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LIFTING N-DIMENSIONAL GALOIS REPRESENTATIONS TO CHARACTERISTIC ZERO

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Abstract. Let F be a number field, let $N \geq 3$ be an integer, and let k be a finite field of characteristic ℓ. We show that if $\overline{\rho} : G_F \longrightarrow GL_N(k)$ is a continuous representation with image of $\overline{\rho}$ containing $SL_N(k)$ then, under moderate conditions at primes dividing $\ell\infty$, there is a continuous representation $\rho : G_F \longrightarrow GL_N(W(k))$ unramified outside finitely many primes with $\overline{\rho} \sim \rho$ mod ℓ. Stronger results are presented for $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_3(k)$.

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1. Introduction. A celebrated result of Khare and Wintenberger [**8**] proves that every odd, irreducible, continuous representation $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{F}}_\ell)$ is modular, that is, $\bar{\rho}$ is the mod-ℓ reduction of an ℓ-adic Galois representation $G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}}_\ell)$ attached to a modular form. The statement, commonly referred to as Serre's (modularity) conjecture, was initially known only when $\bar{\rho}$ had solvable image following work of Langlands and Tunnell. A key evidence for Serre's conjecture was provided by Ramakrishna in [**13**] by proving the existence of an ℓ-adic lift of $\bar{\rho}$. Ramakrishna's construction and subsequent refinements (see [**1**, **2**, **15**]) play a crucial role in Khare and Winterberger's proof; for earlier applications of Ramakrishna's lifting resutls to modularity of $GL_2(\mathbb{F}_7)$ and $GL_2(\mathbb{F}_9)$ valued representations, see [**9**] and [**6**].

Now, let F be a number field, let $N \geq 3$ be an integer, and suppose we are given a continuous representation $\overline{\rho} : G_F \longrightarrow GL_N(\overline{\mathbb{F}}_\ell)$. Just as in the two-dimensional case, we then expect $\bar{\rho}$ to satisfy some version of modularity. In particular, we should be able to find a finite extension K of $\mathbb{Q}_\ell$ and a continuous representation $\rho : G_F \longrightarrow GL_N(\mathcal{O}_K)$ with values in the integer ring of K which is unramified outside finitely many primes and whose reduction modulo the maximal ideal of $\mathcal{O}_K$ is equivalent to $\bar{\rho}$. In this paper, we generalise the method of Ramakrishna, [**13**], to $N \geq 3$ and provide an answer to the finding such characteristic zero lifts when the image of $\overline{\rho}$ and the residue characteristic ℓ are 'big'.

Before we describe the main result, we recall some terminology. Let A be a commutative ring and let $\rho : G_F \longrightarrow GL_N(A)$ be a representation. Then, adρ is the $A[G_F]$-module consisting of $N \times N$ matrices over A with the action of $g \in G_F$ on a matrix M given by $\rho(g)M\rho(g)^{-1}$, and ad$^0\rho$ denotes the $A[G_F]$-submodule of $N \times N$ matrices over A with trace 0. Also, we will call the representation $\rho : G_F \longrightarrow GL_N(A)$ *totally even* if the projective image of the decomposition group at each infinite place of F is trivial. (Equivalently, any choice of complex conjugation acts trivially on adρ.)

We now state the main result of this paper; for definitions of terms involved, see Section 2. Essentially, the result states that a residual Galois representation with big image (including the assumption that ℓ is large) and good properties at ℓ admits characteristic zero liftings.

MAIN THEOREM. *Fix an integer $N \geq 3$. Let k be a finite field of characteristic ℓ, and let $\overline{\rho} : G_F \longrightarrow GL_N(k)$ be a continuous representation of the absolute Galois group of a number field F. Let $W := W(k)$ denote the Witt ring of k, and fix a continuous character $\chi : G_F \longrightarrow W^{\times}$ lifting the determinant of $\overline{\rho}$ (i.e., $\chi \pmod{\ell} = \det \overline{\rho}$). Assume that*

(1) *The image of $\overline{\rho}$ contains $SL_N(k)$;*
(2) *$\overline{\rho}$ is not totally even;*
(3) *If v is a place of F lying above ℓ then $H^0\big(G_{F_v}, ad^0\overline{\rho}(1)\big) = (0)$.*

Suppose that $\ell > N^{3[F:\mathbb{Q}]N}$. There then exists a global deformation condition $\mathcal{D}$ with determinant χ for $\overline{\rho}$ such that the universal deformation ring for type $\mathcal{D}$ deformations of $\overline{\rho}$ is a power series ring over W in at least $N-2$ variables. In particular, there is a continuous representation $\rho : G_F \longrightarrow GL_N(W)$ with determinant χ satisfying the following properties:

- *$\rho \pmod{\ell} \sim \overline{\rho}$; and,*
- *ρ is unramified outside finitely many primes.*

We can remove the local hypothesis at ℓ and say more when the number field is Q and $N = 3$. More precisely, let $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_3(k)$ satisfy the first two conditions of the main theorem (so, $\overline{\rho}$ is odd and its image contains $SL_3(k)$). Then, $\overline{\rho}$ has a lifting to $GL_3(W(k))$ whenever $\ell \geq 11$, or $\ell = 7$ and the fixed field of $\mathrm{ad}^0\overline{\rho}$ does not contain $\cos(2\pi/7)$. See Theorem 6.2.

The basic organisational principle underlying our approach is a beautiful result of Böckle relating the structure of a universal deformation ring to its local (uni)versal components. See [**1**, **2**]; for a precise statement of the result we need, see Theorem 2.2 in Section 2.2. The problem thus becomes one of finding a global deformation condition with smooth local components and trivial dual Selmer group. It is perhaps worth noting here that the two requirements are not completely independent of each other (as can be seen from the discussion in Section 2.2). Ramakrishna's great insight, in the GL_2 case, is to show how to reduce the size of the dual Selmer group by a clever tweaking of the global deformation condition at some primes. We will adapt Ramakrishna's strategy so that the sizes of dual Selmer groups can be controlled (and reduced) when $N \geq 3$.

There are two key ingredients in being able to make such an extension. First, we prove a cohomological result which gives conditions under which a subspace of $H^1(G_F, M)$ can be distinguished by its restriction at a prime. This provides us with a collection of primes where an adjustment of the local condition can result in a smaller dual Selmer group. The second component is local: for each prime $v \nmid \ell$, we need to produce a smooth deformation condition of sufficiently large dimension for the restriction of $\overline{\rho}$ to a local decomposition group at the prime v. There are complications when the residue characteristic of F_v is relatively small (for instance, when the residue characteristic is not bigger than N), and we avoid these by assuming $[F_v(\zeta_\ell) : F_v] \geq 3N$. (See Theorem 4.3.) The condition $\ell > N^{3[F:\mathbb{Q}]N}$ is an easy—but not an economic—bound that allows us to avoid local complications at small primes for general N, ℓ.

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While the hypothesis at primes above ℓ ensures that we do not have to deal with the more difficult problem of studying local deformations at ℓ, it does still cover a wide range of examples. Note that the hypothesis at a prime $v|\ell$ is equivalent to the assumption that the only G_{F_v}-equivariant homomorphism from $\overline{\rho}$ to $\overline{\rho}(1)$ is the zero map. The exceptions can be easily classified for small N, and we do so for the case when $N = 3$ and $F = \mathbb{Q}$. We do not attempt to put any geometric condition as the representations we are looking at might not even have the right duality property (to link up with automorphic forms).

A similar generalisation of Ramakrishna's lifting technique to GL_N was also obtained by Hamblen, **[7]**, about the same time when an earlier version of this paper was first prepared. Even so, we hope that this paper still carries an interest for the following reasons. First, the study of local deformations presented here, in particular the existence of smooth deformations of right dimension, has independent merit. Although some of the local analysis also appears in **[4]**, there is a difference in approach (for instance in the study of tamely ramified deformations and also in the role of tensor product of deformations). Second, there is a slight difference in the method: we rely on Böckle's result to produce smooth universal deformation rings, and make use of different local conditions. Consequently we are able to prove existence of characteristic 0 lifts for general number fields, and strong lifting results when the base field is $\mathbb{Q}$ and $N = 3$.

This paper is organised as follows. After setting out the requisite terminology, Section 2 describes the overall strategy of the proof following Böckle's result (see Theorem 2.2 and the ensuing paragraphs). Section 3, then, establishes a result in Galois cohomology (see Theorem 3.1) which allows us to show that the new deformation conditions we consider reduce the size of the dual Selmer group, while Section 4 proves the existence smooth local deformation rings. These two sections are independent of each other. Section 5, then, applies the results of Sections 3 and 4 to produce deformation conditions with trivial dual Selmer group. Finally, we complete the construction of characteristic 0 liftings in Section 6, and show how our earlier discussion extends to proving stronger lifting results for GL_3.

1.1. Notation. The ℓ-adic cyclotomic character is always denoted by ω and $\overline{\omega}$ is the mod ℓ-cyclotomic character. The term 'prime' on its own always indicates a finite prime except when the context makes it clear that we are also including infinte primes. If F is a number field, we assume we are given fixed embeddings $\overline{F} \hookrightarrow \overline{F}_v$ for each prime v (including the infinite ones). If F is unramified at the prime v we shall view Frob_v as element of G_F via the embedding $\overline{F} \hookrightarrow \overline{F}_v$. If A is a topological ring and $\rho : G_F \longrightarrow GL_N(A)$ is a continuous representation, we shall denote the restriction of ρ to a decomposition group at v by ρ_v. We shall frequently use $H^*(F, M)$ to denote $H^*(G_F, M)$. The group of unramified cohomology classes at a prime is indicated by the presence of a subscript (as in H^*_{nr}).

If k is a finite field, then the Witt ring of k will be denoted by $W(k)$ and $\widehat{x} \in W(k)$ denotes the Teichmüller lift of $x \in k$. A CNL $W(k)$-algebra, or simply a CNL algebra if the finite field k is clear, is shorthand for a complete, Noetherian, local algebra with residue field k. If χ (resp. ρ) is a $W(k)$ valued character (resp. homomorphism), then we will use the same letters for their extension to a CNL $W(k)$-algebra.

2. Preliminaries. In this section, we give a brief summary of deformation theory and recall the definitions of some of the key objects used in the analysis of universal deformation rings attached to global deformation conditions. This leads on to a description of Böckle's result and an outline of the main steps needed to prove our main theorem (see Theorem 2.2 and the paragraphs following it). Aside from setting out key terminology and notation, we hope that the discussion in this section will make transparent the basic argument and structure of this paper.

2.1. Deformation conditions in general. We begin with a sketch of deformation theory for group representations as developed by Mazur (see [**10**,**11**]). The presentation closely follows Sections 23 and 26 of [**10**] apart from some minor adjustments. In particular, we specify what the term 'a deformation condition' precisely means since, for the most part, we shall be involved in checking that the properties we specify at a local decomposition group determine a deformation condition. (See [**4**] for a slightly different approach using more explicit descriptions of the conditions (DC0), (DC1), and (DC2) below. For pro/near representability in a general context, see [**14**].)

Let Π be a profinite group satisfying the 'finiteness at ℓ' property of Mazur (Section 1 of [**10**]). For our purposes, a representation of Π is a continuous homomorphism $\rho : \Pi \longrightarrow GL_N(A)$, where A is a topological ring. The underlying free A-module on which Π acts will be denoted by $V(\rho)$. Given two representations

$$\rho_A : \Pi \longrightarrow GL_N(A), \qquad \rho_B : \Pi \longrightarrow GL_N(B),$$

and a morphism $f : A \longrightarrow B$ in the relevant category, we say that ρ_A is a lift of ρ_B if $f\rho_A = \rho_B$.

If $\rho_1 : \Pi \longrightarrow GL_n(A)$, $\rho_2 : \Pi \longrightarrow GL_m(A)$ are two representations, then $\mathrm{Hom}(V(\rho_1), V(\rho_2))$, or just simply $\mathrm{Hom}(\rho_1, \rho_2)$, is shorthand for the $A[\Pi]$-module of A-linear maps from $V(\rho_1)$ to $V(\rho_2)$. As a representation $\mathrm{Hom}(\rho_1, \rho_2)$ can be described as the group of $m \times n$ matrices over A with Π action given by $(g, M) \longrightarrow \rho_2(g)M\rho_1(g)^{-1}$. We shall take $\rho_1 \otimes \rho_2 : G_F \longrightarrow GL_{mn}(A)$ to mean the representation (gotten from $V(\rho_1) \otimes V(\rho_2)$) expressed with respect to the basis $v_1 \otimes w_1, \ldots, v_1 \otimes w_m, \ldots, v_n \otimes w_1, \ldots, v_n \otimes w_m$ where $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ are the bases for ρ_1 and ρ_2 respectively. Note that $\mathrm{Hom}(\rho_1, \rho_2)$ is naturally isomorphic to $\rho_1^* \otimes \rho_2$ where ρ_1^* is the dual representation for ρ_1.

Let $\mathrm{Rep}_N(\Pi; k)$ denote the following category:

- Objects are pairs (A, ρ_A) where A is a CNL $W(k)$-algebra and $\rho_A : \Pi \longrightarrow GL_N(A)$ is a representation.
- A morphism from (A, ρ_A) to (B, ρ_B) is a pair (f, M) where $f : A \longrightarrow B$ is a morphism of local rings and $M \in GL_N(B)$ satisfies $f\rho_A = M\rho_B M^{-1}$.

Given a representation $\overline{\rho} : \Pi \longrightarrow GL_N(k)$, a *deformation condition* $\mathcal{D}$ for $\overline{\rho}$ is a full subcategory $\mathcal{D} \subseteq \mathrm{Rep}_N(\Pi; k)$ satisfying the following properties:

(DC0) $(k, \overline{\rho}) \in \mathcal{D}$, and if $(A, \rho_A) \in \mathcal{D}$, then $\overline{\rho} \sim \rho_A \mod \mathfrak{m}_A$.

(DC1) If (A, ρ_A) is an object in $\mathcal{D}$ and $(f, M) : (A, \rho_A) \longrightarrow (B, \rho_B)$ is a morphism, then (B, ρ_B) is also in $\mathcal{D}$.

(DC2) Let $\alpha : A \longrightarrow C$ and $\beta : B \longrightarrow C$ be morphisms of Artinian CNL algebras. Assume that β is a *small extension,* that is, $\beta : B \longrightarrow C$ is surjective and $\ker \beta$ is a non-zero principal ideal killed by the maximal ideal $\mathfrak{m}_B$.

Then, in the cartesian diagram

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & C \end{array},$$

an object $(A \times_C B, \rho)$ of $\mathrm{Rep}_N(\Pi; k)$ is in $\mathcal{D}$ if and only if $(A, \pi_A \rho)$, $(B, \pi_B \rho)$ are in $\mathcal{D}$.

We say that $\rho : \Pi \longrightarrow GL_N(A)$, or (A, ρ), is of type $\mathcal{D}$ if (A, ρ) is in $\mathcal{D}$. If $\chi : \Pi \longrightarrow W^\times$ is a character, we say that $\mathcal{D}$ has determinant χ if $\det \rho = \chi$ for any $(A, \rho) \in \mathcal{D}$. The deformation condition $\mathcal{D}$ is said to be smooth if for any surjection $f : A \longrightarrow B$ and an object (B, ρ_B) of type $\mathcal{D}$, there is an object (A, ρ_A) in $\mathcal{D}$ such that $f\rho_A = \rho_B$. It is sufficient to verify the smoothness condition for small extensions only. The tangent space of $\mathcal{D}$ will be denoted by $T\mathcal{D}$, and will be viewed as a k-subspace of $H^1(\Pi, \mathrm{ad}\overline{\rho})$ (it is a subspace of $H^1(\Pi, \mathrm{ad}^0\overline{\rho})$ if the determinant is fixed).

In practice, conditions (DC0), (DC1), and the only if part of condition (DC2), will almost always be immediate. If $\mathcal{D}$ is a deformation condition for $\overline{\rho} : \Pi \longrightarrow GL_N(k)$, the functor

$$\mathcal{D}(A) := \{\text{type } \mathcal{D} \text{ liftings } \rho : \Pi \longrightarrow GL_N(A) \text{ of } \overline{\rho}\} / \text{strict equivalence}$$

is nearly representable. If $\mathcal{D}$ is smooth then the (uni)versal deformation ring is a power series ring.

Our objective is to produce (uni)versal deformation rings, which are power series rings. In view of the following lemma, one can make use of extension of scalars to produce such (uni)versal deformation rings.

LEMMA 2.1. *Let $k_0 \subset k_1$ be finite fields of characteristic ℓ, and let $\overline{\rho}_0 : \Pi \longrightarrow GL_n(k_0)$ be a representation. Denote by $\overline{\rho}_1 : \Pi \longrightarrow GL_n(k_1)$ the extension of scalars of $\overline{\rho}_0$ to $GL_n(k_1)$.*

Given a deformation condition $\mathcal{D}_1 \subseteq Rep_n(\Pi; k_1)$, let $\mathcal{D}_0$ be the full subcategory of $Rep_n(\Pi; k_0)$ consisting of those objects $(A, \rho) \in Rep_n(\Pi; k_0)$ such that $\big(A \otimes_{W(k_0)} W(k_1), \rho \otimes W(k_1)\big) \in \mathcal{D}_1$. Then

(1) *$\mathcal{D}_0$ is a deformation condition for $\overline{\rho}_0$, and $\dim_{k_0} T\mathcal{D}_0 = \dim_{k_1} T\mathcal{D}_1$;*
(2) *Let R_0, R_1 be the (uni)versal deformation rings of type $\mathcal{D}_0, \mathcal{D}_1$. Then, there is an isomorphism $R_1 \longrightarrow R_0 \otimes_{W(k_0)} W(k_1)$. In particular, if R_1 is a power series ring then so is R_0.*

Proof. Checking that $\mathcal{D}_0$ is a deformation condition is straightforward. Extension of scalars give a natural isomorphism between $H^1(\Pi, \mathrm{ad}\overline{\rho}_0) \otimes k_1$ and $H^1(\Pi, \mathrm{ad}\overline{\rho}_1)$. Thus, there is a subspace $L \subseteq H^1(\Pi, \mathrm{ad}\overline{\rho}_0)$ such that $L \otimes k_1 = T\mathcal{D}_1$. One then checks that L has to be the tangent space for $\mathcal{D}_0$.

