\frac{1}{2}}(1+|X|)^{\frac{3}{4}}}.$$

Hence, it suffices to consider the case $|X_2| \leqslant \kappa|X_1|$. In this case (A 1) implies

$$g(X) \leqslant \frac{2|X_1|^{\frac{1}{2}}}{|X|^{\frac{1}{2}}|X_2|(1+|X|-X_1)^{\frac{1}{4}}} \leqslant \frac{2}{|X_2|(1+|X|-X_1)^{\frac{1}{4}}},$$

which gives $|X_2| g(X) \in L^\infty(\mathbb{R}^2)$. Thus we have $(|X|^{\frac{1}{2}}+|X_2|)g(X) \in L^\infty(\mathbb{R}^2)$, and

$$(|X|^{\frac{1}{2}}+|X_2|)g(X) \leqslant \frac{C}{(1+|X|-X_1)^{\frac{1}{4}}} \leqslant C\left(\frac{1}{1+|X|^{\frac{1}{4}}}+\min\left\{1, \frac{|X_1|^{\frac{1}{4}}}{|X_2|^{\frac{1}{2}}}\right\}\right).$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

Thus, estimate (A 2) holds. Estimate (A 3) is also proved in the same manner. Finally, it is clear that $(|X|^{\frac{1}{2}}+|X_2|)^{-1}(1+|X|^{\frac{1}{4}})^{-1}$ belongs to $L^\infty_{X_1}L^1_{X_2}$, and since

$$\int_{\mathbb{R}} \frac{1}{(|X_1|^{\frac{1}{2}}+|X_2|)|X_2|^{\frac{1}{2}}}\,dX_2 = 2\int_0^\infty \frac{1}{(|X_1|^{\frac{1}{2}}+r)r^{\frac{1}{2}}}\,dr$$
$$= \frac{2}{|X_1|^{\frac{1}{4}}}\int_0^\infty \frac{1}{(1+s)s^{\frac{1}{2}}}\,ds = \frac{C}{|X_1|^{\frac{1}{4}}},$$

the right-hand side of (A 2) belongs to $L^\infty_{X_1}L^1_{X_2}$, and hence, so does g. The proof is complete. □

Appendix B. Estimate of localized terms R

We give a proof of (4.13). Note that the support of $\nabla\chi$ is included in $\overline{D_0}=\{1\leqslant |x|\leqslant 2\}$. To simplify the notation we set $u(t)=e^{-t\mathbb{A}_{\alpha,\Omega}}f$. In virtue of the estimate for the Bogovskii operator stated in (4.7), it is easy to see

$$\|(\Delta\chi)u(s)+2\nabla\chi\cdot\nabla u(s)-\alpha(\partial_1\chi)u(s)-(\nabla\chi)p_\Omega(s)-L_\alpha\mathbb{B}[\nabla\chi\cdot u(s)]\|_{L^4}$$
$$\leqslant C(\|u(s)\|_{W^{1,4}(D_0)}+\|p_\Omega(s)\|_{L^4(D_0)}).$$

Thus, what is nontrivial in the estimates of $R(t)$ defined by (4.9) is the estimate of the term $\partial_s\mathbb{B}[\nabla\chi\cdot u(s)]$. To estimate this we observe that

$$\begin{aligned}\partial_t\mathbb{B}[\nabla\chi\cdot u] &= \mathbb{B}[\nabla\chi\cdot(\Delta u-\alpha\partial_1 u-\nabla p_\Omega)]\\ &= \mathbb{B}[\nabla\cdot(\nabla\chi\nabla u)]+\mathbb{B}[(\nabla^2\chi)\nabla u]\\ &\quad -\alpha\mathbb{B}[\nabla\chi\cdot\partial_1 u]-\mathbb{B}[\nabla(p_\Omega\nabla\chi)]-\mathbb{B}[(\nabla^2\chi)p_\Omega].\end{aligned}$$

