ON THE OPTIMALITY OF A STRAIGHT DEDUCTIBLE UNDER BELIEF HETEROGENEITY

BY

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ABSTRACT

This article attempts to extend Arrow's theorem of the deductible to the case of belief heterogeneity, which allows the insured and the insurer to have different beliefs about the distribution of the underlying loss. Like Huberman *et al.* [(1983) *Bell Journal of Economics* **14**(2), 415–426], we preclude *ex post* moral hazard by asking both parties in the insurance contract to pay more for a larger realization of the loss. It is shown that, *ceteris paribus*, full insurance above a constant deductible is always optimal for any chosen utility function of a risk-averse insured if and only if the insurer appears more optimistic about the conditional loss given non-zero loss than the insured in the sense of monotone hazard rate order. We derive the optimal deductible level explicitly and then examine how it is affected by the changes of the insured's risk aversion, the insurance price and the degree of belief heterogeneity.

KEYWORDS

Arrow's theorem of the deductible, deductible insurance, *ex post* moral hazard, heterogeneous beliefs, monotone hazard rate order.

1. INTRODUCTION

Since the seminal work of Arrow (1963), the study of optimal insurance design has attracted great attention from both academics and practitioners, and has become a cornerstone in insurance economics. Within the framework of von Neumann–Morgenstern expected utility (EU) theory, Arrow (1963) studies the optimal insurance policy from the perspective of a risk-averse insured, assuming that the insurance premium is calculated based only upon the actuarial value of the insurance coverage. He finds that the deductible insurance, which is full insurance above a constant deductible, is optimal under the criterion of maximizing the EU of the insured's final wealth. This finding is

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so well known that it is called Arrow's theorem of the deductible in the literature. Arrow's result has subsequently been extended to an optimization criterion that preserves second-degree stochastic dominance; see, for example, Van Heerwaarden *et al.* (1989), Gollier and Schlesinger (1996), Schlesinger (1997) and Chi and Lin (2014).

In all the aforementioned studies, it is implicitly assumed that the insured and the insurer have the same probabilistic belief about the underlying random loss. In particular, the loss randomness is often assumed to be objective and irrelevant to the personal views of both parties. However, this assumption of belief homogeneity usually deviates from the actual practice and has been questioned in decision theory. Savage (1972) asserts that an individual often makes a decision by using a personal view of probability. Especially, in the insurance contract both parties possess different information about the underlying loss and hence have heterogeneous beliefs.

Nevertheless, the study of optimal insurance problems with belief heterogeneity has not received full attention, and only very few papers have been devoted to this topic. Marshall (1992) is the first to study these problems and obtains several interesting results. Precisely, using heuristic analysis, he demonstrates that optimal solutions with the non-negative indemnity constraint can have almost any form and even do not resemble insurance contracts if no limitation is put on the belief heterogeneity. Marshall then focuses on a very special form of belief heterogeneity, where both parties have the same conditional loss distribution given non-zero loss and the insurer assigns a smaller probability mass to zero loss than the insured. He extends the result of Raviv (1979) to this form of belief heterogeneity and obtains a similar result as Arrow (1963) in favor of the deductible insurance when the insurer is risk-neutral. By further restricting the marginal indemnity to be non-negative, Huang *et al.* (2001) obtain a semi-analytic optimal insurance solution under a very general assumption of belief heterogeneity when the insurance premium is calculated based on the actuarial value of the coverage. Although their optimal contract is found to contain a deductible minimum, they encounter the same problem as Marshall that the optimal contract may violate the *principle of indemnity*,[1] which is well-accepted in the theory of insurance demand.

Notably, even if satisfying the principle of indemnity, the insurance contract may remain inapplicable in practice because of *ex post moral hazard*. In order to preclude *ex post* moral hazard, Huberman *et al.* (1983) suggest that the insurance contract should ask the marginal indemnity to be non-negative and less than unity. However, this condition often fails to be met by optimal contracts with belief heterogeneity obtained in the literature. For example, under the assumption that the insurer is more pessimistic about the loss than the insured in the sense of monotone likelihood ratio (MLR) order, Gollier (2013) shows that the marginal indemnity of the optimal solution is less than unity, but unfortunately it is unclear whether the marginal indemnity is non-negative. This non-negative marginal indemnity constraint can prevent downward misrepresentation of the damage by the insured. On the other hand, assuming that the insurer's probability measure is compatible with that of the insured, Ghossoub (2016, 2017) obtains the optimality of a variable deductible insurance schedule, whose marginal indemnity is always non-negative but may be strictly larger than unity such that the insured has an incentive to enlarge some loss.

Surprisingly, this condition imposed on the marginal indemnity to preclude moral hazard has usually been neglected by academics in the design of an optimal insurance policy. One of the reasons may be that this condition often has an insignificant impact when the insured and the insurer share the same probabilistic belief in the underlying random loss. In particular, Carlier and Dana (2005) show that this condition is satisfied by optimal insurance solutions in the vast majority of EU or non-EU models with homogeneous beliefs. Another reason may be that it becomes very challenging or even impossible to solving the related optimization problems once this condition is introduced. When the beliefs of both parties become heterogeneous, the above discussions, however, indicate that this condition plays a critical role in the study of optimal insurance problems. To the best of our knowledge, there is hardly any paper that is dedicated to the EU-based optimal insurance design with heterogeneous beliefs by taking this condition into account. Thus, our objective is to fill this gap and shed some light on this topic.

In this article, we revisit the optimal insurance design with belief heterogeneity by restricting admissible insurance contracts to satisfy the principle of indemnity and the marginal indemnity condition suggested by Huberman *et al.* (1983), and focus on extending Arrow's theorem of the deductible to the case of belief heterogeneity. As in the literature, we assume that the insurance premium is calculated by the expected value principle; that is, the insurance premium is proportional to the net premium calculated from the insurer's point of view. Under the criterion of maximizing the EU of the final wealth of the risk-averse insured, it is shown that the admissible insurance contract is always suboptimal to the deductible insurance policy with the same insurance premium for any chosen utility function of the insured if and only if the insurer appears more optimistic about the conditional loss given non-zero loss than the insured in the sense of monotone hazard rate (MHR) order.[2] In addition, the optimal deductible level is derived explicitly and is found to decrease in the degree of the insured's risk aversion but increase in the insured's zero-loss probability mass. However, the change of the insurance price or the insurer's zero-loss probability has an ambiguous effect on the optimal deductible level.

The main contributions of this article are threefold. First, we extend Arrow's theorem of the deductible to the form of belief heterogeneity satisfying the MHR condition, while alternative insurance contracts are restricted to follow the principle of indemnity and have the marginal indemnity being non-negative but less than unity. Further, our results manifest that the optimality of a straight deductible relies only upon the relative optimism of the insurer about the conditional loss distribution given positive loss and is irrelevant to the belief disagreement in the zero-loss probability. Therefore, the optimality of deductible insurance is obtainable for a more general form of belief heterogeneity than that assumed in Marshall (1992).

