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CONSTANCY OF NEWTON POLYGONS OF F-ISOCRYSTALS ON ABELIAN VARIETIES AND ISOTRIVIALITY OF FAMILIES OF CURVES

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Abstract We prove constancy of Newton polygons of all convergent F-isocrystals on Abelian varieties over finite fields. Applying the constancy, we prove the isotriviality of proper smooth families of curves over Abelian varieties. More generally, we prove the isotriviality over projective smooth varieties on which any convergent F-isocrystal has constant Newton polygons.

Keywords: F-isocrystal; Newton polygon; isotriviality of families of curves

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1. Introduction

The variation of Newton polygons of convergent F-isocrystals on algebraic varieties of characteristic $p > 0$ is mysterious. It may depend on arithmetic and geometry of varieties. In this paper we study constancy (i.e., non-variation) of Newton polygons in projective smooth cases. We prove constancy of Newton polygons of all convergent F-isocrystals on Abelian varieties over finite fields. We also give several examples of projective smooth varieties such that any convergent F-isocrystal on the variety has constant Newton polygons. Applying the constancy, we prove the isotriviality of proper smooth families of curves on such varieties. The author wishes this study gives a new aspect on global property of Frobenius slopes.

1.1. Problems

Let k be a perfect field of characteristic p, R a complete discrete valuation ring with residue field $k = R/\mathbf{m}$ where $\mathbf{m}$ is the maximal ideal of R, and K the field of fractions of R which is of mixed characteristic $(0, p)$. Let ord_p be the discrete valuation on K normalized by $\mathrm{ord}_p(p) = 1$. Let σ be a q-Frobenius on K for a positive power q of p, that is, a continuous lift of the q-power Frobenius on k. (See Remark 2.1 for the detail of Frobenius σ.)

Let X be a scheme separated of finite type over $\operatorname{Spec} k$, and $\mathcal{M}$ a convergent F-isocrystal on X/K with respect to Frobenius σ (we simply say a convergent F-isocrystal on X/K if there is no ambiguity of the choice of Frobenius σ) which is introduced by Berthelot [2–4]. For a point $x \in X$ with a geometric point $\overline{x}$ above x, we define a polygon $\mathrm{NP}(\mathcal{M}, x)$, called Newton polygon of $\mathcal{M}$ at x, by the Newton polygon of the F-space $i_{\overline{x}}^*\mathcal{M}$ over $K(\overline{x})$ in the theory of Dieudonné–Manin. Here $k(\overline{x})$ is a function field of $\overline{x}$ and $K(\overline{x})$ is an extension of K with the residue field $k(\overline{x})$ as a discrete valuation field with the same valuation group, and $i_{\overline{x}} : \operatorname{Spec} k(\overline{x}) \to X$ is the canonical morphism. We normalize slopes as $\mathrm{ord}_p(a)/\mathrm{ord}_p(q)$ for the Frobenius $\varphi = a\sigma$ $(a \in K, a \neq 0)$ on an F-space of rank one over K (note that q is the p-power of Frobenius σ). We regard the application $x \mapsto \mathrm{NP}(\mathcal{M}, x)$ as a function on the scheme-theoretic points of X. The smallest slope of $\mathrm{NP}(\mathcal{M}, x)$ is called the initial slope of $\mathcal{M}$ at x.

Definition 1.1. Let $f : X \to \operatorname{Spec} k$ be a morphism separated of finite type.

(1) A convergent F-isocrystal $\mathcal{M}$ on X/K is said to be constant if $\mathcal{M} \cong f^*\mathcal{N}$ for some F-isocrystal $\mathcal{N}$ on $\operatorname{Spec} k/K$. We denote by $F\text{-Isoc}(X/K)^{\mathrm{CST}}$ the full subcategory of constant convergent F-isocrystals in the category $F\text{-Isoc}(X/K)$ of convergent F-isocrystals on X/K.

(2) A convergent F-isocrystal $\mathcal{M}$ on X/K is said to have constant Newton polygons if the application $x \mapsto \mathrm{NP}(\mathcal{M}, x)$ is constant on X. We denote by $F\text{-Isoc}(X/K)^{\mathrm{CNP}}$ the full subcategory of convergent F-isocrystals with constant Newton polygons in $F\text{-Isoc}(X/K)$.

Our problems are as follows.

Problem 1.2. Let X be a smooth geometrically connected scheme over $\operatorname{Spec} k$, and $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ the base change for an algebraic closure $\overline{k}$ of k.

(1) Is the condition $F\text{-Isoc}(X/K) = F\text{-Isoc}(X/K)^{\mathrm{CST}}$ equivalent to the triviality of geometric etale fundamental group of X, i.e., $\pi_1^{\mathrm{et}}(X_{\overline{k}}) = \{1\}$? Here $\pi_1^{\mathrm{et}}(-)$ denotes the etale fundamental group and we usually omit to indicate a base point because we do not need the specific base point in this paper.

(2) Classify varieties X such that any convergent F-isocrystal on X/K has constant Newton polygons, i.e.,

$$F\text{-Isoc}(X/K) = F\text{-Isoc}(X/K)^{\mathrm{CNP}},$$

and study properties of such varieties.

By definition there are natural inclusion relations

$$F\text{-Isoc}(X/K)^{\mathrm{CST}} \subset F\text{-Isoc}(X/K)^{\mathrm{CNP}} \subset F\text{-Isoc}(X/K).$$

When $X = \mathbb{P}_k^1$ is the projective line, any convergent F-isocrystal on X/K is constant. Indeed, there is a natural isomorphism

$$\mathcal{M} \cong H^0_{\mathrm{rig}}(\mathbb{P}_k^1/K, \mathcal{M}) \otimes_K \mathcal{O}_{]\mathbb{P}_k^1[}$$

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as F-isocrystals. Note that the isomorphism above holds without assuming the existence of Frobenius structures. On the other hand there exists a projective smooth curve C over a finite field which has a convergent F-isocrystal on C/K with nonconstant Newton polygons (see § 4.5 for the detail), and hence the inclusion relations

$$F\text{-Isoc}(C/K)^{\mathrm{CST}} \subsetneq F\text{-Isoc}(C/K)^{\mathrm{CNP}} \subsetneq F\text{-Isoc}(C/K)$$

hold. The first inequality holds when $\pi_1^{\mathrm{et}}(C_{\overline{k}})$ is nontrivial by Katz–Crew equivalence between the category of p-adic continuous representations of $\pi_1^{\mathrm{et}}(C)$ and that of unit-root convergent F-isocrystals on C/K [11, Theorem 2.1].