For the second part, there is a surjection $R_1 \longrightarrow R_0 \otimes W(k_1)$. Since the extension $W(k_1)/W(k_0)$ is smooth, the tangent space for $R_0 \otimes W(k_1)$ has the same dimension as the tangent space for R_0. Hence, the surjection is an isomorphism. □

2.2. Global deformations. Now, let F be a number field and let k be a finite field of characteristic ℓ. Fix an absolutely irreducible representation $\overline{\rho} : G_F \longrightarrow GL_N(k)$ and a character $\chi : G_F \longrightarrow W^\times$ such that $\chi \pmod{\ell} = \det \overline{\rho}$.

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Informally, a global deformation condition specifies that we consider liftings of $\overline{\rho}: G_F \longrightarrow GL_N(k)$ with prescribed local behaviour. More precisely: Suppose we are given, for each prime v of F, a deformation condition $\mathcal{D}_v$ for $\overline{\rho}|_v$ with determinant χ. Furthermore, we require that the deformation condition $\mathcal{D}_v$ is *unramified* (i.e., all representations in $\mathcal{D}_v$ are unramified) for almost all primes v. The global deformation condition $\{\mathcal{D}_v\}$ with determinant χ for $\overline{\rho}$ is then the full subcategory of $\mathrm{Rep}_N(G_F; k)$ consisting of those objects $(A, \rho) \in \mathrm{Rep}_N(G_F; k)$ such that $\det \rho = \chi$ and $(A, \rho|_v) \in \mathcal{D}_v$ for all primes v.

For a global deformation condition $\mathcal{D}$ with determinant χ for $\overline{\rho}$, we shall denote the local condition at v by $\mathcal{D}_v$ (so $\mathcal{D} = \{\mathcal{D}_v\}$). We define the ramification set $\Sigma(\mathcal{D})$ to be the finite set consisting of those primes v of F where $\mathcal{D}_v$ is not unramified, primes lying above ℓ and ∞, and primes where $\overline{\rho}$ and χ are ramified. Thus, $\mathcal{D}$ is precisely a deformation condition for $\overline{\rho}|_{\mathrm{Gal}(F_{\Sigma(\mathcal{D})}/F)}$ with prescribed local components (cf. Section 26 of [**10**]). The tangent space for $\mathcal{D}$ is the Selmer group

$$H^1_{\{T\mathcal{D}_v\}}\left(F, \mathrm{ad}^0\overline{\rho}\right) = \ker\left(H^1(G_F, \mathrm{ad}^0\overline{\rho}) \longrightarrow \prod H^1(F_v, \mathrm{ad}^0\overline{\rho})/T\mathcal{D}_v\right).$$

The dual Selmer group for $\mathcal{D}$ is defined as follows. For each prime v of F the pairing $\mathrm{ad}^0\overline{\rho} \times \mathrm{ad}^0\overline{\rho}(1) \longrightarrow k(1)$ obtained by taking trace induces a perfect pairing

$$H^1\left(F_v, \mathrm{ad}^0\overline{\rho}\right) \times H^1\left(F_v, \mathrm{ad}^0\overline{\rho}(1)\right) \longrightarrow H^2(F_v, k(1)).$$

Let $T\mathcal{D}_v^{\perp} \subseteq H^1\left(F_v, \mathrm{ad}^0\overline{\rho}(1)\right)$ be the annihilator of $T\mathcal{D}_v$ under the above pairing. The dual Selmer group $H^1_{\{T\mathcal{D}_v^{\perp}\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right)$ is then determined by the local conditions $\{T\mathcal{D}_v^{\perp}\}$, that is,

$$H^1_{\{T\mathcal{D}_v^{\perp}\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right) := \ker\left(H^1(G_F, \mathrm{ad}^0\overline{\rho}(1)) \longrightarrow \prod H^1(F_v, \mathrm{ad}^0\overline{\rho}(1))/T\mathcal{D}_v^{\perp}\right).$$

While the tangent space for $\mathcal{D}$ is a very difficult object to get a handle on, remarkably a quantitative comparison with the dual Selmer group is possible by the following formula of Wiles (Theorem 8.6.20 in [**12**]):

$$\dim H^1_{\{T\mathcal{D}_v\}}\left(F, \mathrm{ad}^0\overline{\rho}\right) - \dim H^1_{\{T\mathcal{D}_v^{\perp}\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right) = \sum_v \left(\dim T\mathcal{D}_v - \dim H^0(F_v, \mathrm{ad}^0\overline{\rho})\right). \quad (2.1)$$

Note that the summation runs over all primes, *including* the primes at infinity.

We now describe a beautiful result of Böckle which allows one to relate the global (uni)versal deformation ring in terms of local deformation rings. Let $\overline{\rho}$, χ and $\mathcal{D}$ be as above. For each prime v, let R_v be the (uni)versal deformation ring for type $\mathcal{D}_v$ deformations, and let R be the (uni)versal global deformation ring for type $\mathcal{D}$ deformations of $\overline{\rho}$.

Now, choose presentations

$$R_v \cong W(k)[[T_{v,1}, \ldots, T_{v,n_v}]]/J_v, \quad R \cong W(k)[[T_1, \ldots, T_n]]/J$$

of R_v, R as quotients of power series rings in *minimal number of generators*. Thus, $n_v = \dim T\mathcal{D}_v$ and $n = \dim H^1_{\{T\mathcal{D}_v\}}(F, \mathrm{ad}^0\overline{\rho})$; the ideal $J_v = (0)$ if $v \notin \Sigma(\mathcal{D})$. Restriction

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of the (uni)versal deformation to a decomposition group at v induces a map $R_v \longrightarrow R$, which can be then lifted to a map

$$\alpha_v : W(k)[[T_{v,i}]] \longrightarrow W(k)[[T_i]],$$

of local rings. Of course α_v, and even $R_v \longrightarrow R$, might not be unique at all.

THEOREM 2.2. *(Böckle, Theorem 4.2 of [**2**]) With notation as in the preceding paragraphs, the ideal J is generated by the images $\alpha_v J_v$ together with at most* $\dim H^1_{\{T\mathcal{D}_v^{\perp}\}}(F, ad^0\overline{\rho}(1))$ *other elements. Thus,*

$$gen(J) \leq \sum_{v \in \Sigma(\mathcal{D})} gen(J_v) + \dim H^1_{\{T\mathcal{D}_v^{\perp}\}}(F, ad^0\overline{\rho}(1)), \tag{2.2}$$

where $gen(J)$ (resp. $gen(J_v)$) is the minimal number of elements required to generate the ideal J (resp. J_v).

Theorem 2.2 above allows us to prove our main theorem provided we can find a global deformation condition with smooth local conditions and trivial dual Selmer group. For in this case, the right hand side of (2.2) is 0; consequently, the global deformation ring has trivial ideal of relations and therefore is smooth. The question now is how to get to such nice global deformation conditions.

Suppose we start off with a global deformation problem $\mathcal{D}$ with smooth local deformation conditions. By (2.2) the number of global relations is then bounded by the dimension of the dual Selmer group. The critical step then is to tweak one of the local conditions $\mathcal{D}_v$ at some prime so that the new deformation condition has smaller dual Selmer group. We shall show that this can be done in Section 5 provided

$$\dim H^1_{\{T\mathcal{D}_v\}}\left(F, \mathrm{ad}^0\overline{\rho}\right) \geq N - 2 + \dim H^1_{\{T\mathcal{D}_v^{\perp}\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right). \tag{2.3}$$

We then need to resolve two issues. For our choice of primes where the local deformation condition should be—and how it should be—changed, we use a direct generalisation of the one used in [**13**]. However, the verification that this choice indeed reduces the size of the dual Selmer group requires effort. The relevant result, established in Section 3, follows from a careful analysis of the cohomology of G_F with coefficients in various modules associated to the residual representation $\overline{\rho}$.

The second issue is that by Wiles' formula (2.1), the above inequality (2.3) will fail if the local deformation conditions are 'small'. To ensure this doesn't happen, we make sure that $\mathcal{D}_v$ is smooth in $\dim H^0(F_v, \mathrm{ad}^0\overline{\rho})$ variables at primes not dividing ℓ. The required constructions are carried out in Section 4; the precise statement we need is presented in Theorem 4.3. Given these local conditions, the hypotheses at ℓ and ∞ allows us to ensure that (2.3) is satisfied.

3. Galois cohomology. Our aim in this section is to prove a result in Galois cohomology which allows us to show how sizes of dual Selmer groups can be controlled, and be decreased, by careful changes in local conditions. But, before we proceed any further, we record the following running assumption in place for the rest of the section:

Throughout this section, K/F is a finite Galois extension of number fields with Galois group $G := \mathrm{Gal}(K/F)$ and k be a finite extension of $\mathbb{F}_{\ell}$.

The main result of this section, stated below, gives conditions under which a subspace of $H^1(G_F, M)$ can be distinguished by its restriction at a prime.

THEOREM 3.1. *Let $M_1, \ldots, M_n$ be n mutually non-isomorphic, absolutely irreducible $k[G]$ modules with $H^1(G,\ M_i) = 0,\ 1 \leq i \leq n$. We assume that we are given a place v of F and k-subspaces $V_i \subseteq H^1(G_F,\ M_i)$ with the following properties:*

- *$M_1 \oplus \cdots \oplus M_n$ is unramified at v, and that $Frob_v$ acts semi-simply on each M_i;*
- $\dim V_i \leq \dim H^1_{nr}(F_v,\ M_i)$ *for $i = 1, \ldots, n$.*

Under the above assumptions, we can find infinitely many places w such that:

- *$M_1 \oplus \cdots \oplus M_n$ is unramified at w and the images of $Frob_w$, $Frob_v$ in G are the same;*
- *Any cohomology class in V_i is unramified at w;*
- *The restriction map $V_1 \oplus \cdots \oplus V_n \longrightarrow H^1_{nr}(F_w,\ M_1) \oplus \cdots \oplus H^1_{nr}(F_w,\ M_n)$ is injective.*

For clarity, we record the following (generally standard) notation. If M is a $k[G]$-module and $\xi \in H^1(G_F,\ M)$, then the restriction of ξ to G_K is a group homomorphism. We denote by $K(\xi)$ the field through which this homomorphism factorises. Note that the extension $K(\xi)/F$ is Galois. For $\xi_i \in H^1(G_F,\ M)$, $i = 1, \ldots, n$, the compositum of $K(\xi_1), \ldots, K(\xi_n)$ will be denoted by $K(\xi_1, \ldots, \xi_n)$.

We will derive Theorem 3.1 from two propositions, the first of which is as follows.

PROPOSITION 3.2. *Let M be a finite $k[G]$-module satisfying the following two conditions:*

- *M is a simple $\mathbb{F}_\ell[G_F]$-module with $End_{\mathbb{F}_\ell[G_F]}(M) = k$;*
- *$H^1(G,\ M) = 0$.*

If $\psi_1, \psi_2, \ldots, \psi_n$ are n linearly independent classes in the k-vector space $H^1(G_F,\ M)$, then $K(\psi_1), K(\psi_2), \ldots, K(\psi_n)$ are linearly disjoint over K.

The proof of the above proposition relies on the following observation, recorded as a lemma.

LEMMA 3.3. *Let M be as in Proposition 3.2, and let $0 \neq \xi \in H^1(G_F,\ M)$. Then:*

(*a*) *The restriction $\xi : Gal(K(\xi)/K) \longrightarrow M$ is an isomorphism of G-modules.*
(*b*) *If L is a Galois extension of F with $K \subseteq L$ then either $K(\xi) \subseteq L$ or $K(\xi) \cap L = K$.*

Proof. The images of $\mathrm{Gal}(K(\xi)/K)$ and $\mathrm{Gal}(K(\xi)/(K(\xi) \cap L))$ under ξ are subspaces of M stable under the action of G. The lemma follows as M is simple. □

Proof of Proposition 3.2 We first do the case $n = 2$. If $K(\psi_1)$ and $K(\psi_2)$ are not linearly disjoint over K, then by the above lemma $K(\psi_1) = K(\psi_2)$. The composite

$$M \xrightarrow{\psi_1^{-1}} \mathrm{Gal}(K(\psi_1)/K) = \mathrm{Gal}(K(\psi_2)/K) \xrightarrow{\psi_2} M$$

is a G-module automorphism of M. Since k is the endomorphism ring of M, the composite $\psi_2\psi_1^{-1}$ must be a non-zero element of k, and so ψ_1 and ψ_2 are linearly dependent—a contradiction.

We use induction for the general case. Suppose, we have proved that the fields $K(\psi_1), \ldots, K(\psi_{n-1})$ are linearly disjoint. We then need to show that $K(\psi_n)$ and $K(\psi_1, \ldots, \psi_{n-1})$ are linearly disjoint over K where $K(\psi_1, \ldots, \psi_{n-1})$ is the compositum of $K(\psi_1), \ldots, K(\psi_{n-1})$.

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Suppose they are not linearly disjoint. Then, Lemma 3.3 implies that $K(\psi_n)$ is a subfield of $K(\psi_1, \dots, \psi_{n-1})$ with $\mathrm{Gal}(K(\psi_n)/K) \cong M$. If we can now show that $K(\psi_n) = K(a_1\psi_1 + \cdots + a_{n-1}\psi_{n-1})$ for some $a_1, \dots, a_n \in k$ then, appealing to the case $n = 2$ of the proposition, we see that ψ_n is a linear combination of $\psi_1, \dots, \psi_{n-1}$—which is a contradiction.

Let $\mathcal{E}$ be the set of Galois extensions E/F with $K \subseteq E \subseteq K(\psi_1, \dots, \psi_{n-1})$ and $\mathrm{Gal}(E/K)$ isomorphic to M as G modules, and let V be the k-subspace of $H^1(G_F, M)$ spanned by $\psi_1, \dots, \psi_{n-1}$. We claim that the map $\mathbb{P}(V) \longrightarrow \mathcal{E}$ given by $\psi \longrightarrow K(\psi)$ is a bijection. This will complete the proof of the inductive step as $K(\psi_n) \in \mathcal{E}$.

That the map $\mathbb{P}(V) \longrightarrow \mathcal{E}$ is an injection follows from the case $n = 2$ of the proposition. Now, by our hypothesis, we have identifications

$$\mathrm{Gal}(K(\psi_1, \dots, \psi_{n-1})/K) \cong \mathrm{Gal}(K(\psi_1)/K) \times \cdots \times \mathrm{Gal}(K(\psi_{n-1})/K) \cong M^{n-1}$$

of G-modules. Using the simplicity of M, we observe that elements of $\mathcal{E}$ correspond to G-submodules of M^{n-1} which are isomorphic to M^{n-2}, that is, kernels of non-trivial G module homomorphisms from M^{n-1} to M. Since

$$\begin{aligned} \mathrm{Hom}_G(M \times \cdots \times M, M) &\cong \mathrm{Hom}_G(M, M) \times \cdots \times \mathrm{Hom}_G(M, M) \\ &\cong k \times \cdots \times k, \end{aligned}$$

we deduce $|\mathcal{E}| = |\mathbb{P}(k^{n-1})| = |\mathbb{P}(V)|$, and this establishes the claim. □

The second proposition needed to prove Theroem 3.1 requires a small degree of preparation. We fix an absolutely irreducible $k[G]$-module M with $H^1(G, M) = 0$, along with an element $g \in G$ which acts semi-simply on M. We denote by M^g the kernel of multiplication by $g - 1$ on M. Note that we have a decomposition $M = M^g \oplus (g-1)M$.

Let us also fix a non-trivial subgroup $L \subseteq M$ invariant under G_F with minimal dimension as an $\mathbb{F}_\ell$-vector space. It is then straightforward to check that L is simple, that k contains $\mathrm{End}_{\mathbb{F}_\ell[G_F]}(L) =: k'$ (say), and that $M \cong L \otimes_{k'} k$. Furthermore, we have $M^g = L^g \otimes_{k'} k$ and $(g-1)M = (g-1)L \otimes_{k'} k$.

Proposition 3.4. *With assumptions and notations as in the previous two paragraphs, let V be a finite dimensional k-subspace of $H^1(G_F, M)$. If $\dim M^g \geq \dim V$ we can find a lift $\widetilde{g} \in G_F$ of g such that the restriction map*

$$V \hookrightarrow H^1(G_F, M) \longrightarrow H^1(\langle \widetilde{g} \rangle, M)$$

is injective.

Proof. Set $n := \dim V$. Since $H^1(G_F, M) \cong H^1(G_F, L) \otimes_{k'} k$, we can find the following:

- a basis $\xi_1, \dots, \xi_n$ of V,
- m linearly independent cocycles $\psi_1, \dots, \psi_m$ in the k'-vector space $H^1(G_F, L)$ with $m \geq n$ and such that $\xi_i := \psi_i + \sum_{j>n} a_{ij}\psi_j$ for some $a_{ij} \in k$, $i = 1, \dots, n$.

Fix a lift $g' \in G_F$ of g. We can identify $H^1(\langle g' \rangle, M)$ with M^g. For ease of notation, we set

$$K_0 := K(\psi_j, j > n), \text{ and } K_i := K(\psi_i, \psi_j, j > n), \ i = 1, \dots, n.$$

By Proposition 3.2, the extensions K_i, $i = 1, \ldots, n$ are linearly disjoint over K_0.

For each $1 \leq i \leq n$, the cocycle ξ_i restricts to ψ_i on K_0. Since $\psi_i(\mathrm{Gal}(K_i/K_0)) = L$ and $\xi_i(xg') = \psi_i(x) + \xi_i(g')$ for any $x \in \mathrm{Gal}(K_i/K_0)$, we see that the k-subspace of M generated by $\psi_i(xg')$ is M.

