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ON OZAKI CLOSE-TO-CONVEX FUNCTIONS

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Abstract

Let f be analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. We give sharp bounds for the initial coefficients of the Taylor expansion of such functions in the class of strongly Ozaki close-to-convex functions, and of the initial coefficients of the inverse function, together with some growth estimates.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions f analytic in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with Taylor series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent (that is, one-to-one) functions. A function $f \in \mathcal{A}$ is called starlike (with respect to the origin) if $f(\mathbb{D})$ is starlike with respect to the origin and convex if $f(\mathbb{D})$ is convex. Let $\mathcal{S}^*(\alpha)$ and $C(\alpha)$ denote respectively the classes of starlike and convex functions of order α for $0 \le \alpha < 1$ in $\mathcal{S}$. It is well known that a function $f \in \mathcal{A}$ belongs to $\mathcal{S}^*(\alpha)$ if and only if $\mathrm{Re}\,(zf'(z)/f(z)) > \alpha$ for $z \in \mathbb{D}$, and $f \in C(\alpha)$ if and only if $\mathrm{Re}(1 + zf''(z)/f'(z)) > \alpha$. Similarly, a function $f \in \mathcal{A}$ belongs to $\mathcal{K}$, the class of close-to-convex functions, if and only if there exists $g \in \mathcal{S}^*$ such that $\mathrm{Re}\,[e^{i\tau}(zf'(z)/g(z))] > 0$ for $z \in \mathbb{D}$ and $\tau \in (-\pi/2, \pi/2)$. Thus, $C \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$. When $\tau = 0$, the resulting subclass of close-to-convex functions is denoted by $\mathcal{K}_0$.

Although the class $\mathcal{K}$ was first formally introduced by Kaplan [5] in 1952, already in 1941 Ozaki [9] considered functions in $\mathcal{A}$ satisfying the condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \quad (z \in \mathbb{D}). \tag{1.2}$$

It follows from the original definition of Kaplan [5] that functions satisfying (1.2) are close-to-convex and therefore members of $\mathcal{S}$.

Kargar and Ebadian [6] considered the following generalisation to (1.2).

DEFINITION 1.1. Let $f \in \mathcal{A}$ be locally univalent for $z \in \mathbb{D}$ and let $-1/2 < \lambda \leq 1$. Then $f \in \mathcal{F}(\lambda)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2} - \lambda \quad (z \in \mathbb{D}). \tag{1.3}$$

Clearly, when $-1/2 < \lambda \leq 1/2$, functions defined by (1.3) provide a subset of C, with $\mathcal{F}(1/2) = C$, and, since $1/2 - \lambda \geq -1/2$ when $\lambda \leq 1$, functions in $\mathcal{F}(\lambda)$ are close-to-convex when $1/2 \leq \lambda \leq 1$. We shall call members of $f \in \mathcal{F}(\lambda)$ when $1/2 \leq \lambda \leq 1$ *Ozaki close-to-convex functions* and denote this class by $\mathcal{F}_O(\lambda)$.

For $0 < \beta \leq 1$, the classes $\mathcal{S}^{**}(\beta)$ of strongly starlike functions and $C^{**}(\beta)$ of strongly convex functions are defined for $f \in \mathcal{A}$ and $z \in \mathbb{D}$, respectively, by

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2}$$

and

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\beta\pi}{2}.$$

Functions in $\mathcal{S}^{**}(\beta)$ and $C^{**}(\beta)$ are more difficult to deal with than those in $\mathcal{S}^*$ and C, and relatively few exact coefficient bounds are known. Sharp bounds are known only for functionals involving the coefficients a_2, a_3 and a_4 (see [1–3] and [17]).

Even more elusive are sharp bounds for the class $\mathcal{K}^{**}(\beta)$ of strongly close-to-convex functions, defined for $f \in \mathcal{A}$ and $z \in \mathbb{D}$, by

$$\left|\arg \frac{zf'(z)}{g(z)}\right| < \frac{\beta\pi}{2},$$

where $0 < \beta \leq 1$ and $g \in \mathcal{S}^*$. It is a relatively simple exercise to obtain sharp bounds for the coefficients $|a_2|$ and $|a_3|$ when $f \in \mathcal{K}^{**}(\beta)$, but finding sharp bounds for $|a_4|$ appears to be a more difficult problem.