Our main interest is the mysterious gap of two categories $F\text{-Isoc}(X/K)^{\mathrm{CNP}} \subset F\text{-Isoc}(X/K)$. In this paper we give a new example of Problem 1.2(2), that is, we prove that the inclusion relations

$$F\text{-Isoc}(X/K)^{\mathrm{CST}} \subsetneq F\text{-Isoc}(X/K)^{\mathrm{CNP}} = F\text{-Isoc}(X/K)$$

hold for any Abelian variety X over a spectrum $\mathrm{Spec}\, k$ of a finite field k (see § 1.2).

Concerning Problem 1.2(1), T.Abe–H.Esnault and K.S.Kedlaya gave the affirmative answer when k is a finite field applying the companion theorem [1, 35, 36]. In the case of convergent isocrystals (without Frobenius structures) Esnault–Shiho studied the constancy problem, called de Jong conjecture [19, Conjecture 2.1]. They proved the de Jong conjecture under certain hypotheses for a general perfect field k [19, 20, 52]. Moreover the constancy of geometric convergent isocrystals (see Definition 2.10) is proved in [20, Theorem 1.3]. As mathematical statements, the case of convergent isocrystals is much stronger than that of convergent F-isocrystals. However, our interests in this paper are the variation of Newton polygons of Frobenius structures and its application to geometry. Returning to our problems, if X is proper smooth, then the triviality $\pi_1^{\mathrm{et}}(X_{\overline{k}}) = \{1\}$ is equivalent to the coincidence

$$F\text{-Isoc}(X/K)^{\mathrm{CST}} = F\text{-Isoc}(X/K)^{\mathrm{CNP}}$$

of categories (Corollary 2.17). At this moment the author does not know whether Problem 1.2(1), i.e., $F\text{-Isoc}(X/K)^{\mathrm{CNP}} = F\text{-Isoc}(X/K)$, is valid or not in general even when $\pi_1^{\mathrm{et}}(X_{\overline{k}}) = \{1\}$ except the case where k is finite. Moreover, he would like to find a proof by using only p-adic methods.

The author is ignorant of classifications of projective smooth curves of genus $\geqslant 2$ in this aspect. He does not know whether the constancy problem of Newton polygons depends on the base field k or not in general.

Remark 1.3. For the following projective smooth and connected variety X over an arbitrary field k, any convergent F-isocrystal on X/K is constant:

(1) X admits a projective smooth lift $\mathcal{X}$ over $\mathrm{Spec}\, R$ such that $\pi_1^{\mathrm{et}}(\mathcal{X}_{\overline{K}}) = \{1\}$ for the geometric generic fiber $\mathcal{X}_{\overline{K}}$.

(2) X is separably rationally connected (see the definition in [37, IV, Definition 3.2]).

Indeed, in the case (1) any convergent isocrystal on X/K is constant, by rigid and complex GAGA principles of de Rham cohomologies and [25, 40] (see Introduction of [19]), so that

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it is also constant for convergent F-isocrystals. In the case (2) we may assume that X is geometrically connected, and then any two points are connected by rational curves [37, IV,Theorem 3.9]. Hence any convergent F-isocrystal has constant Newton polygons. Since the works of de Jong–Starr [14] and Kóllar (see [12, Corollary 3.6]) imply the simply connectedness $\pi_1^{\mathrm{et}}(X_{\bar{k}}) = \{1\}$, the assertion follows from Corollary 2.17.

1.2. Constancy results

Our main result is the following:

Theorem 1.4 (Theorem 3.7). *Let k be a finite field, and X an Abelian variety over* $\mathrm{Spec}\, k$*. Then any convergent F-isocrystal on X/K has constant Newton polygons, i.e.,*

$$F\text{-Isoc}(X/K)^{\mathrm{CNP}} = F\text{-Isoc}(X/K).$$

The crucial idea of the proof is as follows. Let C be a projective smooth and geometrically connected curve of genus $\geqslant 1$ over $\mathrm{Spec}\, k$, $\mathcal{M}$ a convergent F-isocrystal on C/K, and $D_{\mathcal{M}}$ a reduced divisor of C consisting of points x at which the Newton polygon $\mathrm{NP}(\mathcal{M}, x)$ is different from that at the generic point. Estimating the degree of $D_{\mathcal{M}}$ in two ways, the congruence of the L-function $L(X/k, \mathcal{M}; t)$ of $\mathcal{M}$ modulo $\mathbf{m}$ and the Euler–Poincaré formula (Proposition 3.1), we have an inequality

$$\deg(D_{\mathcal{M}}) \leqslant B$$

for the degree of $D_{\mathcal{M}}$ by a constant B which depends only on the cardinality of k, the genus of C and the rank of $\mathcal{M}$ (Theorem 3.3 and Remark 3.5).

Now let X be an Abelian variety over $\mathrm{Spec}\, k$. Let $\mathcal{M}$ be a convergent F-isocrystal on X/K, and $D_{\mathcal{M}}$ a set of X consisting of points at which the Newton polygon of $\mathcal{M}$ is different from that at the generic point of X. When $D_{\mathcal{M}} \neq \emptyset$, $D_{\mathcal{M}}$ is a closed subscheme in X purely of codimension 1 by de Jong–Oort purity theorem [13, Theorem 4.1]. Suppose $D_{\mathcal{M}} \neq \emptyset$. Then there exists a projective smooth and geometrically connected curve C of genus $\geqslant 1$ in X (after we replace k by a finite extension) such that the degrees of the reduced divisors $(C \cap D_{[n]^*\mathcal{M}})_{\mathrm{red}}$ are unbounded on n, where $[n]$ means the morphism of multiplication on X with a positive integer n. Since the rank of $[n]^*\mathcal{M}$ is stable for n, this contradicts the existence of bound with respect to the k-curve C.

