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About a cavitation model including bubbles in thin film lubrication: A first mathematical analysis

ALFREDO JARAMILLO[1], GUY BAYADA[2], IONEL CIUPERCA[3] and MOHAMMED JAI[2]

[1] *Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, 13560-970 São Carlos, Brazil*
email: ajaramillopalma@gmail.com
[2] *CNRS, INSA de Lyon, Institut Camille Jordan UMR 5208, Université de Lyon, F-69621 Villeurbanne, France*
emails: guy.bayada@insa-lyon.fr; mohammed.jai@insa-lyon.fr
[3] *CNRS, Institut Camille Jordan UMR 5208, Université de Lyon, F-69622 Villeurbanne, France*
email: ciuperca@math.univ-lyon1.fr

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In lubrication problems, which concern thin film flow, cavitation has been considered as a fundamental element to correctly describe the characteristics of lubricated mechanisms. Here, the well-posedness of a cavitation model that can explain the interaction between viscous effects and micro-bubbles of gas is studied. This cavitation model consists of a coupled problem between the compressible Reynolds partial differential equation (PDE) (that describes the flow) and the Rayleigh–Plesset ordinary differential equation (that describes micro-bubbles evolution). A simplified form without bubbles convection is studied here. This coupled model seems never to be studied before from its mathematical aspects. Local times existence results are proved and stability theorems are obtained based on the continuity of the spectrum for bounded linear operators. Numerical results are presented to illustrate these theoretical results.

Key words: Cavitation modelling, thin film lubrication, Reynolds equation, Rayleigh–Plesset equation

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1 Introduction

Cavitation is observed in various engineering devices, ranging from hydraulic systems to turbo pumps for space applications. It is a challenging issue linked with various phenomena: acoustic, thermodynamic and fluid dynamics. In the lubrication area, which is concerned with thin film flow, cavitation has been considered as a fundamental element to correctly describe the characteristics of lubricated mechanisms [1, 2]. Cavitation has often been primarily associated with a diminution of the pressure p in the liquid falling below the vapour pressure. Numerous models have been introduced to couple this unilateral condition with the Reynolds equation, which is usually used to model the pressure evolution in thin film flow. Mathematical studies of these models can be found in [3, 4, 5, 6, 7, 8] in which existence and uniqueness results are given for both the stationary and transient cases. Another approach has been proposed in [9] by considering cavitation as a multifluid problem with a free boundary between two immiscible

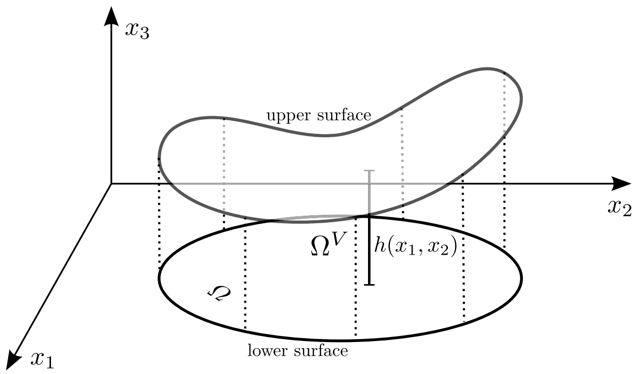

FIGURE 1. Three-dimensional scheme of the physical framework.

fluids. However, it is physically recognised that the cavitation phenomenon is linked with the existence and evolution of micro-bubbles in a liquid. This aspect has not been taken into account in these models. It is, however, taken into account in the well-known FLUENT software for fluid mechanics [10, 11, 12] in which micro-bubbles evolution is coupled with the Navier–Stokes system for a three-dimensional flow. In the lubrication area, this phenomenon has been ignored until the works of Someya's group [13, 14] who proposed to couple the full Rayleigh–Plesset equation (which describes the evolution of a bubble) with the Reynolds equation (which describes the fluid). Numerous works follow in the lubrication literature using simplified forms of the Rayleigh–Plesset equation for various kinds of applications [15, 16, 17, 18, 19, 20]. The paper of Snyder *et al.* [21] can be considered as a review paper in this field.

1.1 The Reynolds–Rayleigh–Plesset coupling

The fluid is contained in a domain $\Omega^V \subset \mathbb{R}^3$, limited by a domain $\Omega \subset \mathbb{R}^2$ in the x_1-x_2 plane, an upper surface given by the gap function $h\,(x_1, x_2)$ defined on Ω and by a vertical lateral boundary as shown in Figure 1. The surfaces are in relative motion along the x_1-x_2 plane at velocity $\mathbf{U} \in \mathbb{R}^2$. It is also assumed that there is no relative motion of the surfaces along the x_3 axis. In this work, theoretical results on the well-posedness of the Reynold–Rayleigh–Plesset (RRP) cavitation model for the flow of a fluid multicomponent mixture are presented. Here, a brief description of that mathematical model is given, and the physical hypotheses and a heuristic justification are given in the Appendix.

The mixture is composed by two phases: an incompressible liquid phase (with known density ρ_ℓ and viscosity μ_ℓ) and a gas phase (with known density ρ_g and viscosity μ_g). It is assumed that the gas phase is composed by a distribution of bubbles immersed on the liquid and that around a point $\mathbf{x}$ at time t there can be bubbles of only one certain radius $R(\mathbf{x}, t)$. In addition, the dynamics of the field $R(\mathbf{x}, t)$ are governed by the Rayleigh–Plesset equation (e.g., [22]):

$$\rho_\ell \left[\frac{3}{2} \left(\frac{\mathrm{D}R}{\mathrm{D}t} \right)^2 + R \frac{\mathrm{D}^2 R}{\mathrm{D}t^2} \right] = P_0 \left(\frac{R_0}{R} \right)^{3k} - p_m - \frac{2\sigma}{R} - 4 \left(\frac{\mu_\ell + \kappa^s / R}{R} \right) \frac{\mathrm{D}R}{\mathrm{D}t}, \tag{1.1}$$

where the terms at the left-hand side are called *inertial terms*, p_m is the mixture's pressure around the bubble, P_0 is the inner pressure of the bubble when its radius is equal to R_0, k is the polytropic

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exponent (see the Appendix); σ is the coefficient of surface tension, κ^s is the surface dilatational viscosity [21]; and

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\mathbf{u}_b \cdot \nabla), \tag{1.2}$$

with $\mathbf{u}_b \in \mathbb{R}^3$ corresponding to the transport velocity of the bubbles. In the right-hand side of equation (1.1), the first term models the pressure of the gas contained in the bubble.

As a first step in the mathematical analysis of this system, in this work, the transport velocity of the bubbles is assumed to be null, $\mathbf{u}_b = 0$. This simplification can be found in some numerical works involving squeeze film dampers [16]. In other studies, $\mathbf{u}_b$ is taken as the average of the surfaces velocities [23, 24] or as the average of the thin film velocity field [17, 18, 19, 21]. In consequence, equation (1.1) relates two unknown fields: R and p_m. A second equation is obtained by introducing the local gas fraction ($\frac{\text{volume of gas}}{\text{total volume}}$) in terms of R:

$$\alpha = \alpha\,(R(\mathbf{x}, t)), \tag{1.3}$$

for instance, assuming the number of bubbles per liquid volume unit to be a constant, denoted by n_b^ℓ, statistical arguments (e.g., [25] Section 10.1.2) can be given to show that:

$$\alpha(\mathbf{x}, t) = \frac{n_b^\ell \frac{4\pi R(\mathbf{x},t)^3}{3}}{1 + n_b^\ell \frac{4\pi R(\mathbf{x},t)^3}{3}}. \tag{1.4}$$

Then, the averaged mixture density $\bar{\rho}$ can be related to α by means of (e.g., [25]):

$$\bar{\rho}\,(R(\mathbf{x}, t)) = \rho_g\,\alpha\,(R(\mathbf{x}, t)) + \rho_\ell\,(1 - \alpha\,(R(\mathbf{x}, t))). \tag{1.5}$$

It is also assumed some model for the mixture effective viscosity field, denoted by μ_{eff}, in terms of the gas fraction α, so one may write $\mu_{\text{eff}} = \mu_{\text{eff}}(\alpha(R))$. The RRP model assumes the mixture pressure p_m accomplishes the compressible Reynolds equation:

$$\nabla_x \cdot \left(\frac{\bar{\rho} h^3}{12\mu_{\text{eff}}} \nabla p_m \right) = \nabla_x \cdot \left(\frac{\mathbf{U}}{2}\,\bar{\rho} h \right) + \frac{\partial \bar{\rho} h}{\partial t} \qquad \text{in } \Omega. \tag{1.6}$$

The RRP cavitation model consists in the coupling of equations (1.1) and (1.6) along with suitable boundary conditions.

It is noteworthy that there exist many works in Mechanics' literature concerning the numerical resolution and modelling aspects of the coupling of the Rayleigh–Plesset equation with fluid flow equations (e.g., [26, 27, 28, 29, 30, 31]). The well-known software FLUENT for fluid mechanics uses also this type of modelling [10, 11, 12]. On the other hand, in the mathematical field, few works are concerned with this problem. The Rayleigh–Plesset equation alone without coupling (in which the pressure is a known data) has been subject of interest as differential equations with singularities [32, 33]. However, to the knowledge of the authors, no mathematical analysis of the full coupling of the Rayleigh–Plesset equation with a flow equation (Euler, Stokes or Reynolds) so far appeared.

In this work, a mathematical analysis is carried on by first writing an abstract form of the coupling RRP by means of auxiliary functions that depend on the unknown radii field R. Then, some general properties of these auxiliary functions are identified from the physics and held as hypotheses (e.g., positiveness, monotonicity, existence of critical points). Informally, a first step

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of the study consists in writing the coupled model as an ordinary differential equation (ODE) on a Banach space and making use of a suitable version of the Cauchy–Lipschitz theorem. For this, it is shown that the unknown p can be *eliminated* by writing it in terms of R and its derivatives. A second step regards the well-posedness of the stationary problem: the existence of a trivial stationary solution is established and then non-trivial solutions are found by continuity arguments; finally, continuity arguments are also used to extend the stability of the trivial solution to the stability of non-trivial cases. These two analysis steps are independently performed for two scenarios: (1) including or (2) disregarding the inertial terms in the Rayleigh–Plessset equation.

The structure of this document is as it follows: after the introduction section, the mathematical framework is described in Section 2 where notations and some previous required results are given. Section 3 is devoted to the study of the full system (1.1)–(1.6) including inertial terms; existence of a stationary solution is gained by way of the Implicit Function theorem around some particular data for which a stationary solution is easy to compute; a stability result is obtained with a small data assumption by studying the spectrum of a differential operator, and the continuity of that spectrum around the particular data; at last, an instability result is gained by means of the Routh–Hurwitz theorem. In Section 4, a simplified Rayleigh–Plesset equation neglecting the inertial terms is considered; unlike the previous section, existence of the (local) solution of the system is not obvious and requires to use the Freedholm Alternative theorem; stability results of the stationary solution for small data are obtained using also the spectrum's continuity of a differential operator. In Section 5, some numerical examples are shown where time convergence towards stationary solutions is observed. Finally, a heuristic justification of both equation (1.6) and the RRP coupling is given in the Appendix.

2 Mathematical framework

In this section, we introduce some notations and previous results to be used along this document.