We claim that we can find $x_i \in \mathrm{Gal}(K_i/K_0)$, $1 \leq i \leq n$, such that $\xi_1(x_1g'), \ldots, \xi_n(x_ng')$ generate an n-dimensional subspace of $M/(g-1)M$. To see this, first pick $x_1 \in \mathrm{Gal}(K_1/K_0)$ such that $\xi_1(x_1g')$ is non-trivial when projected to $M/(g-1)M$. Having found $x_i \in \mathrm{Gal}(K_i/K_0)$, $i = 1, \ldots, j$ with $j < n$ and such that $\xi_1(x_1g'), \ldots, \xi_j(x_jg')$ generate a j-dimensional subspace of $M/(g-1)M$ we can find an $x_{j+1} \in \mathrm{Gal}(K_{j+1}/K_0)$ with the property that $\xi_{j+1}(x_{j+1}g')$ does not lie in the subspace of M spanned by $\xi_1(x_ig'), \ldots, \xi_j(x_jg')$ and $(g-1)M$. This is possible as this latter subspace has dimension $j + \dim_k(g-1)M < \dim_k M$.

Finally, using Proposition 3.2, we can find x in the Galois group of K_0 which acts as x_i on each extension K_i/K_0. Set $\widetilde{g} = xg'$. Then, as $\xi_1(\widetilde{g}), \ldots, \xi_n(\widetilde{g})$ generate an n-dimensional subspace of $M/(g-1)M$, we see that the images of ξ_i when restricted to $H^1(\langle\widetilde{g}\rangle, M)$ are linearly independent. □

Proof of Theorem 3.1 Denote by $K(V_i)$ the splitting field for V_i over K, and by $K(V_1, \ldots, V_n)$ the compositum of $K(V_i)$. We claim that the extensions $K(V_i)$ are linearly disjoint over K. To see this, we observe that each $\mathrm{Gal}(K(V_i)/K)$ is isomorphic to a subgroup of M_i as a G-module and therefore $\mathrm{Gal}(K(V_i)/K) \otimes_{\mathbb{F}_p} k$ is a direct sum of copies of M_i as a $k[G]$-module. Thus, if $K(V_i)$ and $K(V_j)$ are not linearly disjoint over K for some $i \neq j$, then the semi-simplifications of $\mathrm{Gal}(K(V_i)/K) \otimes_{\mathbb{F}_p} k$ and $\mathrm{Gal}(K(V_i)/K) \otimes_{\mathbb{F}_p} k$ will have a common irreducible factor. But, this cannot happen as M_i and M_j are absolutely irreducible and non-isomorphic.

Take $g \in G$ to be an element which Frob_v lifts and let $g' \in \mathrm{Gal}(K(V_1, \ldots, V_n)/F)$ be a lift of g. By Proposition 3.4, we can find $x_i \in \mathrm{Gal}(K(V_i)/K)$ such that $V_i \longrightarrow H^1(\langle x_ig'\rangle, M_i)$ is injective. Using disjointness of the $K(V_i)$'s, we can find an $x \in \mathrm{Gal}(K(V_1, \ldots, V_n)/K)$ such that x acts on $K(V_i)$ as x_i. By the Chebotarev density theorem, we can then find a place w of F lifting xg' and unramified in $K(V_1, \ldots, V_n)$. It is now immediate such a w satisfies the properties asked for. □

4. Local deformation conditions. Our objective in this section is to construct examples of local deformation conditions which admit a sufficiently large (uni)versal deformation ring. *Throughout this section, k is a finite field of characteristic ℓ and p is a prime different from ℓ.*

DEFINITION 4.1. Let F be a finite extension of $\mathbb{Q}_p$ and let $\overline{\rho} : G_F \longrightarrow GL_N(k)$ be a representation. We say that a deformation condition $\mathcal{D}$ for $\overline{\rho}$ is *well-behaved* if $\mathcal{D}$ is smooth and $\dim\ T\mathcal{D} = \dim\ H^0(G_F, \mathrm{ad}\overline{\rho})$.

EXAMPLE 4.2. Let F be a finite extension of $\mathbb{Q}_p$ and let $\overline{\rho} : G_F \longrightarrow GL_N(k)$ be a representation. If $\overline{\rho}$ is unramified, then the class of unramified liftings is a well-behaved deformation condition. The unrestricted deformation condition is well-behaved if $H^2(G_F, \mathrm{ad}\overline{\rho}) = (0)$.

We can now state our main result asserting the existence of well-behaved deformation conditions.

THEOREM 4.3. *Let F be a finite extension of $\mathbb{Q}_p$, let k be a finite field of characteristic $\ell \neq p$, and let $\overline{\rho} : G_F \longrightarrow GL_N(k)$ be a representation. Assume that all irreducible*

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components occurring in the semi-simplification of $\overline{\rho}$ are absolutely irreducible. If $p \leq N$ and $\overline{\rho}$ is wildly ramified assume that $[F(\zeta_\ell) : F] \geq 3N$ where ζ_ℓ is an ℓth root of unity. Then, the following hold:

(*a*) *There exists a well-behaved deformation condition $\mathcal{D}$.*
(*b*) *Suppose $\chi : G_F \longrightarrow W^\times$ is a character lifting $\det \overline{\rho}$. Assume that N, ℓ are co-prime. Then, liftings of type $\mathcal{D}$ and determinant χ is a smooth deformation condition for $\overline{\rho}$ and the dimension of its tangent is equal to* $\dim H^0(G_F, ad^0\overline{\rho})$.

To construct a well-behaved deformation condition $\mathcal{D}$ as claimed (and also to outline the structure of this section), we proceed as follows:

(I) We would like to build up $\mathcal{D}$ from well-behaved deformation conditions for some decomposition of $\overline{\rho}$. In section 4.1 we show that a good way of decomposing $\overline{\rho}$ is to make sure that the basic blocks have no common irreducible components, even after taking Tate twists.
(II) The blocks can then be analysed separately. There are essentially three cases we need to consider.
   (i) First, the case when a given residual representation is tamely ramified. The deformation condition in this case is to obtained by specifying a Jordan–Holder decomposition for a generator of tame inertia. See Section 4.2.
   (ii) The residual representation is a tensor product of two smaller representations. In Section 4.3 we study when we can construct the candidate well-behaved deformation by using tensor products.
   (iii) The residual representation is induced, in which case we try to induce a known well-behaved deformation condition. This is done in Section 4.4
(III) Finally, we verify that the hypotheses of Theorem 4.3 guarantee applicability of the preceding steps and complete the proof Theorem 4.3 in Section 4.5.

As indicated in Section 1, local deformation conditions for a class of residual representations are constructed in [**4**]. While there is some overlap in the treatment of induced and tamely ramified deformations, the results here do not follow directly from [**4**]. Moreover, the approaches are different (and quite significantly in the case of tamely ramified deformations).

The second part of Theorem 4.3 is straightforward *given* the first part, and we deal with it right away. As indicated earlier, the first part of Theorem 4.3 will be proved in Section 4.5.

Proof of Theorem 4.3 (b). We need only check smoothness, and for that it suffices to check that any deformation $\rho : G_F \longrightarrow GL_N(A)$ of type $\mathcal{D}$ can be twisted to a deformation with determinant χ. If $\psi : G_F \longrightarrow A^\times$ is a character and we want $\chi = \det(\psi\rho)$, then $\psi^N = \chi \det \rho^{-1}$. We can find such a character ψ because $\chi \det \rho^{-1} : G_F \longrightarrow 1 + \mathfrak{m}_A$ and

$$1 + \mathfrak{m}_A \xrightarrow{x \to x^N} 1 + \mathfrak{m}_A$$

is an isomorphism. □

4.1. Direct sums of deformation conditions. In this subsection, we show that given a favourable decomposition of the residual representation, taking direct sum of liftings of the components determines a deformation condition.

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We will keep the following assumptions for the rest of this subsection. We assume we are given a finite extension $F/\mathbb{Q}_p$ and representations $\overline{\rho}_i : G_F \longrightarrow GL_{d_i}(k)$, $i = 1, \ldots, n$ satisfying

$$\mathrm{Hom}_{k[G_F]}\left(\overline{\rho}_i, \overline{\rho}_j(r)\right) = (0) \tag{4.1}$$

for $i \neq j$, $r \in \mathbb{Z}$. We also assume that we are given a deformation condition $\mathcal{F}_i$ for each residual representation $\overline{\rho}_i$, $i = 1, \ldots, n$.

We will also keep the following notation for the rest of this subsection.

- We set $\overline{\rho} := \overline{\rho}_1 \oplus \cdots \oplus \overline{\rho}_n$ and $N := d_1 + \ldots + d_n$. Thus, the representation $\overline{\rho}$ takes values in $GL_N(k)$.
- We denote by $\mathcal{F} := \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_n$ the full subcategory of $\mathrm{Rep}_N(G_F; k)$ consisting of objects (A, ρ) such that the representation $\rho \sim \rho_1 \oplus \cdots \oplus \rho_n$ with $(A, \rho_i) \in \mathcal{F}_i$. In other words, but perhaps less formally, we are restricting attention to those representations which split completely as a direct sum of representations of type $\mathcal{F}_1, \ldots, \mathcal{F}_n$

We then have the following theorem.

THEOREM 4.4. *$\mathcal{F}$ is a deformation condition for $\overline{\rho}$. The natural map*

$$((A, \rho_i) \in \mathcal{F}_i)_{i=1}^n \longrightarrow (A, \rho_1 \oplus \cdots \oplus \rho_n)$$

induces an isomorphism of tangent spaces

$$T\mathcal{F} \cong T\mathcal{F}_1 \oplus \cdots \oplus T\mathcal{F}_n,$$

and $\mathcal{F}$ is well-behaved if each $\mathcal{F}_i$ is well-behaved.

Theorem 4.4 is an immediate consequence of the following proposition:

PROPOSITION 4.5. *Let R be a CNL algebra, and let $\rho : G_F \longrightarrow GL_N(R)$ be a lift of $\overline{\rho}$. We then have, up to strictly equivalence, a unique decomposition $\rho \cong \rho_1 \oplus \cdots \oplus \rho_n$ where $\rho_i : G_F \longrightarrow GL_{d_i}(R)$ is a lift of $\overline{\rho}_i$.*

The proof of Proposition 4.5 relies on there being no cohomological relations between lifts of $\overline{\rho}_i$ and $\overline{\rho}_j$ when $i \neq j$. More precisely, we need the following lemma:

LEMMA 4.6. *Let $* = 0$, 1 or 2.*

(1) *If $i \neq j$ then $H^*\left(G_F, \mathrm{Hom}(\overline{\rho}_i, \overline{\rho}_j)\right) = (0)$ if $i \neq j$. Consequently, we have*

$$H^*\left(G_F, \mathrm{Hom}(\overline{\rho}, \overline{\rho})\right) \cong H^*\left(G_F, \mathrm{Hom}(\overline{\rho}_1, \overline{\rho}_1)\right) \oplus \cdots \oplus H^*\left(G_F, \mathrm{Hom}(\overline{\rho}_n, \overline{\rho}_n)\right).$$

(2) *Let A be an Artinian CNL algebra, and let $\rho_i : G_F \longrightarrow GL_{d_i}(A)$, $\rho_j : G_F \longrightarrow GL_{d_j}(A)$ be lifts of $\overline{\rho}_i, \overline{\rho}_j$, $i \neq j$. Then,*

$$H^*(G_F, \mathrm{Hom}(\rho_i, \rho_j)) = (0).$$

Proof. The first part follows easily from the triviality of relevant Hom groups (by assumption 4.1), local duality and the local Euler characteristic formula.

For the second part, let J be an ideal of A with $\mathfrak{m}_A J = (0)$. Then,

$$0 \longrightarrow \mathrm{Hom}(\rho_i, \rho_j) \otimes J \longrightarrow \mathrm{Hom}(\rho_i, \rho_j) \longrightarrow \mathrm{Hom}(\rho_i \bmod J, \rho_j \bmod J) \longrightarrow 0$$

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is an exact sequence of G_F-modules. Induction along with the first part then completes the proof. □

Proof of Proposition 4.5 We can take R to be Artinian. Let $\mathfrak{m}$ be its maximal ideal, and let $J \neq (0)$ be an ideal of R killed by $\mathfrak{m}$. Suppose that

$$\rho \pmod{J} = \rho_1' \oplus \cdots \oplus \rho_n'$$

with $\rho_i' : G_F \longrightarrow GL_{d_i}(R/J)$ lifting $\overline{\rho}_i$. The obstruction to lifting ρ_i' to a representation $G_F \longrightarrow GL_{d_i}(R)$ is a cohomology class

$$c_i \in H^2(G_F, \mathrm{Hom}(\overline{\rho}_i \otimes J, \overline{\rho}_i \otimes J)) = H^2(G_F, \mathrm{Hom}(\overline{\rho}_i, \overline{\rho}_i)) \otimes J.$$

Since $\rho \pmod{J}$ lifts to R, $c_1 + \ldots + c_n$ vanishes in $H^2(G_F, \mathrm{Hom}(\overline{\rho}, \overline{\rho})) \otimes J$. Hence, $c_1, \ldots, c_n$ are trivial by the first part of Lemma 4.6.

We can therefore lift each $\rho_i' : G_F \longrightarrow GL_{d_i}(R/J)$ to $\widetilde{\rho}_i : G_F \longrightarrow GL_{d_i}(R)$. If we set $\widetilde{\rho} := \widetilde{\rho}_1 \oplus \cdots \oplus \widetilde{\rho}_n$, then $\rho = (I + \xi)\widetilde{\rho}$ with $\xi \in H^1(G_F, \mathrm{Hom}(\overline{\rho} \otimes J, \overline{\rho} \otimes J))$. By the first part of Lemma 4.6, we see that $\xi = \xi_1 + \ldots + \xi_n$ with $\xi_i \in H^1(G_F, \mathrm{Hom}(\overline{\rho}_i \otimes J, \overline{\rho}_i \otimes J))$. The required decomposition for ρ follows. The uniqueness part follows from the second part of Lemma 4.6. □

4.2. Tamely ramified representations. We now consider the problem of constructing a well-behaved deformation condition when the residual representation is tamely ramified. Our objective is to study liftings obtained by specifying a Jordan–Holder decomposition for a generator of tame inertia. The Jordan–Holder decomposition together with Frobenius action on the tame generator allow us to study tamely ramified liftings algebraically and produce the required well behaved deformation condition.

Throughout this subsection, F is a fixed finite extension of $\mathbb{Q}_p$ with residue field of order q. We denote by F^{nr} and F^{tr} the maximal unramified and the maximal tamely ramified extensions of F, and fix

- a topological generator τ of $\mathrm{Gal}(F^{\mathrm{tr}}/F^{\mathrm{nr}})$,
- a lift σ of Frobenius to $\mathrm{Gal}(F^{\mathrm{tr}}/F)$.

The letter T denotes a fixed indeterminate. For a tamely ramified representation $\rho : G_F \longrightarrow GL_n(R)$, we shall view the underlying module $V(\rho)$ as an $R[T]$-module where T acts via τ. (We shall freely identify tamely ramified representations with representations of $\mathrm{Gal}(F^{\mathrm{tr}}/F)$.) Note that the action of σ provides added structure.

To describe this further, we first fix some notation:

- $\phi_q : R[T] \longrightarrow R[T]$ is the injective homomorphism which sends T to T^q (and is the identity on R).
- If M is an $R[T]$-module, then $\phi_q^* M$ is the $R[T]$-module with underlying set M and action twisted by ϕ_q, that is, $(f(T), m) \longrightarrow f(T^q)m$ for all $f(T) \in R[T]$.

Then, with notation as before, specifying the action of σ on $V(\rho)$ is equivalent to specifying an isomorphism $V(\rho) \longrightarrow \phi_q^* V(\rho)$ of $R[T]$-modules. Conversely, these determine the representation completely.

We fix a tamely ramified representation $\overline{\rho} : G_F \longrightarrow GL_n(k)$ throughout this subsection and let (a_{ij}) be the (upper triangular) Jordan normal form of $\overline{\rho}(\tau)$ (so $a_{ij} = 0$ if $i < j$ or $i > j + 1$, and $a_{i,i+1}$ is 0 or 1). We define the $n \times n$ matrix $J(\overline{\rho})$ by

$$J(\overline{\rho}) := (\widehat{a_{ij}}) \quad \text{where } \widehat{a_{ij}} \text{ is the Teichmüller lift of } a_{ij}.$$

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Finally, let $\mathcal{D}_{J(\overline{\rho})}$ be the full subcategory of $\mathrm{Rep}_n(G_F; k)$ consisting of objects (A, ρ) with $\rho : G_F \longrightarrow GL_n(A)$ tamely ramified and $\rho(\tau) \sim J(\overline{\rho})$. We then have the following:

PROPOSITION 4.7. *$\mathcal{D}_{J(\overline{\rho})}$ is a well-behaved deformation condition for $\overline{\rho}$.*

We'd like to study deformations (R, ρ) in $\mathcal{D}_{J(\overline{\rho})}$ using the linear algebra data '$R[T]$-module with added structure', and for that we need a convenient description of $J(\overline{\rho})$ in terms of $R[T]$-modules.

Recall that k is a finite of characteristic $\ell \neq p$. We denote by $k_{(q)}$ the orbits of the action $\alpha \longrightarrow \alpha^q$ on the set of elements in $k^\times$ which have order prime to q. For $\alpha \in k^\times$ with order prime to q we define the polynomial

$$P_\alpha(T) := \left(T - \widehat{\alpha}\right)\left(T - \widehat{\alpha}^q\right) \cdots \left(T - \widehat{\alpha}^{q^d}\right)$$

where d is the smallest non-negative integer with $\alpha^{q^{d+1}} = \alpha$. As usual, $\widehat{\alpha} \in W$ denotes the Teichmüller lift of $\alpha \in k$. Equivalently, P_α is the polynomial whose roots are the Teichmüller lifts of elements in the orbit of α. Finally, if $\mathbf{x} \in k_{(q)}$ is the orbit of α then $P_{\mathbf{x}} := P_\alpha$.

DEFINITION 4.8.