1.3. Application to the isotriviality of families of curves

We will apply our study on constancy of Newton polygons to the isotriviality problem of families of curves in characteristic $p > 0$. At first we recall the definition and review the case of complex algebraic varieties.

Definition 1.5. Let S be a scheme separated of finite type over $\mathrm{Spec}\, k$. A smooth family X over S is isotrivial if, for any geometric points $s, t \in S(\bar{k})$, the geometric fibers X_s and X_t of X at s and t, respectively, are isomorphic to each other as schemes over $\mathrm{Spec}\, \bar{k}$. Here $\bar{k}$ is an algebraic closure of k.

In the case of complex algebraic varieties, the following results are known. Let X be a proper smooth family of connected curves of genus $g \geqslant 2$ over a complex algebraic

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variety S. If a projective smooth S is either a simply connected variety (more generally the topological fundamental group $\pi_1^{\mathrm{top}}(S)$ is finite), an Abelian variety, a uniruled surface over a curve of genus $\leqslant 1$, or a surface of Kodaira dimension 0, then the family X is isotrivial over S (see the detail in [15]). Note that when $S = \mathbb{C}$ or $\mathbb{C}^\times$ as a complex algebraic variety, the isotriviality also holds since the coarse moduli space $\mathfrak{M}_g$ of curves of genus g is hyperbolic.

When k is a field of characteristic $p > 0$, the isotriviality of families of curves over a projective smooth curve of genus $\leqslant 1$ is known by L.Szpiro [53, Théorème 4]. More generally he studied the isotriviality problem for families of semistable curves.

Our result is as follows.

Theorem 1.6 (Theorem 4.5). *Let S be a projective smooth and connected scheme over* Spec k*, and $f : X \to S$ a proper smooth family of connected curves. Suppose that any geometric convergent F-isocrystal $\mathcal{M}$ on S/K (see Definition 2.10) has constant Newton polygons. Then the family X over S is isotrivial.*

If there exists at least a fiber which is an ordinary curve (see Definition 4.1) in the family X over S, then X is a family of ordinary curves over S by the constancy hypothesis of Theorem 1.6. In this case the Torelli theorem implies the isotriviality of the family since the ordinary locus of coarse moduli space of Abelian varieties is quasi-affine [44, XI, Théorème 5.2]. When there exists no ordinary point in S, we apply A.Tamagawa and M.Saïdi's works [49, 54] : there exists a finite etale covering $f' : Y' \to X'$ after a finite etale base change S' of S such that the new part $J(Y', X') = J(Y')/(f')^*J(X')$ of the relative Jacobian variety over S' is ordinary. Then $J(Y', X')$ over S' is isotrivial, and hence Torelli theorem for new parts implies the isotriviality of X over S [54, Corollary 4.7].

Corollary 1.7. *Suppose that S is a projective smooth and geometrically connected scheme over* Spec k *satisfying one of the following:*

(1) $\pi_1^{\mathrm{et}}(S_{\bar{k}})$ *is finite.*

(2) (Corollary 4.6) *S is an Abelian variety.*

(3) *S is either a ruled surface over a curve of genus $\leqslant 1$ or a projective smooth surface of Kodaira dimension* 0*.*

Then any proper smooth family of connected curves over S is isotrivial.

In order to prove (1) it is sufficient to prove the assertion when S is simply connected, i.e., the geometric etale fundamental group $\pi_1^{\mathrm{et}}(S_{\bar{k}})$ is trivial. Then any geometric convergent isocrystal on S/K is constant by [20, Theorem 1.3], so that it is also constant as a convergent F-isocrystal. The isotriviality follows from Theorem 4.5. (3) follows from the classification theorem of surfaces (see [39], for example), the previous (1), (2) and Propositions 2.8 and 2.9.

The converse of Theorem 1.6 is also an interesting problem.

The content of the paper is as follows. We study basic properties of variation of Newton polygons in § 2, and give a proof of existence of slope filtrations of F-isocrystals with constant Newton polygons in our context in Appendix A (see § 2.3). In § 3 we prove the

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constancy of Newton polygons for any convergent F-isocrystals on an Abelian variety over a finite field. In § 4 we apply our results on the constancy of Newton polygons to the isotriviality of families of curves.

2. Newton polygon

In this section we study several properties of Newton polygons.

2.1. Frobenius σ

Let k be a perfect field of characteristic $p > 0$, R a complete discrete valuation ring with residue field $k = R/\mathbf{m}$, and K the field of fractions of R which is of mixed characteristic $(0, p)$. We fix a discrete valuation ord_p on K (and its extension as valuation fields) which is normalized by $\mathrm{ord}_p(p) = 1$. Let σ be a q-Frobenius on K for a positive power q of p, that is, a continuous lift of the q-power Frobenius on k.

When we consider Katz–Crew equivalence between continuous p-adic representations and unit-root F-isocrystals [11, Theorem 2.1], we assume that

(i) $\mathbb{F}_q \subset k$ and

(ii) $K_\sigma \otimes_{W(\mathbb{F}_q)} W(k) \cong K$,

where $\mathbb{F}_q$ is the finite field of q-elements, K_σ is the σ-invariant subfield of K and $W(k)$ is the ring of Witt vectors with coefficients in k. In this case we denote the ring of integers of K_σ (respectively the maximal ideal, respectively a uniformizer of K_σ) by R_σ (respectively $\mathbf{m}_\sigma$, respectively π).