We note that in contrast to the definition of $\mathcal{K}$, the definition of $\mathcal{F}(\lambda)$ does not involve an independent starlike function g, but, as was shown in [11], members of $\mathcal{F}(1)$ have coefficients which grow at the same rate as those in $\mathcal{K}$, that is, $O(n)$ as $n \to \infty$.

We make the following definition, which extends (1.3), the special case with $\beta = 1$.

DEFINITION 1.2. Let $f \in \mathcal{A}$ for $z \in \mathbb{D}$, with $0 < \beta \leq 1$ and $1/2 \leq \lambda \leq 1$. Then f is called strongly Ozaki close-to-convex if and only if

$$\left|\arg\left(\frac{2\lambda - 1}{2\lambda + 1} + \frac{2}{2\lambda + 1}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)\right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{D}). \tag{1.4}$$

We denote this class of functions by $\mathcal{F}_O(\lambda, \beta)$.

The primary object of this paper is to obtain sharp bounds for the coefficients $|a_2|$, $|a_3|$ and $|a_4|$, and the corresponding inverse coefficients, for strongly Ozaki close-to-convex functions, thus providing sharp inequalities for the fourth coefficient of a class of strongly close-to-convex functions. We also give some distortion theorems.

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2. Lemmas

We will use the following lemmas (see, for example, [1]) for functions $p \in \mathcal{P}$, the class of functions with positive real part in $\mathbb{D}$, given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

LEMMA 2.1. *If $p \in \mathcal{P}$, then $|p_n| \le 2$ for $n \ge 1$ and*

$$\left|p_2 - \frac{\mu}{2}p_1^2\right| \le \max\{2, 2|\mu - 1|\} = \begin{cases} 2, & 0 \le \mu \le 2, \\ 2|\mu - 1|, & \textit{elsewhere}. \end{cases}$$

Also,

$$|p_2 - \tfrac{1}{2}p_1^2| \le 2 - \tfrac{1}{2}|p_1^2|.$$

LEMMA 2.2. *Let $p \in \mathcal{P}$. If $0 \le B \le 1$ and $B(2B - 1) \le D \le B$, then*

$$|p_3 - 2Bp_1p_2 + Dp_1^3| \le 2.$$

LEMMA 2.3. *If $p \in \mathcal{P}$, then*

$$|p_3 - (\mu + 1)p_1p_2 + \mu p_1^3| \le \max\{2, 2|2\mu - 1|\} = \begin{cases} 2, & 0 \le \mu \le 1, \\ 2|2\mu - 1|, & \textit{elsewhere}. \end{cases}$$

We will also use the following result from the theory of differential subordination (see [8]).

LEMMA 2.4. *Let $\Omega \subset \mathbb{C}$ and suppose that the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfies $\psi(ix, y; z) \notin \Omega$ for all $x \in \mathbb{R}$, $y \le -n(1 + x^2)/2$ and $z \in \mathbb{D}$. If p is analytic in $\mathbb{D}$, $p(0) = 1$ and $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{D}$, then* Re $p(z) > 0$ *for $z \in \mathbb{D}$.*

The following result (see [12] and [4, page 67]) is often useful and we will need it in Theorem 3.4.

LEMMA 2.5. *Suppose that $f \in \mathcal{S}$ and that $z = re^{i\theta} \in \mathbb{D}$. If*

$$m'(r) \le |f'(z)| \le M'(r),$$

where $m'(r)$ and $M'(r)$ are real-valued functions of r in $[0, 1)$, then

$$\int_0^r m'(t)\,dt \le |f(z)| \le \int_0^r M'(r)\,dt.$$

Although functions in $\mathcal{F}(\lambda)$ are close-to-convex when $1/2 \le \lambda \le 1$, Ponnusamy *et al.* [11] gave an example to show that when $\lambda = 1$, they are not necessarily starlike. On the other hand, we will show in this paper that when the second coefficient of the Taylor expansion for $f(z)$ is zero, functions in $\mathcal{F}(1)$ are starlike of order $1/2$, that is, $\mathrm{Re}(zf'(z)/f(z)) > 1/2$.

In the next section, we consider the class $\mathcal{F}(\lambda)$, that is, when $-1/2 \le \lambda \le 1$. The following sections will be concerned with Ozaki close-to-convex functions, that is, when $1/2 \le \lambda \le 1$.