(1) A type function $\boldsymbol{t}$ is a map $\boldsymbol{t} : k_{(q)} \times \mathbb{N} \longrightarrow \mathbb{Z}$ such that
  - $\boldsymbol{t}(\mathbf{x}, m) \geq \boldsymbol{t}(\mathbf{x}, m + 1)$ for all $\mathbf{x} \in k_{(q)}$, $m \in \mathbb{N}$, and
  - $\boldsymbol{t}(\mathbf{x}, m) = 0$ for almost all $\mathbf{x}$, m.

(2) Let R be a CNL W-algebra, and let $\boldsymbol{t}$ be a type function. The *standard $R[T]$ module of type $\boldsymbol{t}$*, denoted by $J(R, \boldsymbol{t})$, is

$$\bigoplus_{\mathbf{x} \in k_{(q)}} \left(\frac{R[T]}{\left(P_{\mathbf{x}}^{\boldsymbol{t}(\mathbf{x},1)}\right)} \oplus \frac{R[T]}{\left(P_{\mathbf{x}}^{\boldsymbol{t}(\mathbf{x},2)}\right)} \oplus \cdots \right).$$

An $R[T]$ module M is said to be of type $\boldsymbol{t}$ if M is isomorphic to $J(R, \boldsymbol{t})$. A tamely ramified representation $\rho : G_F \longrightarrow GL_n(R)$ is said to be of type $\boldsymbol{t}$ if the underlying module $V(\rho)$ is of type $\boldsymbol{t}$.

We make the following observation. Let $\overline{\rho} : G_F \longrightarrow GL_n(k)$ be our given tamely ramified representation. Because $\sigma\tau\sigma^{-1} = \tau^q$, the uniqueness of Jordan normal form implies that $V(\overline{\rho})$ is a $k[T]$-module of type $\boldsymbol{t}$ for some type function $\boldsymbol{t}$. Fix one such type function $\boldsymbol{t}$. Then, (A, ρ) is in $\mathcal{D}_{J(\overline{\rho})}$ if and only if ρ is of type $\boldsymbol{t}$.

We now establish some results that will be needed in the proof of our key proposition 4.7.

LEMMA 4.9. *Let $\alpha, \beta \in k^\times$ have orders prime to q and let $f : R \longrightarrow S$ be a surjective homomorphism of Artinian CNL algebras. Given $m, n \geq 1$ and $\phi \in \mathrm{Hom}_{S[T]}\left(S[T]/(P_\alpha^m), S[T]/(P_\beta^n)\right)$, there exists $\widetilde{\phi} \in \mathrm{Hom}_{R[T]}\left(R[T]/(P_\alpha^m), R[T]/(P_\beta^n)\right)$ such that the diagram*

$$\begin{array}{ccc} R[T]/(P_\alpha^m) & \xrightarrow{\widetilde{\phi}} & R[T]/(P_\beta^n) \\ \downarrow & & \downarrow \\ S[T]/(P_\alpha^m) & \xrightarrow{\phi} & S[T]/(P_\beta^n) \end{array}$$

commutes.

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Proof. The lemma holds trivially if $\alpha \neq \beta^{q^j}$ for any $j \geq 0$ because

$$\mathrm{Hom}_{R[T]}\left(R[T]/(P_\alpha^m), R[T]/(P_\beta^n)\right) = (0)$$

in this case.

Suppose now that $\alpha = \beta$. To give an $S[T]$-module homomorphism $\phi : S[T]/(P_\alpha^m) \longrightarrow S[T]/(P_\alpha^n)$ is equivalent to finding a $g(T) \in S[T]$ such that $P_\alpha^m g(T) \in (P_\alpha^n)$ (and $\phi(1) = g(T) \pmod{P_\alpha^n}$). If $m \geq n$, take $\widetilde{g}(T) \in R[T]$ to be a lift of $g(T)$, and define

$$\widetilde{\phi} : R[T]/(P_\alpha^m) \longrightarrow R[T]/(P_\alpha^n)$$

by setting $\widetilde{\phi}(1) = \widetilde{g}(T) \pmod{P_\alpha^n}$. If $m < n$, we have $g(T) = P_\alpha^{n-m}h(T)$ for some $h(T) \in S[T]$. In this case, define

$$\widetilde{\phi}(1) := P_\alpha^{n-m}\widetilde{h}(T) \pmod{P_\alpha^n}$$

where $\widetilde{h}(T) \in R[T]$ is a lift of $h(T)$. □

Proposition 4.10. *Let R be an Artinian CNL algebra, and let I be an ideal of R. If M, N are $R[T]$-modules of type $\boldsymbol{t}_M, \boldsymbol{t}_N$, respectively, then any $R[T]$-module homomorphism $M/IM \longrightarrow N/IN$ lifts to a homomorphism $M \longrightarrow N$.*

Proof. Fix isomorphisms

$$\theta_M : M \longrightarrow \bigoplus \frac{R[T]}{P_\alpha^{\boldsymbol{t}_M(\alpha,i)}}, \; \theta_N : N \longrightarrow \bigoplus \frac{R[T]}{P_\alpha^{\boldsymbol{t}_N(\alpha,i)}},$$

and let $\overline{\theta}_M, \overline{\theta}_N$ be their reductions modulo I. Given a homomorphism of $R[T]$-modules $\overline{\phi} : M/IM \longrightarrow N/IN$, we can apply Lemma 4.9 to find a lift

$$\psi : \bigoplus \frac{R[T]}{P_\alpha^{\boldsymbol{t}_M(\alpha,i)}} \longrightarrow \bigoplus \frac{R[T]}{P_\alpha^{\boldsymbol{t}_N(\alpha,i)}}$$

of $\overline{\theta}_N\overline{\phi}\overline{\theta}_M^{-1}$. If we now take $\phi : M \longrightarrow N$ to be $\theta_N^{-1}\psi\theta_M$, then $\phi \pmod{I} = \overline{\phi}$. □

Proposition 4.11. *Let R be a CNL W-algebra. Let $\phi_q : R[T] \longrightarrow R[T]$ be the injective homomorphism sending T to T^q. Then, ϕ_q induces an isomorphism*

$$\frac{R[T]}{P_\alpha^n} \longrightarrow \frac{R[T]}{P_\alpha^n}$$

of R algebras for any $\alpha \in k^\times$ of order coprime to q, $n \geq 1$.

Consequently, if M is an $R[T]$-module of type $\boldsymbol{t}$ then ϕ_q^*M is also of type $\boldsymbol{t}$.

Proof. First, suppose that R is Artinian. Suppose, we have a polynomial $f(T) \in R[T]$ with

$$f(T^q) = P_\alpha(T)^n g(T)$$

for some $g(T) \in R[T]$. Then,

$$f(\widehat{\alpha}) = f(\widehat{\alpha}^q) = \cdots = 0.$$

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Since $\widehat{\alpha}^{q^i} - \widehat{\alpha}^{q^j}$ is a unit if $0 \leq i < j < d_\alpha$, we have $f(T) = P_\alpha(T)h(T)$ for some $h(T) \in R[T]$. Now,

$$\begin{aligned} P_\alpha(T^q) &= \prod_{i=0}^{d_\alpha-1} \left(T^q - \widehat{\alpha}^{q^i q}\right) \\ &= P_\alpha(T) \prod_{\substack{\zeta^q=1 \\ \zeta\neq 1}} \prod_{i=0}^{d_\alpha-1} \left(T - \zeta\widehat{\alpha}^{q^i}\right) \\ &= P_\alpha(T) \prod_{\substack{\zeta^q=1 \\ \zeta\neq 1}} P_\alpha(\zeta T), \end{aligned}$$

and therefore

$$P_\alpha(T)^{n-1} g(T) = h(T^q) \prod_{\substack{\zeta^q=1 \\ \zeta\neq 1}} P_\alpha(\zeta T).$$

Since $\widehat{\alpha}^{q^j} - \zeta\widehat{\alpha}^{q^i}$ are units, we have

$$h(\widehat{\alpha}) = h(\widehat{\alpha}^{q^2}) = \cdots = 0.$$

We can now conclude (by induction) that ϕ_q induces an injection

$$\frac{R[T]}{P_\alpha^n} \longrightarrow \frac{R[T]}{P_\alpha^n},$$

and therefore induces an isomorphism.

The non-Artinian case follows on taking inverse limits. □

Proof of Proposition 4.7. To show that $\mathcal{D}_{J(\overline{\rho})}$ determines a deformation condition, we need only verify condition (DC2) as (DC0) and (DC1) are obvious. Fix a type function $\boldsymbol{t}$ so that $\overline{\rho}$ is of type $\boldsymbol{t}$. Let

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & C \end{array}$$

be a Cartesian diagram of Artinian local W-algebras with β small, and suppose that we are given an object $(A \times_C B, \rho)$ in $\mathrm{Rep}_n(G_F; k)$ such that the projections $\pi_A\rho$ and $\pi_B\rho$ are also of type $\boldsymbol{t}$. We then need to show that ρ is of type $\boldsymbol{t}$.

Denote by $\rho_{\boldsymbol{t}} : I_F \longrightarrow GL_n(W)$ the tamely ramified representation that sends the fixed tame generator τ to the matrix $J(\overline{\rho})$. If R is a CNL W-algebra then we will continue to use $\rho_{\boldsymbol{t}}$ for the representation that sends τ to $J(\overline{\rho})$ viewed now as a matrix over R via the W-algebra structure; the context will always make clear where $\rho_{\boldsymbol{t}}$ is valued in.

Let (b) be the kernel of β. Then, π_A is small with kernel generated by $(0, b)$. We may then suppose that $\pi_A\rho|_{I_F} = \rho_{\boldsymbol{t}}$, and so $\rho|_{I_F} = (I + (0, b\xi))\rho_{\boldsymbol{t}}$ with ξ a 1-cocycle representing an element of $H^1(I_F, \mathrm{ad}\overline{\rho})$. We need to show that ξ is trivial.

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Now, $\pi_B\rho|_{I_F} = (I + b\xi)\rho_t$, and also $M\pi_B\rho M^{-1}|_{I_F} = \rho_t$ for some $M \in GL_n(B)$. Going down to $C = B/(b)$ and using $\beta\pi_B = \alpha\pi_A$, we obtain $\beta(M)\rho_t\beta(M)^{-1} = \rho_t$, that is, $\beta(M)$ commutes with ρ_t. Using Proposition 4.10, we can find $M' \in GL_n(B)$ such that $M'\rho_t M'^{-1} = \rho_t$ and $M \equiv M' \pmod{b}$. Write $M'^{-1}M = I + bX$ with X an $n \times n$ matrix over k. Then,

$$\rho_t = (I + bX)\pi_B\rho(I - bX)|_{I_F}, \quad \text{that is,} \quad (I - bX)\rho_t(I + bX) = (I + b\xi)\rho_t$$

and hence ξ is trivial.

We now consider smoothness of the deformation condition. Let $R \longrightarrow S$ be a surjective morphism of Artinian local W-algebras, and let $\rho_S : G_F \longrightarrow GL_n(S)$ be a representation of type $\boldsymbol{t}$ lifting $\bar{\rho}$. Conjugating ρ_S by a matrix congruent to the identity modulo the maximal ideal of S, we may suppose that $V(\rho_S)$ is $J(S, \boldsymbol{t})$. The action of σ specifies a morphism

$$\theta_S : J(S, \boldsymbol{t}) \longrightarrow \phi_q^* J(S, \boldsymbol{t})$$

of $S[T]$-modules which can then be lifted, by Proposition 4.10, to

$$\theta_R : J(R, \boldsymbol{t}) \longrightarrow \phi_q^* J(R, \boldsymbol{t}).$$

Hence, $\mathcal{D}_{J(\overline{\rho})}$ is smooth.

Finally, we consider the tangent space of $\mathcal{D}_{J(\overline{\rho})}$. The deformations of $\overline{\rho}$ to $k[\epsilon]/\epsilon^2$ are uniquely determined by $H^1(G_F, \mathrm{ad}\overline{\rho})$. For $\xi \in H^1(G_F, \mathrm{ad}\overline{\rho})$, the lift $(I + \epsilon\xi)\overline{\rho}$ is of type $\boldsymbol{t}$ if and only if the restriction of ξ to inertia is trivial. Thus, the tangent space for $\mathcal{D}_{J(\overline{\rho})}$ is $H^1\left(G_F/I_F, (\mathrm{ad}\overline{\rho})^{I_F}\right)$, and hence $\mathcal{D}_{J(\overline{\rho})}$ is a well behaved deformation condition. $\square$

4.3. Deformations for tensor products. We now consider step II(ii) in the outline plan of the proof of Theorem 4.3. Thus, our starting point will be a residual representation which is the tensor product of two smaller representations. We then want to determine if taking tensor products of classes of liftings of the two components gives a deformation condition for the bigger residual representation.

Let F is a finite extension of $\mathbb{Q}_p$ and fix, for the rest of this subsection, a residual representation $\overline{\theta} : G_F \longrightarrow GL_n(k)$ such that

- $\overline{\theta}$ is absolutely irreducible,
- $\ell \nmid n$, and
- $\overline{\theta}$ is not equivalent to its Tate twist $\overline{\theta}(1)$.

We set s to be the smallest positive integer such that $\overline{\theta}(s) \sim \overline{\theta}$. (So, $s \geq 2$ by our assumption.) We then have the following.

THEOREM 4.12. *Suppose that* $1 \leq m \leq s - 2$, *and let* $\overline{\rho} : G_F \longrightarrow GL_{mn}(k)$ *be a representation such that* $\overline{\rho}^{ss} \cong \overline{\theta}(a_1) \oplus \cdots \oplus \overline{\theta}(a_m)$ *for some integers* $a_1, \ldots, a_m$. *There is then a deformation condition* $\mathcal{E}$ *for* $\overline{\rho}$ *with the following properties:*

- *If* $(A, \rho_A) \in \mathcal{E}$, *then* $\det \rho_A$ *restricted to the inertia subgroup of* G_F *is the Teichmüller lift of* $\det \overline{\rho}$;
- $\mathcal{E}$ *is a smooth deformation condition;*
- *The dimension of the tangent space for* $\mathcal{E}$ *is equal to* $\dim H^0(G_F, ad\overline{\rho})$.

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We make the following definition for convenience: A representation $r : G_F \longrightarrow GL_d(k)$ is said to be *s-small* if

$$r^{ss} \cong k(i_1) \oplus \cdots \oplus k(i_d)$$

with $0 \leq i_1, \ldots, i_m \leq s-2$.

We shall make use of the natural isomorphism between $\mathrm{Hom}(V, W)$ and $V^\vee \otimes W$ for k-vector spaces V, W in what follows without any further qualification. Also, the identity map on U naturally identifies $\mathrm{Hom}(V, W)$ as a subspace of $\mathrm{Hom}(V \otimes U, W \otimes U)$. If $\ell \nmid \dim U$, then $\mathrm{Hom}(V \otimes U, W \otimes U)$ is naturally identified with $\mathrm{Hom}(V, W) \oplus \mathrm{Hom}(V, W) \otimes \mathrm{ad}^0 U$ where $\mathrm{ad}^0 U$ is the vector space of trace zero endomorphisms of U.

LEMMA 4.13.

(*a*) *If* $|j| \leq s-2$ *then* $H^i\left(G_F, ad^0\overline{\theta}(j)\right) = (0)$ *for all* $i \geq 0$.

(*b*) *If* $0 \leq a, b \leq s-2$ *then we have natural isomorphisms*

$$H^i\left(G_F, \mathrm{Hom}\left(\overline{\theta}(a), \overline{\theta}(b)\right)\right) \cong H^i\left(G_F, k(b-a)\right)$$

for all $i \geq 0$.

(*c*) *If* $\overline{\rho}_1, \overline{\rho}_2$ *are two s-small representations then the natural inclusion* $\mathrm{Hom}\left(\overline{\rho}_1, \overline{\rho}_2\right) \hookrightarrow \mathrm{Hom}\left(\overline{\rho}_1 \otimes \overline{\theta}, \overline{\rho}_2 \otimes \overline{\theta}\right)$ *induces isomorphisms*

$$H^i\left(G_F, \mathrm{Hom}\left(\overline{\rho}_1 \otimes \overline{\theta}, \overline{\rho}_2 \otimes \overline{\theta}\right)\right) \cong H^i\left(G_F, \mathrm{Hom}\left(\overline{\rho}_1, \overline{\rho}_2\right)\right)$$

for all $i \geq 0$.

Proof. For part (a), one checks that the statement holds for $|j| \leq s-1$ when $i = 0$. The full result then follows after an application of local Tate duality and the Euler characteristic formula. Part (b) of the lemma is then immediate from part (a) via the natural identifications

$$\mathrm{Hom}\left(\overline{\theta}(a), \overline{\theta}(b)\right) \cong \mathrm{Hom}\left(\overline{\theta}, \overline{\theta}\right)(b-a) \cong \mathrm{ad}^0\overline{\theta}(b-a) \oplus k(b-a).$$

For part (c), we have

$$\mathrm{Hom}\left(\overline{\rho}_1 \otimes \overline{\theta}, \overline{\rho}_2 \otimes \overline{\theta}\right) \cong \mathrm{Hom}(\overline{\rho}_1, \overline{\rho}_2) \oplus \mathrm{Hom}(\overline{\rho}_1, \overline{\rho}_2) \otimes \mathrm{ad}^0\overline{\theta},$$

and $H^i\left(G_F, \mathrm{Hom}(\overline{\rho}_1, \overline{\rho}_2) \otimes \mathrm{ad}^0\overline{\theta}\right)$ is trivial by part a. □

Let $\theta : G_F \longrightarrow GL_n(W)$ be the unique (up to equivalence) lifting of $\overline{\theta}$ with determinant the Teichmüller lift of $\det\overline{\theta}$. (The existence and uniqueness of such a representation is an immediate consequence of the above lemma.) Fix also an s-small representation $\overline{\rho}_0 : G_F \longrightarrow GL_m(k)$ and a deformation condition $\mathcal{D}$ for $\overline{\rho}_0$.

Define $\mathcal{D} \otimes \theta$ to be the full subcategory of $\mathrm{Rep}_{mn}(G_F)$ whose objects are pairs (A, ρ_A) with $\rho_A \sim \rho_0 \otimes \theta$ for some $(A, \rho_0) \in \mathcal{D}$.