Remark 2.1. The above hypotheses (i) and (ii) on K and σ always hold if we replace K by a finite unramified extension K' of K. Indeed, for any q-Frobenius σ, there is a uniformizer π of the p-adic completion $\widehat{K}^{\mathrm{ur}} = K \otimes_{W(k)} W(\overline{k})$ of a maximal unramified extension K^{ur} of K such that π is σ-fixed, i.e., $\sigma(\pi) = \pi$. Here σ also denotes the unique extension of Frobenius σ on $\widehat{K}^{\mathrm{ur}}$. Indeed, for a uniformizer π' of K, one can solve the equation $\sigma(u\pi') = u\pi'$ on u in $\widehat{K}^{\mathrm{ur}}$. Since $\widehat{K}^{\mathrm{ur}}$ is a finite extension of $W(\overline{k})[1/p]$ generated by π and the σ-invariant subring of $W(\overline{k})$ is $W(\mathbb{F}_q)$, $(\widehat{K}^{\mathrm{ur}})_\sigma$ is a finite extension of $\mathbb{Q}_p$ and π is algebraic over K. If we put a composite $K' = K(\widehat{K}^{\mathrm{ur}})_\sigma$ of fields, then K' is unramified over K by $K' \subset \widehat{K}^{\mathrm{ur}}$ and $(K')_\sigma \otimes_{W(\mathbb{F}_q)} W(k') \cong K'$ for some finite extension k' of k with $\mathbb{F}_q \subset k'$. The existence of a σ-invariant uniformizer after a finite unramified extension also holds even when k is not perfect (see the beginning of A.2 in the appendix).

2.2. Variation of Newton polygons

Let us recall Dieudonné–Manin classification and Newton polygons of F-spaces over a local field briefly (see [41, Chapter 2, Theorem 2.1] and [16]).

Let M be an F-space over K with respect to Frobenius σ, that is, a K-vector space M of finite dimension with a σ-linear bijection

$$F : M \to M$$

which is called Frobenius. Suppose that the residue field k is algebraically closed. Then the category of F-spaces is semi-simple with simple objects

$$E_{s/r} = K[F]/(F^r - \pi^s) \quad \text{for } (r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z} \quad \text{with } (r, s) = 1$$

where the relation $Fa = \sigma(a)F$ holds for any $a \in K$ in $K[F]$, by Dieudonné–Manin's classification. The rational number $s\mathrm{ord}_p(\pi)/r\mathrm{ord}_p(q)$ is called slope of $E_{r,s}$ and its rank over K is r. If $M = \bigoplus_i E_{r_i/s_i}^{m_i}$ with $s_1/r_1 < s_2/r_2 < \cdots < s_l/r_l$, then the Newton polygon of M is the lower convex hull of points

$$(0,0), (m_1r_1, m_1s_1\mathrm{ord}_p(\pi)/\mathrm{ord}_p(q)), (m_1r_1 + m_2r_2, (m_1s_1 + m_2s_2)\mathrm{ord}_p(\pi)/\mathrm{ord}_p(q)), \ldots,$$
$$(m_1r_1 + \cdots + m_lr_l, (m_1s_1 + \cdots + m_ls_l)\mathrm{ord}_p(\pi)/\mathrm{ord}_p(q)).$$

Note that if $F = q\sigma$ of rank 1, then it is of slope 1. Our definition of slopes is stable under any extension of the field K and the change of power of Frobenius σ.

Let X be a scheme separated of finite type over $\mathrm{Spec}\, k$ and $\mathcal{M}$ a convergent F-isocrystal on X/K. For a point $x \in X$ (not necessary a closed point) and for a geometric point $\overline{x}$ with a natural morphism $i_{\overline{x}} : \overline{x} \to X$, we define the Newton polygon of $\mathcal{M}$ at x by that of F-space $i_{\overline{x}}^*\mathcal{M}$ over $K(\overline{x})$. Here $k(\overline{x})$ is the field of functions of $\overline{x}$, and $K(\overline{x})$ is the extension of K as complete discrete valuation fields such that the residue field is $k(\overline{x})$. It is independent of the choice of geometric point $\overline{x}$ above x. We denote Newton polygon of $\mathcal{M}$ at x by $\mathrm{NP}(\mathcal{M}, x)$. Then

$$x \in X \mapsto \mathrm{NP}(\mathcal{M}, x)$$

is a function on the set of points of X.

Let α and β be Newton polygons. We say that α is above β, denote it by $\alpha \prec \beta$, if α and β have same endpoints and all polygons of α are upper than or equal to that of β.

Proposition 2.2 (Grothendieck's specialization theorem [29, Theorem 2.3.1] [10, Theorem 2.1]). *With the notation as above we have the following.*

(1) *Let $x, y \in X$. If x is a specialization of y, then* $\mathrm{NP}(\mathcal{M}, x) \prec \mathrm{NP}(\mathcal{M}, y)$.

(2) *Suppose X is irreducible. Then the set*

$$U = \left\{ x \in X \;\middle|\; \begin{array}{l}\text{The initial slopes of } \mathcal{M} \text{ at } x \text{ and} \\ \text{that at the generic point coincide.}\end{array} \right\}$$

is open in X.

Proof. (2) Let $y \in X \setminus U$, and x any specialization of y. Then the initial slope at x is greater than or equal to the initial slope at y by (1). Hence $x \in X \setminus U$. Since y is arbitrary, $X \setminus U$ is closed. □

Another important property of variation of Newton polygons is the purity theorem by de Jong and Oort.

Theorem 2.3 [13, Theorem 4.1]. *Let X be a smooth irreducible scheme separated of finite type and η the generic point of X. For a convergent F-isocrystal $\mathcal{M}$ on X/K, any generic point of the set-theoretical complement of the open subscheme*

$$U_{\mathcal{M}} = \{x \in X \mid \mathrm{NP}(\mathcal{M}, x) = \mathrm{NP}(\mathcal{M}, \eta)\}$$

is of codimension 1 in X.

2.3. Slope filtrations

Definition 2.4. Let X be a scheme separated of finite type over $\operatorname{Spec} k$, and $\mathcal{M}$ a convergent F-isocrystal on X/K.