Second, it is highlighted that quite different from the case of belief homogeneity, the optimal insurance form with heterogeneous beliefs is very sensitive to the introduction of the marginal indemnity condition suggested by Huberman *et al.* (1983). Specifically, comparing the results in Ghossoub (2017) with ours, it is easy to find that the optimal solution changes from a variable deductible insurance policy to a deductible insurance contract once this condition is imposed.

Third, this article is the first to derive the optimal deductible level explicitly without a second-order condition, which requires the optimization objective to be concave in the deductible level and is usually assumed in the literature. It is worthwhile noting that this condition often fails to be satisfied, as pointed out by Schlesinger (1981).

The rest of the article is organized as follows. In Section 2, we introduce an optimal insurance model with heterogeneous beliefs where admissible insurance contracts satisfy the principle of indemnity and have the marginal indemnity being non-negative but less than unity. Because the assumption of belief heterogeneity plays a critical role in the optimal insurance design, some interesting forms are introduced and their relationship is discussed in detail in Section 3. A necessary and sufficient condition on the belief heterogeneity is obtained for the optimality of a straight deductible, and the optimal deductible level is derived explicitly in Section 4. Section 5 carries out a comparative analysis and investigates how the optimal deductible level is affected by the insured's risk aversion and the belief deviance, and some concluding remarks are provided in Section 6. Finally, Appendix A gives a detailed discussion of belief compatibility introduced by Ghossoub (2017) and Appendix B collects the proofs to the theorems and propositions established in the article.

2. The model

Let Ω be the set of states of the world. Suppose that over a fixed time period, an insured endowed with an initial wealth W is facing an uncertain loss, which is modeled by a non-negative bounded random variable X. Denote by $\mathscr{F}$ the sigma algebra generated by X, and by $\mathbb{P}$ the insured's subjective probability measure defined on the measurable space $(\Omega, \mathscr{F})$. To avoid trivial cases, we assume $\mathbb{E}^{\mathbb{P}}[X] > 0$ in this article.

In order to reduce its risk exposure, the insured plans to purchase an insurance contract, in which the insured cedes an amount of risk $I(X)$ to an insurer and retains the residual loss $X - I(X)$. Functions $I(x)$ and $x - I(x)$ are thus called the insured's ceded and retained loss functions, respectively. The principle of indemnity, which is well-accepted in the theory of insurance demand, requires the ceded risk to be non-negative and less than the underlying loss. Mathematically, we should have $0 \le I(x) \le x$. The ceded loss function $I(x)$ is further restricted to be increasing in Huang *et al.* (2001), who consider to prevent downward misrepresentation of the damage by the insured. However, this restriction is insufficient to eliminate *ex post* moral hazard, as pointed out by Huberman *et al.* (1983). To reduce *ex post* moral hazard, they suggest that both parties should pay more for a larger realization of the underlying loss. In other words, both $I(x)$ and $x - I(x)$ should be increasing functions,[3] that is,

$$0 \le I(x_2) - I(x_1) \le x_2 - x_1 \quad \text{for all} \quad 0 \le x_1 \le x_2. \tag{2.1}$$

It is further equivalent to $0 \le I'(x) \le 1$, a.e., where $I'(x)$ is the derivative of $I(x)$. In this article, we follow the way of Huberman *et al.* (1983) to constrain alternative insurance contracts, and derive the optimal ceded loss function among the set

$$\mathfrak{C} = \{0 \le I(x) \le x : \text{ both } I(x) \text{ and } x - I(x) \text{ are increasing functions}\}. \tag{2.2}$$

Covering part of the loss for the insured, the insurer will be compensated with a payment in the name of insurance premium. As in the literature, we assume the insurer is risk-neutral such that the insurance premium can be calculated by the expected value principle. Recall that the insurer and the insured may have different probabilistic beliefs about the underlying loss. Without loss of generality, we denote by $\mathbb{Q}$ the insurer's subjective probability measure defined on the measurable space $(\Omega, \mathcal{F})$. Using $\pi(\cdot)$ to represent the insurance premium principle, we have

$$\pi(I(X)) = (1 + \rho)\mathbb{E}^{\mathbb{Q}}[I(X)] \tag{2.3}$$

for some safety loading coefficient $\rho \ge 0$. It is necessary to point out that this assumed premium principle plays a critical role in determining the optimal contract later.

In the presence of the insurance contract $I(x)$, the insured's final wealth, which is no longer $W - X$, has to take into account the insurance premium and the indemnity. More precisely, using $w_I(X)$ to represent the insured's final wealth, we have

$$w_I(X) = W - X + I(X) - \pi(I(X)). \tag{2.4}$$

To avoid the bankruptcy issues, we make the following assumption:

Assumption 2.1. $w_I(X) \ge 0$ *almost surely for any* $I \in \mathfrak{C}$ *under probability measure* $\mathbb{P}$.

This assumption is naturally satisfied if the insured's initial wealth W is larger than $M_{\mathbb{P}}(X) + \pi(X)$, where $M_{\mathbb{P}}(X)$ is the essential supremum of X under $\mathbb{P}$. As in the literature, we further assume that the insured is risk averse and wants to maximize the EU of its final wealth. More specifically, if the risk-averse insured has a utility function $U(\cdot)$ satisfying $U'(\cdot) > 0$ and $U''(\cdot) < 0$, then the optimal insurance model can be formulated by

$$\max_{I(\cdot)\in\mathfrak{C}} \mathbb{E}^{\mathbb{P}}\left[U(w_I(X))\right]. \tag{2.5}$$

Obviously, the optimal solution to the above maximization problem depends heavily on the assumption of belief heterogeneity. It is quite challenging to obtain an explicit optimal indemnity schedule for a general form of belief heterogeneity, and hence the previous studies are often confined to consider some special forms. On the other hand, it is known from Arrow's theorem of the deductible that the optimal solution contains a straight deductible if the beliefs are homogeneous, that is, $\mathbb{P}=\mathbb{Q}$. Therefore, it is very interesting to investigate whether Arrow's theorem of the deductible can be extended to some kind of belief heterogeneity.

3. Forms of belief heterogeneity

To obtain the optimality of a straight deductible, we have to make some assumptions on the belief heterogeneity. Before presenting our assumptions, we will introduce some interesting forms of belief heterogeneity used in the literature and discuss their relationship.

Marshall (1992) introduces a very special form of belief heterogeneity, where

(1) the conditional distribution function of X given $X>0$ under probability measure $\mathbb{P}$ is same with that under probability measure $\mathbb{Q}$, and
(2) the insured assigns a larger probability mass to zero loss than the insurer, that is, $\mathbb{P}(X=0)>\mathbb{Q}(X=0)$.

In the above setting, the belief disagreement only appears in the zero-loss probability and the insured seems more optimistic about the loss than the insurer. Obviously, this assumption of belief heterogeneity is so strict that Marshall's result lacks generality.

Gollier (2013) considers another form of belief heterogeneity, in which the insured is more optimistic about the loss than the insurer in the sense of MLR order. More specifically, the ratio $\frac{f_X^{\mathbb{Q}}(t)}{f_X^{\mathbb{P}}(t)}$ is assumed to be increasing in t, where $f_X^{\mathbb{P}}(t)$ and $f_X^{\mathbb{Q}}(t)$ are probability density functions of X under probability measures $\mathbb{P}$ and $\mathbb{Q}$, respectively. Although MLR order is very powerful to rank random variables, unfortunately this assumption of belief heterogeneity cannot cover Marshall's special form.