Since $\nabla\chi$ vanishes on ∂D_0 we can apply the estimate for the Bogovskii operator in the Sobolev space with a negative order (cf. [8]), which yields

$$\begin{aligned}&\|\mathbb{B}[\nabla\cdot(\nabla\chi\nabla u)]\|_{L^4(D_0)}+\|\mathbb{B}[\nabla(p_\Omega\nabla\chi)]\|_{L^4(D_0)}\\ &\quad\leqslant C(\|\nabla\chi\nabla u\|_{L^4(D_0)}+\|p_\Omega(\nabla\chi)\|_{L^4(D_0)})\\ &\quad\leqslant C(\|\nabla u\|_{L^4(D_0)}+\|p_\Omega\|_{L^4(D_0)}).\end{aligned}$$

The other terms are estimated by applying the standard estimate for the Bogovskii operator, and (4.13) follows. The proof is complete.

Appendix C. Estimate of Finn and Smith solutions

In this appendix we show that the stationary solutions constructed by Finn and Smith [22] obey the condition **(A1)**. The result and the argument stated here are not essentially new, so we only give a sketch of the proof. We note that the *subcritical* bound **(A2)** easily follows from [22] and thus the details are omitted here. First we observe that the solution (U,P) to (1.3) of Finn and Smith satisfies the global bound

$$\|\nabla U\|_{L^2(\Omega)}\leqslant C, \tag{C 1}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

where C is independent of small α [22, Lemma 5.2]. Then from the bootstrap argument and the local elliptic regularity for the Stokes system it is not difficult to see that U is smooth on $\overline{\Omega}$ and satisfies the local bound

$$\|U\|_{W^{2,4}(\Omega\cap B_4(0))} + \|P\|_{W^{1,4}(\Omega\cap B_4(0))} \leqslant C, \tag{C 2}$$

where C is independent of small α. Furthermore, we have already known from [22] that U obeys the pointwise estimate

$$|U_i(x)| \leqslant \frac{C}{|\log\alpha|} h_i(\alpha x), \tag{C 3}$$

where

$$\begin{aligned} h_i(X) &= \log\frac{2}{|X|}, \quad 0 < |X| \leqslant 1,\\ h_1(X) &= |X|^{-\frac{1}{2}}, \quad h_2(X) = |X|^{-\frac{5}{6}},\ |X| > 1. \end{aligned}$$

Therefore, it suffices to consider the estimate of $U(x)$ for $|\alpha x| = |X| \geqslant 1$. To this end let us recall the representation formula

$$\begin{aligned} U_j(x) &= -\alpha \sum_{1\leqslant i,l\leqslant 2} \int_{\Omega} \partial_l E_{ij}(x-y;\alpha) U_l(y) U_i(y)\, dy \\ &\quad - \sum_{1\leqslant i,l\leqslant 2} \int_{\partial\Omega} E_{ij}(x-y;\alpha) T_{il}(U,P)(y) n_l(y)\, d\sigma_y \\ &\quad + \alpha \int_{\partial\Omega} E_{1j}(x-y;\alpha) n_1(y)\, d\sigma_y. \end{aligned} \tag{C 4}$$

Here we have used the fact $U|_{\partial\Omega} = -\mathbf{e}_1$, and $E(x-y;\alpha) = (E_{ij}(x-y;\alpha))_{1\leqslant i,j\leqslant 2}$ is the Oseen fundamental solution. The tensor $T(U,P)$ is defined as $T(U,P) = (\nabla U + \nabla U^{\top}) - P\mathbb{I}$, where $\mathbb{I}$ is the identity matrix. Thus U solves the integral equation

$$U = U^0 + \mathcal{N}_\alpha(U,U),$$

where

$$\begin{aligned} U_j^0(x) &= -\alpha \sum_{1\leqslant i,l\leqslant 2} \int_{\Omega\cap|\alpha y|\leqslant 1} \partial_l E_{ij}(x-y;\alpha) U_l(y) U_i(y)\, dy \\ &\quad - \sum_{1\leqslant i,l\leqslant 2} \int_{\partial\Omega} E_{ij}(x-y;\alpha) T_{il}(U,P)(y) n_l(y)\, d\sigma_y \\ &\quad + \alpha \int_{\partial\Omega} E_{1j}(x-y;\alpha) n_1(y)\, d\sigma_y \end{aligned}$$

and

$$\mathcal{N}_\alpha(U,U) = \left(-\alpha \sum_{1\leqslant i,l\leqslant 2} \int_{|\alpha y|>1} \partial_l E_{ij}(x-y;\alpha) U_l(y) U_i(y)\, dy\right)_j.$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