Let $\Omega \subset \mathbb{R}^N$, $N = 1, 2$ be a regular domain, and introduce the change of variables

$$p(\mathbf{x}, t) = p_m(\mathbf{x}, t)/\rho_\ell \;, \tag{2.1}$$

we consider the abstract problem of finding $p(\mathbf{x}, t)$, $R(\mathbf{x}, t) > 0$, with $\mathbf{x} \in \Omega$ and $t \geq 0$, such that

$$\frac{3}{2}\frac{1}{R}\left(\frac{\partial R}{\partial t}\right)^2 + \frac{\partial^2 R}{\partial t^2} = \frac{f_1(R) - p}{R} - \frac{\partial R}{\partial t} f_2(R) \tag{2.2}$$

and

$$\begin{gathered}\nabla_{\mathbf{x}} \cdot \left(f_3(R)\, h^3\, \nabla p\right) = \nabla_{\mathbf{x}} \cdot \left(f_4(R)\, \mathbf{U}\, h\right) + h f_5(R)\, \frac{\partial R}{\partial t}, \\ p = 0 \qquad \text{on } \partial\Omega,\end{gathered} \tag{2.3}$$

where $\mathbf{U} \in \mathbb{R}^N$. Along the initial conditions for every $\mathbf{x} \in \bar{\Omega}$:

$$\begin{aligned} R(\mathbf{x}, 0) &= r_1(\mathbf{x}), \\ \frac{\partial R}{\partial t}(\mathbf{x}, 0) &= r_2(\mathbf{x}), \end{aligned} \tag{2.4}$$

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and r_1, r_2 are regular known functions. The terms in the left-hand side of equation (2.2) are named *inertial terms*. In the next sections, we study the well-posedness of problem (2.2)–(2.3)–(2.4) when including or disregarding the inertial terms.

For $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, we define

$$B_{\alpha,\beta} = \left\{ w \in L^\infty(\Omega) : \alpha \le w \le \beta \text{ a.e. on } \Omega \right\}.$$

We make also the following hypotheses:

H1: $f_1 \in C^2\left(\mathbb{R}_*^+; \mathbb{R}\right)$, $\exists \bar{R}, \delta_1 \in \mathbb{R}_*^+$ such that $f_1\left(\bar{R}\right) = 0$ and $f_1'(R) < 0$ $\forall R \in [\bar{R} - \delta_1, \bar{R} + \delta_1]$. We denote $m_1 = \min_{R \in [\bar{R}-\delta_1, \bar{R}+\delta_1]} \left|f_1'(R)\right|$ and $M_1 = \max_{R \in [\bar{R}-\delta_1, \bar{R}+\delta_1]} \left|f_1'(R)\right|$;
H2: $f_2 \in C^2\left(\mathbb{R}_*^+; \mathbb{R}_*^+\right)$;
H3: $f_3 \in C^2\left(\mathbb{R}_*^+; \mathbb{R}\right)$ and $\exists m_3, M_3 > 0$ such that $m_3 \le f_3(r) \le M_3$ $\forall r \in \mathbb{R}^+$;
H4: $f_4 \in C^2\left(\mathbb{R}_*^+; \mathbb{R}^+\right)$, $f_4'(r) < 0$ $\forall r > 0$;
H5: $f_5 \in C^2\left(\mathbb{R}_*^+; \mathbb{R}_*^-\right)$;
H6: $h \in B_{m_0, M_0}$ for $0 < m_0 < M_0$ constants. We denote $h_0 = \operatorname{ess-inf}_{\Omega} h$.

Remark 1 *The physical model given by equations (1.1)–(1.6) is a particular case of problem (2.2)–(2.4) for which*

$$f_1(R) = \frac{1}{\rho_\ell}\left(P_0\left(\frac{R_0}{R}\right)^{3k} - \frac{2\sigma}{R}\right), \tag{2.5}$$

$$f_2(R) = \frac{4}{\rho_\ell}\left(\frac{\mu_\ell + \kappa^s/R}{R^2}\right), \qquad f_3(R) = \frac{1}{12}\frac{(1-\alpha(R)) + \alpha(R)\rho_g/\rho_\ell}{(1-\alpha(R))\mu_\ell + \alpha(R)\mu_g},$$

$$f_4(R) = \frac{1}{2}\left[1 + \alpha(R)\left(\rho_g/\rho_\ell - 1\right)\right], \qquad f_5(R) = f_4'(\alpha(R))\,\alpha'(R).$$

The hypothesis (H1) is related to the well-known (e.g., [22]) shape of function f_1 (see Figure 3), having a unique critical point R_{crit}.

The next result is a particular case of Theorem 4.2 in [34].

Proposition 1 *Let Ω be a smooth domain on $\mathbb{R}^N$, $f \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$ be the unique solution of the elliptic problem*[1]

$$\begin{aligned} \nabla \cdot (a \nabla u) &= f \quad , \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

for $a \in B_{\alpha,\beta}$, $0 < \alpha < \beta$. Then there exists $q > 2$ (which depends on α, β, Ω and the dimension N) such that, if $f \in W^{-1,q}(\Omega)$, then u belongs to $W_0^{1,q}(\Omega)$ and satisfies

$$\|u\|_{W_0^{1,q}(\Omega)} \le C\,\|f\|_{-1,q},$$

where $C = C(\alpha, \beta, \Omega, N)$.

[1] Henceforth, we denote '$\nabla_{\mathbf{x}} \cdot$ ' by '$\nabla \cdot$ '.

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Now, to fix henceforth a Sobolev space $W^{1,q}(\Omega)$, we define the open subset $Q \subset C\left(\bar{\Omega}\right)$ as

$$Q = \left\{R \in C\left(\bar{\Omega}\right) : R(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in \bar{\Omega}\right\}, \tag{2.6}$$

and set $q > 2$ given by Proposition 1 with $\alpha = m_0^3\, m_3 \min\{m_1, 1\}$ and $\beta = M_0^3\, M_3 \max\{M_1, 1\}$.

We define also the mapping

$$\begin{array}{rcl} A : Q \times C\left(\bar{\Omega}\right) & \longrightarrow & C\left(\bar{\Omega}\right) \\ (R_1, R_2) & \longrightarrow & A_1(R_1) + A_2\left(R_1, R_2\right), \end{array} \tag{2.7}$$

where $A_1 : Q \mapsto C\left(\bar{\Omega}\right)$ is such that $A_1\left(R_1\right)$ is the unique solution of the elliptic problem

$$\begin{array}{rl} \nabla \cdot \left(h^3 f_3\left(R_1\right) \nabla A_1\left(R_1\right)\right) = \nabla \cdot \left(\mathbf{U}\, h f_4(R_1)\right) & \text{in } \Omega, \\ A_1\left(R_1\right) = 0 & \text{on } \partial\Omega, \end{array} \tag{2.8}$$

and $A_2 : Q \times C\left(\bar{\Omega}\right) \mapsto C\left(\bar{\Omega}\right)$ is such that $A_2\left(R_1, R_2\right)$ is the unique solution of the elliptic problem

$$\begin{array}{rl} \nabla \cdot \left(h^3 f_3\left(R_1\right) \nabla A_2\left(R_1, R_2\right)\right) = h f_5\left(R_1\right) R_2 & \text{in } \Omega, \\ A_2\left(R_1, R_2\right) = 0 & \text{on } \partial\Omega. \end{array} \tag{2.9}$$

Remark 2 *Both the solutions of* (2.8) *and* (2.9) *are in* $C\left(\bar{\Omega}\right)$ *since* $W^{1,p}(\Omega) \subset C\left(\bar{\Omega}\right)$ *continuously for any* $p > N$.

Remark 3 *For any* $R_1 \in Q$, $A_2\left(R_1, \cdot\right)$ *is a bounded linear operator.*

Lemma 1 *The application* A *is of class* C^2 *from* $Q \times C\left(\bar{\Omega}\right)$ *into* $C\left(\bar{\Omega}\right)$.

Proof Let us define $\phi : Q \times C\left(\bar{\Omega}\right) \times W_0^{1,q}(\Omega) \mapsto W^{-1,q}(\Omega)$ by

$$\phi\left(R_1, R_2, p\right) = \nabla \cdot \left(h^3 f_3\left(R_1\right) \nabla p\right) - \nabla \cdot \left(\mathbf{U}\, h f_4(R_1)\right) - h f_5\left(R_1\right) R_2. \tag{2.10}$$

We show first that ϕ is of class C^2. Since f_3, f_4 and f_5 are of class C^2, it is enough to prove that the application $\phi^1 : C\left(\bar{\Omega}\right)^4 \times W_0^{1,q}(\Omega) \mapsto W^{-1,q}(\Omega)$ defined by

$$\phi^1\left(\xi_1, \xi_2, \xi_3, \xi_4, w\right) = \nabla \cdot \left(h^3 \xi_1 \nabla w\right) - \nabla \cdot \left(\mathbf{U}\, h\, \xi_2\right) - h\, \xi_3\, \xi_4$$

is of class C^2, which follows from observing that its first and third terms are quadratic and the second one is linear.

By the Lax–Milgram theorem and Proposition 1, we have also that the partial derivative

$$\begin{array}{rcl} \frac{\partial \phi}{\partial p}\left(R_1, R_2, p\right)(z) : W_0^{1,q}(\Omega) & \longrightarrow & W^{-1,q}(\Omega) \\ z & \longrightarrow & \nabla \cdot \left(h^3 f_3\left(R_1\right) \nabla z\right) \end{array}$$

is an isomorphism. Therefore, the result follows from noticing that $\phi\left(R_1, R_2, A\left(R_1, R_2\right)\right) = 0 \;\; \forall \left(R_1, R_2\right) \in Q \times C\left(\bar{\Omega}\right)$ and applying the Implicit Function theorem (e.g., [35]) to the application ϕ. $\square$

For a linear operator $\mathcal{L}$ we denote by $\mathrm{Vp}\left(\mathcal{L}\right)$ its set of eigenvalues and by $\mathrm{Sp}\left(\mathcal{L}\right)$ its spectrum. Let us recall the following classical results on ODE in Banach spaces:

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Proposition 2 *Let X be a Banach space, let A be a bounded linear operator on X and $\epsilon > 0$. Then there exists $\delta > 0$ such that, if B is a bounded linear operator on X and $\|A - B\| < \delta$, then for every $\lambda \in \mathrm{Sp}\,(B)$ there exists $\xi \in \mathrm{Sp}\,(A)$ such that $|\lambda - \xi| < \epsilon$.*

For a detailed proof of the previous result, the reader is referred to Lemma 3 in [36].

Proposition 3 (Cauchy–Lipschitz) *Let $f \in C(U; E)$, where U is an open set of E and $u_0 \in U$, and assume that f is of class C^r, $r \in \mathbb{N}^*$. Then the next properties hold*

- *There exists $T > 0$ and u in $C^1([t_0 - T, t_0 + T]; U)$ solution to the Cauchy problem:*

$$\begin{cases} u' = f(u)\,, \\ u(t_0) = u_0\,. \end{cases} \tag{2.11}$$

- *If v is another solution of (2.11). Then $v = u$ on the intersection of the intervals of definition of v and u.*
- *u is of class C^{r+1}.*

Definition 1 A solution $u \in C^1([0, T]; E)$ of the autonomous Cauchy problem $u' = f(u)$, $u(0) = u_0$ is called **maximal** if u cannot be extended to a solution on an interval containing $[0, T]$.