(1) $\mathcal{M}$ is said to be isoclinic of slope $\gamma \in \mathbb{Q}$ if, for any geometric point $\overline{x}$ of X with a natural morphism $i_{\overline{x}} : \overline{x} \to X$, the F-space $i_{\overline{x}}^{*}\mathcal{M}$ over $K(\overline{x})$ is a direct sum of copies of $E_{\gamma \operatorname{ord}_p(q)/\operatorname{ord}_p(\pi)}$. $\mathcal{M}$ is said to be unit-root if it is isoclinic of slope 0.

(2) An increasing filtration $\{S_\lambda \mathcal{M}\}_{\lambda \in \mathbb{Q}}$ of $\mathcal{M}$ as convergent F-isocrystals on X/K is called the slope filtration if it satisfies the following conditions

   (i) $S_\lambda \mathcal{M} = 0$ for $\lambda \ll 0$, $S_\lambda \mathcal{M} = \mathcal{M}$ for $\lambda \gg 0$ and $S_{\lambda+}\mathcal{M} = S_\lambda \mathcal{M}$ for any λ;

   (ii) $S_\lambda \mathcal{M}/S_{\lambda-}\mathcal{M}$ is either 0 or isoclinic of slope λ for any λ,

   where $S_{\lambda-}\mathcal{M} = \bigcup_{\mu<\lambda} S_\mu \mathcal{M}$ and $S_{\lambda+}\mathcal{M} = \bigcap_{\mu>\lambda} S_\mu \mathcal{M}$.

The existence of slope filtrations of F-crystals on $\operatorname{Spec} k[[t]]$ having constant Newton polygons up to isogenies was proved in [29, Corollary 2.6.3], and the case of unipotent convergent F-isocrystals was proved in [6, Théorème 3.2.3]. The existence of generic slope filtration for convergent F-isocrystals is proved in [51, Proposition 5.8]. In general case the existence of slope filtration is given in [35, Corollary 4.2] with brief indication of the proof. The author does not find a detailed proof in any published literature. So we prove the following in Appendix A.

Theorem 2.5. *Let X be a smooth scheme separated of finite type over k, and $\mathcal{M}$ a convergent F-isocrystal on X/K. Suppose that the initial slope (i.e., the smallest slope) of $\mathcal{M}$ of any generic point of X is γ and the rank of slope γ of F-isocrystal $i_x^{*}\mathcal{M}$ $(x \in X)$ is constant on X, say the rank is r. Then there exists a convergent sub F-isocrystal $\mathcal{L}$ of $\mathcal{M}$ on X/K which is isoclinic of slope γ and of rank r.*

Corollary 2.6. *Let X be a smooth scheme separated of finite type over* $\operatorname{Spec} k$, *and $\mathcal{M}$ a convergent F-isocrystal on X/K which has constant Newton polygons. Then $\mathcal{M}$ admits a unique slope filtration $\{S_\lambda \mathcal{M}\}_{\lambda \in \mathbb{Q}}$.*

2.4. Properties of constancy of Newton polygons

Let X be a scheme separated of finite type over $\operatorname{Spec} k$, and $\mathcal{M}$ a convergent F-isocrystal on X/K. We define the convergent cohomology of X by

$$H^i_{\mathrm{conv}}(X/K, \mathcal{M}) = \mathbb{H}^i(]X[_{\mathcal{P}}, \mathcal{M} \otimes \Omega^\bullet_{]X[_{\mathcal{P}}/K}).$$

Here we take a closed immersion $X \to \mathcal{P}$ into a smooth formal scheme $\mathcal{P}$ over $\operatorname{Spf} R$, $]X[_{\mathcal{P}}$ is the associated rigid analytic tube [3, Section 1.1], and $\mathbb{H}^i(]X[_{\mathcal{P}}, \mathcal{M} \otimes \Omega^\bullet_{]X[_{\mathcal{P}}/K})$ is the hypercohomology of the de Rham complex $\mathcal{M} \otimes \Omega^\bullet_{]X[_{\mathcal{P}}/K}$ associated to the convergent isocrystal $\mathcal{M}$. The convergent cohomology is independent of the choice of the closed immersion $X \to \mathcal{P}$. In [7, 58] the convergent cohomology $H^i_{\mathrm{conv}}(X/K, \mathcal{M})$ is denoted by $H^i_{\mathrm{rig}}((X, X)/K, \mathcal{M})$, the rigid cohomology of X overconvergent along $\emptyset$, namely, we do not consider the overconvergent regularity along boundary. When $\mathcal{M} = \mathcal{O}_{]X[}$ (the unit convergent F-isocrystal), we simply denote the convergent cohomology by $H^i_{\mathrm{conv}}(X/K)$.

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If X is proper over $\operatorname{Spec} k$, then the convergent cohomology $H^i_{\mathrm{conv}}(X/K, \mathcal{M})$ is nothing but the rigid cohomology $H^i_{\mathrm{rig}}(X/K, \mathcal{M})$. The convergent cohomology is furnished with a σ-linear homomorphism

$$F : H^i_{\mathrm{conv}}(X/K, \mathcal{M}) \to H^i_{\mathrm{conv}}(X/K, \mathcal{M})$$

which is called Frobenius. In general $H^i_{\mathrm{conv}}(X/K, \mathcal{M})$ may not be of finite dimension over K and F may not act on it bijectively except $i = 0$. For a K-vector space with σ-linear homomorphism $F : V \to V$, we define a K-space V_{fin} by the subspace of V consisting of elements w such that there exist elements $a_1, a_2, \ldots, a_n \in K$ with $a_n \neq 0$ satisfying $F^n w + a_1 F^{n-1} w + \cdots + a_n w = 0$. If $W \subset V$ is a finite dimensional F-stable subspace of V such that $F : W \to W$ is surjective, then $W \subset V_{\mathrm{fin}}$. Hence, V_{fin} is an F-stable K-subspace of V. In the case of rigid cohomology we have an equality $H^i_{\mathrm{rig}}(X/K, \mathcal{M})_{\mathrm{fin}} = H^i_{\mathrm{rig}}(X/K, \mathcal{M})$ by finite dimensionality and the bijectivity of Frobenius F ([57, Theorem 5.1.1] for constant cases, and [31, Theorem 1.2.1, 1.2.3] for general coefficients). Note that the bijectivity of Frobenius on rigid cohomology follows from the finiteness and Poincaré duality.