Noting the homogeneous property $E_{ij}(x;\alpha) = E_{ij}(X;1) =: E_{ij}(X)$ for $X = \alpha x$ and setting $V(X) = U(x)$ and $V^0(X) = U^0(x)$, we see that the above integral equation is equivalent with

$$V = V^0 + N(V, V),$$

where

$$N(V, V) = \left(- \sum_{1 \leqslant i,l \leqslant 2} \int_{|Y|>1} \partial_l E_{ij}(X - Y) V_l(Y) V_i(Y)\, dY \right)_j .$$

Let us write $V = V^0 + \tilde{V}$, then the equation for $\tilde{V}$ is

$$\tilde{V} = N(V, V^0) + N(V, \tilde{V}). \tag{C 5}$$

We now regard V and V_0 in (C 5) as given terms, and then try to reconstruct $\tilde{V}$ as the solution to (C 5) in $\{|X| > 1\}$. To this end let introduce the norm

$$\begin{aligned} \|\tilde{V}\| &= \sup_{|X|>1} |X_1|^{\frac14} |\tilde{V}(X)|, \\ \|\tilde{V}\|' &= \sup_{|X|>1} |X|^{\frac12} (1 + |X| - X_1)^{\frac34} |\tilde{V}(X)|. \end{aligned} \tag{C 6}$$

Our aim is to solve (C 5) in the closed ball $\{f \in L^\infty(\{|X| > 1\}) \mid \|f\|' \leqslant \frac{C}{|\log\alpha|}\}$ and also to show the uniqueness in $\{f \in L^\infty(\{|X| > 1\}) \mid \|f\| < \infty\}$ for small α. The key is the following pointwise estimate of $\nabla E(X)$:

$$\begin{aligned} \sum_{(i,j,l)\neq(1,1,2)} |\partial_l E_{ij}(X)| &\leqslant \frac{C}{|X|(1 + |X|)^{\frac12} (1 + |X| - X_1)^{\frac12}}, \\ |\nabla E(X)| &\leqslant \frac{C}{|X|(1 + |X| - X_1)}. \end{aligned} \tag{C 7}$$

This can be shown from the expansion of $E_{ij}(X)$ for $|X| \gg 1$; see, e.g., [49, equation (13)] and also [11, equation (22)] for details. The crucial point is that $\partial_l E_{ij}$ with $(i, j, l) \neq (1, 1, 2)$ has a slightly faster isotropic decay than $\partial_2 E_{11}$, while V_2 decays isotropically faster than V_1 as stated in (C 3).

Step 1. $|V^0(X)| \leqslant \frac{C}{|\log\alpha|} \Big(\frac{1}{|X|^{\frac12} (1+|X|-X_1)^{\frac34}} + \frac{1}{1+|X|} \Big)$ for $|X| > 1$.

By the definition of U^0 the velocity V^0 is written as $V^0 = V^{0,1} + V^{0,2} + V^{0,3}$ in the natural manner, and we observe from (C 3) that

$$|V^{0,1}(X)| \leqslant \frac{C}{|\log\alpha|^2} \int_{|Y|\leqslant 1} |\nabla E(X - Y)| (1 + |\log|Y||^2)\, dY.$$