3. The class $\mathcal{F}(\lambda)$

Theorem 3.1. *Let $\mathcal{A}_n$ be the set of functions in $\mathcal{A}$ given by*

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots .$$

If $f \in \mathcal{F}(\lambda)$ for $-1/2 \le \lambda \le 1$, $0 \le \alpha < 1$, $n \in \mathbb{N}$ and $\widehat{\lambda} = \lambda(\alpha, n) = \min\{\lambda_*(\alpha, n), 1\}$, where

$$\lambda_*(\alpha, n) = \begin{cases} \dfrac{1}{2} - \alpha + \dfrac{n}{2} \cdot \dfrac{1-\alpha}{\alpha}, & \alpha \ge \dfrac{1}{2}, \\ \dfrac{1}{2} - \alpha + \dfrac{n}{2} \cdot \dfrac{\alpha}{1-\alpha}, & \alpha < \dfrac{1}{2}, \end{cases}$$

then $\mathcal{A}_n \cap \mathcal{F}(\widehat{\lambda}) \subset \mathcal{S}^*(\alpha)$.

Proof. First note that $-1/2 < \widehat{\lambda} \le 1$. Next, let $f \in \mathcal{A}_n \cap \mathcal{F}(\widehat{\lambda})$ and consider the function

$$p(z) = \frac{1}{1-\alpha}\left[\frac{zf'(z)}{f(z)} - \alpha\right],$$

which is analytic in $\mathbb{D}$ with $p(0) = 1$. For this function, with

$$\psi(r, s) = \frac{s(1-\alpha)}{(1-\alpha)r + \alpha} + (1-\alpha)r + \alpha \quad \text{and} \quad \Omega = \left\{\omega : \operatorname{Re}\omega > \frac{1}{2} - \widehat{\lambda}\right\},$$

we have

$$\psi(p(z), zp'(z)) = 1 + \frac{zf''(z)}{f'(z)} \in \Omega \quad (z \in \mathbb{D}).$$

Therefore, in view of Lemma 2.4, in order to prove that $f \in \mathcal{S}^*(\alpha)$ it is enough to show that $\psi(ix, y; z) \notin \Omega$, that is,

$$\operatorname{Re}\psi(ix, y; z) = \frac{y\alpha(1-\alpha)}{(1-\alpha)^2x^2 + \alpha^2} + \alpha \le \frac{1}{2} - \widehat{\lambda}$$

or, equivalently,

$$y \le \left(\frac{1}{2} - \widehat{\lambda} - \alpha\right)\left(\frac{\alpha}{1-\alpha} + \frac{1-\alpha}{\alpha} \cdot x^2\right) \tag{3.1}$$

for all $x \in \mathbb{R}$, $y \le -n(1 + x^2)/2$ and $z \in \mathbb{D}$. This happens only when

$$-\frac{n}{2}(1 + x^2) \le \left(\frac{1}{2} - \widehat{\lambda} - \alpha\right)\left(\frac{\alpha}{1-\alpha} + \frac{1-\alpha}{\alpha} \cdot x^2\right),$$

that is, when

$$\left(\frac{1}{2} - \widehat{\lambda} - \alpha\right)\frac{\alpha}{1-\alpha} + \frac{n}{2} + \left[\left(\frac{1}{2} - \widehat{\lambda} - \alpha\right)\frac{1-\alpha}{\alpha} + \frac{n}{2}\right]x^2 \ge 0$$

for all $x \in \mathbb{R}$. The last inequality holds if and only if

$$\left(\frac{1}{2} - \widehat{\lambda} - \alpha\right)\frac{\alpha}{1-\alpha} + \frac{n}{2} \ge 0$$

and

$$\left(\frac{1}{2} - \widehat{\lambda} - \alpha\right)\frac{1-\alpha}{\alpha} + \frac{n}{2} \ge 0.$$

Finally, it easy to verify that $\widehat{\lambda}$ satisfies the two inequalities above. □

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By specifying values of α and n in Theorem 3.1, we deduce the following results.

COROLLARY 3.2.

(i) $\mathcal{C} = \mathcal{F}(1/2) \subset \mathcal{S}^*$ (since $\widehat{\lambda} = \lambda(0, 1) = 1/2$);
(ii) $\mathcal{A}_n \cap \mathcal{F}(\widehat{\lambda}) \subset \mathcal{S}^*(1/2)$ for $\widehat{\lambda} = \min\{n/2, 1\}$;
(iii) $\mathcal{C} = \mathcal{F}(1/2) \subset \mathcal{S}^*(1/2)$ (taking $n = 1$ in (ii));
(iv) $\mathcal{A}_2 \cap \mathcal{F}(1) \subset \mathcal{S}^*(1/2)$ (taking $n = 2$ in (ii)).