Proposition 2.7. *Let $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$ be an exact sequence of $F\text{-Isoc}(X/K)$.*

(1) *$\mathcal{M}$ is an object in $F\text{-Isoc}(X/K)^{\mathrm{CNP}}$ if and only if so are both $\mathcal{L}$ and $\mathcal{N}$.*

(2) *If $\mathcal{M}$ is an object in $F\text{-Isoc}(X/K)^{\mathrm{CST}}$, then so are both $\mathcal{L}$ and $\mathcal{N}$. Suppose furthermore that $H^1_{\mathrm{conv}}(X/K)_{\mathrm{fin}} = 0$. Then the converse holds.*

In particular, the category $F\text{-Isoc}(X/K)^{\mathrm{CNP}}$ (respectively $F\text{-Isoc}(X/K)^{\mathrm{CST}}$) is an Abelian subcategory of $F\text{-Isoc}(X/K)$.

Proof. (1) Suppose that $\mathcal{M}$ has constant Newton polygons. The slope filtration of $\mathcal{M}$ in Corollary 2.6 induces the slope filtrations of $\mathcal{L}$ and $\mathcal{N}$ since homomorphisms of F-isocrystals are strict with respect to slopes. Hence, both $\mathcal{L}$ and $\mathcal{N}$ have constant Newton polygons. The converse is trivial.

(2) In the case of a connected X, the inequality $\dim_K H^0_{\mathrm{conv}}(X/K, \mathcal{M}) \leqslant \operatorname{rank} \mathcal{M}$ holds in general. Moreover, $\mathcal{M}$ is constant if and only if $\dim_K H^0_{\mathrm{conv}}(X/K, \mathcal{M}) = \operatorname{rank} \mathcal{M}$. Hence, the constancy of $\mathcal{M}$ implies the constancy of $\mathcal{L}$ and $\mathcal{N}$.

Since $\mathcal{L} \cong H^0_{\mathrm{conv}}(X/K, \mathcal{L}) \otimes \mathcal{O}_{]X[}$ for a constant convergent F-isocrystal $\mathcal{L}$, we have an isomorphism $H^1_{\mathrm{conv}}(X/K, \mathcal{L}) \cong H^0_{\mathrm{conv}}(X/K, \mathcal{L}) \otimes_K H^1_{\mathrm{conv}}(X/K)$. Hence the exact sequence

$$0 \to H^0_{\mathrm{conv}}(X/K, \mathcal{L}) \to H^0_{\mathrm{conv}}(X/K, \mathcal{M}) \to H^0_{\mathrm{conv}}(X/K, \mathcal{N}) \to H^1_{\mathrm{conv}}(X/K, \mathcal{L})_{\mathrm{fin}}$$

implies the converse since $\dim_K H^0_{\mathrm{conv}}(X/K, \mathcal{L}) = \operatorname{rank} \mathcal{L}$ and the same holds for $\mathcal{N}$. □

We give several properties of constancy of Newton polygons.

Proposition 2.8. *Let $f : X \to Y$ be a morphism of schemes separated of finite type over* $\operatorname{Spec} k$.

(1) *Suppose that Y is smooth and the set-theoretical complement of the image $f(X)$ in Y is of codimension $\geqslant 2$. If any convergent F-isocrystal on X/K has constant Newton polygons, then the same holds for any convergent F-isocrystal on Y/K.*

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(2) *Suppose that X is connected and the morphism f is finite etale. If any convergent F-isocrystal on Y/K has constant Newton polygons, then the same holds for any convergent F-isocrystal on X/K.*

Proof. (1) follows from Theorem 2.3.

(2) Let $f_{\mathrm{conv}*} : F\text{-Isoc}(X/K) \to F\text{-Isoc}(Y/K)$ be the push forward induced by the finite etale morphism $f : X \to Y$. For a convergent F-isocrystal $\mathcal{M}$ on X/K and the point $x \in X$, the Newton polygon of $f_{\mathrm{conv}*}\mathcal{M}$ at the point $f(x)$ is a $\deg(f)$ time Newton polygon $\mathrm{NP}(\mathcal{M}, x)$. □

Proposition 2.9. *Let X and Y be projective smooth and connected schemes over* $\operatorname{Spec} k$*. Suppose that X and Y are binational over* $\operatorname{Spec} k$*. Any convergent F-isocrystal on X/K is constant (respectively has constant Newton polygons) if and only if the same holds for any convergent F-isocrystal on Y/K.*

Proof. Since the set of fundamental points of binational transformations is closed of codimension $\geqslant 2$ [27, V, Lemma 5.1], the assertion follows from Theorem 2.3 and Kedlaya's extension theorem [32, Proposition 5.3.3] (via the extension of overconvergent F-isocrystals). □

2.5. Geometric F-isocrystals

Definition 2.10. Let S be a smooth scheme separated of finite type over $\operatorname{Spec} k$. A convergent isocrystal (respectively F-isocrystal) $\mathcal{M}$ on S/K is geometric if there exists a proper smooth morphism $f : X \to S$ such that $\mathcal{M}$ is isomorphic to a subquotient of the relative rigid cohomology $R^i f_{\mathrm{rig}*}\mathcal{O}_{]X[}$ of X over S for some integer i as a convergent isocrystal (respectively F-isocrystal).

We do not care about overconvergence along a boundary of S in the rigid cohomology. Since $f : X \to S$ is proper smooth, the rigid cohomology is nothing but the convergent cohomology of Ogus in [46] and the finiteness theorem below holds. Hence the above definition makes sense.

Theorem 2.11 [46, § 3]. *Let S be a smooth scheme separated of finite type over* $\operatorname{Spec} k$*, and $f : X \to S$ be a proper smooth morphism. Then the relative rigid cohomology $R^i f_{\mathrm{rig}*}\mathcal{O}_{]X[}$ is a convergent F-isocrystal on S/K.*

Remark 2.12. In this paper we use the finiteness theorem only in the case of constant coefficients. Recently Xu proved that the relative convergent cohomology $R^i f_{\mathrm{rig}*}\mathcal{M}$ is convergent F-isocrystal on S/K for an arbitrary proper smooth morphism f and an arbitrary convergent F-isocrystal $\mathcal{M}$ [59, Theorem 1.9] (see the detail around Berthelot's conjecture on the coherence of relative rigid cohomology in [38]).