PROPOSITION 4.14. *With notation as above,* $\mathcal{D} \otimes \theta$ *is a deformation condition for* $\overline{\rho}_0 \otimes \overline{\theta}$*. The tangent space for* $\mathcal{D} \otimes \theta$ *is naturally identified with* $\mathcal{D}$.

Proof. We first show that $\mathcal{D} \otimes \theta$ is a deformation condition, and for that we need only verify that a lifting $\rho : G_F \longrightarrow A \times_B C$ is in $\mathcal{D} \otimes \theta$ if the projections of ρ to A and C are in $\mathcal{D} \otimes \theta$.

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Claim 1: If $\rho : G_F \longrightarrow GL_{mn}(A)$ is a lifting of $\overline{\rho}_0 \otimes \overline{\theta}$, then ρ is strictly equivalent to $\rho_0 \otimes \theta$ for some lifting $\rho_0 : G_F \longrightarrow GL_m(A)$ of $\overline{\rho}_0$.

Proof of claim: We use induction on length for A Artinian. Let J be an ideal of A killed by the maximal ideal $\mathfrak{m}$ of A. Then, ρ mod J is strictly equivalent to $\rho_1 \otimes \theta$ for some lift to A/J of $\overline{\rho}_0$. The obstruction to lifting ρ_1 to $GL_m(A)$ lies in $H^2(G_F, \mathrm{ad}\overline{\rho}_0) \otimes J$, and the obstruction vanishes by Lemma 4.13, part *c*. We can therefore find a lifting $\rho_0' : G_F \longrightarrow GL_m(A)$ of $\overline{\rho}_0$ such that ρ mod $J = \rho_0' \otimes \theta$ mod J. It follows that $\rho = \rho_0' \otimes \theta\,(1 + \xi)$ for some $\xi \in H^1(G_F, \mathrm{ad}\overline{\rho}_0 \otimes \overline{\theta}) \otimes J$, and the claim follows from Lemma 4.13, part *c*.

Claim 2: If $\rho_1, \rho_2 : G_F \longrightarrow GL_m(A)$ are two liftings of $\overline{\rho}_0$ and $\rho_1 \otimes \theta \sim_s \rho_2 \otimes \theta$, then $\rho_1 \sim_s \rho_2$.

Proof of claim: With A, J as in the proof of claim 1 and using induction on length, one deduces that assuming ρ_1 mod $J = \rho_2$ mod J, we have $\rho_1 \otimes \theta = \rho_2 \otimes \theta(1 + \xi)$ with $\xi \in H^1(G_F, \mathrm{ad}\overline{\rho}_0 \otimes \overline{\theta}) \otimes J$. Lemma 4.13 again completes the proof.

Now, let $(A \times_B C, \rho)$ be a lifting of $\overline{\rho}_0 \otimes \overline{\theta}$. We may assume by claim 1 that $\rho = \rho_0 \otimes \theta$ for ρ_0 a lifting of $\overline{\rho}_0$. If the projections of ρ to A and C are in $\mathcal{D} \otimes \theta$, then claim 2 implies that the projections of ρ_0 to A and C are in $\mathcal{D}$. Hence, $(A \times_B C, \rho_0) \in \mathcal{D}$, thus proving the theorem.

The statement about tangent spaces is immediate from Lemma 4.13. □

Proof of Theorem 4.12. Twisting $\overline{\rho}$ by a power of the cyclotomic character, we may assume that $0 \leq a_1, \ldots, a_m \leq s - 2$. It is then easy to see, using Lemma 4.13, that $\overline{\rho} \sim \overline{\rho}_0 \otimes \overline{\theta}$ where $\overline{\rho}_0$ is a s-small representation with $\overline{\rho}_0^{\mathrm{ss}} \cong k(a_1) \oplus \cdots \oplus k(a_m)$. Now, let $\mathcal{E}_0$ be the deformation condition for the tamely ramified representation $\overline{\rho}_0$ constructed in Section 4.2, and take $\mathcal{E}$ to be the deformation condition $\mathcal{E}_0 \otimes \theta$. All claims then follow from Proposition 4.14 and properties of $\mathcal{E}_0$. □

4.4. Induced representations. We now consider the final part in the analysis of blocks that make up a residual representation given in the outline plan for proof of Theorem 4.3. Thus, we need to consider when representations induced from liftings of a given residual representation determine a deformation condition for the induced residual representation.

The set up for this subsection is as follows. Let $F \subsetneq L$ be fixed finite extensions of $\mathbb{Q}_p$, and set $n = [L : F]$. We assume we are given a representation $\overline{\rho} : G_F \longrightarrow GL_{mn}(k)$, which is induced from $\overline{\theta} : G_L \longrightarrow GL_m(k)$. We also fix, throughout this subsection, a coset decomposition

$$G_F = g_1 G_L \sqcup \cdots \sqcup g_n G_L$$

with $g_1 = e$.

Now, $V(\overline{\rho})$ has a G_L invariant vector subspace M, such that

- $V(\overline{\theta}) \cong M$ as G_L-modules, and
- $V(\overline{\rho}) = g_1 M \oplus \cdots \oplus g_n M$.

The subspace $N := g_2 M + \cdots g_n M$ is G_L invariant and $V = M \oplus N$ as G_L-modules. Let $\overline{\vartheta} : G_L \longrightarrow GL_{(n-1)m}(k)$ be a representation given by (some fixed choice of basis of) N. Assume that:

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- $\overline{\rho}|_{G_L} = \overline{\theta} \oplus \overline{\vartheta}$, and
- $\mathrm{Hom}_{G_L}(M, N(r)) = (0)$ for all $r \in \mathbb{Z}$.

Under these assumptions, we have canonical isomorphisms

$$H^i(G_F, \mathrm{ad}\overline{\rho}) \cong H^i(G_L, \mathrm{ad}\overline{\theta})$$

by Shapiro's lemma. Furthermore, Proposition 4.5 shows that any lift $\rho : G_F \longrightarrow GL_{mn}(R)$ of $\overline{\rho}$ restricted to G_L is strictly equivalent to $\theta \oplus \vartheta$, where θ, ϑ are lifts of $\overline{\theta}$ and $\overline{\vartheta}$.

LEMMA 4.15. *Let A be an Artinian CNL W-algebra, and let $\rho : G_F \longrightarrow GL_{mn}(A)$ be a lift of $\overline{\rho}$. If*

$$\rho|_{G_L} = \theta \oplus \vartheta$$

with θ, ϑ lifts of $\overline{\theta}, \overline{\vartheta}$, then ρ is equivalent to $\mathrm{Ind}\,\theta$.

Proof. We fix a basis for $V(\overline{\rho})$ as follows: View $V(\overline{\theta})$ as a subspace of $V(\overline{\rho})$ via $V(\overline{\rho}) = V(\overline{\theta}) \oplus V(\overline{\vartheta})$, and take the basis $\{g_i\overline{\mathrm{e}}_j \mid 1 \le i \le n,\ 1 \le j \le m\}$ with $\{\overline{\mathrm{e}}_1, \ldots, \overline{\mathrm{e}}_m\}$ a basis of $V(\overline{\theta})$. Now, $V(\rho) = V(\theta) \oplus V(\vartheta)$ as $A[G_L]$-modules, and so we can pick a basis $\{\mathrm{e}_1, \ldots, \mathrm{e}_m\}$ of $V(\theta)$ such that e_i is a lift of $\overline{\mathrm{e}}_i$. It is now clear that

$$V(\rho) = g_1 V(\theta) + \cdots + g_n V(\theta) + \mathfrak{m}_A V(\rho),$$

and therefore, by Nakayama's lemma, one sees that

$$V(\rho) = g_1 V(\theta) \oplus \cdots \oplus g_n V(\theta).$$

This completes the proof (using, for instance, Proposition 10.5 of **[5]**). □

Now, let $\mathcal{F}$ be a deformation condition for θ, and denote by $\mathrm{Ind}\mathcal{F}$ the full subcategory of $\mathrm{Rep}_{mn}(G_F; k)$ whose objects are $(A, \rho) \in \mathrm{Rep}_{mn}(G_F; k)$ with $V(\rho) \cong \mathrm{Ind}\,V(\theta)$ for some $(A, \theta) \in \mathcal{F}$.

PROPOSITION 4.16. *$\mathrm{Ind}\mathcal{F}$ is a deformation condition for $\overline{\rho}$. If $\mathcal{F}$ is well-behaved then so is $\mathrm{Ind}\mathcal{F}$.*

Proof. To show that $\mathrm{Ind}\mathcal{F}$ is a deformation condition, we need only check (DC2). Suppose, given $\alpha : A \longrightarrow C$, $\beta : B \longrightarrow C$, with β small, and a lift

$$\rho : G_F \longrightarrow GL_{mn}(A \times_C B)$$

of $\overline{\rho}$ with $(A, \alpha\rho)$, $(B, \beta\rho)$ in $\mathrm{Ind}\mathcal{F}\mathrm{Rep}_{mn}$. Conjugating by an element of $GL_{mn}(A \times_C B)$, we can take ρ to be a lift of $\overline{\rho}$, and that $\rho|_{G_L} = \theta \oplus \vartheta$ where θ, ϑ are lifts of $\overline{\theta}$ and $\overline{\vartheta}$. Since $\rho \sim \mathrm{Ind}\,\theta$ by Lemma 4.15, we need to verify that $(A \times_C B, \theta)$ is in $\mathcal{F}\mathrm{Rep}_m$.

Let (A, θ') be an object of $\mathcal{F}\mathrm{Rep}_m$ with $\mathrm{Ind}\theta' \sim \alpha\rho$. By Proposition 4.5, the composite

$$V(\theta') \hookrightarrow V(\alpha\rho) \cong V(\alpha\theta) \oplus V(\alpha\vartheta) \longrightarrow V(\alpha\theta)$$

is an isomorphism of $A[G_L]$-modules. Hence, $(A, \alpha\theta)$ is an object of $\mathcal{F}\mathrm{Rep}_m$. Similarly, $(B, \beta\theta)$ is an object of $\mathcal{F}\mathrm{Rep}_m$, and hence $(A \times_C B, \theta)$ is in $\mathcal{F}\mathrm{Rep}_m$.

Clearly, $\mathrm{Ind}\mathcal{F}$ is smooth if $\mathcal{F}$ is, and the tangent space for $\mathrm{Ind}\mathcal{F}$ is the image of $T\mathcal{D}$ under the Shapiro isomorphism. The (uni)versal deformation ring for $\mathrm{Ind}\mathcal{F}$ is a

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power series ring over W, the restriction of the determinant of the (uni)versal Ind$\mathcal{F}$ deformation is the Teichmüller lift of $\det\overline{\rho}$. The second statement of the proposition now follows. □

4.5. Proof of Theorem 4.3. We will now complete the proof of Theorem 4.3 by decomposing a given residual representation suitably so that the results we have discussed apply.

Recall we are assuming that our representation $\overline{\rho}: G_F \longrightarrow GL_N(k)$ has all irreducible components occurring in the semi-simplification of $\overline{\rho}$ absolutely irreducible, and that $[F(\zeta_\ell):F] \geq 3N$ if $p \leq N$ and $\overline{\rho}$ is wildly ramified. Our task is to construct a well-behaved deformation condition for $\overline{\rho}$. Throughout this subsection, we fix absolutely irreducible continuous representations

$$\overline{\theta}_i : G_F \longrightarrow GL_{n_i}(k), \ \ i = 1, \ldots, n$$

occurring in the semi-simplification of $\overline{\rho}$, such that

- if $i \neq j$, then $\overline{\theta}_i$ and $\overline{\theta}_j(r)$ are not equivalent for any $r \in \mathbb{Z}$;
- $\overline{\rho}^{ss}$ is a direct sum of $\overline{\theta}_i$, $i = 1, \ldots, n$, and Tate twists of $\overline{\theta}_i$'s.

LEMMA 4.17. *Let V be the underlying $k[G_F]$-module for $\overline{\rho}$. Then V has a submodule isomorphic to $V(\overline{\theta}_i)$ for each i. If V_i denotes the maximal submodule of V whose composition series consists only of $\overline{\theta}_i$ and Tate twists of $\overline{\theta}_i$, then $V = V_1 \oplus \cdots \oplus V_n$. Furthermore, for any $r \in \mathbb{Z}$, $i \neq j$, we have*

$$\mathrm{Hom}_{G_F}\left(V_i, V_j(r)\right) = (0).$$

Proof. We may suppose that V has a submodule U isomorphic to $\overline{\theta}_1$. Using induction, we get an exact sequence of $k[G_F]$ modules

$$0 \longrightarrow U \longrightarrow V \longrightarrow M_1 \oplus \cdots \oplus M_n \longrightarrow 0,$$

where each M_i composition series consisting only of $\overline{\theta}_i$ and Tate twists of $\overline{\theta}_i$. Thus, V corresponds to an element of

$$H^1\left(G_F, \mathrm{Hom}(M_1 \oplus \cdots \oplus M_n, U)\right).$$

By Tate local duality, $H^1(G_F, \mathrm{Hom}(M_i, U))$ is trivial if $i \neq 1$, and the proposition follows. □

By Theorem 4.4 and the above lemma, we can assume that the semi-simplification of $\overline{\rho}$ is a direct sum of Tate twists of a single absolutely irreducible representation $\overline{\theta}$: $G_F \longrightarrow GL_n(k)$. If $\overline{\theta}$ is tamely ramified, we proceed as in Section 4.2, Proposition 4.7.

Now, assume that $\overline{\theta}$ is wildly ramified. We shall deal with the case when $p \leq N$ first. Let s be the smallest positive integer such that $\theta \sim \theta(s)$, and let m be the number irreducible components of $\overline{\rho}^{ss}$ isomorphic to some Tate twist of $\overline{\theta}$. The inequalities $ns \geq 3N$ (obtained by comparing determinants of θ and $\theta(s)$) and $nm \leq N$ imply that $1 \leq m \leq s-2$. The existence of a well-behaved deformation condition then follows from Theorem 4.12.

Finally, assume from here on that $\overline{\theta}$ is wildly ramified and $p > N$. Let $\overline{\rho}^{ss} \cong \overline{\theta}(i_1) \oplus \cdots \oplus \overline{\theta}(i_m)$, and denote by $F(\overline{\rho})$ the extension of F through which $\overline{\rho}$ factorises.

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Since $n < p$, twisting by a character $G_F \longrightarrow k^\times$ if necessary, we can assume that the p-part of the determinant of $\overline{\theta}$ is trivial. A consideration of ramification subgroups shows that we can find an abelian normal, wildly ramified, p-subgroup $Z \lhd \mathrm{Gal}(F(\overline{\rho})/F)$. Our assumption that the determinant has no p-part then shows that $\overline{\theta}|_Z$ is not central.

We now give a characterisation of $\overline{\rho}$ as an induced module. The representation $\overline{\rho}$ when restricted to Z splits as a direct sum of characters. Clearly, if $\overline{\theta}|_Z \sim \chi_1 \oplus \cdots \oplus \chi_d$, then $\overline{\rho}|_Z \sim (\chi_1 \oplus \cdots \oplus \chi_d)^{mn/d}$. We fix one such character χ and set

$$V[\chi] := \{v \in V(\overline{\rho}) \,|\, \overline{\rho}(z)(v) = \chi(z)v \quad \text{for all } z \in Z\}\,.$$

If $g \in \mathrm{Gal}\,(F(\overline{\rho})/F)$, then the character ${}^g\chi$, defined by

$${}^g\chi(z) := \chi(gzg^{-1}),$$

is also a constituent character of $\overline{\theta}|_Z$, and we have $V[{}^g\chi] = gV[\chi]$. Thus, $\mathrm{Gal}\,(F(\overline{\rho})/F)$ acts transitively on the distinct constituent characters of $\overline{\theta}|_Z$ and there are at least two distinct constituent characters. Let L be the finite extension of F inside $F(\overline{\rho})$ cut out by the stabiliser of χ, and fix a coset decomposition

$$G_F = g_1 G_L \sqcup \cdots \sqcup g_n G_L,$$

with $g_1 = e$. Then, $V = g_1 V[\chi] \oplus \cdots \oplus g_n V[\chi]$, and so V is induced from the G_L-module $V[\chi]$. Since χ is a wildly ramified character,

$$\mathrm{Hom}_Z\left(V[{}^g\chi], V[{}^{g'}\chi]\right) = (0),$$

if $gG_L \neq g'G_L$, and so for any $r \in \mathbb{Z}$, we have

$$\mathrm{Hom}_{G_L}\,(V[\chi], (g_2 V[\chi] \oplus \cdots \oplus g_n V[\chi])(r)) = (0).$$

Finally, inductively on N, one can find a well-behaved deformation condition for the representation of G_L arising from $V[\chi]$. Using Theorem 4.16, the induced deformation condition is a well-behaved deformation condition for $\overline{\rho}$.

4.6. Deformations at special unramified primes. We conclude this section with a look at a special class of smooth local deformations, which are of great significance in reducing dimensions of (global) dual Selmer groups.

Let F be a finite extension of $\mathbb{Q}_p$ and let $\overline{\rho} : G_F \longrightarrow GL_n(k)$ be the diagonal representation

$$\overline{\rho} = \begin{pmatrix} \bar{\omega}^{n-1} & & & \\ & \bar{\omega}^{n-2} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

We assume that the order of the mod ℓ cyclotomic character $\bar{\omega}$ is greater than n. Fix a basis $\{\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n\}$ with $\overline{\rho}$ acting on e_i by the character $\bar{\omega}^{n-i}$ and an identification

$$\mathrm{ad}\overline{\rho} \cong \bigoplus_{1\le i,j\le n} \mathrm{Hom}(k\mathrm{e}_j, k\mathrm{e}_i) \cong \bigoplus_{1\le i,j\le n} k(i-j).$$

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LEMMA 4.18. *Any lifting of $\overline{\rho}$ is strictly equivalent to an upper triangular representation.*

Proof. We use Artinian induction. So, let $\rho : G_F \longrightarrow GL_n(A)$ be a lift of $\overline{\rho}$ with A Artinian, and let J be a non-zero ideal of A killed by $\mathfrak{m}_A$. We assume that $\rho_J : G_F \longrightarrow GL_n(A/J)$, the reduction of ρ modulo J, is upper triangular (after conjugating by a matrix that reduces to the identity modulo $\mathfrak{m}_A$ if necessary).