Recently, Ghossoub (2017) introduces a novel form of belief heterogeneity, which asks probability measure $\mathbb{Q}$ to be compatible with $\mathbb{P}$. More precisely, it is assumed that $\mathbb{E}^{\mathbb{Q}}[h_1(X)]\le\mathbb{E}^{\mathbb{Q}}[h_2(X)]$ for any two Borel measurable functions $h_1(x)$ and $h_2(x)$ satisfying the following properties:

(1) $h_i(x)\ge 0$ for $i=1,2$;
(2) $h_1(x)$ is an increasing function;
(3) $h_1(X)$ and $h_2(X)$ have the same distribution under $\mathbb{P}$.

As shown in Ghossoub (2017), this form of belief heterogeneity is quite general in the sense that it includes Marshall's special form and the belief

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deviance reflecting the relative optimism of the insurer in MLR order. Further discussions of belief compatibility are given in Appendix A. Interestingly, Theorem A1 shows that if some weak condition is satisfied, this assumption of belief heterogeneity is a little stricter than the one satisfying the MHR condition, which is formally defined as follows:

Definition 3.1. *The belief heterogeneity is called to satisfy MHR condition if the distribution function of the loss X given $X > 0$ under $\mathbb{Q}$ is smaller than that under $\mathbb{P}$ in MHR order, that is,*

$$Hr(t) = \frac{\mathbb{Q}(X > t)}{\mathbb{P}(X > t)} \textit{ is decreasing over } \left[0, \max\{M_{\mathbb{P}}(X), M_{\mathbb{Q}}(X)\}\right). \tag{3.1}$$

The notion of belief heterogeneity satisfying MHR condition concerns the relative optimism of the insurer about the probability assessment of the tail event. Precisely, the insurer becomes increasingly optimistic about the occurrence of the tail event as the threshold level t increases. In fact, this assumption of belief heterogeneity is not quite strict and includes many forms as special cases. More specifically, noting that MHR condition puts no limit on the belief disagreement in zero-loss probability, it includes Marshall's special form of belief heterogeneity. Furthermore, noting that MHR order is weaker than MLR order (Theorem 1.C.1, Shaked and Shanthikumar, 2007), the belief heterogeneity reflecting the relative optimism of the insurer in MLR order is thus a special case. It even includes the belief compatibility once some weak condition is satisfied, as discussed above.

4. Optimality of a straight deductible

With the help of the form of belief heterogeneity established in Definition 3.1, we can obtain a necessary and sufficient condition for the optimality of deductible insurance in the following theorem.

Theorem 4.1. *In the insurance model (2.5), the admissible insurance contract is always suboptimal to the deductible insurance policy with the same insurance premium for any increasing concave utility function $U(\cdot)$ if and only if the belief heterogeneity satisfies MHR condition.*

Notably, the above result is applicable to a more general premium principle assumed in Gollier and Schlesinger (1996), who set the insurance premium to be a general function of the actuarial value of coverage and obtain the optimality of a straight deductible under homogeneous beliefs. Their results are thus extended by the above theorem to the form of belief heterogeneity satisfying MHR condition. Theorem 4.1 also covers the result for the very special form of belief heterogeneity considered in Marshall (1992). In addition, comparing

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it with the result of Ghossoub (2017), we can find that the marginal indemnity condition suggested by Huberman *et al.* (1983) greatly affects the optimal insurance design with belief heterogeneity. In particular, when the insurer is more optimistic about the underlying loss than the insured in MLR order, the optimal solution is changed from the variable deductible insurance to the deductible insurance once this condition is introduced.

Under this assumption of belief heterogeneity, the optimal deductible level can be derived explicitly in the following theorem.

Theorem 4.2. *Assume that the belief heterogeneity satisfies MHR condition. Define*

$$\varphi(d) = \frac{\mathbb{E}^{\mathbb{P}}[U'(w_{(x-d)_+}(X))]}{U'(W - d - \pi((X-d)_+))}, \quad 0 \le d < M_{\mathbb{Q}}(X), \tag{4.1}$$

where $(x)_+ = \max\{x, 0\}$*, then* $\varphi(d)$ *is decreasing over* $\left[VaR^{\mathbb{Q}}_{\frac{1}{1+\rho}}(X), M_{\mathbb{Q}}(X)\right)$*, and an optimal solution to the insurance model* (2.5) *is the deductible insurance with the deductible level*

$$d^* = \sup\left\{VaR^{\mathbb{Q}}_{\frac{1}{1+\rho}}(X) \le d < M_{\mathbb{Q}}(X) : \varphi(d) \ge \frac{1}{(1+\rho)Hr(d)}\right\} \vee VaR^{\mathbb{Q}}_{\frac{1}{1+\rho}}(X), \tag{4.2}$$

where $\sup \emptyset = -\infty$ *by convention,* $x \vee y = \max\{x, y\}$*,* $Hr(t)$ *is given in* (3.1) *and* $VaR^{\mathbb{Q}}_{\frac{1}{1+\rho}}(X)$ *is the value at risk of* X *at a confidence level* $\frac{\rho}{1+\rho}$ *under* $\mathbb{Q}$ *defined as*

$$VaR^{\mathbb{Q}}_{\frac{1}{1+\rho}}(X) = \inf\left\{x \ge 0 : \mathbb{Q}(X > x) \le \frac{1}{1+\rho}\right\}. \tag{4.3}$$

Especially, full insurance is optimal if and only if $Hr(0) \le \frac{1}{1+\rho}$*.*

If the insured and the insurer share the same probabilistic belief in the underlying loss, then the above theorem shows that full insurance is an optimal choice of a risk-averse insured if and only if the safety loading coefficient ρ is equal to zero. In other words, under the belief homogeneity, the necessary and sufficient condition for the optimality of full insurance is fair premium. This is the well-known *Mossin's theorem*. In the presence of belief heterogeneity, we can find that even if the safety loading coefficient is positive, full insurance still can be an optimal solution as long as the belief heterogeneity satisfies MHR condition and the insurer assigns a much larger probability mass to zero loss than the insured.

It is worthwhile noting that Theorem 4.2 derives the optimal deductible level explicitly without the second-order condition. This condition asks the optimization objective $\mathbb{E}^{\mathbb{P}}[U(w_{(x-d)_+}(X))]$ to be concave in d and is often assumed to derive the optimal level of deductible insurance in the literature.

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However, Schlesinger (1981) points out that it is not always satisfied even under homogeneous beliefs.

5. Comparative statics

From Equation (4.2), it is easy to see that the optimal deductible level relies heavily on the insured's risk preference, the insurance price and the belief heterogeneity in the loss distribution. It is of great interest to analyze the effects of these factors on the insured's demand for insurance.