We note that (iii) is the well-known Marx–Strohhäcker theorem [13] and that (iv) corresponds to [8, Theorem 2.6i, page 68].

3.1. Coefficients. In [11], Ponnusamy *et al.* gave sharp coefficient bounds and some distortion theorems for $f \in \mathcal{F}(1)$. It was also shown that every partial sum (or section) $s_n(z) = z + \sum_{k=2}^{n} a_k z^k$ of a function $f \in \mathcal{F}(1)$ given by (1.1) belongs to $\mathcal{C}$ in the disc $|z| < 1/6$ and that this radius is the best possible. We extend the coefficient result by finding sharp bounds for the coefficients of the Ozaki close-to-convex functions $\mathcal{F}_O(\lambda)$.

THEOREM 3.3. *Let $f \in \mathcal{F}_O(\lambda)$ be given by (1.1). Then, for $n \geq 2$,*

$$|a_n| \leq \frac{1}{n!} \prod_{k=2}^{n} (k + 2\lambda - 1).$$

The inequality is sharp when $f(z) = f_\lambda(z) = (1/2\lambda)((1/(1 - z)^{2\lambda}) - 1)$.

PROOF. Write

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n := h(z)$$

and let

$$p(z) = \frac{2}{1 + 2\lambda}\Big[h(z) - \frac{1}{2} + \lambda\Big] = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Then $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$, $\operatorname{Re} h(z) > 1/2 - \lambda$ and $|p_n| \leq 2$ for $n \geq 1$ and, since $c_n = (1/2 + \lambda)p_n$, we have $|c_n| \leq 1 + 2\lambda$ for $n \geq 1$.

For each integer n, the coefficients a_n are polynomials with positive coefficients in c_n, so $|a_n|$ will be less than or equal to the result of replacing $|c_n|$ by $1 + 2\lambda$. Thus, by the principle of majorisation (see, for example, [7]),

$$1 + \frac{zf''(z)}{f'(z)} \ll \frac{1 + 2\lambda z}{1 - z}$$

and

$$f(z) \ll \frac{1}{2\lambda}\Big(\frac{1}{(1 - z)^{2\lambda}} - 1\Big) := z + \sum_{n=2}^{\infty} d_n z^n.$$

Therefore,

$$|a_n| \leq d_n = \frac{1}{n!} \prod_{k=2}^{n} (k + 2\lambda - 1),$$

which is (3.1). □

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3.2. Distortion theorems. We next give distortion results for functions $f \in \mathcal{F}_O(\lambda)$.

THEOREM 3.4. *Let $f \in \mathcal{F}_O(\lambda)$. Then, for $z = re^{i\theta} \in \mathbb{D}$,*

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{(1+2\lambda)r}{1-r},$$

$$\frac{1}{(1+r)^{1+2\lambda}} \le |f'(z)| \le \frac{1}{(1-r)^{1+2\lambda}},$$

$$\frac{1}{2\lambda}\Big(\frac{1}{(1+r)^{2\lambda}} - 1\Big) \le |f(z)| \le \frac{1}{2\lambda}\Big(\frac{1}{(1-r)^{2\lambda}} - 1\Big).$$

PROOF. From (1.3),

$$1 + \frac{zf''(z)}{f'(z)} = \Big(\frac{1}{2} + \lambda\Big)p(z) + \frac{1}{2} - \lambda. \tag{3.2}$$

Thus,

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+2\lambda z}{1-z}$$

and so

$$\frac{zf''(z)}{f'(z)} \prec \frac{(1+2\lambda)z}{1-z}.$$

Hence,

$$\frac{zf''(z)}{f'(z)} = \frac{(1+2\lambda)\omega(z)}{1-\omega(z)},$$

where $|\omega(z)| \le |z|$. The first inequality in the theorem now follows.

To prove the inequalities for $|f'(z)|$, we use a result of Suffridge [14, Theorem 3], which states that if F is convex and $zG'(z) \prec zF'(z)$, then $G(z) \prec F(z)$. Using this result, we integrate (3.2) to obtain

$$f'(z) \prec \frac{1}{(1-z)^{1+2\lambda}}.$$

The inequalities for $|f'(z)|$ now follow in the same way.