We write B_n for the standard Borel subgroup of GL_n consisting of upper triangular matrices, and let $b\overline{\rho}$ be the subspace of $\mathrm{ad}\overline{\rho}$ consisting of upper triangular matrices. A standard calculation then shows that $H^*(G_F, \mathrm{ad}\overline{\rho}) = H^*(G_F, b\overline{\rho})$ when $* = 0, 1, 2$.

The obstruction to lifting ρ_J to $B_n(A)$ is given by an element of $H^2(G_F, b\overline{\rho}) \otimes J$. The obstruction vanishes because its image in $H^2(G_F, \mathrm{ad}\overline{\rho}) \otimes J$ is trivial (since ρ lifts ρ_J to A). Thus, there is an upper triangular lift $\rho' : G_F \longrightarrow GL_n(A)$ of ρ_J, and we can write

$$\rho = (I + \psi)\rho' \quad \text{with} \quad \psi \in H^1(G_F, \mathrm{ad}\overline{\rho}) \otimes J.$$

Now, deformations of ρ_J to $B_n(A)$ are precisely given by $(I + \xi)\rho'$ where ξ is a cocycle in $H^1(G_F, b\overline{\rho}) \otimes J$. The lemma follows since $H^1(G_F, \mathrm{ad}\overline{\rho}) \otimes J = H^1(G_F, b\overline{\rho}) \otimes J$. □

Let $\mathfrak{B}$ be the full subcategory of $\mathrm{Rep}_n(G_F; k)$ with objects (A, ρ) satisfying $\rho \bmod \mathfrak{m}_A = \overline{\rho}$ and

$$\rho \sim \begin{pmatrix} \omega^{n-1} & * & * \\ & \omega^{n-2} & \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}.$$

Lemma 4.18 then readily implies that $\mathfrak{B}$, which we shall refer to as the *Ramakrishna condition*, is in fact a deformation condition for $\overline{\rho}$. (When $n = 2$, these are the deformation conditions discussed in Section 3 of [**13**].)

PROPOSITION 4.19. *$\mathfrak{B}$ is smooth and its tangent space is*

$$\bigoplus_{i=1}^{n-1} H^1(G_F, \mathrm{Hom}(ke_{i+1}, ke_i)).$$

Proof. Let $\rho : G_F \longrightarrow GL_n(B)$ be a representation, say $\rho = \left(b_{ij}\omega^{j-i}\right)$, where $b_{ij} : G_F \longrightarrow B$ are functions with

$$b_{ij}(\sigma) = \begin{cases} 0, & \text{if } i > j, \\ 1, & \text{if } i = j \end{cases}$$

for any $\sigma \in G_F$. Each $b_{i,i+1} \in H^1(G_F, B(1))$. The calculation in example E4 of [**15**] shows that for a surjection $f : A \longrightarrow B$, the map

$$H^1(G_F, A(1)) \longrightarrow H^1(G_F, B(1))$$

is surjective. If we assume f to be small and identify the $k[G_F]$-module of $n \times n$ matrices with entries from $\ker(f)$ with $\mathrm{ad}\overline{\rho}$, it follows that the obstruction to there being a lift of type $\mathfrak{B}$ of ρ to A is given by an element of

$$H^2\left(G_F, \oplus_{j-i\geq 2}\operatorname{Hom}(k e_j, k e_i)\right).$$

But, this cohomology group vanishes because

$$\dim_k H^0(G_F, k(j-i)) = \dim_k H^2(G_F, k(j-i)) = \dim_k H^0(G_F, k(i-j+1)) = (0)$$

for $j-i\geq 2$ as $\bar{\omega}$ has order greater than n. Consequently

$$H^1\left(G_F, \oplus_{j-i\geq 1}\operatorname{Hom}(k e_j, k e_i)\right) = \oplus_i H^1\left(G_F, \operatorname{Hom}(k e_{i+1}, k e_i)\right),$$

from which the statement about the tangent space follows. □

5. Constructing global deformation conditions with trivial dual Selmer group. In this section, we show how to transform a given global deformation condition in such a way that the dual Selmer group decreases in size while at the same time retaining smoothness properties of local components (of the original deformation condition).

We begin by fixing a number field F and a finite field k of characteristic ℓ. We also fix, throughout this section, a representation $\overline{\rho}: G_F \longrightarrow GL_N(k)$ and a character $\chi: G_F \longrightarrow W^\times$ lifting $\det\overline{\rho}$ (so $\chi \pmod{\ell} = \det\overline{\rho}$).

We shall say that a global deformation condition $\mathcal{D}$ with determinant χ for $\overline{\rho}$ *satisfies the tangent space inequality* if the inequality

$$\sum_{v\in\Sigma(\mathcal{D})}\dim T\mathcal{D}_v \geq (N-2) + \sum_{v\in\Sigma(\mathcal{D})}\dim H^0\left(G_v, \operatorname{ad}^0\overline{\rho}\right) \tag{5.1}$$

holds. Recall that $\Sigma(\mathcal{D})$ is the finite set consisting of those primes v of F where $\mathcal{D}_v$ is not unramified, primes lying above ℓ and ∞, and primes where $\overline{\rho}$ and χ are ramified. By Wiles' formula 2.1, $\mathcal{D}$ as satisfies the tangent space inequality if

$$\dim H^1_{\{T\mathcal{D}_v\}}\left(F, \operatorname{ad}^0\overline{\rho}\right) - \dim H^1_{\{T\mathcal{D}_v^\perp\}}\left(F, \operatorname{ad}^0\overline{\rho}(1)\right) \geq N-2.$$

DEFINITION 5.1. The residual representation $\overline{\rho}: G_F \longrightarrow GL_N(k)$ is said to be a *big representation* if the following properties hold:

(R1) $\operatorname{ad}^0\overline{\rho}$ is absolutely irreducible and

$$H^1\left(\operatorname{Gal}(F(\operatorname{ad}^0\overline{\rho})/F), \operatorname{ad}^0\overline{\rho}\right) = H^1\left(\operatorname{Gal}(F(\operatorname{ad}^0\overline{\rho}(1))/F), \operatorname{ad}^0\overline{\rho}(1)\right) = (0).$$

(R2) There is a non-archimedean prime w_0 of F with $w_0 \nmid \ell$, such that

$$\overline{\rho}|_{w_0} \sim \begin{pmatrix} \bar{\omega}^{N-1} & & & \\ & \bar{\omega}^{N-2} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \otimes \overline{\eta},$$

where $\overline{\eta}$ is an unramified character, and the mod ℓ cyclotomic character $\bar{\omega}$ has order strictly greater than N.

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Note that if $\overline{\rho}$ is big, then R2 implies that F does not contain all ℓth roots of unity, that $\mathrm{ad}^0\overline{\rho}$ and $\mathrm{ad}^0\overline{\rho}(1)$ are inequivalent, and that $\ell > N$. Also, if $\overline{\rho}$ is big and k' is a finite extension of k, then the extension of scalars of $\overline{\rho}$ to $GL_N(k')$ is again a big representation. Further examples of big representations are supplied by the following proposition:

PROPOSITION 5.2.

(*i*) *Let F be a number field, and fix an integer $N \geq 2$. There is a constant C such that if k is a finite field of characteristic $\ell > C$, then any representation $\overline{\rho} : G_F \longrightarrow GL_N(k)$ with Im$\overline{\rho}$ containing $SL_N(k)$ is a big representation.*

(*ii*) *Let $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_3(k)$ be a representation with Im$\overline{\rho}$ containing $SL_3(k)$. Assume that ℓ, the characteristic of k, is at least 7. Furthermore, assume that if $\ell = 7$ then the fixed field of ad$^0\overline{\rho}$ does not contain* $\cos(2\pi/7)$. *Then $\overline{\rho}$ is a big representation.*

Proof. We fix the following notation first:

- $\widetilde{\rho} : G_F \longrightarrow PGL_N(k)$ is the projectivization of $\overline{\rho}$ and $\widetilde{\chi} : G_F \longrightarrow k^\times/k^{\times N}$ is the determinant of $\widetilde{\rho}$.
- $F(\widetilde{\chi}, \bar{\omega})$ is the extension of F through which $\widetilde{\chi}$ and $\bar{\omega}$ factors. Similarly $F(\widetilde{\chi})$ (respectively $F(\bar{\omega})$, $F(\widetilde{\rho})$) is the extension of F through which $\widetilde{\chi}$ (respectively $\bar{\omega}$, $\widetilde{\rho}$) factors.

Finally, we set

$$d := [F(\widetilde{\chi}) : F] \quad \text{and} \quad e := (\ell - 1)/[F(\widetilde{\chi}, \bar{\omega}) : F(\widetilde{\chi})].$$

We shall now show that the proposition holds with $C = 2edN + 1$.

The extension $F(\widetilde{\rho})/F(\widetilde{\chi})$ has Galois group $PSL_N(k)$, and so $F(\widetilde{\rho})$, $F(\widetilde{\chi}, \bar{\omega})$ are linearly disjoint over $F(\widetilde{\chi})$. Since $\bar{\omega}\left(G_{F(\widetilde{\chi})}\right) = \mathbb{F}_\ell^{\times e}$, the image of the homomorphism

$$\widetilde{\rho} \times \bar{\omega} : G_F \longrightarrow PGL_N(k) \times \mathbb{F}_\ell^\times$$

contains $PSL_N(k) \times \mathbb{F}_\ell^{\times e}$.

Fix a generator a of the cyclic group $\mathbb{F}_\ell^\times$, and set $b = a^{2ed}$. It is then straightforward to check that

$$X := \text{ the projective image of the diagonal matrix } \begin{pmatrix} b^{N-1} & & & \\ & \ddots & & \\ & & b & \\ & & & 1 \end{pmatrix}$$

is an element of $PSL_N(k)$. By the Chebotarev density theorem, there is an unramified prime v such that $\widetilde{\rho}(\mathrm{Frob}_v) = X$ and $\bar{\omega}(\mathrm{Frob}_v) = b$. Hence,

$$\overline{\rho}|_{F_v} \sim \begin{pmatrix} \bar{\omega}^{N-1} & & & \\ & \ddots & & \\ & & \bar{\omega} & \\ & & & 1 \end{pmatrix} \otimes \overline{\eta},$$

where $\overline{\eta}$ is an unramified character. Now, the order of $\bar{\omega}|_{F_v}$ is the order of b, and this is greater than N if $2edN < \ell - 1$; so, (R2) holds.

We now verify condition (R1). Note that $\ell \geq 7$ since $C \geq 5$. We use the representation $\overline{\rho} : G_F \longrightarrow GL_N(k)$ to identify $\mathrm{Gal}(F(\mathrm{ad}^0\overline{\rho})/F)$ with a subgroup of $PGL_N(k)$, and view $\mathrm{ad}^0\overline{\rho}$ as a $PGL_n(k)$-module. Since the image of $\overline{\rho}$ contains $SL_N(k)$, we see that $PSL_N(k)$ is a normal subgroup of $\mathrm{Gal}(F(\mathrm{ad}^0\overline{\rho})/F)$ of index coprime to ℓ. Hence,

$$H^1\left(\mathrm{Gal}(F(\mathrm{ad}^0\overline{\rho})/F), \mathrm{ad}^0\overline{\rho}\right) \hookrightarrow H^1\left(PSL_N(k), \mathrm{ad}^0\overline{\rho}\right).$$

Now, the inflation map $H^1\left(PSL_N(k), \mathrm{ad}^0\overline{\rho}\right) \longrightarrow H^1\left(SL_N(k), \mathrm{ad}^0\overline{\rho}\right)$ is an isomorphism because $PSL_N(k)$ is the quotient of $SL_N(k)$ by its centre, which has order coprime to ℓ. Since $H^1\left(SL_N(k), \mathrm{ad}^0\overline{\rho}\right) = (0)$ by Theorem 4.2 of [**3**], we can therefore conclude $H^1\left(\mathrm{Gal}(F(\mathrm{ad}^0\overline{\rho})/F), \mathrm{ad}^0\overline{\rho}\right) = (0)$.

The verification that $H^1\left(\mathrm{Gal}(F(\mathrm{ad}^0\overline{\rho}(1))/F), \mathrm{ad}^0\overline{\rho}(1)\right) = (0)$ is similar but needs an extra step. Set K to be the Galois extension of F generated by $F(\mathrm{ad}^0\overline{\rho}(1))$ and $F(\bar{\omega})$. By considering the extension $K/F(\bar{\omega})$ so that the Tate twist becomes trivial, we see that $\mathrm{Gal}(K/F)$ contains a subgroup of index coprime to ℓ and isomorphic to $PSL_N(k)$. Thus, as in preceding case, we deduce $H^1\left(\mathrm{Gal}(K/F), \mathrm{ad}^0\overline{\rho}(1)\right) = (0)$ and therefore, by the inflation-restriction exact sequence we have

$$H^1\left(\mathrm{Gal}(F(\mathrm{ad}^0\overline{\rho}(1))/F), \mathrm{ad}^0\overline{\rho}(1)\right) = (0).$$

This completes the proof of part (i) of the proposition.

We now prove part (ii), which deals with the case when $N = 3$ and $F = \mathbb{Q}$. Note that $d = [\mathbb{Q}(\tilde{\chi}) : \mathbb{Q}]$ is either 1 or 3, and since $\ell \geq 7$ we must have $[\mathbb{Q}(\tilde{\chi}, \bar{\omega}) : \mathbb{Q}(\tilde{\chi})] \geq 4$ except in the case $\mathbb{Q}(\tilde{\chi}) = \mathbb{Q}(\cos(2\pi/7))$ (which we are excluding). Hence, the image of $\widetilde{\rho} \times \bar{\omega}$ contains an element of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \times a,$$

where $a \in \mathbb{F}_\ell^\times$ has order at least 4. The rest of the proof is then as before. □

Remark 5.3. Keep the notation introduced in the proof of Proposition 5.2. Since $F(\widetilde{\chi}, \bar{\omega}) \supset \mathbb{Q}(\bar{\omega})$, we have $[F(\widetilde{\chi}, \bar{\omega}) : \mathbb{Q}] \geq [\mathbb{Q}(\bar{\omega}) : \mathbb{Q}]$ and so $d[F : \mathbb{Q}] \geq e$. Along with $d \leq N$, we see that if $\ell > 2[F : \mathbb{Q}]N^3 + 1$ then $2edN < \ell - 1$. Hence, if $\ell > 2[F : \mathbb{Q}]N^3 + 1$, then $\mathrm{Im}\overline{\rho}$ contains $SL_N(k)$ and $\overline{\rho} : G_F \longrightarrow GL_N(k)$ is a big representation.

PROPOSITION 5.4. *Let $\overline{\rho} : G_F \longrightarrow GL_N(k)$ be a big representation, and let $\chi : G_F \longrightarrow W^\times$ be a character lifting $\det\overline{\rho}$. Fix a prime w_0 of F such that $\overline{\rho}|_{w_0}$ satisfies condition R2 of Definition 5.1.*

If $\mathcal{D}_0$ is a global deformation condition with determinant χ for $\overline{\rho}$ satisfying the tangent space inequality, then there exists a global deformation condition $\mathcal{D}$ with determinant χ for $\overline{\rho}$ with $\Sigma(\mathcal{D}) \supseteq \Sigma(\mathcal{D}_0)$, such that

- *If $v \in \Sigma(\mathcal{D}_0)$, then $\mathcal{D}_{0v} = \mathcal{D}_v$.*
- *If $v \in \Sigma(\mathcal{D}) - \Sigma(\mathcal{D}_0)$ then $\mathcal{D}_v$ is smooth and $\overline{\rho}(Frob_v) = \overline{\rho}(Frob_{w_0})$. Furthermore, the tangent space $T\mathcal{D}_v$ satisfies*

$$H^1(F_v, ad^0\overline{\rho}) = H^1_{nr}(F_v, ad^0\overline{\rho}) \oplus T\mathcal{D}_v.$$

- *We have $H^1_{\{T\mathcal{D}_v{}^\perp\}}\left(F, ad^0\overline{\rho}(1)\right) = (0)$.*

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Proof. If $H^1_{\{T\mathcal{D}^\perp_{0,v}\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right) = (0)$ then we can take $\mathcal{D} = \mathcal{D}_0$ and there is nothing to check. So, we suppose that we can find

$$0 \neq \xi \in H^1_{\{T\mathcal{D}^\perp_{0,v}\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right).$$

Then, using Wiles' formula 2.1, we see that

$$\dim_k H^1_{\{T\mathcal{D}_{0v}\}}\left(F, \mathrm{ad}^0\overline{\rho}\right) \geq N - 1.$$

We now use Theorem 3.1 to produce a prime $w_1 \notin \Sigma(\mathcal{D}_0)$ such that

(a) $\overline{\rho}(\mathrm{Frob}_{w_1}) = \overline{\rho}(\mathrm{Frob}_{w_0})$ and $\bar{\omega}(\mathrm{Frob}_{w_1}) = \bar{\omega}(\mathrm{Frob}_{w_0})$;

(b) The restriction

$$H^1_{\{T\mathcal{D}_{0v}\}}\left(F, \mathrm{ad}^0\overline{\rho}\right) \longrightarrow H^1_{\mathrm{nr}}\left(F_{w_1}, \mathrm{ad}^0\overline{\rho}\right)$$

is surjective; and,

(c) The image of ξ when restricted to $H^1_{\mathrm{nr}}\left(F_{w_1}, \mathrm{ad}^0\overline{\rho}(1)\right)$ is non-trivial.