First, we study how the optimal deductible level is affected by the change of the insured's risk aversion, the degree of which is often quantified by the Arrow–Pratt measure of absolute risk aversion $\mathcal{A}_U(x) = -U''(x)/U'(x)$. The twice continuously differentiable utility function $V(x)$ is said to be more risk averse than $U(x)$ in the sense of Arrow–Pratt if $\mathcal{A}_U(x) \leq \mathcal{A}_V(x)$ for any $x \geq 0$. Furthermore, if $\mathcal{A}_U(x)$ is decreasing, then the insured is called to exhibit decreasing absolute risk aversion (DARA), which is a very standard assumption in insurance economics.

Proposition 5.1. *If the insured becomes more risk averse in the sense of Arrow-Pratt, a lower deductible level will be chosen. Moreover, if the insured exhibits DARA, then the larger the initial wealth W, the higher the optimal deductible level.*

It is noteworthy that the effect of the insured's risk preference on the optimal deductible level has been extensively studied under the assumption of belief homogeneity before. The above proposition manifests that this effect is still valid even if the beliefs are heterogeneous and satisfy MHR condition. It should be emphasized that the insured's initial wealth W considered above is restricted to satisfy Assumption 2.1.

Next, we analyze the effect of the degree of belief heterogeneity on the insured's demand for insurance. To simplify the analysis, we will take a similar way as Marshall (1992) to focus on the change of optimal deductible level with respect to the zero-loss probability. In particular, under probability measure $\mathbb{P}$, we rewrite the cumulative distribution function of X by

$$\mathbb{P}(X \leq t) = p + (1-p)\mathbb{P}(\tilde{X} \leq t) \quad \text{for any non-negative } t, \tag{5.1}$$

where $p = \mathbb{P}(X = 0) \in [0, 1)$ and $\tilde{X}$, which is a positive random variable, is equally distributed with X given $X > 0$ under $\mathbb{P}$. When $\tilde{X}$ is fixed, the effect of the value change of p on the optimal deductible level is presented in the following proposition.

Proposition 5.2. *The optimal deductible level is increasing in the insured's subjective zero-loss probability p.*

If a larger probability mass is assigned to zero loss, the insured will appear more optimistic about the underlying loss such that a higher deductible level is acceptable. Therefore, the result in the above proposition is consistent with intuition.

The similar way will be used to discuss the effect of the insurer's zero-loss probability on the insured's demand for insurance. In particular, we let

$$\mathbb{Q}(X \le t) = q + (1-q)\mathbb{Q}(\hat{X} \le t)$$

for some $q \in [0, 1)$ and a positive random variable $\hat{X}$. Under the expected value premium principle, we have

$$\pi((X-d)_+) = (1+\rho)(1-q)\mathbb{E}^{\mathbb{Q}}[(\hat{X}-d)_+],\ \forall d \ge 0.$$

Therefore, given the probability distribution of $\hat{X}$ under $\mathbb{Q}$, the change of q has an opposite effect on the insurance premium compared to the safety loading coefficient ρ. Further, as pointed out by Gollier (2001), raising ρ makes the insurance more costly such that the insured decreases the demand for insurance, but at the same time reduces the insured's wealth such that it has more desire to purchase insurance under the assumption of DARA. As a result, the change of q or ρ has an ambiguous effect on the optimal deductible level.

6. Concluding remarks

In this article, we conduct a comparison between the deductible insurance and the insurance contract satisfying the principle of indemnity and the marginal indemnity condition suggested by Huberman *et al.* (1983), and thereby obtain the optimality of a straight deductible under the belief heterogeneity satisfying MHR condition. Focusing on the deductible insurance, we derive the optimal deductible level explicitly, and then investigate how it is affected by the changes of the insured's risk preference, the insurance price and the degree of belief heterogeneity.

Admittedly, there are unsolved problems. The optimal insurance form is still unclear when the belief heterogeneity fails to satisfy MHR condition. Furthermore, the assumption of expected value premium principle plays a critical role in deriving the optimal insurance solution in this article. It is of great interest to analyze this problem with other insurance premium principles. We leave these for future research exploration. Notably, this problem with an extended Wang's premium principle and a general assumption of belief heterogeneity is solved completely by Boonen (2016) when the EU-based optimization criterion is changed to be the one based on dual utility.

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Notes

1. The principle of indemnity, which asks the indemnity to be non-negative but less than the loss, can prevent the insured from making profit by destruction of the insurance object and from falling into a worse financial position after a loss.

2. MHR order, which is weaker than MLR order, has been widely used in economics and finance. For instance, it is adopted by Simsek (2013) to model investors' belief disagreements that affect asset prices in subprime mortgage crisis. For more detailed discussions of stochastic orders used in this article, we refer to Shaked and Shanthikumar (2007).

3. Throughout this article, the terms "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing", respectively.

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APPENDIX A. BELIEF COMPATIBILITY

In this section, an investigation will be carried out to the belief compatibility, which was introduced by Ghossoub (2017) to model the belief heterogeneity.

Lemma A1. *If probability measure $\mathbb{Q}$ is compatible with probability measure $\mathbb{P}$, then there exists a non-negative function $\psi(\cdot)$ satisfying $\mathbb{E}^{\mathbb{P}}[\psi(X)] \le 1$ such that*

$$\mathbb{E}^{\mathbb{Q}}[h(X)] = \mathbb{E}^{\mathbb{P}}[h(X)\psi(X)] \tag{A.1}$$

for any function $h(x)$ with $h(0) = 0$.

Proof. Similar to Ghossoub (2016), we can use Lebesgue decomposition theorem (Theorem 10.61, Aliprantis and Border, 2006) to get a unique pair $(\mathbb{Q}_{ac}, \mathbb{Q}_s)$ of non-negative finite measures on $(\Omega, \mathcal{F})$ such that $\mathbb{Q} = \mathbb{Q}_{ac} + \mathbb{Q}_s$, where $\mathbb{Q}_{ac}$ is absolutely continuous with respect to $\mathbb{P}$, while $\mathbb{Q}_s$ and $\mathbb{P}$ are mutually singular. In other words, there exists a set $A \in \mathcal{F}$ such that $\mathbb{Q}_s(A) = \mathbb{P}(\Omega \backslash A) = 0$, and for any $B \in \mathcal{F}$ it is satisfied that $\mathbb{Q}_{ac}(B) = 0$ whenever $\mathbb{P}(B) = 0$. Therefore, we have

$$\mathbb{P}(A) = 1, \quad \mathbb{Q}_{ac}(\Omega \backslash A) = 0. \tag{A.2}$$

Moreover, it follows from Radon–Nikodym theorem (Theorem 13.18, Aliprantis and Border, 2006) that there exists a $\mathbb{P}$-almost unique and $\mathcal{F}$-measurable random variable $\frac{\mathrm{d}\mathbb{Q}_{ac}}{\mathrm{d}\mathbb{P}}$ satisfying

$$\mathbb{Q}_{ac}(B) = \mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}_{ac}}{\mathrm{d}\mathbb{P}} \mathbb{1}_B\right], \ \forall B \in \mathcal{F},$$

where $\mathbb{1}_B$ is the indicator function of an event B. Recall that $\mathcal{F}$ is the sigma algebra generated by X. There must exist a Borel measurable function ψ such that

$$\frac{d\mathbb{Q}_{ac}}{d\mathbb{P}} = \psi(X), \quad \mathbb{P}-a.s. \tag{A.3}$$

which in turn implies $\mathbb{E}^{\mathbb{P}}[\psi(X)] = \mathbb{Q}_{ac}(\Omega) \leq \mathbb{Q}(\Omega) = 1$.