Definition 2 Let $f \in C(U; E)$ and $v \in U$. The point v such that $f(v) = 0$ is said to be an asymptomatically stable solution for the ODE $u' = f(u)$ if there exist $\epsilon > 0$ such that for any u_0 such that $\|u_0 - v\| \leq \epsilon$, the maximal solution of $u' = f(u)$, $u(0) = u_0$ is well defined for every $t \geq 0$, $\|u(t) - v\| \leq \epsilon$ for every $t \geq 0$ and $\lim_{t\to\infty} \|u(t) - v\| = 0$.

Definition 3 Let $f \in C(U; E)$ and $v \in U$. The point v such that $f(v) = 0$ is said to be an unstable stationary solution for the ODE $u' = f(u)$ if there exists $\epsilon_0 > 0$ such that for every $\eta > 0$ there exists $T > 0$ and a solution $u \in C^1([0, T], E)$ of $u' = f(u)$ that accomplishes $\|u(0) - v\| \leq \eta$ and $\|u(T) - v\| \geq \epsilon_0$.

Proposition 4 *Let $f \in C^2(U; E)$ and $v \in U$ be such that $f(v) = 0$. Assume that* $\mathrm{Sp}\,(\mathrm{D}f(v)) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\}$. *Then v is an asymptomatically stable solution for the ODE $u' = f(u)$.*

Proposition 5 *Let $f \in C^2(U; E)$ and $v \in U$ be such that $f(v) = 0$. Suppose that* $\max\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}\,(\mathrm{D}f(v))\}$ *is reached at an eigenvalue of* $\mathrm{D}f(v)$ *with real part strictly positive. Then v is an unstable solution for the ODE $u' = f(u)$.*

These proofs of propositions 3–5 can be found in [37], Sections 5.4, 8.1 and 8.2.

3 Well-posedness with inertial terms

Due to the fact that the unknown field p in equation (2.2) can be expressed as an operator depending on R and $\frac{\partial R}{\partial t}$ according to (2.3), the theory of ODE on Banach spaces can be applied to study the system (2.2)–(2.4).

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3.1 Existence of a local solution

Let us denote $R_1 = R$, $R_2 = \frac{\partial R}{\partial t}$ and $\tilde{R} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$. Then, the problem (2.2)–(2.3)–(2.4) can be rewritten as

$$\begin{aligned} \frac{d\tilde{R}}{dt} &= F(\tilde{R}), \\ \tilde{R}(0) &= \tilde{R}_0, \end{aligned} \tag{3.1}$$

where $\tilde{R}_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \in Q \times C\left(\bar{\Omega}\right)$ and $F : Q \times C\left(\bar{\Omega}\right) \mapsto \left(C\left(\bar{\Omega}\right)\right)^2$ with

$$F(R_1, R_2) = \begin{pmatrix} R_2 \\ -\frac{3}{2}\frac{R_2^2}{R_1} - R_2 f_2\left(R_1\right) + \frac{f_1\left(R_1\right) - A\left(R_1, R_2\right)}{R_1} \end{pmatrix}. \tag{3.2}$$

By means of Lemma 1, we have that F is of class C^2. Thus, from the Cauchy–Lipschitz theorem we obtain the next local existence and uniqueness result:

Theorem 1 *There exists $T > 0$ such that the problem (3.1) has a unique solution in $C^3\left([0, T]; Q \times C\left(\bar{\Omega}\right)\right)$.*

3.2 Existence of stationary solutions

Observe that a stationary solution (R_s, p_s) of problem (2.2)–(2.3) satisfies $p_s = f_1\left(R_s\right)$. For the next result, we denote $h^+ = h - h_0$ (notice that $h^+ = 0$ if and only if h is constant). Thus, (R_s, p_s) is solution of the system

$$\begin{aligned} \nabla \cdot \left(\left(h^+ + h_0\right)^3 f_3\left(R_s\right) \nabla p_s\right) &= \nabla \cdot \left(\mathbf{U}\left(h^+ + h_0\right) f_4\left(R_s\right)\right) && \text{in } \Omega, \\ p_s &= f_1\left(R_s\right) && \text{in } \Omega, \\ p_s &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

Notice that in the particular case $h^+ = 0$ or $\mathbf{U} = 0$, $(R_s, p_s) = \left(\bar{R}, 0\right)$ is solution of (3.3), with $\bar{R}$ given in (H1).

Theorem 2 *Fix $\mathbf{U} \in \mathbb{R}^2$ and $h_0 > 0$. Then the problem (3.3) has a unique solution (R_s, p_s) with $R_s > 0$ whenever $\left\|h^+\right\|_\infty$ is small enough. Moreover, the solution (R_s, p_s) depends continuously on h^+.*

Proof First, we use the relation $p_s = f_1\left(R_s\right)$ to rewrite the stationary problem. Since $\nabla p_s = f_1'\left(R_s\right) \nabla R_s$, making the change of variable $R_s = \bar{R} + \xi$, the problem (3.3) can be written in terms of ξ as

$$\begin{aligned} -\nabla \cdot \left(\left(h^+ + h_0\right)^3 a_0\left(\xi\right) \nabla \xi\right) &= \nabla \cdot \left(\mathbf{U} h^+ b_0(\xi)\right) + \nabla \cdot \left(\mathbf{U} h_0\, b_0(\xi)\right) && \text{in } \Omega, \\ \xi &> -\bar{R} && \text{in } \Omega, \\ \xi &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.4}$$

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where $a_0(\xi) = -f_3\left(\bar{R}+\xi\right) f_1'\left(\bar{R}+\xi\right)$, and $b_0(\xi) = f_4\left(\bar{R}+\xi\right)$. We introduce the set

$$W = \left\{ \xi \in W_0^{1,q}(\Omega) : \operatorname*{ess\text{-}inf}_{\Omega} \xi > -\bar{R} \right\},$$

which is open since the continuous embedding $W_0^{1,q} \subset C\left(\bar{\Omega}\right)$, and the application

$$\begin{aligned} \phi_2 : W \times L^\infty(\Omega) &\longmapsto W^{-1,q}(\Omega) \\ (\xi, \delta) &\longmapsto \nabla \cdot \left((\delta + h_0)^3 a_0(\xi) \nabla \xi\right) + \nabla \cdot (\mathbf{U}\delta\, b_0(\xi)) + \nabla \cdot (\mathbf{U} h_0\, b_0(\xi)). \end{aligned} \tag{3.5}$$

Using an argument analogous to the one used in Lemma 1 to prove that ϕ^1 is of class C^2, it is possible to prove that ϕ_2 is of class C^2. Now noticing that $\phi_2(0,0) = 0$, let us assume that $\frac{\partial \phi_2}{\partial \xi}(0,0)$ is invertible. Then, by means of the Implicit Function theorem, we have that

- $\exists V_1 \subset W$ neighbourhood of 0 on $W_0^{1,q}(\Omega)$; V_2 neighbourhood of 0 on $L^\infty(\Omega)$;
- $\exists \psi : V_2 \longmapsto V_1$ function of class C^1 such that $\forall \delta \in V_2$, $\psi(\delta)$ is solution of problem (3.4). Equivalently, $(R_s, p_s) = \left(\bar{R} + \psi(\delta), f_1\left(\bar{R} + \psi(\delta)\right)\right)$ is solution of problem (3.3),

which is the result we want as the existence of V_2 can also be described as $\left\|h^+\right\|_\infty$ small enough.

It only remains to show that $\frac{\partial \phi_2}{\partial \xi}(0,0)$ is invertible. Indeed, we have $\forall z \in W_0^{1,q}(\Omega)$

$$\begin{aligned} \frac{\partial \phi_2}{\partial \xi}(0,0)(z) &= \nabla \cdot \left((h_0+\delta)^3 \left(a_0(\xi)\nabla z + a_0'(\xi)\, z\, \nabla \xi\right) + (h_0+\delta)\; b_0'(\xi)\, \mathbf{U} z\right)\Big|_{(\xi,\delta)=(0,0)} \\ &= \nabla \cdot \left(h_0^3\, a_0(0)\nabla z + h_0\, b_0'(0)\, \mathbf{U} z\right). \end{aligned}$$

Fixing an arbitrary $g \in W^{-1,q}(\Omega)$ and denoting $\ell = h_0 b_0'(0)\, \mathbf{U} \in \mathbb{R}^2$, we will prove that there exists a unique $z \in W_0^{1,q}(\Omega)$ such that

$$\nabla \cdot \left(h_0^3\, a_0(0)\nabla z + \ell\, z\right) = g \qquad \text{in } \Omega.$$

Since $g \in H^{-1}(\Omega)$, $a_0(0) > 0$ and $h_0 > 0$ (see (H1) and (H3)), by means of the Lax–Milgram theorem the variational problem

$$-\int_\Omega \left(h_0^3\, a_0(0)\nabla z + \ell\, z\right) \cdot \nabla \phi\, d\Omega = \int_\Omega g\, \phi\, d\Omega \qquad \forall \phi \in H_0^1(\Omega),$$

has a unique solution $z \in H_0^1(\Omega)$. Moreover, from the continuous inclusion $H^1(\Omega) \subset L^q(\Omega)$, we have $\nabla \cdot (\ell\, z) \in W^{-1,q}(\Omega)$ and thus by Proposition 1, we obtain $z \in W_0^{1,q}(\Omega)$. □

A proof analogous to the one of Theorem 2 may be written for the next result:

Theorem 3 *Fix* $h \in B_{m_0,M_0}$, $0 < m_0 < M_0$. *Then there exists* $\epsilon(h) > 0$ *such that the problem* (3.3) *has a unique solution* (R_s, p_s) *with* $R_s > 0$ *whenever* $\|\mathbf{U}\| < \epsilon(h)$. *Moreover, the solution* (R_s, p_s) *depends continuously on* $\mathbf{U}$.

3.3 Stability analysis

Recalling the application F given by (3.2) and the stationary solution (R_s, p_s) introduced in the previous section, we denote by $\mathcal{L}_F$ the differential of F at $(R_s, 0)$, i.e.,

$$\begin{aligned} \mathcal{L}_F : \left(C\left(\bar{\Omega}\right)\right)^2 &\longmapsto \left(C\left(\bar{\Omega}\right)\right)^2 \\ (S_1, S_2) &\longmapsto DF(R_s, 0)(S_1, S_2). \end{aligned} \tag{3.6}$$

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We will show the stability of the stationary solution in some particular cases. For this, it is proved that, for such particular cases, the spectrum of $\mathcal{L}_F$ is such that Re $(\lambda) < 0$ $\forall\lambda \in$ Sp $(\mathcal{L}_F) \setminus \{0\}$. Previously, we perform some computations.