Hence it is easy to see from (C 7) that $\|V^{0,1}\|' \leqslant \frac{C}{|\log\alpha|^2}$. As for $V^{0,2}$, we see

$$V^{0,2}_j(X) = - \sum_{1 \leqslant i,l \leqslant 2} \int_{\partial\Omega} E_{ij}(X - \alpha y) T_{il}(U, P)(y) n_l(y)\, d\sigma_y,$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

and thus, when $|X| \geqslant 1$ and α is small enough, we can see that the leading term in the estimate of $V_j^{0,2}$ is given by the term

$$\sum_{1\leqslant i,l\leqslant 2} E_{ij}(X) \int_{\partial\Omega} T_{il}(U,P)(y) n_l(y)\, d\sigma_y.$$

The key point here is that we have

$$\left| \sum_{1\leqslant l\leqslant 2} \int_{\partial\Omega} T_{il}(U,P)(y) n_l(y)\, d\sigma_y \right| \leqslant \frac{C}{|\log\alpha|}, \tag{C 8}$$

which reflects the Stokes paradox; see [22, equation (5.4)]. On the other hand, we have the pointwise estimate for $E_{ij}(X)$ such that

$$|E_{ij}(X)| \leqslant C\left(\frac{1}{|X|^{\frac{1}{2}}(1+|X|-X_1)^{\frac{3}{4}}} + \frac{1}{1+|X|} \right), \quad |X| \geqslant 1, \tag{C 9}$$

which can be checked from the expansion of $E_{ij}(X)$ for $|X| \gg 1$; see, e.g., [49, equation (13)]. Thus, $V^{0,2}$ satisfies the desired pointwise estimate stated above. The estimate for $V_j^{0,3}(X) = \alpha \int_{\partial\Omega} E_{1j}(X-\alpha y) n_1(y)\, d\sigma_y$ also follows from (C 9). Step 1 is proved.

Step 2: $\|N(V,f)\|' \leqslant \frac{C}{|\log\alpha|}\|f\|'$.

We have from (C 3) for V and (C 7) for ∇E,

$$\begin{aligned} &|N(V,f)(X)| \\ &\quad \leqslant \frac{C}{|\log\alpha|} \int_{|Y|>1} \frac{1}{|X-Y|(1+|X-Y|)^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}} \\ &\qquad \times \frac{1}{|Y|(1+|Y|-Y_1|)^{\frac{3}{4}}}\, dY \|f\|' \\ &\qquad + \frac{C}{|\log\alpha|} \int_{|Y|>1} \frac{1}{|X-Y|(1+|X-Y|-(X_1-Y_1))} \frac{1}{|Y|^{\frac{4}{3}}(1+|Y|-Y_1|)^{\frac{3}{4}}}\, dY \|f\|'. \end{aligned}$$

The triangle inequality for the weight $|X|-X_1$ implies that

$$\begin{aligned} &(1+|X|-X_1)^{\frac{3}{4}} \int_{|Y|>1} \frac{1}{|X-Y|(1+|X-Y|)^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}} \\ &\qquad \times \frac{1}{|Y|(1+|Y|-Y_1|)^{\frac{3}{4}}}\, dY \\ &\quad \leqslant C \int_{|Y|>1} \frac{1}{|X-Y|(1+|X-Y|)^{\frac{1}{4}}} \frac{1}{|Y|(1+|Y|-Y_1|)^{\frac{3}{4}}}\, dY \\ &\qquad + C \int_{|Y|>1} \frac{1}{|X-Y|(1+|X-Y|)^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}} \frac{1}{|Y|}\, dY \\ &\quad = I_1 + I_2, \end{aligned}$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

and

$$
\begin{aligned}
&(1+|X|-X_1)^{\frac{3}{4}} \int_{|Y|>1} \frac{1}{|X-Y|(1+|X-Y|-(X_1-Y_1))} \frac{1}{|Y|^{\frac{4}{3}}(1+|Y|-Y_1|)^{\frac{3}{4}}}\, dY \\
&\leqslant C \int_{|Y|>1} \frac{1}{|X-Y|(1+|X-Y|-(X_1-Y_1))^{\frac{1}{4}}} \frac{1}{|Y|^{\frac{4}{3}}(1+|Y|-Y_1|)^{\frac{3}{4}}}\, dY \\
&\quad + C \int_{|Y|>1} \frac{1}{|X-Y|(1+|X-Y|-(X_1-Y_1))} \frac{1}{|Y|^{\frac{4}{3}}}\, dY \\
&= I_3 + I_4.
\end{aligned}
$$