From (A.2), we can see that indemnities X and $X\mathbb{1}_A$ satisfy the principle of indemnity and have the same distribution under probability measure $\mathbb{P}$. If $\mathbb{Q}$ is compatible with $\mathbb{P}$, then the definition of belief compatibility in Section 3 will imply

$$0 \geq \mathbb{E}^{\mathbb{Q}}[X] - \mathbb{E}^{\mathbb{Q}}[X\mathbb{1}_A] = \mathbb{E}^{\mathbb{Q}}[X\mathbb{1}_{\Omega\backslash A}] = \int_{\Omega\backslash A} X d\mathbb{Q}_s = \int_{\Omega} X d\mathbb{Q}_s,$$

where the last two equalities are derived from the facts $\mathbb{Q}_{ac}(\Omega\backslash A) = 0$ and $\mathbb{Q}_s(A) = 0$. Noting that $X \geq 0$ and $\mathbb{Q}_s$ is a non-negative finite measure, we can get from the above inequality that $\mathbb{Q}_s(X > 0) = 0$, which in turn implies

$$\mathbb{E}^{\mathbb{Q}}[h(X)] = \int h(X)d\mathbb{Q}_s + \int h(X)d\mathbb{Q}_{ac} = \mathbb{E}^{\mathbb{P}}[h(X)\psi(X)]$$

for any function $h(\cdot)$ satisfying $h(0) = 0$, where the last equality follows from (A.3). This completes the proof. ■

From the above lemma, it is easy to see that the impact of belief compatibility on the insurance premium is realized by the function $\psi(\cdot)$. If some weak condition is satisfied, this function is shown to have a monotone property in the following proposition.

Proposition A1. *Denote by $S_{\mathbb{P}}(X)$, $\partial S_{\mathbb{P}}(X)$ and $\overset{\circ}{S_{\mathbb{P}}}(X)$ the support of X, its boundary and the interior under probability measure $\mathbb{P}$, respectively. Assume that under $\mathbb{P}$ the loss X is a continuous random variable with $\mathbb{P}(X \in \partial S_{\mathbb{P}}(X)) = 0$, and that probability measure $\mathbb{Q}$ is compatible with $\mathbb{P}$ and the corresponding density function $\psi(x)$ is left-continuous or right-continuous at any point of $\overset{\circ}{S_{\mathbb{P}}}(X)$. Then $\psi(x)$ has a version that is decreasing over $S_{\mathbb{P}}(X)$ (function $\tilde{\psi}(x)$ is called to be a version of $\psi(x)$ if Equation* (A.1) *still holds when $\psi(X)$ is replaced by $\tilde{\psi}(X)$.)*

Proof. First, we shall prove by contradiction that $\psi(x)$ is decreasing over $\overset{\circ}{S_{\mathbb{P}}}(X)$. More specifically, if $\psi(x)$ does not decrease over $\overset{\circ}{S_{\mathbb{P}}}(X)$, then there must exist two interior points x_1, x_2 such that $0 < x_1 < x_2$ and $\psi(x_1) < \psi(x_2)$. As it is assumed that $\psi(x)$ is right-continuous or left-continuous at x_1 and x_2 and X is a continuous random variable under $\mathbb{P}$, we can construct two disjoint closed intervals $[y_1, z_1]$ and $[y_2, z_2]$ such that

$$0 < y_1 < z_1 < y_2 < z_2, \quad x_i \in [y_i, z_i] \subset S_{\mathbb{P}}(X) \text{ for } i = 1, 2$$

and

$$\mathbb{P}(X \in [y_1, z_1]) = \mathbb{P}(X \in [y_2, z_2]) > 0, \quad \psi(x) < \psi(y), \ \forall y_1 \leq x \leq z_1, \ y_2 \leq y \leq z_2. \tag{A.4}$$

Based on the above construction, we define

$$I(x) = \begin{cases} \min\{y_1, x\}/2, & x \in [0, y_2]; \\ y_1, & x \in (y_2, z_2]; \\ x, & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{I}(x) = \begin{cases} x/2, & x \in [0, y_1]; \\ y_1, & x \in (y_1, z_1]; \\ y_1/2, & x \in (z_1, z_2]; \\ x, & \text{otherwise,} \end{cases}$$

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then we have

- $\tilde{I}(x) \geq 0$ and $I(x) \geq 0$;
- $I(x)$ is an increasing function;
- $I(X)$ and $\tilde{I}(X)$ are equally distributed under probability measure $\mathbb{P}$.

If $\mathbb{Q}$ is compatible with $\mathbb{P}$, we can get from the definition of belief compatibility and Lemma A1 that

$$0 \geq \mathbb{E}^{\mathbb{Q}}[I(X)] - \mathbb{E}^{\mathbb{Q}}[\tilde{I}(X)] = \frac{y_1}{2}\left(\mathbb{E}^{\mathbb{P}}\left[\psi(X)\mathbb{1}_{\{X\in(y_2,z_2]\}}\right] - \mathbb{E}^{\mathbb{P}}\left[\psi(X)\mathbb{1}_{\{X\in(y_1,z_1]\}}\right]\right) > 0,$$

where the last inequality is derived from (A.4). Consequently, a contradiction leads, and hence $\psi(x)$ must be decreasing over the interior of $S_{\mathbb{P}}(X)$.

Next, we extend the monotone property of $\psi(x)$ from $\overset{\circ}{S_{\mathbb{P}}}(X)$ to the set $S_{\mathbb{P}}(X)$ by defining

$$\tilde{\psi}(x) = \sup_{\substack{y\in \overset{\circ}{S_{\mathbb{P}}}(X)\\ y\geq x}} \psi(y),\ \forall x \in S_{\mathbb{P}}(X). \tag{A.5}$$

Trivially, $\tilde{\psi}(x)$ is decreasing over $S_{\mathbb{P}}(X)$ and $\tilde{\psi}(x) = \psi(x)$ for any $x \in \overset{\circ}{S_{\mathbb{P}}}(X)$. Using the fact $S_{\mathbb{P}}(X) = \partial S_{\mathbb{P}}(X) \cup \overset{\circ}{S_{\mathbb{P}}}(X)$ and the assumption of $\mathbb{P}(X \in \partial S_{\mathbb{P}}(X)) = 0$, we must have $\mathbb{P}\left(\psi(X) = \tilde{\psi}(X)\right) = 1$, which in turn implies $\mathbb{E}^{\mathbb{P}}[\psi(X)h(X)] = \mathbb{E}^{\mathbb{P}}[\tilde{\psi}(X)h(X)]$ for any Borel measurable function $h(x)$. This completes the proof. ■

It is noteworthy that X is always assumed to be continuously distributed under $\mathbb{P}$ in Ghossoub (2016, 2017). Furthermore, the assumption of $\psi(x)$ in the above proposition is quite weak because a Borel measurable function is "almost" continuous in a measure theoretic sense according to Lusin's theorem (Theorem 12.8, Aliprantis and Border, 2006). However, if this assumption is violated, it is unclear whether Proposition A1 still holds.