We will use the following proposition to reduce the problem to the case where k is a finite field. The similar argument for constant convergent isocrystals is discussed in [20, § 5.4].

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Proposition 2.13. *Let κ be a perfect field, $\mathcal{T}$ a smooth integral scheme separated of finite type over* $\operatorname{Spec}\kappa$*, and $\mathfrak{f} : \mathcal{X} \to \mathcal{S}$ a proper smooth morphism of smooth schemes separated of finite type over $\mathcal{T}$. Let k be a perfect field which includes the function field $\kappa(\mathcal{T})$ of $\mathcal{T}$, and $f : X \to S$ the base change of $\mathfrak{f} : \mathcal{X} \to \mathcal{S}$ by the natural morphism* $\operatorname{Spec} k \to \mathcal{T}$ *via the generic point of $\mathcal{T}$. Suppose that, for any closed point t of $\mathcal{T}$ over* $\operatorname{Spec}\kappa$*, the relative rigid cohomology $R^i\mathfrak{f}_{t,\mathrm{rig}*}\mathcal{O}_{]\mathcal{X}_t[}$ on $\mathcal{S}_t/K(t)$ has constant Newton polygons, where $\kappa(t)$ is the function field of t and $K(t) = K_\sigma \otimes_{W(\mathbb{F}_q)} W(\kappa(t))$. Then the relative rigid cohomology $R^i f_{\mathrm{rig}*}\mathcal{O}_{]X[}$ on S/K has constant Newton polygons.*

Proof. We may assume that $\mathcal{S}$ is irreducible. Let $\mathcal{U}$ be an open subscheme of $\mathcal{S}$ consisting of points at which the Newton polygon of the relative rigid cohomology $R^i\mathfrak{f}_{\mathrm{rig}*}\mathcal{O}_{]\mathcal{X}[}$ coincides with that at the generic point of $\mathcal{S}$, and hence with that at the generic point of S. The open subscheme $\mathcal{U}$ is defined over $\operatorname{Spec}\kappa$ by Grothendieck's specialization theorem. If $\mathfrak{g} : \mathcal{S} \to \mathcal{T}$ denotes the structure morphism, then $\mathfrak{g}(\mathcal{U})$ is an open subscheme of $\mathcal{T}$ over $\operatorname{Spec}\kappa$ and $\mathfrak{g}^{-1}(\mathfrak{g}(\mathcal{U})) = \mathcal{U}$ since Newton polygons of $R^i\mathfrak{f}_{t,\mathrm{rig}*}\mathcal{O}_{]\mathcal{X}_t[}$ are constant on $\mathcal{S}_t$ by our hypothesis. Hence $S \subset \mathcal{U} \times_{\mathcal{T}} \operatorname{Spec} k$, and $R^i f_{\mathrm{rig}*}\mathcal{O}_{]X[}$ has constant Newton polygons on S. Note that, if $i_t : \mathcal{S}_t \to \mathcal{S}$ and $j_S : S \to \mathcal{S}$ are the canonical structure morphisms, the base change homomorphisms

$$i_t^* R^i\mathfrak{f}_{\mathrm{rig}*}\mathcal{O}_{]\mathcal{X}[} \to R^i\mathfrak{f}_{t,\mathrm{rig}*}\mathcal{O}_{]\mathcal{X}_t[} \quad \text{and} \quad j_S^* R^i\mathfrak{f}_{\mathrm{rig}*}\mathcal{O}_{]\mathcal{X}[} \to R^i f_{\mathrm{rig}*}\mathcal{O}_{]X[}$$

are isomorphisms since the relative cohomologies are coherent [58, Corollary 2.3.3]. □

2.6. Constancy versus constant Newton polygons

Proposition 2.14. *Let X be a smooth geometrically connected scheme separated of finite type over* $\operatorname{Spec} k$*. Then the following conditions (i) and (ii) are equivalent.*

(i) $F\text{-Isoc}(X/K)^{\mathrm{CST}} = F\text{-Isoc}(X/K)^{\mathrm{CNP}}$.

(ii) $\pi_1^{\mathrm{et}}(X_{\overline{k}}) = \{1\}$ *and* $H^1_{\mathrm{conv}}(X/K)_{\mathrm{fin}} = 0$ *(see the definition before Proposition 2.7).*

Proof. (i) ⇒ (ii): Recall that there exists a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_{K_\sigma}(\mathrm{Gal}(\overline{k}/k)) & \xrightarrow{\cong} & F\text{-Isoc}(\operatorname{Spec} k/K)^0 \\ \downarrow & & \downarrow \\ \mathrm{Rep}_{K_\sigma}(\pi_1^{\mathrm{et}}(X)) & \xrightarrow{\cong} & F\text{-Isoc}(X/K)^0, \end{array}$$

where the horizontal arrows are Katz–Crew equivalences between the category of p-adic continuous representations of etale fundamental groups and the category of unit-root (i.e., isoclinic of slope 0) convergent F-isocrystals [11, Theorem 2.1]. Let $\rho : \pi_1^{\mathrm{et}}(X) \to \mathrm{GL}_r(K_\sigma)$ be a p-adic continuous representation, and $\mathcal{M}$ the unit-root convergent F-isocrystal on X/K associated to ρ by the equivalence above. Under the fiber sequence

$$1 \to \pi_1^{\mathrm{et}}(X_{\overline{k}}) \to \pi_1^{\mathrm{et}}(X) \to \mathrm{Gal}(\overline{k}/k) \to 1,$$

we have $\rho(\pi_1^{\mathrm{et}}(X_{\overline{k}})) = \{1\}$ since $\mathcal{M}$ is constant by our hypothesis. In order to prove that (i) implies $\pi_1^{\mathrm{et}}(X_{\overline{k}}) = \{1\}$, it is sufficient to show the existence of a p-adic representation ρ of $\pi_1^{\mathrm{et}}(X)$ with $\rho(\pi_1^{\mathrm{et}}(X_{\overline{k}})) \neq \{1\}$ under the hypothesis $\pi_1^{\mathrm{et}}(X_{\overline{k}}) \neq \{1\}$. Suppose now that