In order to do this, let K/F be the extension of F through which $\mathrm{ad}^0\overline{\rho}$ and $\bar{\omega}$ split, and set $G := \mathrm{Gal}(K/F)$. Then, $\mathrm{ad}^0\overline{\rho}$ and $\mathrm{ad}^0\overline{\rho}(1)$ are non-isomorphic absolutely irreducible $k[G]$-modules. Using the inflation-restriction exact sequence together with property (R1) of big representations and the observation that $[K : F(\mathrm{ad}^0\overline{\rho})]$ and $[K : F(\mathrm{ad}^0\overline{\rho}(1))]$ are coprime to ℓ, we conclude that

$$H^1(G, \mathrm{ad}^0\overline{\rho}) = H^1(G, \mathrm{ad}^0\overline{\rho}(1)) = (0).$$

We now apply Theorem 3.1 to the $k[G]$-module $\mathrm{ad}^0\overline{\rho} \oplus \mathrm{ad}^0\overline{\rho}(1)$ and place w_0 of F as follows. Fix subspaces

$$V_1 \subseteq H^1_{\{T\mathcal{D}_{0v}\}}\left(F, \mathrm{ad}^0\overline{\rho}\right) \subseteq H^1(G_F, \mathrm{ad}^0\overline{\rho})$$

of dimension $N - 1$ and

$$V_2 := k\xi \subseteq H^1_{\{T\mathcal{D}^\perp_{0,v}\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right) \subseteq H^1(G_F, \mathrm{ad}^0\overline{\rho}(1)).$$

Now, $\dim_k H^1_{\mathrm{nr}}\left(F_{w_1}, \mathrm{ad}^0\overline{\rho}\right) = \dim_k H^1_{\mathrm{nr}}\left(F_{w_1}, \mathrm{ad}^0\overline{\rho}(1)\right) = N - 1$. Take w_1 to be a place of F given by the conclusion of Theorem 3.1. Then, condition (a) above follows since the images of Frob_{w_0}, Frob_{w_1} in G are the same. The injectivity of

$$V_1 \oplus V_2 \longrightarrow H^1_{\mathrm{nr}}\left(F_{w_1}, \mathrm{ad}^0\overline{\rho}\right) \oplus H^1_{\mathrm{nr}}\left(F_{w_1}, \mathrm{ad}^0\overline{\rho}(1)\right)$$

then ensures conditions (b) and (c) also hold. (For condition (b) one needs to use that the restriction $V_1 \longrightarrow H^1_{\mathrm{nr}}\left(F_{w_1}, \mathrm{ad}^0\overline{\rho}\right)$ is an isomorphism, which follows from the injectivity by a dimension count.)

We now use the prime w_1 and define a new deformation condition $\mathcal{D}_1$ for $\overline{\rho}$ with determinant χ with the following local conditions: At primes not equal to w_1, the local deformation conditions $\mathcal{D}_{0v}$ and $\mathcal{D}_{1v}$ are the same. At the prime w_1, the local deformation condition $\mathcal{D}_{1w_1}$ is determined by a Ramakrishna condition (cf. Section 4.6). Thus, $\mathcal{D}_1$ is smooth at w_1.

The proof now proceeds as in Lemma 1.2 of **[15]**: Denote by $\{\mathcal{S}_v\}$ the local Selmer conditions

$$\mathcal{S}_v = \begin{cases} T\mathcal{D}_{0v}, & \text{if } v \neq w_1; \\ (0), & \text{if } v = w_1. \end{cases}$$

Using Wiles' formula 2.1,

$$\begin{aligned} &\dim H^1_{\{\mathcal{S}_v\}}\left(F, \mathrm{ad}^0\overline{\rho}\right) - \dim H^1_{\{\mathcal{S}_v^\perp\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right) \\ &= \sum_{v\nmid\infty}(\dim \mathcal{S}_v - \dim H^0(F_v, \mathrm{ad}^0\overline{\rho})) - \sum_{v\mid\infty} H^0\left(F_v, \mathrm{ad}^0\overline{\rho}\right) \\ &= \dim H^1_{\{T\mathcal{D}_{0v}\}}\left(F, \mathrm{ad}^0\overline{\rho}\right) - \dim H^1_{\{T\mathcal{D}_{0v}^\perp\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right) - \dim H^1_{\mathrm{nr}}\left(F_{w_1}, \mathrm{ad}^0\overline{\rho}\right), \end{aligned}$$

and by (b), the sequence

$$0 \longrightarrow H^1_{\{\mathcal{S}_v\}}\left(F, \mathrm{ad}^0\overline{\rho}\right) \longrightarrow H^1_{\{\mathcal{L}_{0v}\}}\left(F, \mathrm{ad}^0\overline{\rho}\right) \longrightarrow H^1_{\mathrm{nr}}\left(F_{w_1}, \mathrm{ad}^0\overline{\rho}\right) \longrightarrow 0$$

is exact. Hence, we have

$$H^1_{\{\mathcal{S}_v^\perp\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right) = H^1_{\{T\mathcal{D}_{0v}^\perp\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right).$$

Using condition (c) along with $H^1(F_{w_1}, \mathrm{ad}^0\overline{\rho}(1)) = H^1_{\mathrm{nr}}\left(F_{w_1}, \mathrm{ad}^0\overline{\rho}(1)\right) \oplus T\mathcal{D}_{1w_1}^\perp$, we see that

$$0 \neq \xi \notin H^1_{\{T\mathcal{D}_{1,v}^\perp\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right) \subseteq H^1_{\{T\mathcal{D}_{0,v}^\perp\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right).$$

Thus, $\dim H^1_{\{T\mathcal{D}_{1,v}^\perp\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right) \lneqq \dim H^1_{\{T\mathcal{D}_{0,v}^\perp\}}\left(F, \mathrm{ad}^0\overline{\rho}(1)\right)$, and the proposition follows inductively. □

An application of Theorem 2.2 then gives the following:

THEOREM 5.5. *We keep the notations and assumptions of Proposition 5.4 above. If for each $v \in \Sigma(\mathcal{D}_0)$ the local deformation condition $\mathcal{D}_{0v}$ is smooth, then the universal deformation ring for deformations of type $\mathcal{D}$ is a power series ring over W in*

$$\sum_{v\in\Sigma(\mathcal{D}_0)} \dim_k T\mathcal{D}_{0v} - \sum_{v\in\Sigma(\mathcal{D}_0)} \dim_k H^0(F_v, ad^0\overline{\rho})$$

variables.

6. Lifting Galois representations to characteristic 0.

In this section, we complete the proof of the main theorem. We also show how the general arguments we have used, with some care, produce strong lifting results in the GL_3 case.

6.1. Proof of the main theorem.

Recall that we are given a continuous representation $\overline{\rho} : G_F \longrightarrow GL_N(k)$ and a character $\chi : G_F \longrightarrow W^\times$ lifting the determinant of $\overline{\rho}$ such that

(1) the image of $\overline{\rho}$ contains $SL_N(k)$;
(2) $\overline{\rho}$ is not totally even;
(3) if v is a place of F lying above ℓ then $H^0\left(G_{F_v}, \mathrm{ad}^0\overline{\rho}(1)\right) = (0)$.

We are assuming that the characteristic of k satisfies the inequality $\ell \geq N^{3[F:\mathbb{Q}]N}$ with $N \geq 3$. We then need to produce a smooth global deformation condition $\mathcal{D}$ with determinant χ for $\overline{\rho}$ and dimension of tangent space at least $N-2$.

Twisting by an Nth root of the pro-ℓ part of χ if necessary (possible as $\ell > N$) and extending scalars, it follows from Proposition 5.2 and Lemma 2.1 that it suffices to prove the following.

PROPOSITION 6.1. *Suppose we are given a representation* $\overline{\rho} : G_F \longrightarrow GL_N(k)$ *satisfying the following hypotheses:*

- (H0) *for any open subgroup* $H \leq G_F$ *all irreducible components of the semi-simplification of* $\overline{\rho}|_H$ *are absolutely irreducible;*
- (H1) $\overline{\rho} : G_F \longrightarrow GL_N(k)$ *is a big representation;*
- (H2) $\overline{\rho}$ *is not totally even; and,*
- (H3) *for every prime* $v|\ell$, *we have* $H^0(F_v, ad^0\overline{\rho}(1)) = (0)$.

Assume that the characteristic of k satisfies the inequality $\ell \geq N^{3[F:\mathbb{Q}]N}$ *with* $N \geq 3$, *and let* $\chi : G_F \longrightarrow W^\times$ *be a character lifting* $\det\overline{\rho}$ *and minimally ramified away from* ℓ.

Under the above assumptions, there is a global deformation condition $\mathcal{D}$ *with determinant* χ *for* $\overline{\rho}$ *such that the universal deformation ring is a power series ring over* W *in*

$$\sum_{v\in\Sigma(\mathcal{D})} \dim_k \mathcal{D}_v - \sum_{v\in\Sigma(\mathcal{D})} \dim_k H^0(F_v, ad^0\overline{\rho}) \geq N-2$$

variables.

Proof. Observe that $\ell \geq N^{3[F:\mathbb{Q}]N}$ implies $[F_v(\zeta_\ell) : F_v] \geq 3N$ for every $v|N!$. Now, let $\mathcal{D}_0$ be the deformation condition with determinant χ for $\overline{\rho}$ given by the following local conditions:

- At a prime $v|\ell$, the local deformation condition is given by the single restriction that the determinant is χ.
- At a prime v where $\overline{\rho}$ is ramified, the local condition $\mathcal{D}_{0v}$ is the one given by Theorem 4.3.
- $\mathcal{D}_{0v}$ is unramified at all other primes.

Let v be a prime of F lying above ℓ. By assumption H3 and local duality, we have

$$\dim H^2\left(F_v, \mathrm{ad}^0\overline{\rho}\right) = \dim H^0\left(F_v, \mathrm{ad}^0\overline{\rho}(1)\right) = 0.$$

Hence, the deformation condition $\mathcal{D}_{0v}$ is smooth and, by the local Euler characteristic formula, we have

$$\dim T\mathcal{D}_{0v} - \dim H^0\left(F_v, \mathrm{ad}^0\overline{\rho}\right) = [F_v : \mathbb{Q}_\ell](N^2-1).$$

Adding up over primes above ℓ, we get

$$\sum_{v|\ell} \dim T\mathcal{D}_{0,v} - \sum_{v|\ell} \dim H^0\left(F_v, \mathrm{ad}^0\overline{\rho}\right) = [F : \mathbb{Q}](N^2-1).$$

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We are assuming that $\overline{\rho}$ is not totally even. We can therefore find a real prime $\infty_{\mathbb{R}}$ of F, a choice $c \in G_F$ of complex conjugation under the embedding given by $\infty_{\mathbb{R}}$ such that $\overline{\rho}(c)$ is not a scalar. Let m be the number of $+1$ eigenvalues of $\overline{\rho}(c)$. Then,

$$\begin{aligned}&\sum_{v|\infty} \dim H^0\left(F_v, \operatorname{ad}^0\overline{\rho}\right)\\ &\leq ([F:\mathbb{Q}]-1)(N^2-1) + \dim H^0\left(F_{\infty_{\mathbb{R}}}, \operatorname{ad}^0\overline{\rho}\right)\\ &= ([F:\mathbb{Q}]-1)(N^2-1) + m^2 + (N-m)^2 - 1.\end{aligned}$$

At a finite prime $v \in \Sigma(\mathcal{D}_0)$, which is coprime to ℓ, we have $\dim T\mathcal{D}_{0v} = \dim H^0(F_v, \operatorname{ad}^0\overline{\rho})$. Hence,

$$\begin{aligned}&\sum_{v\in\Sigma(\mathcal{D}_0)} \dim T\mathcal{D}_{0v} - \sum_{v\in\Sigma(\mathcal{D}_0)} \dim H^0\left(F_v, \operatorname{ad}^0\overline{\rho}\right)\\ &\geq [F:\mathbb{Q}](N^2-1) - ([F:\mathbb{Q}]-1)(N^2-1) - m^2 - (N-m)^2 + 1\\ &= 2m(N-m).\end{aligned}$$

From $(m-1)(N-m-1) \geq 0$, we get $m(N-m) \geq N-1$, and consequently $\mathcal{D}_0$ satisfies the tangent space inequality.

Applying Theorem 5.5, we obtain a deformation condition $\mathcal{D}$ with determinant χ such that the universal deformation ring is a power series ring over W in

$$\begin{aligned}&\sum_{v\in\Sigma(\mathcal{D})} \dim_k \mathcal{D}_v - \sum_{v\in\Sigma(\mathcal{D})} \dim_k H^0(F_v, \operatorname{ad}^0\overline{\rho})\\ &= \sum_{v\in\Sigma(\mathcal{D}_0)} \dim_k \mathcal{D}_v - \sum_{v\in\Sigma(\mathcal{D}_0)} \dim_k H^0(F_v, \operatorname{ad}^0\overline{\rho})\\ &\geq N-2\end{aligned}$$

variables. □

6.2. A lifting result when $N = 3$ and $F = \mathbb{Q}$. We now discuss how to improve on the main theorem for the case when $N = 3$ and $F = \mathbb{Q}$. From here on, k is a finite field of characteristic ℓ. An odd representation is one with complex conjugation having two distinct eigenvalues.

THEOREM 6.2. *Let $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_3(k)$ be an odd representation with image of $\overline{\rho}$ containing $SL_3(k)$ and let $\chi : G_{\mathbb{Q}} \longrightarrow W^\times$ be a character lifting the determinant of $\overline{\rho}$. Suppose that $\ell \geq 7$, and further assume that if $\ell = 7$ then the fixed field of $\operatorname{ad}^0\overline{\rho}$ does not contain* $\cos(2\pi/7)$. *Then, there is a continuous representation $\rho : G_{\mathbb{Q}} \longrightarrow GL_3(W)$ with determinant χ, unramified outside finitely many primes, such that $\rho \pmod{\ell} = \overline{\rho}$.*

A significant feature of Theorem 6.2, distinguishing it from other lifting results, is that there are no restrictions imposed at ℓ. Although not stated explicitly Theorem 6.2 constructs families of characteristic 0 liftings (because the universal deformation ring in play is a power series ring in at least $N-2=1$ variable). For explicit examples, we need to be able to write down odd representations $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_3(k)$ with $\overline{\rho}(G_{\mathbb{Q}}) \supseteq SL_3(k)$; this can be done for certain residue fields k (see [**16**]).

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Proof of Theorem 6.2. As in the proof of the main theorem, we may extend scalars and assume that $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_3(k)$ satisfies the three conditions H0, H1 (by Proposition 5.2) and H2 of the preceding Section 6.1. Thus, $\overline{\rho}$ is a big odd representation such that for any open subgroup $H \leq G_{\mathbb{Q}}$ all irreducible components in the semi-simplification of $\overline{\rho}|_H$ are absolutely irreducible.

Let $\overline{\rho}_p$ denote the restrictions of $\overline{\rho}$ to $G_{\mathbb{Q}_p}$. We will now find for each prime p a smooth local deformation condition $\mathcal{D}_{0p}$ with determinant χ for $\overline{\rho}_p$ such that $\dim T\mathcal{D}_{0p} = \dim H^0(\mathbb{Q}_p, \mathrm{ad}^0\overline{\rho})$ if $p \neq \ell$ and

$$\dim T\mathcal{D}_{0\ell} \geq \dim H^0(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho}) + 5. \tag{6.1}$$

There is no issue at a primes away from 2, 3 and ℓ: If $p > 3$ and $p \neq \ell$ we take $\mathcal{D}_{0p}$ to be the one obtained through Theorem 4.3.

If we can find the above local conditions at 2, 3 and ℓ, then Theorem 6.2 follows immediately from Theorem 5.5 once we verify that the global deformation condition $\mathcal{D}_0 = \{\mathcal{D}_{0p}\}$ with determinant χ satisfies the tangent space inequality (5.1). By our assumption on tangent spaces away from ℓ, we therefore need to check if the inequality

$$\dim T\mathcal{D}_{0\ell} \geq 1 + \dim H^0(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho}) + \dim H^0(\mathbb{R}, \mathrm{ad}^0\overline{\rho})$$

holds. But, this follows from (6.1) since $\dim H^0(\mathbb{R}, \mathrm{ad}^0\overline{\rho}) = 4$ as $\overline{\rho}$ is not totally even. □

Now, let p be one of 2, 3 or ℓ. We then have the following descriptions of the local representation $\overline{\rho}_p$.

Proposition 6.3. *Let $\overline{\rho}$ be as above and let p be one of* 2, 3 *or ℓ. If $H^2(\mathbb{Q}_p, ad^0\overline{\rho}_p) \neq$* (0) *then, after conjugating if necessary, we can put $\overline{\rho}_p$ into one of the following forms.*

Type A $\overline{\rho}_p = \begin{pmatrix} 1 & * & * \\ 0 & \bar{\omega} & * \\ 0 & 0 & \bar{\omega}^2 \end{pmatrix} \eta.$

Type B $\overline{\rho}_p = \begin{pmatrix} 1 & x & y \\ 0 & \varepsilon & z \\ 0 & 0 & \bar{\omega} \end{pmatrix} \eta,$

where x is non-split if $\varepsilon = \bar{\omega}^{-1}$ and z is non-split if $\varepsilon = \bar{\omega}^2$. (The non-split condition ensures that $\overline{\rho}_p$ is not of Type A.)

Type C *$\overline{\rho}_p$ is absolutely irreducible and induced from a character of $G_{\mathbb{Q}_p(\zeta_\ell)}$. This case occurs only when* $(p, \ell) = (2, 7)$ *or* $(3, 13)$.

Proof. Let V be the underlying k-vector space for the representation $\overline{\rho}$. We write elements of $V(1) = V \otimes k(1)$, the underlying space for $\overline{\rho}(1)$, simply as $v(1)$ for $v \in V$. Thus, $g \in G_{\mathbb{Q}_p}$ acts on $v(1)$ by sending it to $w(1)$ where $w = \bar{\omega}(g)\overline{\rho}(g)(v) \in V$. The assumption $H^2(\mathbb{Q}_p, \mathrm{ad}^0\overline{\rho}) \neq (0)$ then implies that

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}}(V, V(1)) \cong H^0(\mathbb{Q}_p, \mathrm{ad}^0\overline{\rho}(1)) \neq (0),$$

and we can therefore find a non-zero homomorphism $\phi : V \longrightarrow V(1)$ of $G_{\mathbb{Q}_p}$-modules.