Now, we can establish the relationship between the belief compatibility and the belief heterogeneity satisfying MHR order in the following theorem.

Theorem A1. *Under the same assumption of Proposition A1, the belief heterogeneity satisfies MHR condition.*

Proof. Under the belief compatibility, it follows from (A.1) and Proposition A1 that

$$\mathbb{Q}(X > t) = \mathbb{E}^{\mathbb{P}}[\psi(X)\mathbb{1}_{\{X>t\}}],\ \forall t \geq 0 \tag{A.6}$$

for some function $\psi(x)$ which is non-negative and decreasing over $S_{\mathbb{P}}(X)$. As a result, we have $\mathbb{Q}(X > t) = 0$ whenever $\mathbb{P}(X > t) = 0$, and hence $M_{\mathbb{Q}}(X) \leq M_{\mathbb{P}}(X)$.

By virtue of (A.6), we can rewrite the hazard rate function in (3.1) as

$$Hr(t) = \frac{\mathbb{E}^{\mathbb{P}}\left[\psi(X)\mathbb{1}_{\{X>t\}}\right]}{\mathbb{P}(X > t)} = \mathbb{E}^{\mathbb{P}}[\psi(X)|X > t],\ \forall 0 \leq t < M_{\mathbb{P}}(X),$$

which implies that $Hr(t)$ is decreasing over $[0, M_{\mathbb{P}}(X))$. From Definition 3.1, we know that the belief compatibility satisfies MHR condition. The proof is finally completed. ■

APPENDIX B. PROOFS

This section collects the proofs to the theorems and propositions established in the article. For notational convenience, $M_{\mathbb{Q}}(X)$ and $M_{\mathbb{P}}(X)$ will be abbreviated to $M_{\mathbb{Q}}$ and $M_{\mathbb{P}}$, respectively.

B.1. Proof of Theorem 4.1

(i) We first assume that MHR condition is satisfied by the belief heterogeneity. According to Definition 3.1, we must have $M_{\mathbb{Q}} \leq M_{\mathbb{P}}$ and $Hr(t)$ is decreasing over $[0, M_{\mathbb{P}})$, where $Hr(t)$ is defined in (3.1). For any admissible ceded loss function $I(x) \in \mathfrak{C}$, if the insurance premium $\pi(I(X))$ is equal to $\pi(X)$, then it follows from (2.4) and the fact $0 \leq I(x) \leq x$ that $w_I(X) \leq W - \pi(X)$. Therefore, the insurance contract with the ceded loss function $I(x)$ is suboptimal to full insurance.

Else if $\pi(I(X)) = 0$, then it follows from (2.3) that $\mathbb{E}^{\mathbb{Q}}[I(X)] = 0$, which together with (2.1) implies $I(x) = 0$ for any $0 \leq x \leq M_{\mathbb{Q}}$. Therefore, we have $I(x) \leq (x - M_{\mathbb{Q}})_+$ for any $x \geq 0$ and $\pi(I(X)) = \pi((X - M_{\mathbb{Q}})_+) = 0$. As a consequence, we have $w_I(X) \leq w_{(x-M_{\mathbb{Q}})_+}(X)$. That is, the insurance contract with $I(x)$ is suboptimal to the deductible insurance policy with the same insurance premium.

Otherwise, if $\pi(I(X)) \in (0, \pi(X))$, we can obtain from (2.1) that

$$\mathbb{E}^{\mathbb{Q}}[I(X)] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\infty} \mathbb{1}_{\{X>t\}} \mathrm{d}I(t)\right] = \int_0^{\infty} \mathbb{Q}(X > t) I'(t) \mathrm{d}t, \tag{B.1}$$

where the first equality follows from $I(0) = 0$ and the last equality is derived by Fubini's theorem. Note that $0 \leq I(X) \leq X$ and that the stop-loss function $\mathbb{E}^{\mathbb{Q}}[(X - t)_+]$ is strictly decreasing and continuous over $[0, M_{\mathbb{Q}})$. There must exist a unique $d \in (0, M_{\mathbb{Q}})$ such that

$$\mathbb{E}^{\mathbb{Q}}[I(X)] = \mathbb{E}^{\mathbb{Q}}[(X - d)_+], \tag{B.2}$$

which, together with (B.1), would imply

$$\int_d^{\infty} \mathbb{Q}(X > t) \mathrm{d}t = \int_0^{\infty} \mathbb{Q}(X > t) I'(t) \mathrm{d}t. \tag{B.3}$$

Following, using the arguments similar to (B.1), we can get

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[I(X)] - \mathbb{E}^{\mathbb{P}}[(X - d)_+] &= \int_0^{\infty} \mathbb{P}(X > t) \left(I'(t) - \mathbb{1}_{\{t>d\}}\right) \mathrm{d}t \\
&\leq \int_0^{M_{\mathbb{Q}}} \mathbb{P}(X > t) \left(I'(t) - \mathbb{1}_{\{t>d\}}\right) \mathrm{d}t \\
&= \int_0^{M_{\mathbb{Q}}} \frac{1}{Hr(t)} \mathbb{Q}(X > t) \left(I'(t) - \mathbb{1}_{\{t>d\}}\right) \mathrm{d}t \\
&\leq \frac{1}{Hr(d)} \int_0^{M_{\mathbb{Q}}} \mathbb{Q}(X > t) \left(I'(t) - \mathbb{1}_{\{t>d\}}\right) \mathrm{d}t = 0,
\end{aligned}$$

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where the first inequality is derived by the facts $I'(t) \leq 1$ and $d < M_{\mathbb{Q}}$, the second inequality follows from the decreasing property of $Hr(t)$ and the last equality is obtained from (B.3). As a consequence, we have

$$\mathbb{E}^{\mathbb{P}}[\min\{X, d\}] = \mathbb{E}^{\mathbb{P}}[X] - \mathbb{E}^{\mathbb{P}}[(X-d)_+] \leq \mathbb{E}^{\mathbb{P}}[X - I(X)].$$

Furthermore, because $0 \leq I(X) \leq X$, it is easy to verify

$$\mathbb{P}(\min\{X, d\} > t) - \mathbb{P}(X - I(X) > t) \begin{cases} \leq 0, & t \geq d; \\ \geq 0, & t < d. \end{cases}$$

Therefore, using the Karlin–Novikoff cut criterion (Theorem 3.2.4, Rolski *et al.*, 1999), we have

$$\mathbb{E}^{\mathbb{P}}[v(\min\{X, d\})] \leq \mathbb{E}^{\mathbb{P}}[v(X - I(X))]$$

for any increasing convex function $v(t)$ provided that the expectations exist. The above equation, together with (2.4) and (B.2), would lead to $\mathbb{E}^{\mathbb{P}}\left[U(w_{(x-d)_+}(X))\right] \geq \mathbb{E}^{\mathbb{P}}[U(w_I(X))]$ for any increasing concave utility function $U(\cdot)$.

In short, if the belief heterogeneity satisfies MHR condition, the admissible insurance contract is always suboptimal to the deductible insurance policy with the same insurance premium for any increasing concave utility function $U(\cdot)$ endowed by the insured.