We see from Lemma A.1 that

$$
\begin{aligned}
|X|^{\frac{1}{2}} I_1(X) &\leqslant C \int_{|Y|>1} \frac{1}{|X-Y|^{\frac{3}{4}}} \frac{1}{|Y|(1+|Y|-Y_1)^{\frac{3}{4}}}\, dY \\
&\quad + C\left(\int_{|Y|>1,|X-Y|<1} + \int_{|Y|>1,|X-Y|>1}\right) \\
&\quad \times \frac{1}{|X-Y|(1+|X-Y|)^{\frac{1}{4}}} \frac{1}{|Y|^{\frac{1}{2}}(1+|Y|-Y_1)^{\frac{3}{4}}}\, dY \\
&\leqslant C \int_{\mathbb{R}} \frac{1}{|X_1-Y_1|^{\frac{3}{4}}} \frac{1}{(1+|Y_1|)^{\frac{1}{2}}}\, dY_1 \\
&\quad + C + C \int_{\mathbb{R}} \frac{1}{(1+|X_1-Y_1|)^{\frac{5}{4}}}\, dY_1 \\
&\leqslant C < \infty.
\end{aligned}
$$

Similarly, by using the bound $\frac{1}{|X-Y|^{\frac{1}{3}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}} \in L^{\infty}_{Y_1}(\mathbb{R}; L^{\frac{3}{2}}_{Y_2}(\mathbb{R}))$, which follows again from Lemma A.1, it is not difficult to see that $|X|^{\frac{1}{2}} I_2(X) \leqslant C$. The bound for I_3 and I_4 is obtained in the same manner, and we have

$$
|X|^{\frac{1}{2}}(I_3(X) + I_4(X)) \leqslant C < \infty.
$$

The details are omitted here. Step 2 is proved.

Step 3: $\|N(V, V^0)\|' \leqslant \frac{C}{|\log \alpha|^2}$.

Thanks to Step 1 and the proof of Step 2 it suffices to show

$$
\left\| \int_{|Y|>1} \sum_{(i,j,l)\neq(1,1,2)} |\partial_l E_{ij}(X-Y)|(1+|Y|)^{-\frac{1}{2}-1}\, dY \right\|' \leqslant C < \infty, \tag{C 10}
$$

$$
\left\| \int_{|Y|>1} |\partial_2 E_{11}(X-Y)|(1+|Y|)^{-\frac{5}{6}-1}\, dY \right\|' \leqslant C < \infty. \tag{C 11}
$$

Here we only give a proof of (C 10), for (C 11) is easier. In order to show (C 10) we observe from (C 7) that

$$
|X|^{\frac{1}{2}}(1+|X|-X_1)^{\frac{3}{4}} \int_{|Y|>1} \sum_{(i,j,l)\neq(1,1,2)} |\partial_l E_{ij}(X-Y)|(1+|Y|)^{-\frac{1}{2}-1}\, dY
$$

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

$$\leqslant C \int_{|Y|>1} \frac{1}{|X-Y|^{\frac{3}{4}}} \frac{1}{(1+|Y|)^{\frac{3}{2}}}\, dY + C \int_{|Y|>1} \frac{1}{|X-Y|(1+|X-Y|)^{\frac{1}{4}}} \frac{1}{(1+|Y|)}\, dY$$
$$+ C \int_{|Y|>1} \frac{1}{|X-Y|^{\frac{1}{2}}(1+|X-Y|)^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}} \frac{1}{(1+|Y|)^{\frac{3}{4}}}\, dY$$
$$+ C \int_{|Y|>1} \frac{1}{|X-Y|(1+|X-Y|)^{\frac{1}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}} \frac{1}{(1+|Y|)^{\frac{1}{4}}}\, dY.$$