Recalling that $f_1(R_s) = p_s = A(R_s, 0)$, we obtain:

$$(\mathcal{L}_F(S_1, S_2))_1 = S_2, \tag{3.7}$$

$$(\mathcal{L}_F(S_1, S_2))_2 = \frac{f_1'(R_s)\, S_1}{R_s} - \frac{1}{R_s}\left(D_1A(R_s, 0)(S_1) + D_2A(R_s, 0)(S_2)\right) + \frac{1}{R_s^2}A(R_s, 0)\, S_1 - f_2(R_s)\, S_2. \tag{3.8}$$

Now, since $A_2(R, 0) = 0$ for any R in Q, we have that $D_1A(R_s, 0) = DA_1(R_s)$. With this, deriving (2.8) with respect to R_1 and denoting $\pi_1(S_1) = D_1A(R_s, 0)(S_1)$ we obtain that $\pi_1(S_1)$ satisfies

$$\begin{aligned} -\nabla \cdot \left(h^3 f_3(R_s)\ \nabla\pi_1(S_1)\right) &= \nabla \cdot \left(h^3 f_3'(R_s)\, S_1\, \nabla A_1(R_s) - \mathbf{U}hf_4'(R_s)\, S_1\right), \\ \pi_1(S_1) &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.9}$$

Similarly, we have $D_2A(R_s, 0)(S_2) = D_2A_2(R_s, 0)(S_2) = A_2(R_s, S_2)$. Thus, denoting $\pi_2(S_2) = D_2A(R_s, 0)(S_2)$ we have that $\pi_2(S_2)$ accomplishes

$$\begin{aligned} -\nabla \cdot \left(h^3 f_3(R_s)\ \nabla\pi_2(S_2)\right) &= -hf_5(R_s)\, S_2 \qquad \text{in } \Omega, \\ \pi_2(S_2) &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.10}$$

For the next results, denote $b_1 = -f_1'(\bar{R})\bar{R}^{-1} > 0$, $b_2 = f_2\left(\bar{R}\right) > 0$, $b_r = 1/\bar{R}$, $b_3 = f_3\left(\bar{R}\right)$, $b_4 = -f_4'\left(\bar{R}\right)$ and $b_5 = -f_5\left(\bar{R}\right)$, all positive constants as follows from (H1) to (H5).

Remark 4 *If $h^+ = 0$ or $\mathbf{U} = 0$, we have $A_1(R_s) = 0$. Thus $A(R_s, 0) = p_s = 0$, $R_s = \bar{R}$ and $\mathcal{L}_F(S_1, S_2)$ can be written*

$$\mathcal{L}_F(S_1, S_2) = B\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} - b_r\begin{pmatrix} 0 \\ \pi_1(S_1) + \pi_2(S_2) \end{pmatrix}, \tag{3.11}$$

where $B = \begin{pmatrix} 0 & 1 \\ -b_1 & -b_2 \end{pmatrix}$, and equation (3.9) reads

$$\begin{aligned} -b_3\nabla \cdot \left(h^3\ \nabla\pi_1(S_1)\right) &= b_4\nabla \cdot (\mathbf{U}hS_1) \qquad \text{in } \Omega, \\ \pi_1(S_1) &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.12}$$

We denote by $\{\lambda_1^B, \lambda_2^B\}$ the set of eigenvalues of B and notice that Re $\left(\lambda_1^B\right) < 0$ and Re $\left(\lambda_2^B\right) < 0$.

Lemma 2 *Let $h^+ = 0$ or $\mathbf{U} = 0$. Then*

$$\text{Sp}(\mathcal{L}_F) \subset \text{Vp}(\mathcal{L}_F) \cup \{\lambda_1^B, \lambda_2^B\}.$$

Moreover, if $\lambda \in \text{Vp}(\mathcal{L}_F) \setminus \{\lambda_1^B, \lambda_2^B\}$ with associated eigenfunction $(S_1, S_2) \in C\left(\bar{\Omega}\right)^2$, then $(S_1, S_2) \in H_0^1(\Omega)^2$, $S_2 = \lambda S_1$ and S_1 is solution of the problem

$$\frac{b_3}{b_r}\xi(\lambda)\ \nabla \cdot \left(h^3\nabla S_1\right) = b_4\mathbf{U} \cdot \nabla(hS_1) + \lambda\, b_5 h\, S_1 \qquad \text{in } \Omega, \tag{3.13}$$

$$S_1 = 0 \qquad \text{on } \partial\Omega, \tag{3.14}$$

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where $\xi(\lambda) = \lambda^2 + b_2\lambda + b_1$ *with roots* $\{\lambda_1^B, \lambda_2^B\}$.

Proof Remind that $p_s = A(R_s, 0) = 0$ and $R_s = \bar{R}$. For any $\lambda \in \mathbb{C} \setminus \{\lambda_1^B, \lambda_2^B\}$, from equation (3.11) we have

$$(\mathcal{L}_F - \lambda I)\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = (B - \lambda I)\left[\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} - b_r (B - \lambda I)^{-1}\begin{pmatrix} 0 \\ \pi_1(S_1) + \pi_2(S_2) \end{pmatrix}\right].$$

Since the map $(S_1, S_2) \mapsto \pi_1(S_1) + \pi_2(S_2)$ is compact, by means of the Fredholm's Alternative theorem the mapping at the right-hand side of this equation (from $C(\bar{\Omega})^2$ into itself) is injective if and only if it is surjective, from where we have the inclusion $\mathrm{Sp}(\mathcal{L}_F) \subset \mathrm{Vp}(\mathcal{L}_F) \cup \{\lambda_1^B, \lambda_2^B\}$.

Fix now $\lambda \in \mathrm{Vp}(\mathcal{L}_F) \setminus \{\lambda_1^B, \lambda_2^B\}$ with associated eigenvector $(S_1, S_2) \neq (0, 0)$, so we can write

$$\begin{aligned} S_2 &= \lambda S_1, \\ -b_1 S_1 - b_2 S_2 - b_r[\pi_1(S_1) + \pi_2(S_2)] &= \lambda S_2. \end{aligned}$$

Then we obtain

$$\pi_2(\lambda S_1) + \pi_1(S_1) = -\frac{\xi(\lambda)}{b_r} S_1.$$

Since $\xi(\lambda) \neq 0$ and from the definitions of π_1 and π_2, we deduce that $(S_1, S_2) \in H_0^1(\Omega)^2$. Thus, using this last equation, equations (3.10) and (3.12), we obtain the equations (3.13) and (3.14). □

Theorem 4 *Let h be as in Theorem 3. Then there exists* $\epsilon = \epsilon(h) > 0$ *such that if* $\|\mathbf{U}\|_\infty < \epsilon$, *the solution* (R_s, p_s) *of problem* (3.3) *is asymptotically stable for the evolution problem* (3.1).

Proof Assume first $\mathbf{U} = 0$ and denote $\mathcal{L}_F^0 = \mathcal{L}_F|_{\mathbf{U}=0}$. Then due to Lemma 2 it is enough to study the eigenvalues of $\mathcal{L}_F$. Thus, take $\lambda \in \mathrm{Vp}(\mathcal{L}_F) \setminus \{\lambda_1^B, \lambda_2^B\}$ with associated eigenfunction (S_1, S_2), from Lemma 2 we have $S_2 = \lambda S_1$ and $S_1 \in H_0^1(\Omega)$ accomplishing equations (3.13) and (3.14), which read

$$\begin{aligned} \frac{b_3}{b_r}\xi(\lambda)\,\nabla\cdot\left(h^3\nabla S_1\right) &= \lambda\, b_5 h\, S_1 \qquad \text{in } \Omega, \\ S_1 &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Since $\xi(\lambda)$ is not null we deduce that $\lambda \neq 0$, otherwise (S_1, S_2) would be null. Then we obtain that S_1 accomplishes the next variational formulation

$$-\frac{b_3}{b_r}\frac{\xi(\lambda)}{\lambda}\int_\Omega h^3 \nabla S_1 \cdot \nabla\phi \, d\Omega = b_5 \int_\Omega h S_1\,\phi\, d\Omega \qquad \forall \phi \in H_0^1(\Omega). \tag{3.15}$$

Taking $\phi = \bar{S_1}$ we obtain that $\gamma = -\xi(\lambda)/\lambda \in \mathbb{R}^+$ and since λ accomplishes the equation $\lambda^2 + (\gamma + b_2)\lambda + b_1 = 0$ we conclude that $\mathrm{Re}(\lambda) < 0$. We have shown the result for the case $\mathbf{U} = 0$.

For the general case, we observe from Theorem 3 that the mapping $\mathbf{U} \mapsto R_s(\mathbf{U})$ is continuous in a neighbourhood $V_1 \ni 0$ in $\mathbb{R}^2$, thus if $\mathbf{U} \to 0$ in $\mathbb{R}^2$ then $\left\| DF(R_s(\mathbf{U}), 0) - DF\left(\bar{R}, 0\right)\right\| \to 0$ in the space of linear continuous operators from $C(\bar{\Omega})^2$ into itself. Then the result follows from Proposition 2. □

We give now a result of instability for $\|\mathbf{U}\|$ big enough.

Theorem 5 *Let us assume* $h^+ = 0$ *and* $\Omega =]0,1[\times]0,1[$. *Then the solution* (R_s, p_s) *of problem* (3.3) *is asymptotically unstable for the evolution problem* (3.1) *for* $\|\mathbf{U}\|$ *big enough.*

Proof Due to Lemma 2 it is enough to study the eigenvalues of $\mathcal{L}_F$. Fix now $\lambda \in \mathrm{Vp}(\mathcal{L}_F) \setminus \{\lambda_1^B, \lambda_2^B\}$ with associated eigenvector $(S_1, S_2) \neq (0,0)$. Now defining $\gamma_1, \gamma_2 \in \mathbb{C}$ by

$$\gamma_1 = -\frac{b_4\, b_r}{h_0^2 b_3 \xi\,(\lambda)}, \qquad \gamma_2 = -\frac{b_5\, b_r}{h_0^2 b_3 \xi\,(\lambda)},$$

then from equations (3.13) to (3.14), we have

$$\begin{aligned} \Delta S_1 + \gamma_1 \mathbf{U}\cdot\nabla\,(S_1) + \lambda\gamma_2\, S_1 &= 0 && \text{in } \Omega,\\ S_1 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We deduce from this that $\lambda \neq 0$. In fact, if $\lambda = 0$ then one may compute that $S_1 = 0, S_2 = \lambda S_1 = 0$, which is a contradiction. Assuming $S_1(x_1, x_2) = \varphi_1(x_1)\,\varphi_2(x_2)$ with both φ_1 and φ_2 non-nulls and $\varphi_1(0) = \varphi_1(1) = \varphi_2(0) = \varphi_2(1) = 0$ it is possible to obtain

$$\frac{\varphi_1''(x_1)}{\varphi_1(x_1)} + \gamma_1 U_1 \frac{\varphi_1'(x_1)}{\varphi_1(x_1)} = -\frac{\varphi_2''(x_2)}{\varphi_2(x_2)} - \gamma_1 U_2 \frac{\varphi_2'(x_2)}{\varphi_2(x_2)} - \lambda\gamma_2.$$

Therefore, there exists $\mu \in \mathbb{C}$ such that

$$\varphi_1''(x_1) + \gamma_1 U_1\, \varphi_1'(x_1) - \mu\,\varphi_1(x_1) = 0, \tag{3.16}$$

$$\varphi_2''(x_2) + \gamma_1 U_2\, \varphi_2'(x_2) + (\lambda\gamma_2 + \mu)\varphi_2(x_2) = 0. \tag{3.17}$$

Denote by r_1, r_2 the roots of the characteristic polynomial $P\,(r) = r^2 + \gamma_1 U_2 r + \lambda\gamma_2 + \mu$ of the last equation. Then $r_1 \neq r_2$, otherwise φ_2 would be null, and so φ_2 can be written

$$\varphi_2\,(x_2) = C_1 \exp\,(r_1 x_2) + C_2 \exp\,(r_2 x_2)\,.$$

Thus, the conditions $\varphi_2(0) = \varphi_2(1) = 0$ imply

$$\begin{aligned} C_1 + C_2 &= 0,\\ C_1 \exp r_1 + C_2 \exp r_2 &= 0. \end{aligned}$$