An application of Lemma 2.5 gives the bounds for $|f(z)|$. □

3.3. Growth and area estimates. For $f \in \mathcal{S}$, $z = re^{i\theta} \in \mathbb{D}$, let $M(r) = \max_{|z|=r} |f(z)|$, $C(r)$ be the curve $f(|z| = r)$, $L(r)$ the length of $C(r)$ and $A(r)$ the area enclosed by $C(r)$. A long-standing problem for functions in $\mathcal{K}$ is whether $M(r)$ can be replaced by $\sqrt{A(r)}$ in the growth estimate $L(r) = O(M(r)\log(1/(1-r)))$ as $r \to 1$, a result already known for functions in $\mathcal{S}^*$. Similarly, replacing $M(r)$ by $\sqrt{A(r)}$ in the known estimate $na_n = O(M((n+1)/n))$ as $n \to \infty$ for functions in $\mathcal{K}$ remains an open question [15, 16].

Since the definition of Ozaki close-to-convex functions does not include an independent starlike function, it is relatively easy to show that both these growth estimates can be improved when $f \in \mathcal{F}_O(\lambda)$, as follows.

THEOREM 3.5. *Let $f \in \mathcal{F}_O(\lambda)$ be given by (1.1), with $M(r)$, $L(r)$ and $A(r)$ defined as above. Then*

$$L(r) = O\Big(\sqrt{A(r)}\log\frac{1}{1-r}\Big) \quad \text{as } r \to 1$$

and

$$na_n = O\Big(\sqrt{A((n+1)/n)}\Big) \quad \text{as } n \to \infty.$$

PROOF. For $z = re^{i\theta}$,

$$L(r) = \int_0^{2\pi} |zf'(z)|\,d\theta \le \int_0^r \int_0^{2\pi} |zf''(z) + f'(z)|\,d\theta\,d\rho,$$

where now $z = \rho e^{i\theta}$. Thus, from (3.2),

$$\begin{aligned} L(r) &\le \Big(\frac{1}{2} + \lambda\Big)\int_0^r \int_0^{2\pi} |f'(z)p(z)|\,d\theta\,d\rho + \Big(\lambda - \frac{1}{2}\Big)\int_0^r \int_0^{2\pi} |f'(z)|\,d\theta\,d\rho \\ &= \Big(\frac{1}{2} + \lambda\Big)I_1(r) + \Big(\lambda - \frac{1}{2}\Big)I_2(r), \quad \text{say.} \end{aligned}$$

We first deal with $I_1(r)$. The Cauchy–Schwarz inequality gives

$$\begin{aligned} I_1(r) &\le \Big(\int_0^r \int_0^{2\pi} |f'(z)|^2\,d\theta\,d\rho\Big)^{1/2}\Big(\int_0^r \int_0^{2\pi} |p(z)|^2\,d\theta\,d\rho\Big)^{1/2} \\ &= O\Big(\sqrt{A(r)}\log\frac{1}{1-r}\Big) \quad \text{as } r \to 1, \end{aligned}$$

since the first integral is $\sqrt{A(r)}$ and since $\int_0^{2\pi} |p(z)|^2\,d\theta \le 2\pi(1+3r^2)/(1-r^2)$ when $p \in \mathcal{P}$ (see, for example, [10]). Applying the Cauchy–Schwarz inequality to $I_2(r)$ gives $\sqrt{A(r)}$, which therefore establishes the first estimate in Theorem 3.4.

For the second estimate, we use Cauchy's theorem to write, with $z = re^{i\theta}$,

$$n^2 a_n = \frac{1}{2\pi r^n}\int_0^{2\pi} z(zf'(z))' e^{-in\theta}\,d\theta$$

and so

$$\begin{aligned} n^2|a_n| &\le \frac{1+2\lambda}{4\pi r^{n-1}}\int_0^{2\pi} |f'(z)p(z)|\,d\theta + \frac{2\lambda - 1}{4\pi r^{n-1}}\int_0^{2\pi} |f'(z)|\,d\theta \\ &= \frac{1+2\lambda}{4\pi r^{n-1}}J_1(r) + \frac{2\lambda-1}{4\pi r^{n-1}}J_2(r), \quad \text{say.} \end{aligned}$$