First, suppose $\dim \ker \phi = 1$. Set $\ker \phi = \langle u \rangle$ and write $\phi(V) = U(1)$ where U is a two-dimensional $G_{\mathbb{Q}_p}$-submodule of V. We then claim $u(1) \in \phi(V)$, or equivalently $\ker \phi \subset U$. To see this, we note that if the claim is not true then the restriction

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$\phi|_U : U \longrightarrow U(1)$ is an isomorphism of $G_{\mathbb{Q}_p}$-modules. On taking determinants of the underlying two-dimensional representations, we then obtain $\bar{\omega}^2 = 1$. This is not possible as quadratic extensions of $\mathbb{Q}_p$ for $p = 2, 3, \ell$ cannot contain all ℓth roots of 1 when $\ell \geq 7$.

We therefore obtain $G_{\mathbb{Q}_p}$ stable filtrations

$$\langle u \rangle \subsetneq \phi^{-1}(\langle u(1) \rangle) \subsetneq V \quad \text{and} \quad \langle u(1) \rangle \subsetneq \phi^{-1}(\langle u(1) \rangle)(1) \subsetneq V(1).$$

Using these filtrations, we can assume that

$$\overline{\rho}_p = \begin{pmatrix} \alpha & * & * \\ 0 & \beta & * \\ 0 & 0 & \gamma \end{pmatrix} \quad \text{and} \quad \overline{\rho}_p(1) = \begin{pmatrix} \alpha\bar{\omega} & * & * \\ 0 & \beta\bar{\omega} & * \\ 0 & 0 & \gamma\bar{\omega} \end{pmatrix}.$$

The isomorphism $\phi^{-1}(\langle u(1) \rangle)/\langle u \rangle \longrightarrow \langle u(1) \rangle$ shows that $\beta = \alpha\bar{\omega}$. The injection $V/\phi^{-1}(\langle u(1) \rangle) \longrightarrow V(1)/\langle u(1) \rangle$ then implies that $\gamma = \beta\bar{\omega}$ or $\gamma = \gamma\bar{\omega}$. As $\bar{\omega}$ is non-trivial, we must have $\gamma = \beta\bar{\omega} = \alpha\bar{\omega}^2$. Consequently $\overline{\rho}$ is of Type A.

We now consider the case when $\dim \ker \phi = 2$. Let $\phi(V) = \langle u(1) \rangle$. Then, $u \in \ker \phi$ for otherwise ϕ induces an isomorphism $\langle u \rangle \longrightarrow \langle u(1) \rangle$ of $G_{\mathbb{Q}_p}$-modules. Using the $G_{\mathbb{Q}_p}$ stable filtrations

$$\langle u \rangle \subsetneq \ker \phi \subsetneq V \quad \text{and} \quad \langle u(1) \rangle \subsetneq \ker \phi(1) \subsetneq V(1),$$

we can assume

$$\overline{\rho}_p = \begin{pmatrix} \alpha & * & * \\ 0 & \beta & * \\ 0 & 0 & \gamma \end{pmatrix} \quad \text{and} \quad \overline{\rho}_p(1) = \begin{pmatrix} \alpha\bar{\omega} & * & * \\ 0 & \beta\bar{\omega} & * \\ 0 & 0 & \gamma\bar{\omega} \end{pmatrix}.$$

The isomorphism $V/\ker \phi \longrightarrow \langle u(1) \rangle$ implies that $\gamma = \alpha\omega$. Hence, $\overline{\rho}$ is of either of Type B or of Type A.

Finally, suppose that $\dim \ker \phi = 0$. Thus, $\overline{\rho} \sim \overline{\rho}(1)$. Taking determinants, we obtain $\bar{\omega}^3 = 1$. Hence, $[\mathbb{Q}_p(\zeta_\ell) : \mathbb{Q}_p] = 3$ and $(p, \ell) = (2, 7)$ or $(3, 13)$.

If $\overline{\rho}$ is not absolutely irreducible then its semi-simplification must contain a character χ. The isomorphism $\overline{\rho} \sim \overline{\rho}(1)$ then implies that $\overline{\rho}^{ss} = \chi \oplus \chi\bar{\omega} \oplus \chi\bar{\omega}^2$ and so $\overline{\rho}$ will be of Type A or Type B.

So, let us now suppose $\overline{\rho}$ is absolutely irreducible and $M\overline{\rho}M^{-1} = \overline{\rho}(1)$ for some invertible matrix $M \in GL_3(k)$. If the restriction of $\overline{\rho}$ to $G_{\mathbb{Q}_p(\zeta_\ell)}$ is still absolutely irreducible then M is a scalar matrix and $\overline{\rho} = \overline{\rho}(1)$ (equality of matrices!), which is clearly false. Thus, V has an absolutely irreducible $G_{\mathbb{Q}_p(\zeta_\ell)}$ stable subspace U of dimension 1 or 2.

Let $g \in G_{\mathbb{Q}_p}$ be a lift of the generator of $\mathrm{Gal}(\mathbb{Q}_p(\zeta_\ell)/\mathbb{Q}_p)$. If $\dim U = 2$ then $U \cap gU$ is a non-zero $G_{\mathbb{Q}_p(\zeta_\ell)}$ stable subspace and so, by irreducibility, we have $U = gU$. Thus, U is in fact stable under $G_{\mathbb{Q}_p}$-action, contradicting absolute irreducibility of V. So, $\dim U = 1$ and $V = U + gU + g^2U$. Hence, $\overline{\rho}$ is induced from a character of $G_{\mathbb{Q}_p(\zeta_\ell)}$. □

We can now proceed with our construction of suitable local deformation conditions.

Local conditions when $p = 2$ or 3. If $H^2(\mathbb{Q}_p, \mathrm{ad}^0\overline{\rho}_p) = (0)$ then we take $\mathcal{D}_{0p}$ to be the class of liftings with determinant χ (cf. Example 4.2). Thus, $\mathcal{D}_{0p}$ is smooth and $\dim T\mathcal{D}_{0p} = \dim H^0(\mathbb{Q}_p, \mathrm{ad}^0\overline{\rho})$.

Suppose, now $H^2(\mathbb{Q}_p, \mathrm{ad}^0\overline{\rho}) \neq (0)$. We assume that $\overline{\rho}_p$ is in the matrix forms specified by Proposition 6.3, and specify local deformations as follows.

First, assume that $\overline{\rho}_p$ is either of Type A, or of Type B with ε unramified. Thus, $\overline{\rho}_p$ is a twist of a tamely ramified representation. We then take $\mathcal{D}_{0p}$ to be any smooth deformation condition $\mathcal{D}_{0p}$ with determinant χ and $\dim T\mathcal{D}_{0p} = \dim H^0(\mathbb{Q}_p, \mathrm{ad}^0\overline{\rho})$. The existence of such a deformation condition is assured by Theorem 4.3.

We now consider the remaining cases. Thus, $\overline{\rho}_p$ is of Type B with ε ramified, or of Type C. Note that ℓ does not divide the order of the image of inertia under $\overline{\rho}_p$. (If $\overline{\rho}_p$ is Type B with ε ramified then we can assume $x = z = 0$ since $H^1(\mathbb{Q}_p, k(\varepsilon^{-1}))$ and $H^1(\mathbb{Q}_p, k(\varepsilon\bar{\omega}^{-1}))$ are both trivial, and then we can make $y = 0$ because $H^1(\mathbb{Q}_p, k(\bar{\omega}^{-1})) = (0)$.)

The construction and argument now proceeds as in [**15**, Example E1]. Take K to be fixed field of $\overline{\rho}_p$ over $\mathbb{Q}_p^{\mathrm{nr}}$, the maximal unramified extension of $\mathbb{Q}_p$, and then take $\mathcal{D}_{0p}$ to be lifts of $\overline{\rho}_p$ with determinant χ which factor through $\mathrm{Gal}(K/\mathbb{Q}_p)$. Since ℓ does not divide cardinality of $\mathrm{Gal}(K/\mathbb{Q}_p^{\mathrm{nr}})$, we have

$$H^n_{\mathrm{nr}}(\mathbb{Q}_p, \mathrm{ad}^0\overline{\rho}) \cong H^n(\mathrm{Gal}(K/\mathbb{Q}_p), \mathrm{ad}^0\overline{\rho}),$$

for all $n \geq 1$. It follows that $\mathcal{D}_{0p}$ is a smooth deformation condition and its tangent space has dimension $\dim H^0(\mathbb{Q}_p, \mathrm{ad}^0\overline{\rho})$.

Local conditions at ℓ. Our target is to find a smooth local deformation condition $\mathcal{D}_{0\ell}$ for $\overline{\rho}_\ell$ with determinant χ_ℓ and satisfying inequality 6.1:

$$\dim T\mathcal{D}_{0\ell} \geq \dim H^0(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho}) + 5.$$

If $H^2(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho}) = (0)$ then there aren't any obstructions and, following Example 4.2, we take $\mathcal{D}_{0\ell}$ to be the class of liftings with determinant χ_ℓ. This is smooth and

$$\dim T\mathcal{D}_{0\ell} - \dim H^0(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho}) = \dim \mathrm{ad}^0\overline{\rho} = 8.$$

Assume now that $H^2(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho}) \neq (0)$ and that $\overline{\rho}_\ell$ is of the form specified in Proposition 6.3. We now describe the choice of deformations and specify a $G_{\mathbb{Q}_\ell}$ subspace N of $\mathrm{ad}^0\overline{\rho}$ where the tangent space can be computed as follows. (Essentially we only allow those liftings which can be conjugated to certain parabolic subgroups of GL_3 and N is the corresponding adjoint. The same constructions work when $p = 2$ or 3 provided $\bar{\omega}^3 \neq 1$.)

(a) Suppose $\overline{\rho}_\ell$ is either of Type A or of Type B with ε different from 1 or $\bar{\omega}$ or $\bar{\omega}^{-1}$ or $\bar{\omega}^2$. Take $\mathcal{D}_{0\ell}$ to be upper triangular deformations of $\overline{\rho}$ with determinant χ and set N to be the space of trace 0 upper triangular matrices in $\mathrm{ad}^0\overline{\rho}$.
(b) Suppose $\overline{\rho}_\ell$ is of Type B and ε is 1 or $\bar{\omega}^{-1}$. Take $\mathcal{D}_{0\ell}$ to be deformations of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

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with determinant χ, and set N to be the matrices of the same form in $\mathrm{ad}^0\overline{\rho}$.

(c) Suppose $\overline{\rho}_\ell$ is of Type B and ε is $\bar{\omega}$ or $\bar{\omega}^2$. Take $\mathcal{D}_{0\ell}$ to be deformations of the form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

with determinant χ, and set N to be the matrices of the same form in $\mathrm{ad}^0\overline{\rho}$.

The composition series for $\mathrm{ad}^0\overline{\rho}/N$ shows that $H^0(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho}/N) = (0)$. Consequently, the exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{ad}^0\overline{\rho} \longrightarrow \mathrm{ad}^0\overline{\rho}/N \longrightarrow 0$$

implies that $H^0(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho}) \cong H^0(\mathbb{Q}_\ell, N)$ and that

$$H^1(\mathbb{Q}_\ell, N) \longrightarrow H^1(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho})$$

is injective. Since the tangent space of $\mathcal{D}_{0\ell}$ is the image of $H^1(\mathbb{Q}_\ell, N)$ in $H^1(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho})$, we obtain

$$T\mathcal{D}_{0\ell} \cong H^1(\mathbb{Q}_\ell, N).$$

We now sketch a verification that $\mathcal{D}_{0\ell}$ is a deformation condition as defined in Section 2.1 for upper triangular deformations, that is, case a above when $\overline{\rho}_\ell$ is either of Type A or of Type B with $\varepsilon \notin \{1, \bar{\omega}, \bar{\omega}^2, \bar{\omega}^{-1}\}$; the other cases are similar. The argument relies on the following two observations.

Claim 6.4. Let $\rho_1, \rho_2 : G_{\mathbb{Q}_\ell} \longrightarrow GL_3(A)$ be two strictly equivalent upper triangular liftings of $\overline{\rho}_\ell$ in $\mathcal{D}_{0\ell}$. Then, there exist an upper triangular matrix $X \equiv I \pmod{\mathfrak{m}_A}$ such that $\rho_1 = X\rho_2 X^{-1}$.

Proof. We can take A to be Artinian. Now, choose an element $t \in A$ with $\mathfrak{m}_A t = (0)$ and $tA = tk$. Using induction on length, we can find an upper triangular matrix $Y \in GL_3(A)$ such that $Y \equiv I \pmod{\mathfrak{m}_A}$ and

$$\rho_1 = Y\rho_2 Y^{-1} \pmod{tA}.$$

Thus, we can write $Y^{-1}\rho_1 Y = (I + t\xi)\rho_2$ with $\xi \in H^1(\mathbb{Q}_\ell, N)$. By assumption the image ξ in $H^1(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho})$ is trivial. The injectivity of $H^1(\mathbb{Q}_\ell, N) \longrightarrow H^1(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho})$ implies that $\xi = 0$, and the claim follows. □

Claim 6.5. Let $\rho : G_{\mathbb{Q}_\ell} \longrightarrow GL_3(A)$ be an upper triangular lifting of $\overline{\rho}_\ell$ and let M be a 3×3 matrix over A, such that

$$\rho(g)M\rho(g)^{-1} = M \quad \text{for all } g \in G_{\mathbb{Q}_\ell}.$$

Then, M is an upper triangular matrix.

Proof. We can assume that A is Artinian and use induction on length. The claim when $A = k$ is the content of $H^0(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho}) \cong H^0(\mathbb{Q}_\ell, N)$.

For the inductive step, choose $t \in A$ with $\mathfrak{m}_A t = (0)$ and $tA = tk$. Let $\mathcal{M}$ and $\mathcal{N}$ be the set of 3×3 matrices over A of the form

$$\begin{pmatrix} * & * & * \\ t* & * & * \\ t* & t* & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

respectively. Thus, $M \in \mathcal{M}$, and $G_{\mathbb{Q}_\ell}$ acts on $\mathcal{M}, \mathcal{N}$ by conjugation via ρ. Now, $\mathcal{M}/\mathcal{N} \cong \mathrm{ad}^0\overline{\rho}/N$ and since $H^0(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho}/N) = (0)$, we obtain

$$H^0(G_{\mathbb{Q}_\ell}, \mathcal{N}) = H^0(G_{\mathbb{Q}_\ell}, \mathcal{M}).$$

Therefore, $M \in \mathcal{N}$, that is, M is upper triangular. □

We now return to the verification that $\mathcal{D}_{0\ell}$ is a deformation condition. The only non-trivial part is to show that $\mathcal{D}_{0\ell}$ satisfies (DC2), and one checks that this justification reduces to the following claim.

Claim 6.6. Let $\pi : A \longrightarrow C$ be a surjection and let $\rho_A : G_{\mathbb{Q}_\ell} \longrightarrow GL_3(A)$ be a representation in $\mathcal{D}_{0\ell}$ with the property that $\rho_C := \pi\rho_A : G_{\mathbb{Q}_\ell} \longrightarrow GL_3(C)$ is already upper triangular. Then, there is an $X \in GL_3(A)$ such that $\pi(X) = I$ and $X\rho_A X^{-1}$ is upper triangular.

Proof. Let $Y \in GL_3(A)$ with $Y \equiv I \pmod{\mathfrak{m}_A}$ be such that the conjugate $Y\rho_A Y^{-1}$ is upper triangular. Thus, ρ_C and $\pi(Y)\rho_C\pi(Y)^{-1}$ are two strictly equivalent upper triangular lifts of $\overline{\rho}$ to $GL_3(C)$. Using Claim 6.4, we can find an upper triangular matrix $\widetilde{Y} \in GL_3(A)$ with $\widetilde{Y} \equiv I \pmod{\mathfrak{m}_A}$, such that

$$\pi(\widetilde{Y})\rho_C\pi(\widetilde{Y})^{-1} = \pi(Y)\rho_C\pi(Y)^{-1}.$$

Set $Z := \widetilde{Y}^{-1}Y$. Then, $Z\rho_A Z^{-1}$ is upper triangular and $\pi(Z)\rho_C\pi(Z)^{-1} = \rho_C$. By Claim 6.6, the matrix $\pi(Z)$ is upper triangular. Let $\widetilde{Z} \in GL_3(A)$ be an upper triangular matrix lifting $\pi(Z)$, and set $X := \widetilde{Z}^{-1}Z$. Then, $X\rho_A X^{-1}$ is upper triangular and $\pi(X) = I$. □

Finally, we need to show that $\mathcal{D}_{0\ell}$ is smooth and that $\dim T\mathcal{D}_{0\ell}$ satisfies inequality 6.1. Smoothness follows from $H^2(\mathbb{Q}_\ell, N) = (0)$ (cf. [**2**, Theorem 1.2]). To verify the vanishing of this second cohomology group, assume otherwise. Then, by local duality,

$$H^0(\mathbb{Q}_\ell, \mathrm{Hom}(N, k(1))) \neq (0),$$

and so N has a quotient isomorphic to $k(1)$. However, consideration of the composition series for N shows that N has no quotient isomorphic to $k(1)$ except possibly when $\overline{\rho}$ is of Type B and ε is $\bar{\omega}^{-1}$ or $\bar{\omega}^2$. These cases can then be discounted using the non-splitting of x and z, respectively.

Now, for inequality 6.1. We know that $H^0(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho}) \cong H^0(\mathbb{Q}_\ell, N)$ and $T\mathcal{D}_{0\ell} \cong H^1(\mathbb{Q}_\ell, N)$. The local Euler characteristic formula now implies that

$$\dim T\mathcal{D}_{0\ell} - \dim H^0(\mathbb{Q}_\ell, \mathrm{ad}^0\overline{\rho}) = \dim N + \dim H^2(\mathbb{Q}_\ell, N) = \dim N \geq 5,$$

and this completes the proof.

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