Thus, since $(C_1, C_2) \neq (0,0)$ we have

$$\det\begin{pmatrix} 1 & 1 \\ \exp r_1 & \exp r_2 \end{pmatrix} = 0,$$

hence r_1 and r_2 satisfy the equation $r_2 - r_1 = 2k_2\pi\, i\ \forall k_2 \in \mathbb{N}^*$, from which we deduce that

$$\gamma_1^2\, U_2^2 - 4(\lambda\gamma_2 + \mu) = -4\, k_2^2\pi^2, \qquad \forall k_2 \in \mathbb{N}^*, \tag{3.18}$$

where we have used the fact that $r_1 + r_2 = -\gamma_1 U_2$ and $r_1 r_2 = \lambda\gamma_2 + \mu$. Analogously, from the characteristic polynomial of equation (3.16), one may obtain

$$\gamma_1^2\, U_1^2 - 4(-\mu) = -4\, k_1^2\pi^2, \qquad \forall k_1 \in \mathbb{N}^*. \tag{3.19}$$

Denoting $k=(k_1,k_2)\in\mathbb{N}^*\times\mathbb{N}^*$, the addition of these two equations implies

$$\gamma_1^2\,\|\mathbf{U}\|^2-4\,\lambda\gamma_2=-4\,\|k\|^2\pi^2.$$

Recalling the definitions of γ_1 and γ_2, one concludes that λ is root of the fourth-degree polynomial given by

$$P_k(\lambda)=4\|k\|^2\pi^2\lambda^4+\left(4\sigma_2+8\pi^2\|k\|^2b_2\right)\lambda^3+\left(4\sigma_2b_2+4\pi^2\|k\|^2\left(b_2^2+2b_1\right)\right)\lambda^2+ \\ +\left(4\sigma_2b_1+8\pi^2\|k\|^2b_1b_2\right)\lambda+4\pi^2\|k\|^2b_1^2+\sigma_1\|\mathbf{U}\|^2,$$

where $\sigma_1=\frac{b_4^2b_r^2}{b_3^2h_0^4}$ and $\sigma_2=\frac{b_5b_r}{b_3h_0^2}$ are both positive constants. Rewriting this polynomial as $P_k(\lambda)=\alpha_0\lambda^4+\beta_0\lambda^3+\alpha_1\lambda^2+\beta_1\lambda+\alpha_2$, let us now denote the Hurwitz determinants associated with P_k:

$$\Delta_1=\det\left(\beta_0\right),\ \Delta_2=\det\begin{pmatrix}\beta_0&\beta_1\\ \alpha_0&\alpha_1\end{pmatrix},\ \Delta_3=\det\begin{pmatrix}\beta_0&\beta_1&0\\ \alpha_0&\alpha_1&\alpha_2\\ 0&\beta_0&\beta_1\end{pmatrix},\ \Delta_4=\det\begin{pmatrix}\beta_0&\beta_1&0&0\\ \alpha_0&\alpha_1&\alpha_2&0\\ 0&\beta_0&\beta_1&0\\ 0&\alpha_0&\alpha_1&\alpha_2\end{pmatrix}.$$

Then, one obtains $\Delta_1=4\sigma_2+8\pi^2\|k\|^2b_2$, $\Delta_2=\left(4\sigma_2+8\pi^2\|k\|^2b_2\right)\left(4\sigma_2b_2+4\pi^2\|k\|^2\left(b_2^2+b_1\right)\right)$,

$$\Delta_3=\left(320\,b_1b_2^2\pi^2\|k\|^2-16\sigma_1\|\mathbf{U}\|^2\right)\sigma_2^2+\left(512b_1b_2^3\pi^4\|k\|^4-64b_2\pi^2\sigma_1\|k\|^2\|\mathbf{U}\|^2\right)\sigma_2- \\ -64b_2^2\pi^4\sigma_1\|k\|^4\|\mathbf{U}\|^2+256\,b_1b_2^4\pi^6\|k\|^6+64b_1b_2\sigma_2^3,$$

and $\Delta_4=\alpha_2\Delta_3$. According to the Routh–Hurwitz theorem [38], the number of roots of the polynomial P_k with positive real part is equal to the total number of changes of sign in the sequence $\{\alpha_0,\Delta_1,\frac{\Delta_2}{\Delta_1},\frac{\Delta_3}{\Delta_2},\frac{\Delta_4}{\Delta_3}\}$. One may compute $\alpha_0>0$, $\Delta_1>0$, $\frac{\Delta_2}{\Delta_1}>0$, $\frac{\Delta_4}{\Delta_3}>0$ and $\frac{\Delta_3}{\Delta_2}<0$ for $\|\mathbf{U}\|$ big enough, which ends the proof using Proposition 5. □

4 Well-posedness without inertial terms

Disregarding the inertial terms in equation (2.2) (as done in [14, 21, 39]), we obtain the following simplified version of the Rayleigh–Plesset equation

$$\frac{\partial R}{\partial t}=\frac{f_1(R)-p}{Rf_2(R)}, \tag{4.1}$$

along the initial condition

$$R(\mathbf{x},0)=r_1(\mathbf{x})\qquad\forall\mathbf{x}\in\Omega, \tag{4.2}$$

where $r_1\in C\left(\bar{\Omega}\right)$ known and $p\in W_0^{1,q}(\Omega)$ is the solution of (2.3).

4.1 Existence of a local solution

Let us prove that we can express $\partial R/\partial t$ as a function of R from (4.1). Denoting $R_1=R$, $R_2=\frac{\partial R}{\partial t}$, we recall the decomposition

$$p=A(R_1,R_2)=A_1(R_1)+A_2(R_1,R_2),$$

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with A_1 and A_2 as in (2.8) and (2.9), respectively. Now, defining $\Pi : Q \times C\left(\bar{\Omega}\right) \mapsto C\left(\bar{\Omega}\right)$ by

$$\Pi\left(R_1, R_2\right) = \frac{f_1\left(R_1\right) - A_1(R_1) - A_2(R_1, R_2)}{R_1 f_2\left(R_1\right)}, \tag{4.3}$$

we have the next result:

Lemma 3 *Given $R \in Q$, there exists a unique $G(R) \in C\left(\bar{\Omega}\right)$ such that*

$$G(R) = \Pi\left(R, G(R)\right),$$

and the mapping $R \mapsto G(R)$ is of class C^2.

Proof Let us fix $R \in Q$, we will show that there exists a unique $S \in C\left(\bar{\Omega}\right)$ such that $S = \Pi(R, S)$. Using (4.3), we first notice that the equation $S = \Pi(R, S)$ is equivalent to

$$S + \frac{A_2(R, S)}{Rf_2(R)} = \frac{f_1(R) - A_1(R)}{Rf_2(R)}.$$

We denote by $J : C\left(\bar{\Omega}\right) \mapsto C\left(\bar{\Omega}\right)$ the linear mapping $S \mapsto S + A_2(R, S) / (Rf_2(R))$. To prove the existence of a unique solution for the last equation we will show that J is bijective, which will give us the existence of G by taking $G(R) = S$. Now, since the mapping $S \mapsto A_2(R, S)$ is compact, by means of the Fredholm's Alternative theorem it is enough to prove that J is injective. Indeed, let us take $w \in C\left(\bar{\Omega}\right)$ such that $J(w) = 0$, then we have

$$Rf_2(R)\, w + A_2(R, w) = 0.$$

Multiplying this equation by $-f_5(R)\, hw$ and integrating by parts, we obtain

$$\int_\Omega Rf_2(R)\left(-f_5(R)\right) hw^2\, d\Omega + \int_\Omega \left(-f_5(R)\right) hA_2(R, w)\; w\, d\Omega = 0. \tag{4.4}$$

Now, multiplying (2.9) by $A_2(R_1, R_2)$, integrating and using (H5), we have for any $(R_1, R_2) \in Q \times C\left(\bar{\Omega}\right)$

$$\int_\Omega \left(-f_5(R_1)\right) h\, A_2(R_1, R_2)\, R_2\, d\Omega \geq 0. \tag{4.5}$$

Taking $R_1 = R$ and $R_2 = w$ in the last equation and carrying that into equation (4.4), we obtain $w = 0$, so we conclude J is injective.

Next, we prove that G is of class C^2. Let us define the mapping $\Phi : Q \times C\left(\bar{\Omega}\right) \mapsto C\left(\bar{\Omega}\right)$ such that

$$\Phi(R, S) = S - \Pi(R, S),$$

which is of class C^2 since all the involved functions are regular enough. Now, fixing some arbitrary $(R_0, S_0) \in Q \times C\left(\bar{\Omega}\right)$ such that $\Phi(R_0, S_0) = 0$ we have for any $w \in C\left(\bar{\Omega}\right)$

$$\frac{\partial \Phi}{\partial S}(R_0, S_0)(w) = w + \frac{A_2(R_0, w)}{R_0 f_2(R_0)} = J(w).$$

From where we obtain that $\frac{\partial \Phi}{\partial S}(R_0, S_0)$ is an automorphism on $C\left(\bar{\Omega}\right)$. Thus, we conclude that G is of class C^2 by means of the Implicit Function theorem. □

Theorem 6 *There exists $T > 0$ such that problem (2.3)–(4.1)–(4.2) has a unique solution in* $C^3([0, T]; Q)$.

Proof The result follows directly from applying the Cauchy–Lipschitz theorem to the equivalent evolution problem

$$\frac{\partial R}{\partial t} = G(R), \tag{4.6}$$

along the initial condition (4.2). □

4.2 Stability analysis

Let us notice the stationary solution of (4.1) is also the couple (R_s, p_s) obtained in Section 3.2. Here we study the stability of that solution for the evolution problem (4.6).

For that, denote the derivative

$$\begin{aligned} \mathcal{L}_G : C\left(\bar{\Omega}\right) &\longmapsto C\left(\bar{\Omega}\right) \\ w \quad &\longmapsto D\,G\left(R_s\right)(w)\,. \end{aligned} \tag{4.7}$$

Using the definition of $\Pi(R, S)$ we compute the derivative with respect to R in the equation $S = \Pi(R, S)$ and make the evaluation at $R = R_s$, $S = 0$, so we obtain that $\mathcal{L}_G(w)$ satisfies

$$R_s f_2(R_s)\,\mathcal{L}_G(w) - f_1'(R_s)\,w + \pi_1(w) + \pi_2(\mathcal{L}_G(w)) = 0, \tag{4.8}$$

with π_1 and π_2 as in equations (3.9) and (3.10), respectively.

For the next results, we denote $d_1 = -f_1'\left(\bar{R}\right) / \left(\bar{R} f_2\left(\bar{R}\right)\right)$, $d_2 = \left(\bar{R} f_2\left(\bar{R}\right)\right)^{-1}$, $d_3 = \bar{R} f_3\left(\bar{R}\right) f_2\left(\bar{R}\right)$, $d_4 = -f_4'\left(\bar{R}\right)$ and $d_5 = -f_5\left(\bar{R}\right)$. All these constants are positive as follows from (H1) to (H5).