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For $J_1(r)$, the Cauchy–Schwarz inequality and Parseval's theorem give

$$\begin{aligned} J_1(r) &\le \Big(\int_0^{2\pi} |f'(z)|^2\, d\theta\Big)^{1/2}\Big(\int_0^{2\pi} |p(z)|^2\, d\theta\Big)^{1/2} \\ &= \Big(2\pi \sum_{k=1}^{\infty} k^2|a_k|^2 r^{2k-2}\Big)^{1/2}\Big(\int_0^{2\pi} |p(z)|^2\, d\theta\Big)^{1/2} \\ &\le \Big(2\pi \sum_{k=1}^{\infty} k|a_k|^2 r^k(\max kr^{k-2})\Big)^{1/2}\Big(\int_0^{2\pi} |p(z)|^2\, d\theta\Big)^{1/2} \\ &\le 2\pi\Big(\frac{A(\sqrt{r})}{er^2(1-r)}\Big)^{1/2}\Big(\frac{1+3r^2}{1-r^2}\Big)^{1/2}, \end{aligned}$$

since $kr^{k-2} \le 1/(er^2(1-r))$, again using $\int_0^{2\pi} |p(z)|^2\, d\theta \le 2\pi(1+3r^2)/(1-r^2)$.

Finally, we note that

$$J_2(r) = \int_0^{2\pi} |f'(z)|\, d\theta \le \sqrt{2\pi}\Big(\int_0^{2\pi} |f'(z)|^2\, d\theta\Big)^{1/2},$$

which is the first expression above. Noting that $A(\sqrt{r}) = O(A(r))$ as $r \to 1$, and choosing $r = (n+1)/n$ in the estimates for $J_1(r)$ and $J_2(r)$, the second estimate in Theorem 3.4 follows. □

4. The initial coefficients of functions in $\mathcal{F}_O(\lambda, \alpha)$

From (1.4), we can write

$$1 + \frac{zf''(z)}{f'(z)} = \Big(\frac{1}{2} + \lambda\Big)p(z)^{\beta} + \frac{1}{2} - \lambda$$

and so, by equating coefficients,

$$\begin{aligned} a_2 &= \frac{\beta}{4}(1+2\lambda)p_1, \\ a_3 &= \frac{\beta}{12}(1+2\lambda)\Big(p_2 - \frac{1}{2}(1-2\beta-2\beta\lambda)p_1^2\Big), \\ a_4 &= \frac{\beta}{24}(1+2\lambda)\Big(p_3 - \frac{1}{4}(4-7\beta-6\beta\lambda)p_1p_2 \\ &\qquad + \frac{1}{24}(8-21\beta+16\beta^2-18\beta\lambda+30\beta^2\lambda+12\beta^2\lambda^2)p_1^3\Big). \end{aligned} \tag{4.1}$$

We now obtain sharp bounds for the coefficients a_2, a_3 and a_4.

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THEOREM 4.1. *Let $f \in \mathcal{F}_O(\lambda,\beta)$ and suppose that f is given by (1.1) for $z \in \mathbb{D}$. Then*

$$|a_2| \le \frac{\beta}{2}(1+2\lambda), \quad |a_3| \le \begin{cases} \dfrac{\beta}{6}(1+2\lambda), & 0<\beta\le \dfrac{1}{2(1+\lambda)}, \\ \dfrac{\beta^2}{3}(1+\lambda)(1+2\lambda), & \dfrac{1}{2(1+\lambda)} \le \beta \le 1, \end{cases}$$

$$|a_4| \le \begin{cases} \dfrac{\beta}{12}(1+2\lambda), & 0<\beta\le \sqrt{\dfrac{2}{8+15\lambda+6\lambda^2}}, \\ \dfrac{\beta}{36}(1+2\lambda)(1+8\beta^2+15\beta^2\lambda+6\beta^2\lambda^2), & \sqrt{\dfrac{2}{8+15\lambda+6\lambda^2}} \le \beta \le 1. \end{cases}$$

All the inequalities are sharp.

PROOF. The inequality for $|a_2|$ is trivial, since $|p_1| \le 2$, and is sharp when $p_1 = 2$.

For a_3, we note that since $0 \le 1-2\beta-2\beta\lambda \le 2$ when $0<\beta\le 1/(2(1+\lambda))$, and $1-2\beta-2\beta\lambda < 0$ when $1/(2(1+\lambda)) < \beta \le 1$, the inequalities for $|a_3|$ follow on applying Lemma 2.1. The first inequality for a_3 is sharp when $p_1 = 0$ and $p_2 = 2$, and the second is sharp when $p_1 = 2$ and $p_2 = 2$.