(ii) In this part, we will prove by contradiction that the belief heterogeneity should satisfy MHR condition if the deductible insurance is always optimal for any utility function of a risk-averse insured. Specifically, if MHR condition fails to be satisfied by the belief heterogeneity, then it must hold that either $M_{\mathbb{P}} < M_{\mathbb{Q}}$ or $Hr(t)$ is not decreasing over $[0, M_{\mathbb{Q}})$.

If $M_{\mathbb{P}} < M_{\mathbb{Q}}$, we define

$$\tilde{I}(x) = \min\{x, M_{\mathbb{P}}\}.$$

Because $\tilde{I}(x) \leq x$ and this inequality is strict for $x > M_{\mathbb{P}}$, there must exist a $d > 0$ such that $\mathbb{E}^{\mathbb{Q}}[(X-d)_+] = \mathbb{E}^{\mathbb{Q}}[\tilde{I}(X)]$. In addition, it is easy to see that $(x-d)_+ < \tilde{I}(x)$ for any $0 < x \leq M_{\mathbb{P}}$, which in turn implies $\mathbb{E}^{\mathbb{P}}[(X-d)_+] < \mathbb{E}^{\mathbb{P}}[\tilde{I}(X)]$. As a consequence, it follows from (2.3) and (2.4) that

$$\pi(\tilde{I}(X)) = \pi((X-d)_+) \quad \text{and} \quad \mathbb{E}^{\mathbb{P}}[w_{\tilde{I}}(X)] > \mathbb{E}^{\mathbb{P}}[w_{(x-d)_+}(X)].$$

Thus, the ceded loss function $(x-d)_+$ cannot dominate $\tilde{I}(x)$ for all increasing concave utility functions.

Otherwise, if $M_{\mathbb{P}} \geq M_{\mathbb{Q}}$ and $Hr(t)$ is not decreasing over $[0, M_{\mathbb{Q}})$, then there must exist two points x_1 and x_2 such that

$$0 \leq x_1 < x_2 < M_{\mathbb{Q}} \quad \text{and} \quad Hr(x_1) < Hr(x_2).$$

Because $Hr(t)$ is right-continuous, we can find a sufficient small $\epsilon > 0$ and an $m \in (Hr(x_1), Hr(x_2))$ such that

$$Hr(x) < m < Hr(y) \text{ for all } x_1 \leq x < x_1 + \epsilon < x_2 < y \leq x_2 + \epsilon < M_{\mathbb{Q}}. \tag{B.4}$$

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Now we construct an insurance contract with the marginal ceded loss function

$$I'(t) = \begin{cases} 1, & t \geq x_2 + \epsilon; \\ 1 - \eta, & t \in (x_1, x_1 + \epsilon) \cup (x_2, x_2 + \epsilon); \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\eta = \frac{\int_{x_1}^{x_1+\epsilon} \mathbb{Q}(X > t)\mathrm{d}t}{\int_{x_1}^{x_1+\epsilon} \mathbb{Q}(X > t)\mathrm{d}t + \int_{x_2}^{x_2+\epsilon} \mathbb{Q}(X > t)\mathrm{d}t} \in (0, 1).$$

It is easy to get

$$\int_{x_2}^{\infty} \mathbb{Q}(X > t)\mathrm{d}t = \int_0^{\infty} \mathbb{Q}(X > t)I'(t)\mathrm{d}t, \tag{B.5}$$

which together with (2.3) and (B.1) implies $\pi(I(X)) = \pi((X - x_2)_+)$.

Furthermore, using (B.1) again, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[(X - x_2)_+] - \mathbb{E}^{\mathbb{P}}[I(X)] \\
&= \int_{x_2}^{\infty} \mathbb{P}(X > t)\mathrm{d}t - \int_0^{\infty} \mathbb{P}(X > t)I'(t)\mathrm{d}t \\
&= \eta \int_{x_2}^{x_2+\epsilon} \mathbb{P}(X > t)\mathrm{d}t - (1 - \eta) \int_{x_1}^{x_1+\epsilon} \mathbb{P}(X > t)\mathrm{d}t \\
&= \eta \int_{x_2}^{x_2+\epsilon} \frac{\mathbb{Q}(X > t)}{Hr(t)}\mathrm{d}t - (1 - \eta) \int_{x_1}^{x_1+\epsilon} \frac{\mathbb{Q}(X > t)}{Hr(t)}\mathrm{d}t \\
&< \frac{1}{m}\left(\eta \int_{x_2}^{x_2+\epsilon} \mathbb{Q}(X > t)\mathrm{d}t - (1 - \eta) \int_{x_1}^{x_1+\epsilon} \mathbb{Q}(X > t)\mathrm{d}t\right) \\
&= \frac{1}{m}\left(\int_{x_2}^{\infty} \mathbb{Q}(X > t)\mathrm{d}t - \int_0^{\infty} \mathbb{Q}(X > t)I'(t)\mathrm{d}t\right) = 0,
\end{aligned}$$

where the inequality is derived by (B.4) and the last equality follows from (B.5). As a consequence, it follows from (2.4) that

$$\mathbb{E}^{\mathbb{P}}[w_{(x-x_2)_+}(X)] < \mathbb{E}^{\mathbb{P}}[w_I(X)].$$

Thus, given the insurance premium $\pi((X - x_2)_+)$, the deductible insurance is impossible to dominate all admissible insurance contracts for any increasing concave utility function.

Finally, if the deductible insurance is always optimal for any risk-averse insured, the belief heterogeneity should satisfy MHR order. This completes the proof.

B.2. Proof of Theorem 4.2

By Theorem 4.1, if the belief heterogeneity satisfies MHR condition, the analysis of optimal insurance model (2.5) can be simplified to deriving the optimal deductible level. Mathematically, it is equivalent to solving the following maximization problem:

$$\max_{d \geq 0} \Phi(d) = \mathbb{E}^{\mathbb{P}}\left[U\left(w_{(x-d)_+}(X)\right)\right]. \tag{B.6}$$

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It is easy to see that $\Phi(d)$ is decreasing over $[M_{\mathbb{Q}}, \infty)$. For any $0 \le d < M_{\mathbb{Q}}$, taking the derivatives of the optimization objective $\Phi(d)$ with respect to d yields

$$\Phi'(d) = (1+\rho)\mathbb{Q}(X > d)U'(W - d - \pi((X-d)_+))\left(\varphi(d) - \frac{1}{Hr(d)(1+\rho)}\right), \text{a.e.} \qquad \text{(B.7)}$$

where $Hr(d)$ and $\varphi(d)$ are given in (3.1) and (4.1), respectively. Furthermore, we can easily obtain $\varphi(d) \ge \mathbb{P}(X > d)$ and get from the definition of value at risk in (4.3) that

$$VaR^{\mathbb{Q}}_{\frac{1}{1+\rho}}(X) \le z \quad \text{is equivalent to} \quad \mathbb{Q}(X > z) \le \frac{1}{1+\rho} \qquad \text{(B.8)}$$

for all non-negative z. Therefore, for each d strictly less than $VaR^{\mathbb{Q}}_{\frac{1}{1+\rho}}(X)$, the above equation leads to $(1+\rho)\mathbb{Q}(X > d) > 1$, which together with (B.7) can imply

$$\Phi'(d) \ge (1+\rho)\mathbb{Q}(X > d)U'(W - d - \pi((X-d)_+)) \\ \times \left(\mathbb{P}(X > d) - \frac{\mathbb{P}(X > d)}{(1+\rho)\mathbb{Q}(X > d)}\right) \ge 0.$$

In other words, $\Phi(d)$ is increasing over the interval $\left[0, VaR^{\mathbb{Q}}_{\frac{1}{1+\rho}}(X)\right]$. As a result, a solution d^* to the maximization problem (B.6) can be located in $\left[VaR^{\mathbb{Q}}_{\frac{1}{1+\rho}}(X), M_{\mathbb{Q}}\right]$.