Lemma 4 *Asumme $h^+ = 0$ or $\mathbf{U} = 0$. Then*

$$\mathrm{Sp}\left(\mathcal{L}_G\right) \subset \mathrm{Vp}\left(\mathcal{L}_G\right) \cup \{-d_1\}.$$

Moreover, if $w \in C\left(\bar{\Omega}\right)$ is an eigenvector of $\mathcal{L}_G$ with associated eigenvalue λ, then $w \in H_0^1(\Omega)$ and it satisfies

$$\begin{aligned} d_3\left(d_1 + \lambda\right)\nabla \cdot \left(h^3 \nabla w\right) &= d_4\,\mathbf{U} \cdot \nabla(hw) + \lambda\, d_5\, hw \quad && \text{in } \partial\Omega, \\ w &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.9}$$

Proof From Remark 4, we have $(R_s, p_s) = \left(\bar{R}, 0\right)$. Putting this into equation (4.8), we obtain that for any $\lambda \in \mathbb{C}$:

$$\mathcal{L}_G(w) - \lambda\, w = (\lambda + d_1)\left[-w - \frac{d_2}{\lambda + d_1}\left[\pi_1(w) + \pi_2\left(\mathcal{L}_G(w)\right)\right]\right], \tag{4.10}$$

with $\pi_1(w)$ given by (3.12). Since the map $w \mapsto \pi_1(w) + \pi_2(\mathcal{L}_G(w))$ is compact, by means of the Fredholm's Alternative theorem, we obtain that $\mathrm{Sp}(\mathcal{L}_G) \subset \mathrm{Vp}(\mathcal{L}_G) \cup \{-d_1\}$.

Take now $w \in C\left(\bar{\Omega}\right)$ eigenvector of $\mathcal{L}_G$ with associated eigenvalue λ, carrying this into equation (4.10) we obtain

$$\frac{\lambda + d_1}{-d_2}\, w = \pi_1(w) + \lambda\, \pi_2(w)\,,$$

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then $w \in H_0^1(\Omega)$ and equation (4.9) follows from this last relation and equations (3.12) and (3.10). □

Theorem 7 *For every* $\mathbf{U} \in \mathbb{R}^2$ *there exists* $\epsilon = \epsilon(\mathbf{U}) > 0$ *such that if* $\|h^+\|_\infty < \epsilon$*, then the solution* (R_s, p_s) *of problem* (3.3) *is asymptotically stable for the evolution problem* (2.3)–(4.1)–(4.2).

Proof Let us assume first that $h^+ = 0$. By Lemma 4, it is enough to study the eigenvalues of $\mathcal{L}_G$. Hence, take $\lambda \in \mathbb{C} \setminus \{-d_1\}$ such that $\mathcal{L}_G(w) = \lambda w$ for some $w \neq 0$. Then (4.9) reads

$$h_0^2 d_3 (d_1 + \lambda) \Delta w = d_4 \mathbf{U} \cdot \nabla w + \lambda d_5 w \quad \text{in } \Omega, \qquad (4.11)$$
$$w = 0 \quad \text{on } \partial\Omega.$$

We notice that $\lambda \neq 0$. In fact, if $\lambda = 0$ then multiplying the equation (4.11) by w and integrating by parts we obtain $w = 0$, which is not possible. Decomposing $\lambda = \lambda_1 + i\lambda_2$ and $w = w_1 + i w_2$, writing the differential equation for the real and imaginary parts we obtain the equations

$$-h_0^2 d_3 (d_1 + \lambda_1) \Delta w_1 + h_0^2 d_3 \lambda_2 \Delta w_2 + d_4 \mathbf{U} \cdot \nabla w_1 + d_5 (\lambda_1 w_1 - \lambda_2 w_2) = 0,$$
$$-h_0^2 d_3 (d_1 + \lambda_1) \Delta w_2 - h_0^2 d_3 \lambda_2 \Delta w_1 + d_4 \mathbf{U} \cdot \nabla w_2 + d_5 (\lambda_1 w_2 + \lambda_2 w_1) = 0.$$

Multiplying the first equation by w_1, the second equation by w_2 and integrating by parts we may obtain

$$h_0^2 d_3 (d_1 + \lambda_1) \int_\Omega |\nabla w_1|^2 \, d\Omega - h_0^2 d_3 \lambda_2 \int_\Omega \nabla w_2 \cdot \nabla w_1 \, d\Omega + d_5 \lambda_1 \int_\Omega w_1^2 \, d\Omega - d_5 \lambda_2 \int_\Omega w_1 w_2 \, d\Omega = 0,$$
$$h_0^2 d_3 (d_1 + \lambda_1) \int_\Omega |\nabla w_2|^2 \, d\Omega + h_0^2 d_3 \lambda_2 \int_\Omega \nabla w_2 \cdot \nabla w_1 \, d\Omega + d_5 \lambda_1 \int_\Omega w_2^2 \, d\Omega + d_5 \lambda_2 \int_\Omega w_1 w_2 \, d\Omega = 0.$$

Adding up both equations we have

$$h_0^2 d_3 (d_1 + \lambda_1) \int_\Omega \left(|\nabla w_1|^2 + |\nabla w_2|^2\right) d\Omega + d_5 \lambda_1 \int_\Omega \left(|w_1|^2 + |w_2|^2\right) d\Omega = 0.$$

Observing that $\lambda_1 \geq 0$ implies $w = 0$, which is not possible, we conclude that $\mathrm{Re}(\lambda) = \lambda_1 < 0$.

We have shown the stability for $h^+ = 0$. Now from Theorem 2, we have that the mapping $h^+ \mapsto R_s(h^+)$ is continuous in a neighbourhood $V_1 \ni 0$ in $L^\infty(\Omega)$. Thus, if $h^+ \to 0$ in $L^\infty(\Omega)$ then

$$\left\| DG\left(R_s\left(h^+\right), 0\right) - DG\left(\bar{R}, 0\right) \right\| \to 0$$

in the space of linear continuous operators from $C\left(\bar{\Omega}\right)$ into itself. Then the result follows from Proposition 2. □

Theorem 8 *Fix* $h \in B_{m_0, M_0}$, $0 < m_0 < M_0$*. Then there exists* $\epsilon > 0$ *such that if* $\|\mathbf{U}\| < \epsilon$ *then the solution* (R_s, p_s) *of problem* (3.3) *is asymptotically stable for the evolution problem* (2.3)–(4.1)–(4.2).

Proof Let us assume first that $\mathbf{U} = 0$. By Lemma 4, it is enough to study the eigenvalues of $\mathcal{L}_G$. Hence, take $\lambda \in \mathbb{C} \setminus \{-d_1\}$ such that $\mathcal{L}_G(w) = \lambda w$ for some $w \neq 0$. If $\lambda = 0$ then from equation (4.9) we obtain $w = 0$, which is a contradiction. Thus, we have $\lambda \neq 0$ and this time equation (4.9)

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in its variational version reads

$$-d_3\frac{(\lambda+d_1)}{\lambda}\int_\Omega h^3\,\nabla w\cdot\nabla\phi\,d\Omega = d_5\int_\Omega hw\phi\,d\Omega \qquad \forall\phi\in H_0^1(\Omega)\,.$$

Along the same arguments used in Theorem 4, this implies $\lambda\in\mathbb{R}^-$. The result follows analogously to the end of Theorem 7 proof, this time using the continuity of the mapping $\mathbf{U}\mapsto R_s(\mathbf{U})$ asserted in Theorem 3. □

Remark 5 *Theorem 7 highlights the difference between the model without or with inertial terms. If $h^+=0$, when disregarding the inertial terms stability is proved by Theorem 7 for any velocity* $\mathbf{U}$*, while when considering the inertial terms instability is gained for* $\mathbf{U}$ *sufficiently large from Theorem 5.*

5 Numerical examples

In this section, we show some numerical examples for the evolution problem (2.3)–(4.1)–(4.2) (disregarding the inertial terms). The computations are separated in two stages: (1) the Reynolds equation (2.3) is integrated by means of a Finite Volume Method and (2) the Rayleigh–Plesset equation (4.1) is integrated by means of the backward Euler scheme. Special care must be taken when discretising the time derivative in equation (2.3). In fact, for this particular system, the simple approximation $\frac{\partial R}{\partial t}\approx\frac{R^{n+1}-R^n}{\Delta t}$ is prone to instabilities. This issue was exposed in [19] where instead of such approximation the time derivative $\frac{\partial R}{\partial t}$ is substituted in equation (2.3) by the right-hand side of equation (4.1). To provide the resulting discretised equations, let us first divide the domain $\Omega=[0,2\pi J_r]\times[0,B]$ into cells of size $\Delta x_1=2\pi J_r/512$ and $\Delta x_2=B/64$ in the x_1 and x_2 axis, respectively, where $J_R=B=25.4\times10^{-3}$ m. The time step was taken as $\Delta t=0.3$ ms. The number of time steps, denoted by N^*, is taken big enough in order to observe temporal convergence for each case. Thus, $t^{N^*}=\Delta t\,N^*$ corresponds to the final time simulated. This way, for a certain time step t^n, one solves the equations:

1. For the discrete pressure field at each cell:

$$\frac{c_{i-\frac12,j}\,p^n_{i-1,j}-\left(c_{i-\frac12,j}+c_{i+\frac12,j}\right)p^n_{i,j}+c_{i+\frac12,j}\,p^n_{i+1,j}}{\Delta x_1^2}$$
$$+\frac{c_{i,j-\frac12}\,p^n_{i,j-1}-\left(c_{i,j-\frac12}+c_{i,j+\frac12}\right)p^n_{i,j}+c_{i,j+\frac12}\,p^n_{i,j+1}}{\Delta x_2^2}+h_{i,j}\frac{f_5\left(R^n_{i,j}\right)}{R^n_{i,j}f_2\left(R^n_{i,j}\right)}p^n_{i,j}=$$
$$=\frac{\omega J_r}{2}\left(\frac{f_4\left(R^n_{i,j}\right)h_{i,j}-f_4\left(R^n_{i-1,j}\right)h_{i-1,j}}{\Delta x_1}\right)+h_{i,j}f_5\left(R^n_{i,j}\right)\left(\frac{f_1\left(R^n_{i,j}\right)}{R^n_{i,j}f_2\left(R^n_{i,j}\right)}-u_b^1\hat{D}_1(R^n_{i,j})-u_b^2\hat{D}_2(R^n_{i,j})\right),\tag{5.1}$$

with

$$c_{i\pm\frac12,j}=\frac{f_3\left(R^n_{i,j}\right)\left(h^n_{i,j}\right)^3+f_3\left(R^n_{i\pm1,j}\right)\left(h^n_{i\pm1,j}\right)^3}{2},$$
$$c_{i,j\pm\frac12}=\frac{f_3\left(R^n_{i,j}\right)\left(h^n_{i,j}\right)^3+f_3\left(R^n_{i,j\pm1}\right)\left(h^n_{i,j\pm1}\right)^3}{2};$$

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and $\hat{D}_1$, $\hat{D}_2$ correspond to upwind discretisations of the partial derivatives $\frac{\partial R}{\partial x_1}$ and $\frac{\partial R}{\partial x_2}$, respectively.

2. Then for the discrete bubbles radii field at each cell:

$$R_{i,j}^{n+1} = R_{i,j}^n + \Delta t \left(\frac{f_1\left(R_{i,j}^n\right) - p_{i,j}^n}{R_{i,j}^n f_2\left(R_{i,j}^n\right)} - u_b^1 \hat{D}_1(R_{i,j}^n) - u_b^2 \hat{D}_2(R_{i,j}^n) \right) . \tag{5.2}$$

It is worth noticing that due to the substitution mentioned above, the negative quantity $h_{i,j}^n \frac{f_5\left(R_{i,j}^n\right)}{R_{i,j}^n f_2\left(R_{i,j}^n\right)} p_{i,j}^n$ (recall that $f_5 < 0$ and $f_2 > 0$) appears at the left-hand side of equation (5.1), so it can be seen as a stabilising term. Notice also that p^n depends only on the values of R at time t^n. For more details on the numerical method, the reader is referred to [19].