For a_4, we will use Lemma 2.2. In the expression for a_4 in (4.1), let

$$B = (4-7\beta-6\beta\lambda)/8 \quad \text{and} \quad D = (8-21\beta+16\beta^2-18\beta\lambda+30\beta^2\lambda+12\beta^2\lambda^2)/24,$$

so that $0 \le B \le 1$ and $B(2B-1) \le D \le B$ when $0<\beta\le\sqrt{2/(8+15\lambda+6\lambda^2)}$. Thus, applying Lemma 2.2 gives the first inequality for $|a_4|$. Next, write

$$a_4 = \tfrac{1}{24}\beta(1+2\lambda)[p_3 - 2Bp_1p_2 + Bp_1^3 + (D-B)p_1^3]$$

and note that $D-B \ge 0$ when $\sqrt{2/(8+15\lambda+6\lambda^2)} \le \beta \le 4/(7+6\lambda)$. Thus, applying Lemma 2.2 in the case $D = B$ gives the second bound for $|a_4|$, provided $\sqrt{2/(8+15\lambda+6\lambda^2)} \le \beta \le 4/(7+6\lambda)$. Finally, noting that the coefficients of p_1p_2 and p_1^3 in the expression for a_4 in (4.1) are positive when $4/(7+6\lambda) \le \beta \le 1$, and using the inequalities $|p_n| \le 2$ for $n = 1, 2$ and 3, gives the second inequality for $|a_4|$ in this interval. The first inequality for a_4 is sharp when $p_1 = 0$, and the second is sharp when $p_1 = p_2 = p_3 = 2$. □

5. Inverse coefficients of functions in $\mathcal{F}_O(\lambda,\beta)$

For any univalent function f, there exists an inverse function f^{-1} defined on some disc $|\omega| < r_0(f)$ with Taylor expansion

$$f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \cdots. \tag{5.1}$$

Since $\mathcal{F}_O(\lambda,\beta) \subset \mathcal{S}$, inverse coefficients exist for functions $f \in \mathcal{F}_O(\lambda,\beta)$. It is an easy exercise to show from (5.1) that

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= 2{a_2}^2 - a_3, \\ A_4 &= -5{a_2}^3 + 5a_2a_3 - a_4, \end{aligned}$$

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which, on substituting from (4.1), produces

$$
\begin{aligned}
A_2 &= -\frac{\beta}{4}(1+2\lambda)p_1,\\
A_3 &= -\frac{\beta}{12}(1+2\lambda)\Big(p_2 - \frac{1}{2}(1+\beta+4\beta\lambda)p_1^2\Big),\\
A_4 &= -\frac{\beta}{24}(1+2\lambda)\Big(p_3 - \frac{1}{4}(4+3\beta+14\beta\lambda)p_1p_2\\
&\qquad + \frac{1}{24}(8+9\beta+\beta^2+42\beta\lambda+30\beta^2\lambda+72\beta^2\lambda^2)p_1^3\Big).
\end{aligned}
\tag{5.2}
$$

We can now prove the following result.

THEOREM 5.1. *Let $f \in \mathcal{F}_O(\lambda,\beta)$, with inverse function f^{-1} given by (5.1). Then*

$$
|A_2| \le \frac{\beta}{2}(1+2\lambda), \quad |A_3| \le \begin{cases} \dfrac{\beta}{6}(1+2\lambda), & 0<\beta\le \dfrac{1}{1+4\lambda},\\[2ex] \dfrac{\beta^2}{6}(1+2\lambda)(1+4\lambda), & \dfrac{1}{1+4\lambda} \le \beta \le 1,\end{cases}
$$

$$
|A_4| \le \begin{cases} \dfrac{\beta}{12}(1+2\lambda), & 0<\beta\le 2\sqrt{\dfrac{1}{1+30\lambda+72\lambda^2}},\\[2ex] \dfrac{\beta}{72}(1+2\lambda)(2+\beta^2+30\beta^2\lambda+72\beta^2\lambda^2), & 2\sqrt{\dfrac{1}{1+30\lambda+72\lambda^2}} \le \beta \le 1.\end{cases}
$$

All the inequalities are sharp.

PROOF. The inequality for $|A_2|$ is obvious and is sharp when $p_1 = 2$.