Further, the derivative of $\varphi(d)$ can be given by

$$\varphi'(d) = \lambda(d)\left[\mathbb{P}(X > d) - \varphi(d)(1 - (1+\rho)\mathbb{Q}(X > d)) \right. \\ \left. -(1+\rho)\mathbb{Q}(X > d)\frac{\mathbb{E}^{\mathbb{P}}\left[U''(w_{(x-d)_+}(X))\right]}{U''(W - d - \pi((X-d)_+))}\right],$$

where

$$\lambda(d) = -\frac{U''(W - d - \pi((X-d)_+))}{U'(W - d - \pi((X-d)_+))} > 0.$$

For any $d \in \left[VaR^{\mathbb{Q}}_{\frac{1}{1+\rho}}(X), M_{\mathbb{Q}}\right)$, it is easy to get

$$\varphi(d) \ge \mathbb{P}(X > d) \quad \text{and} \quad \frac{\mathbb{E}^{\mathbb{P}}\left[U''(w_{(x-d)_+}(X))\right]}{U''(w - d - \pi((X-d)_+))} \ge \mathbb{P}(X > d),$$

and it follows from (B.8) that $1 \ge (1+\rho)\mathbb{Q}(X > d)$. Thus, we have

$$\varphi'(d) \le \lambda(d)\left[\mathbb{P}(X > d) - \mathbb{P}(X > d)(1 - (1+\rho)\mathbb{Q}(X > d)) \right. \\ \left. -(1+\rho)\mathbb{P}(X > d)\mathbb{Q}(X > d)\right] = 0.$$

That is, $\varphi(d)$ is decreasing over $\left[VaR^{\mathbb{Q}}_{\frac{1}{1+\rho}}(X), M_{\mathbb{Q}}\right)$. On the other hand, MHR condition implies that $Hr(d)$ is decreasing over $[0, M_{\mathbb{Q}})$. Therefore, it follows from (B.7) that the maximal value of $\Phi(d)$ is attainable at the point d^* which is defined in (4.2). Consequently, the ceded loss function $(x-d^*)_+$ is a solution to the optimal insurance model (2.5).

Especially, if $Hr(0) \le \frac{1}{1+\rho}$, then we have

$$1 = \varphi(0) \le \frac{1}{(1+\rho)Hr(0)} \quad \text{and} \quad \mathbb{Q}(X>0) \le Hr(0) \le 1/(1+\rho),$$

which together with (B.8) further imply $VaR^{\mathbb{Q}}_{\frac{1}{1+\rho}}(X) = 0$. Therefore, it follows from (4.2) that $d^* = 0$. That is, full insurance is an optimal solution to the insurance model (2.5) for this case. Reversely, if full insurance is optimal, then we must have

$$\Phi'(0) \le 0,$$

which is equivalent to $Hr(0) \le \frac{1}{1+\rho}$ according to (B.7).

B.3. Proof of Proposition 5.1

From (4.1), it is easy to see that the function $\varphi(d)$ relies on the utility function $U(\cdot)$. To emphasize this dependence, we rewrite it as $\varphi_U(d)$. For any increasing utility function $V(\cdot)$ with $\mathcal{A}_U(x) \le \mathcal{A}_V(x)$, Proposition 2 in Gollier (2001) demonstrates there exists an increasing concave function $g(x)$ such that $V(x) = g(U(x))$, then we have

$$\begin{aligned}
&\varphi_V(d)\\
&= \frac{\mathbb{E}^{\mathbb{P}}\left[g'\left(U(w_{(x-d)_+}(X))\right) U'(w_{(x-d)_+}(X))\right]}{g'\left(U(W-d-(1+\rho)\mathbb{E}^{\mathbb{Q}}[(X-d)_+])\right) U'(W-d-(1+\rho)\mathbb{E}^{\mathbb{Q}}[(X-d)_+])}\\
&\le \frac{\mathbb{E}^{\mathbb{P}}\left[U'(w_{(x-d)_+}(X))\right]}{U'(W-d-(1+\rho)\mathbb{E}^{\mathbb{Q}}[(X-d)_+])} = \varphi_U(d),
\end{aligned}$$

where the inequality is derived by the concavity of $g(\cdot)$ and

$$U(w_{(x-d)_+}(X)) \ge U(W-d-(1+\rho)\mathbb{E}^{\mathbb{Q}}[(X-d)_+]), \forall d \ge 0.$$

Therefore, we can get from (4.2) that the solution d^* to the maximization problem (B.6) for the utility function $V(\cdot)$ is smaller than that for $U(\cdot)$. In other words, the optimal deductible level becomes lower for a more risk-averse insured.

Finally, if the insured exhibits DARA, then a larger initial wealth will make him/her less risk averse such that the insured will choose a higher deductible level. This completes the proof.

B.4. Proof of Proposition 5.2

If $p \le 1-(1+\rho)\mathbb{Q}(X>0)$, which is equivalent to $Hr(0) \le \frac{1}{1+\rho}$, then it follows from Theorem 4.2 that the optimal deductible level is equal to zero. Otherwise, if $p > 1-(1+\rho)\mathbb{Q}(X>0)$, Theorem 4.2 shows that the optimal deductible level can be obtained by

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comparing $\varphi(d)$ with $\frac{\mathbb{P}(X>d)}{(1+\rho)\mathbb{Q}(X>d)}$. By simple calculation, we get from (5.1) that $\varphi(d) \geq \frac{\mathbb{P}(X>d)}{(1+\rho)\mathbb{Q}(X>d)}$ is exactly equivalent to

$$\begin{aligned}\frac{p}{1-p}U'(W-\kappa(d)) \geq &\frac{\mathbb{P}(\tilde{X}>d)}{(1+\rho)\mathbb{Q}(X>d)}U'(W-d-\kappa(d)) \\ &- \mathbb{E}^{\mathbb{P}}\Big[U'(W-\min\{\tilde{X},d\}-\kappa(d))\Big]\end{aligned}$$

for any non-negative d, where $\kappa(d) = (1+\rho)\mathbb{E}^{\mathbb{Q}}[(X-d)_+]$ is independent of p. As a consequence, for any $0 \leq p_1 < p_2 < 1$, the above inequality holds for p_2 whenever it holds for p_1. Therefore, we can obtain from Theorem 4.2 that the optimal deductible level is higher for a larger p.

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