Remark 1 *For the results presented here, the bubbles transport velocity* $\mathbf{u}_b$ *has been set to zero. Nevertheless, the numerical procedure has been also used for non-null fields of the form* $\mathbf{u}_b = \delta\, \omega J_r\, \hat{\mathbf{e}}_1$*, with* $\hat{\mathbf{e}}_1$ *being the unit vector along the circumferential direction (thus* $\delta = 1/2$ *corresponds to setting* $\mathbf{u}_b$ *to the average velocity of the surfaces). For small values of* δ *(*$\delta \approx 1/2$*), the numerical simulations appear to be stable and convergence towards a stationary regime is obtained. Moreover, when diminishing* δ *the results tend continuously to the ones for the case* $\mathbf{u}_b = 0$*. However, for larger values of* δ*, numerical instabilities can be observed, particularly at the boundaries, and numerical convergence is lost. Therefore, it is of interest to study whether the loss of numerical well-posedness is due to a numerical phenomenon or is due to a possible ill-posedness of the coupled model for a non-null velocity* $\mathbf{u}_b$*, generalising the analysis made in this work by further research.*

The gap function h is set as $h(x_1, x_2) = h_0\,(1 - \epsilon\, \cos(x_1))$ with $h_0 > 0$ and $\epsilon \in [0, 1[$ (the eccentricity of the journal). Dirichlet boundary conditions are imposed, reading

$$p\,(x_1, 0, t = 0) = p(x_1, B, t = 0) = 0 \qquad \forall x_1 \in [0, 2\pi J_R],$$

and the next periodic conditions for all time $t \geq 0$

$$p\,(0, x_2, t) = p(2\pi J_R, x_2, t), \; \frac{\partial p}{\partial x_1}\,(0, x_2, t) = \frac{\partial p}{\partial x_1}(2\pi J_R, x_2, , t) \qquad \forall x_2 \in [0, B].$$

The initial conditions are $\hat{R}(\mathbf{x}, t = 0) = R(\mathbf{x}, t = 0)/R_0 = 1$ and $\frac{\partial R}{\partial t} = 0$ in Ω.

Here the gas fraction is written as (see Appendix)

$$\alpha(R) = \frac{\alpha_0\,(R/R_0)^3}{1 + \alpha_0\,(R/R_0)^3} \qquad \forall R > 0, \tag{5.3}$$

where α_0 is a data corresponding to the gas fraction for $R = R_0$, and R_0 is a reference radius.

The geometrical setting corresponds to a journal bearing device, which scheme is shown in Figure 2. The physical parameters setting is given in Table 1. For the next results, we will use the non-dimensional variables $\hat{R} = R/R_0$, $\hat{x}_1 = x_1/J_R$ (longitudinal direction) and $\hat{x}_2 = x_2/B$ (transverse direction) and for some function $f(R)$ we denote $\hat{f}(\hat{R}) = f(R_0\,\hat{R})$.

Table 1. *Parameter values for the journal bearing*

Symbol	Value	Units	Description
ρ_ℓ	854	kg/m^3	Liquid density
μ_ℓ	7.1×10^{-3}	Pa·s	Liquid viscosity
ρ_g	1	kg/m^3	Gas density
μ_g	1.81×10^{-5}	Pa·s	Gas viscosity
κ^s	7.85×10^{-5}	Pa·s·m	Surface dilatational viscosity
k	1.4		Gas polytropic exponent
σ	3.5×10^{-2}	N/m	Liquid surface tension
P_0	1	atm	Reference pressure
R_0	3.85×10^{-7}	m	Bubbles' equilibrium radius at 1 atm
α_0	0.1		Reference gas fraction
J_r	25.4×10^{-3}	m	Journal radius
B	25.4×10^{-3}	m	Journal width
h_0	$0.001 \times J_r$	m	Journal clearance
ϵ	$[0, 1[$		Journal eccentricity
ω	$2\pi \frac{1000}{60}$	rad/s	Journal rotational speed

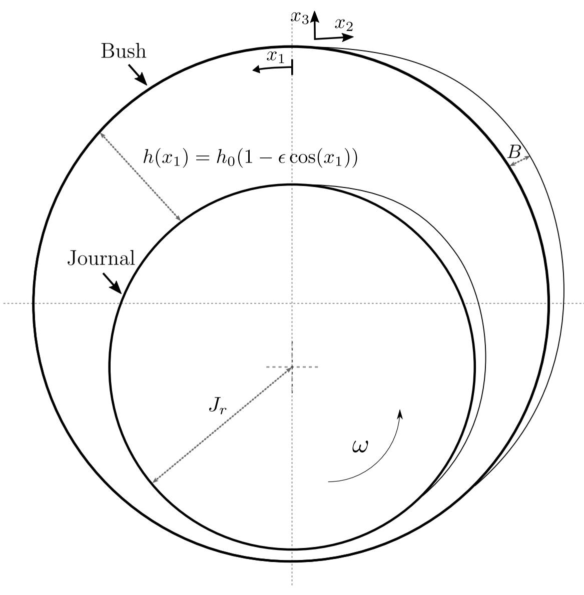

FIGURE 2. Scheme of the journal bearing.

5.1 Time convergence towards a stationary solution

Here the journal eccentricity is fixed to $\epsilon = 0.4$ and the other physical parameters are set as in Table 1. For the physical cases computed here, the function $\hat{f}_1$ has a unique critical point $\hat{R}_{\text{crit}}$ such that $\hat{f}_1'(R) < 0$ for $\hat{R} < \hat{R}_{\text{crit}}$ and $\hat{f}_1'(R) > 0$ for $\hat{R} > \hat{R}_{\text{crit}}$ (see Figure 3), so we denote

$$\hat{p}_{\text{cav}} = \min_{r>0} \hat{f}_1(r) = \hat{f}_1(\hat{R}_{\text{crit}}).$$

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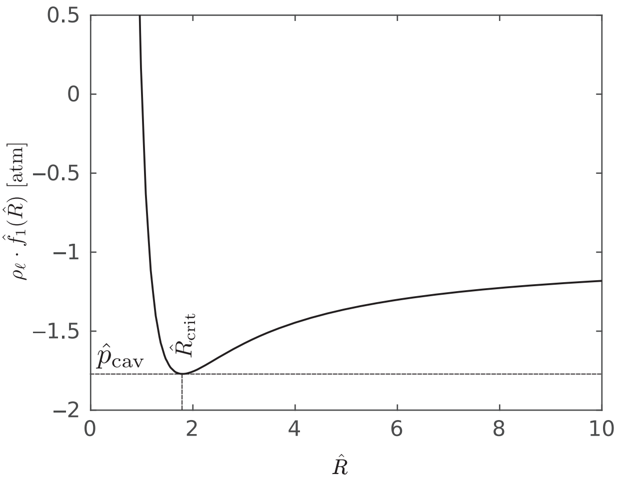

FIGURE 3. Typical shape of $\hat{f}_1$, having a unique critical point $\hat{R}_{crit} \approx 1.8$ for the parameters set in Table 1.

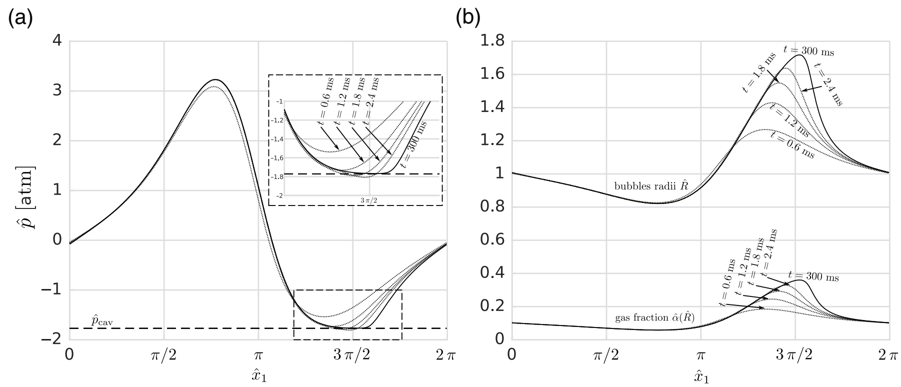

FIGURE 4. Time convergence of the fields pressure (a), and bubbles dimensionless radii and gas fraction (b) for $\epsilon = 0.4$ projected along $\hat{x}_2 = 0.5$.

To simplify the exposition, these two-dimensional pressure and bubbles' radii fields are shown in Figure 4 by fixing $\hat{x}_2 = 0.5$ for different time steps. For the configuration set in this example a numerical convergence in time is obtained, meaning that for $t \geq 300$ ms the profiles are not observed to change. It is worth noticing that for some time steps ($t \approx 2$ ms), there is a region of Ω where $\hat{p} < \hat{p}_{\text{cav}}$ but the converged profile accomplishes $\hat{p}(x_1, x_2, 300 \text{ ms}) \geq \hat{p}_{\text{cav}}$ on Ω. Also, one observes that in the *pressurized* region (where $\hat{p} > 0$), the bubbles radii is such that $\hat{\alpha}(\hat{R})$ is low and then $\hat{\rho}(\hat{R}) \approx \rho_\ell$. On the other hand, in the region where $\hat{p} \approx \hat{p}_{\text{cav}}$ the gas fraction $\hat{\alpha}(\hat{R})$ can reach values as high as 0.4, lowering the mixture average density and effective viscosity (see (A5) and (A9)).

5.2 Stationary solutions varying the eccentricity

A series of simulations were performed for increasing values of the eccentricity ϵ. Until a value of ϵ around 0.41 time convergence of the transient solution towards the stationary one is

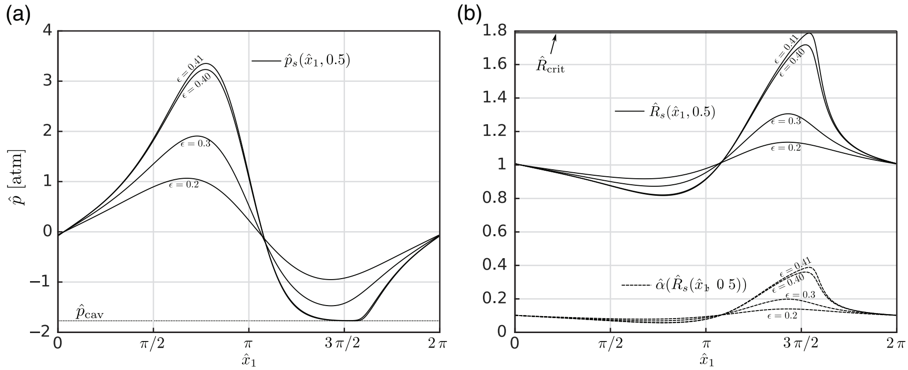

FIGURE 5. Stationary pressure (a), dimensionless radii and gas fraction (b) for different eccentricities of the Journal Bearing.

numerically observed, while for $\epsilon > 0.41$ the time convergence towards a stationary solution is no longer obtained. Let us remark that the same loss of time convergence is numerically observed when increasing the journal rotational speed ω.

As can be observed from Figure 5(b), the maximum value of $\hat{R}$ on the domain increases as the eccentricity increases, reaching the value $\hat{R}_{\mathrm{crit}}$ for $\epsilon \approx 0.41$. Thus, the loss of time convergence could be related to the change in the sign of $\hat{f}_1'$ from negative to positive and then violating hypothesis (H1), which was essential to obtain the stability of the stationary solutions in Section 4.2.

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Conflicts of interest

The authors declare that there is no conflict of interest.