For A_3,

$$|A_3| \le \frac{\beta}{12}(1+2\lambda)\Big|p_2 - \frac{1}{2}(1+\beta+4\beta\lambda)p_1^2\Big|$$

and an application of Lemma 2.1 easily gives the inequalities for $|A_3|$, the first of which is sharp when $p_1 = 0$ and $p_2 = 2$, and the second when $p_1 = 2$ and $p_2 = 2$.

For A_4, from (5.2),

$$
\begin{aligned}
A_4 = -\frac{\beta}{24}(1+2\lambda)\Big[p_3 - \frac{1}{4}(4+3\beta+14\beta\lambda)p_1p_2\\
+ \frac{1}{24}(8+9\beta+\beta^2+42\beta\lambda+30\beta^2\lambda+72\beta^2\lambda^2)p_1^3\Big].
\end{aligned}
$$

We will use Lemma 2.2 with

$$B = \tfrac{1}{8}(4+3\beta+14\beta\lambda) \quad \text{and} \quad D = \tfrac{1}{24}(8+9\beta+\beta^2+42\beta\lambda+30\beta^2\lambda+72\beta^2\lambda^2).$$

Thus, $0 \le B \le 1$ when either

$$0<\beta\le\frac{4}{17} \quad\text{and}\quad \frac{1}{2}\le\lambda\le 1, \quad\text{or}\quad \frac{4}{17}<\beta\le\frac{2}{5} \quad\text{and}\quad \frac{1}{2}\le\lambda\le\frac{4-3\beta}{14\beta}.$$

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Since $1/2 \le \lambda \le (4-3\beta)/(14\beta)$ when $1/2 \le \lambda \le 1$ and $4/17 \le \beta \le 4/(3+14\lambda)$, it follows that $0 \le B \le 1$ is satisfied when $0 < \beta \le 4/(3+14\lambda)$. Also, $B(2B-1) \le D \le B$ when $1/2 \le \lambda \le 1$ and $0 < \beta \le 2\sqrt{1/(1+30\lambda+72\lambda^2)}$. Since

$$2\sqrt{1/(1+30\lambda+72\lambda^2)} \le 4/(3+14\lambda) \quad \text{when } 1/2 \le \lambda \le 1,$$

we can apply Lemma 2.2 to obtain the first inequality in Theorem 5.1 over the range $0 < \beta \le 2\sqrt{1/(1+30\lambda+72\lambda^2)}$.

We next consider the interval $2\sqrt{1/(1+30\lambda+72\lambda^2)} \le \beta \le 4/(3+14\lambda)$. Write

$$A_4 = -\frac{\beta}{24}(1+2\lambda)[p_3 - 2Bp_1p_2 + Bp_1^3 + (D-B)p_1^3]. \tag{5.3}$$

Note that $D - B \ge 0$ when $2\sqrt{1/(1+30\lambda+72\lambda^2)} \le \beta \le 1$. Since $0 \le B \le 1$ is satisfied when $0 < \beta \le 4/(3+14\lambda)$, applying Lemma 2.2 to (5.3) in the case $B = D$ gives the second inequality in Theorem 5.1 when $2\sqrt{1/(1+30\lambda+72\lambda^2)} \le \beta \le 4/(3+14\lambda)$.

Thus, we are left with the interval $4/(3+14\lambda) < \beta \le 1$. We use Lemma 2.3 with $\mu = \beta(3+14\lambda)/4$, so that

$$\begin{aligned} A_4 = -\frac{\beta}{24}(1+2\lambda)\Big[&p_3 - (\mu+1)p_1p_2 + \mu p_1^3 \\ &+ \frac{1}{24}(8 - 9\beta + \alpha^2 - 42\beta\lambda + 30\beta^2\lambda + 72\beta^2\lambda^2)p_1^3\Big]. \end{aligned}$$

Note that $8 - 9\beta + \beta^2 - 42\beta\lambda + 30\beta^2\lambda + 72\beta^2\lambda^2 \ge 0$, when $1/2 \le \lambda \le 1$ and $0 < \beta \le 1$. Also $\mu > 1$ when $4/(3+14\lambda) < \beta \le 1$, and $2\mu - 1 \ge 0$ when $2/(3+14\lambda) \le \beta \le 1$ (which contains the interval $4/(3+14\lambda) < \beta \le 1$). So applying Lemma 2.3 gives the second inequality for $|A_4|$ when $4/(3+14\lambda) < \beta \le 1$.

The first inequality for A_4 is sharp when $p_1 = 0$, and the second is sharp when $p_1 = 2$, $p_2 = 2$ and $p_3 = 2$. □

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