In these four terms the most delicate one is the last term, for which it suffices to consider

$$I_5 = \int_{|Y|>1, |X-Y|>1} \frac{1}{(1+|X-Y|)^{\frac{3}{2}}(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}} \frac{1}{(1+|Y|)^{\frac{1}{4}}}\, dY.$$

Then, from

$$\frac{1}{(1+|X-Y|-(X_1-Y_1))^{\frac{1}{2}}(1+|Y|)^{\frac{1}{4}}} \leqslant \frac{1}{(1+|X-Y|-(X_1-Y_1))^{\frac{3}{4}}(1+|Y|)^{\frac{3}{2}\epsilon}} + \frac{1}{(1+|Y|)^{(\frac{1}{4}-\epsilon)3}}$$

for $0 < \epsilon < \frac{1}{12}$ and from Lemma A.1, one can check that the integral I_5 converges. The other terms are easier to bound and the details are omitted. Step 3 is proved.

Step 4: $\|N(V, f)\| \leqslant \frac{C}{|\log\alpha|}\|f\|$.

We have from (C 3) for V and (C 7) for ∇E,

$$|N(V, f)(X)| \leqslant \frac{C}{|\log\alpha|} \int_{|Y|>1} \frac{1}{|X-Y|(1+|X-Y|-(X_1-Y_1))} \frac{1}{|Y|^{\frac{1}{2}}|Y_1|^{\frac{1}{4}}}\, dY \|f\|.$$

Thus, Lemma A.1 implies

$$|N(V, f)(X)| \leqslant \frac{C}{|\log\alpha|} \int_{\mathbb{R}} \frac{1}{|X_1-Y_1|^{\frac{1}{2}}|Y_1|^{\frac{3}{4}}}\, dY_1 \|f\| \leqslant \frac{C}{|X_1|^{\frac{1}{4}}|\log\alpha|}\|f\|.$$

This proves $\|N(V, f)\| \leqslant \frac{C}{|\log\alpha|}\|f\|$.

In virtue of Step 2 and Step 3 above, we see from the standard fixed point theorem that there exists a unique solution W to (C 5) (i.e., $W = N(V, V^0) + N(V, W)$) satisfying $\|W\|' \leqslant \frac{C}{|\log\alpha|^2}$ if α is small enough. Moreover, Step 4 ensures the uniqueness of the solution to (C 5) in the space $\{f \in L^\infty(\{|X| > 1\}) \mid \|f\| < \infty\}$, which implies $V - V^0 = W$ since we have already known that $V - V^0$ satisfies $\|V - V^0\| \leqslant \|V\| + \|V^0\| < \infty$, in virtue of (C 3) and Step 1. Hence, $V = V^0 + W$ satisfies the bound in **(A1)** by recalling the estimate of V^0 in Step 1.

References

1. K. Abe, On estimates for the Stokes flow in a space of bounded functions, *J. Differential Equations* **261** (2016), 1756–1795.
2. K. Abe, Global well-posedness of the two-dimensional exterior Navier–Stokes equations for non-decaying data, *Arch. Ration. Mech. Anal.* **227** (2018), 69–104.