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A Appendix

Here heuristic arguments to justify equations (1.1) and (1.6) are presented. This presentation is based in the Theory of Multicomponent Fluids [25], which has been used by Carrica and coworkers to study the dynamics of a bubbly mixture around a surface ship [40, 41, 42, 43].

A.1 The mass and momentum conservation equations

The fluid mixture is composed by two phases: an incompressible liquid phase (with density ρ_ℓ and viscosity μ_ℓ) and a gas phase (with reference density ρ_g and viscosity μ_g). The mixture density and velocity vector are denoted by ρ and $\mathbf{u}$, and the pressure field is denoted by p. The characteristic functions of the phases are denoted by $X_k(\mathbf{x}, t)$, $k = \ell, g$ for the liquid and gas phase, respectively. That is, $X_k(\mathbf{x}, t) = 1$ if the phase k is present in $\mathbf{x}$ at time t, and $X_k(\mathbf{x}, t) = 0$ otherwise.

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An ensemble of physical realisations is assumed to exist. Each of these realisations corresponds to an evolution of the physical system (fluid mixture – limiting surfaces – boundary conditions) for which the initial conditions are *near enough* to a set of ideal smooth initial conditions. A probability of occurrence for each realisation is assumed. Here $\langle\cdot\rangle$ denotes the statistical averaging with respect to this distribution function at the point $\mathbf{x} \in \Omega^V$ and time t. The fields

$$\alpha_g\,(\mathbf{x},t) = \langle X_g \rangle, \qquad \alpha_\ell(\mathbf{x},t) = \langle X_\ell \rangle \tag{A1}$$

are called the *gas fraction*, and the *liquid fraction*, respectively. In the statistical model, the phases occupy the whole domain, thus $\alpha_g + \alpha_\ell = 1$. With this, the averaged density, velocity vector and stress tensor for each phase $k = \ell, g$ are defined as

$$\bar\rho_k(\mathbf{x},t) = \frac{\langle \rho X_k \rangle}{\alpha_k}, \qquad \bar{\mathbf{u}}_k(\mathbf{x},t) = \frac{\langle \rho\,\mathbf{u}\,X_k \rangle}{\alpha_k \bar\rho_k}, \qquad \bar{\mathbf{T}}_k = \frac{\langle \mathbf{T}\,X_k \rangle}{\alpha_k}. \tag{A2}$$

Observe that as the liquid phase is incompressible and so one has that $\bar\rho_\ell = \rho_\ell$. Applying the averaging process to the conservation equations of mass and momentum, one obtains [25]

$$\frac{\partial(\alpha_k \bar\rho_k)}{\partial t} + \nabla\cdot(\alpha_k \bar\rho_k\, \bar{\mathbf{u}}_k) = \Gamma_k \qquad \text{in } \Omega^V, \tag{A3}$$

and

$$\frac{\partial(\alpha_k \bar\rho_k\, \mathbf{u}_k)}{\partial t} + \nabla\cdot(\alpha_k \bar\rho_k \bar{\mathbf{u}}_k \otimes \bar{\mathbf{u}}_k) = \nabla\cdot\left(\alpha_k\, \bar{\mathbf{T}}_k\right) + \mathbf{M}_k + v_{ki}\,\Gamma_k \qquad \text{in } \Omega^V, \tag{A4}$$

where $\mathbf{M}_k$ is the interfacial momentum source and, v_{ki} is the interfaces speed and Γ_k is the interfacial mass generation source. In the literature, a series of hypotheses are made to simplify these equations:

- Averaged quantities are smooth;
- Interfacial mass sources are negligible ($\Gamma_k \simeq 0$);
- Interfacial momentum sources are negligible ($\mathbf{M}_k \simeq 0$);
- The gas phase average velocity is equal to the liquid phase average velocity, $\bar{\mathbf{u}}_g = \bar{\mathbf{u}}_\ell$;
- The gas phase density is equal to the gas reference density ρ_g, $\bar\rho_g = \rho_g$.

With these hypotheses, adding up equation (A3) for both phases, simplifying the notation by setting $\alpha \stackrel{\text{def}}{=} \alpha_g$, and introducing the variable

$$\bar\rho\,(\alpha) = \alpha(\mathbf{x},t)\,\rho_\ell + (1-\alpha(\mathbf{x},t))\,\rho_g \tag{A5}$$

yields the conservation law:

$$\frac{\partial \bar\rho}{\partial t} + \nabla\cdot(\bar\rho\,\bar{\mathbf{u}}_\ell) = 0 \qquad \text{in } \Omega^V. \tag{A6}$$

Similarly, adding up equation (A4) for both phases we have

$$\frac{\partial(\bar\rho\,\bar{\mathbf{u}}_\ell)}{\partial t} + \nabla\cdot(\bar\rho\,\bar{\mathbf{u}}_\ell \otimes \bar{\mathbf{u}}_\ell) = -\nabla\bar p + \mu_{\text{eff}}\,\nabla^2 \bar{\mathbf{u}}_\ell \qquad \text{in } \Omega^V, \tag{A7}$$

where it has been assumed the existence of an effective fluid viscosity μ_{eff} (that depends smoothly on α, the liquid viscosity and the gas viscosity) such that

$$\nabla\cdot\left(\alpha\,\bar{\mathbf{T}}_g + (1-\alpha)\bar{\mathbf{T}}_\ell\right) \simeq -\nabla\bar p + \mu_{\text{eff}}\,\nabla^2 \bar{\mathbf{u}}_\ell \qquad \text{in } \Omega^V. \tag{A8}$$

An example off μ_{eff} is given by [44]

$$\mu_{\text{eff}}(\alpha) = \alpha(\mathbf{x},t)\mu_g + (1-\alpha(\mathbf{x},t))\mu_\ell. \tag{A9}$$

To our knowledge, there is a lack of works justifying (A8) or some similar relation, and further research on the topic is needed.

The thin film hypothesis allows to approximate the Navier–Stokes equations (A7) along equation (A6) by the incompressible Reynolds equation [45, 46]:

$$\nabla_{\mathbf{x}} \cdot \left(\frac{\bar{\rho} h^3}{12\mu_{\text{eff}}} \nabla \bar{p} \right) = \nabla_x \cdot \left(\frac{\mathbf{U}}{2} \bar{\rho} h \right) + \frac{\partial \bar{\rho} h}{\partial t} \qquad \text{in } \Omega. \tag{A10}$$

Appendix A.2 The field Rayleigh–Plesset equation

In the literature regarding the Rayleigh–Plesset equation to model the evolution of bubbly fluids, the following hypotheses are generally made:

- The liquid phase of the mixture is continuous while the gas phase corresponds to a high number of spherical bubbles dispersed in the liquid [14, 19, 21, 40, 41, 42, 43];
- The Rayleigh–Plesset procedure is locally applicable;
- The number of bubbles per volume unit, $n_b(\mathbf{x},t)$, is locally constant;
- The evolution of the field $R(\mathbf{x},t)$ is related to the mixture's pressure p_m by means of the Rayleigh–Plesset equation

$$\rho_\ell \left[\frac{3}{2} \left(\frac{\mathrm{D}R}{\mathrm{D}t} \right)^2 + R \frac{\mathrm{D}^2 R}{\mathrm{D}t^2} \right] = P_0 \left(\frac{R_0}{R} \right)^{3k} - p_m - \frac{2\sigma}{R} - 4 \left(\frac{\mu_\ell + \kappa^s/R}{R} \right) \frac{\mathrm{D}R}{\mathrm{D}t}, \tag{A11}$$

where ρ_ℓ and ρ_g (μ_ℓ and μ_g) are the densities (viscosities) of the liquid and the gas, respectively, P_0 is the inner pressure of the bubble when its radius is equal to R_0, k is the polytropic exponent, σ is the coefficient of surface tension and κ^s is the surface dilatational viscosity [21]. Regarding the material derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\mathbf{u}_b \cdot \nabla) . \tag{A12}$$

Let us remark that the convective field $\mathbf{u}_b = (u_b^1, u_b^2, u_b^3)$ is three-dimensional when associated with the Stokes field velocity field. However, as the velocity field associated with the Reynolds equation has third component equal to zero (see [45]), a two-dimensional convective field with $u_b^3 = 0$ is used when transporting the bubbles field R.

In general, the energy equation must be considered in this kind of modelling. However, in this work, the polytropic exponent is set to $k = 1$ or 1.4 (air specific heat), the former value corresponds to an isothermal process while the latter one corresponds to an adiabatic process, in both cases the energy equation is not needed. Nevertheless, it is worth noticing that there exist related works where the energy equation is also considered (e.g., [24, 39]).

It can be proved that the three variables $n_b(\mathbf{x},t)$, $\alpha(\mathbf{x},t)$ and $R(\mathbf{x},t)$ accomplish the geometric relation (e.g., [25] Section 10.1.2)

$$\alpha(\mathbf{x},t) = n_b(\mathbf{x},t) \frac{4\pi R(\mathbf{x},t)^3}{3}. \tag{A13}$$

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To couple (A11) with equation (A10), there remains to give and additional model for $n_b(\mathbf{x}, t)$. Two approaches may be found in the literature:

1. To assume that $n_b(\mathbf{x}, t)$ is a known constant.
2. To assume that the number of bubbles per liquid volume unit denoted by n_b^ℓ, is constant (e.g., [12]) and so

$$n_b(\mathbf{x}, t) = n_b^\ell \alpha_\ell(\mathbf{x}, t) = n_b^\ell(1 - \alpha(\mathbf{x}, t)),$$

that combined with equation (A13) implies

$$\alpha(\mathbf{x}, t) = \frac{n_b^\ell \frac{4\pi R(\mathbf{x},t)^3}{3}}{1 + n_b^\ell \frac{4\pi R(\mathbf{x},t)^3}{3}}. \tag{A14}$$

Notice that this expression is bounded by 1 and it grows monotonically with R. In the literature, it is typically introduced the parameter $\alpha_0 = n_b^\ell \frac{4\pi}{3} R_0^3$ which corresponds to a reference gas fraction. Doing so, this last equation may be written

$$\alpha(R) = \frac{\alpha_0\,(R/R_0)^3}{1 + \alpha_0\,(R/R_0)^3},$$

that corresponds to the formula used in Section 5 for the numerical examples.

The boundedness of α obtained from the second approach is one of the hypotheses made in Section 2. Thus, the theoretical results proved in this work remain valid for other definitions of $\alpha(R)$ accomplishing that property.

Remark 2 *If it is assumed that locally the bubbles can have several possible sizes, one may adapt the multigroup approach used by Carrica and coworkers for the modelling on the interaction of ocean polydispersed air bubbles with a surface ship [41, 42, 43], where a Population Balance Equation is written and a discrete group of possible bubbles radii is assumed. The use of this methodology for the context of the Reynolds–Rayleigh–Plesset cavitation model is an ongoing research topic.*

Remark 3 *The use of the Rayleigh–Plesset equation (1.1) for thin film flow assumes that even for small gaps each spherical bubble is small enough such that there is a sufficiently large domain around it (with respect to the bubbles radii) for which the Rayleigh–Plesset approximation is valid. This assumption is not always fulfilled. For example, in [47], a direct numerical computation of cavitation bubbles using Navier–Stokes equations shows the deformation of such initial spherical bubbles due to large shear flows or interaction with the solid surfaces.*

Remark 4 *It is known that Reynolds equation is valid for homogeneous thin flow only for a small Reynolds number [23, 46]. Thus, the present study cannot consider as high Reynolds number flow.*