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

3. K. Abe and Y. Giga, Analyticity of the Stokes semigroup in spaces of bounded functions, *Acta Math.* **211**(1) (2013), 1–46.
4. K. Abe and Y. Giga, The L^∞-Stokes semigroup in exterior domains, *J. Evol. Equ.* **14**(1) (2014), 1–28.
5. K. Abe, Y. Giga and M. Hieber, Stokes resolvent estimates in spaces of bounded functions, *Ann. Sci. Éc. Norm. Supér. (4)* **48**(3) (2015), 537–559.
6. K. I. Babenko, On stationary solutions of the problem of flow past a body of a viscous incompressible fluid, *Math. USSR Sb.* **20** (1973), 1–25.
7. W. Borchers and T. Miyakawa, L^2 decay for Navier–Stokes flows unbounded domains with application to exterior stationary flows, *Arch. Ration. Mech. Anal.* **118** (1992), 273–295.
8. W. Borchers and H. Sohr, On the equations $\operatorname{rot} v = g$ and $\operatorname{div} u = f$ with zero boundary conditions, *Hokkaido Math. J.* **19** (1990), 67–87.
9. W. Borchers and W. Varnhorn, On the boundedness of the Stokes semigroup in two-dimensional exterior domains, *Math. Z.* **213**(2) (1993), 275–299.
10. I.-D. Chang and R. Finn, On the solutions of a class of equations occurring in continuum mechanics, with application to the Stokes paradox, *Arch. Ration. Mech. Anal.* **7** (1961), 388–401.
11. D. C. Clark, The vorticity at infinity for solutions of the stationary Navier–Stokes equations in exterior domains, *Indiana Univ. Math. J.* **20**(7) (1971), 633–654.
12. W. Dan and Y. Shibata, On the L_q–L_r estimates of the Stokes semigroup in a two-dimensional exterior domain, *J. Math. Soc. Japan* **51** (1999), 181–207.
13. W. Dan and Y. Shibata, Remark on the L_q–L_∞ estimate of the Stokes semigroup in a 2-dimensional exterior domain, *Pacific J. Math.* **189** (1999), 223–239.
14. Y. Enomoto and Y. Shibata, Local energy decay of solutions to the Oseen equations in the exterior domains, *Indiana Univ. Math. J.* **53** (2004), 1291–1330.
15. Y. Enomoto and Y. Shibata, On the rate of decay of the Oseen semigroup in exterior domains and its application to Navier–Stokes equation, *J. Math. Fluid Mech.* **7** (2005), 339–367.
16. R. Farwig, The stationary exterior 3D-problem of Oseen and Navier–Stokes equations in anisotropically weighted Sobolev spaces, *Math. Z.* **211** (1992), 409–447.
17. R. Finn, On steady-state solutions of the Navier–Stokes partial differential equations, *Arch. Ration. Mech. Anal.* **3** (1959), 139–151.
18. R. Finn, *Estimates at Infinity for Steady State Solutions of the Navier–Stokes Equations*, Proceedings of Symposia in Pure Mathematics, Volume IV, pp. 143–148 (American Mathematical Society, Providence, RI, 1961).
19. R. Finn, On the steady-state solutions of the Navier–Stokes equations. III, *Acta Math.* **105** (1961), 197–244.
20. R. Finn, On the exterior stationary problem for the Navier–Stokes equations, and associated perturbation problems, *Arch. Ration. Mech. Anal.* **19** (1965), 363–406.
21. R. Finn and D. R. Smith, On the linearized hydrodynamical equations in two dimensions, *Arch. Ration. Mech. Anal.* **25** (1967), 1–25.
22. R. Finn and D. R. Smith, On the stationary solution of the Navier–Stokes equations in two dimensions, *Arch. Ration. Mech. Anal.* **25** (1967), 26–39.
23. H. Fujita, On the existence and regularity of the steady-state solutions of the Navier–Stokes equations, *J. Fac. Sci. Univ. Tokyo Sect. IA* **9** (1961), 59–102.
24. G. P. Galdi, *Existence and Uniqueness at Low Reynolds Number of Stationary Plane Flow of a Viscous Fluid in Exterior Domains. Recent Developments in Theoretical Fluid Mechanics (Paseky, 1992)*, Pitman Research Notes in Mathematics Series, Volume 291, pp. 1–33 (Longman Scientific and Technical, Harlow, 1993).

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

25. G. P. Galdi, *Stationary Navier–Stokes Problem in a Two-dimensional Exterior Domain* (ed. M. Chipot and P. Quittner), Handbook of Differential Equations, Stationary Partial Differential Equations, Volume I, pp. 71–155 (North-Holland, Amsterdam, 2004).
26. G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations, Steady-State Problems*, 2nd edn (Springer, New York, 2011).
27. G. P. Galdi, J. G. Heywood and Y. Shibata, On the global existence and convergence to steady state of Navier–Stokes flow past an obstacle that is started from rest, *Arch. Ration. Mech. Anal.* **138** (1997), 307–318.
28. L. Grafakos, *Classical and Modern Fourier Analysis* (Pearson Education, Inc., Upper Saddle River, NJ, 2004).
29. J. Guillod, On the asymptotic stability of steady flows with nonzero flux in two-dimensional exterior domains, *Commun. Math. Phys.* **352** (2017), 201–214.
30. J. G. Heywood, On stationary solutions of the Navier–Stokes equations as limits of non-stationary solutions, *Arch. Ration. Mech. Anal.* **37** (1970), 48–60.
31. J. G. Heywood, The exterior nonstationary problem for the Navier–Stokes equations, *Acta Math.* **129** (1972), 11–34.
32. J. G. Heywood, The Navier–Stokes equations: on the existence, regularity and decay of solutions, *Indiana Univ. Math. J.* **29** (1980), 639–681.
33. M. Higaki, Note on the stability of planar stationary flows in an exterior domain without symmetry, Preprint, 2018, arXiv:1805.09262.
34. M. Hillairet and P. Wittwer, Asymptotic description of solutions of the planar exterior Navier–Stokes problem in a half space, *Arch. Ration. Mech. Anal.* **205** (2012), 553–584.
35. T. Hishida, L^q–L^r estimate of the Oseen flow in plane exterior domains, *J. Math. Soc. Japan* **68**(1) (2016), 295–346.
36. H. Iwashita, L^q–L^r estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier–Stokes initial value problems in L^q spaces, *Math. Ann.* **285**(2) (1989), 265–288.
37. T. Kobayashi and Y. Shibata, On the Oseen equation in the three dimensional exterior domains, *Math. Ann.* **310** (1998), 1–45.
38. J. Leray, Étude de diverses équations intégrales non linéaires et de quelques problémes que pose I'hydrodynamique, *J. Math. Pures Appl.* IX **12** (1933), 1–82.
39. Y. Maekawa, On stability of steady circular flows in a two-dimensional exterior disk, *Arch. Ration. Mech. Anal.* **225** (2017), 287–374.
40. Y. Maekawa, On local energy decay estimate of the Oseen semigroup in two dimensions and its application, Preprint.
41. P. Maremonti, Stabilità asintotica in media per moti fluidi viscosi in domini esterni, *Ann. Mat. Pura Appl.* (4) **97** (1985), 57–75.
42. P. Maremonti and S. Shimizu, Global existence of solutions to 2-D Navier–Stokes flow with non-decaying initial data in exterior domains, *J. Math. Fluid Mech.* **20**(3) (2018), 899–927.
43. K. Masuda, On the stability of incompressible viscous fluid motions past objects, *J. Math. Soc. Japan* **27** (1975), 294–327.
44. T. Miyakawa, On nonstationary solutions of the Navier–Stokes equations in an exterior domain, *Hiroshima Math. J.* **12** (1982), 115–140.
45. T. Miyakawa and H. Sohr, On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier–Stokes equations in exterior domains, *Math. Z.* **199** (1988), 455–478.
46. R. O'Neil, Convolution operators and $L(p, q)$ spaces, *Duke Math. J.* **30** (1963), 129–142.

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press

47. Y. Shibata, On an exterior initial-boundary value problem for Navier–Stokes equations, *Quart. Appl. Math.* **57** (1999), 117–155.
48. C. G. Simader and H. Sohr, A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains, in *Mathematical Problems Relating to the Navier–Stokes Equations*, Series on Advances in Mathematics for Applied Sciences, Volume 11, pp. 1–35 (World Scientific Publishing, River Edge, 1992).
49. D. R. Smith, Estimates at infinity for stationary solutions of the Navier–Stokes equations in two dimensions, *Arch. Ration. Mech. Anal.* **20** (1965), 341–372.

https://doi.org/10.1017/S1474748019000240 Published online by Cambridge University Press