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$\pi_1^{\text{et}}(X_{\overline{k}}) \neq \{1\}$. Then there exists a finite Galois extension k' of k and a nontrivial finite etale Galois covering $f : X' \to X \times_{\text{Spec}\, k} \text{Spec}\, k'$ of degree r for some r such that, if $\rho' : \pi_1^{\text{et}}(X') \to \text{GL}_r(K'_{\sigma'})$ is the corresponding representation, then $\rho'(\pi_1^{\text{et}}(X_{\overline{k}})) \neq \{1\}$. Taking the push forward of ρ' (the induced representation and the restriction of scalar), we have a continuous representation $\widetilde{\rho} : \pi_1^{\text{et}}(X) \to \text{GL}_{\widetilde{r}}(K_\sigma)$ for some positive integer $\widetilde{r}$ such that $\widetilde{\rho}(\pi_1^{\text{et}}(X_{\overline{k}})) \neq \{1\}$.

Let us prove that (i) implies $H^1_{\text{conv}}(X/K)_{\text{fin}} = 0$. Suppose $H^1_{\text{conv}}(X/K)_{\text{fin}} \neq 0$. Then there exists a nontrivial F-subspace $L \subset H^1_{\text{conv}}(X/K)_{\text{fin}}$ over K. Let $e_1, \dots, e_r$ be a basis of L and $e_1^\vee, \dots, e_r^\vee$ be a dual basis of the dual F-space $L^\vee$ of L. Then

$$w = e_1 \otimes e_1^\vee + \cdots + e_r \otimes e_r^\vee \in H^1_{\text{conv}}(X/K)_{\text{fin}} \otimes_K L^\vee \subset H^1_{\text{conv}}(X/K, \mathcal{O}_{]X[} \otimes_K L^\vee)$$

is a nontrivial element satisfying $F(w) = w$. Indeed, if $A \in \text{GL}_n(k)$ is a Frobenius matrix of L, i.e., $F(e_1, \dots, e_r) = (e_1, \dots, e_r)A$, then the Frobenius matrix of $L^\vee$ is ${}^tA^{-1}$. Hence w satisfies the relation $F(w) = w$. The nonzero 1-cocycle w determines a nontrivial extension

$$0 \to \mathcal{O}_{]X[} \otimes_K L^\vee \to \mathcal{M} \to \mathcal{O}_{]X[} \to 0$$

of convergent F-isocrystals on X/K by Lemma 2.15. The extension $\mathcal{M}$ has constant Newton polygons by Proposition 2.7(1), but it is not constant. This contradicts our hypothesis.

(ii) $\Rightarrow$ (i): By Katz–Crew's equivalence above the triviality $\pi_1^{\text{et}}(X_{\overline{k}}) = \{1\}$ implies that the natural inverse image functor $F\text{-Isoc}(\text{Spec}\, k/K)^0 \to F\text{-Isoc}(X/K)^0$ is an equivalence. Let $\mathcal{M}$ be a convergent F-isocrystal on X/K which is isoclinic of slope γ, and take a finite extension K' of K whose valuation group contains γ. Let k' be a residue field of K', $X' = X \times_{\text{Spec}\, k} \text{Spec}\, k'$, and $\mathcal{M}'$ the inverse image of $\mathcal{M}$ on X'/K'. Then $\mathcal{M}'$ is a tensor product of a rank 1 constant object and a unit-root object, and hence $\mathcal{M}'$ is constant. Since $H^0_{\text{conv}}(X/K, \mathcal{M}) \otimes_K K' \cong H^0_{\text{conv}}(X'/K', \mathcal{M}')$, $\mathcal{M}$ is constant. The rest is to prove that the slope filtration (Corollary 2.6) is split for any object of $F\text{-Isoc}(X/K)^{\text{CNP}}$. Suppose $\mathcal{L}$ is a constant convergent F-isocrystal on X/K. Then $\mathcal{L} = \mathcal{O}_{]X[} \otimes_K L$ with the F-space $L = H^0_{\text{conv}}(X/K, \mathcal{L})$, and

$$H^1_{\text{conv}}(X/K, \mathcal{L}) \cong H^1_{\text{conv}}(X/K) \otimes_K L.$$

Hence the vanishing $H^1_{\text{conv}}(X/K)_{\text{fin}} = 0$ implies the splitting of the exact sequence

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

with constant convergent F-isocrystals $\mathcal{L}$ and $\mathcal{N}$ since the extension class of F-isocrystals belongs to $H^1_{\text{conv}}(X/K, \mathcal{L}^\vee \otimes \mathcal{N})_{\text{fin}}$ by Lemma 2.15. □

The cohomological interpolation of Hom and Ext for convergent isocrystals in [3, Proposition 2.2.7] and [5, Proposition 1.2.2] implies the lemma below. The last assertion in the lemma below follows from the finite dimensionality of 0th convergent cohomology.

Lemma 2.15. *Let $\mathcal{M}, \mathcal{N}$ be convergent F-isocrystals on X/K, and let us denote a K_σ-space of homomorphisms as convergent F-isocrystals (respectively a K_σ-space of*

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extension classes of convergent F-isocrystals) on X/K by $\mathrm{Hom}_{F\text{-}Isoc}(\mathcal{M},\mathcal{N})$ *(respectively* $\mathrm{Ext}^1_{F\text{-}Isoc}(\mathcal{M},\mathcal{N})$*). Then there is an exact sequence of K_σ-spaces:*

$$\begin{aligned} 0 \to \mathrm{Hom}_{F\text{-Isoc}}(\mathcal{M},\mathcal{N}) &\to H^0_{\mathrm{conv}}(X/K,\mathcal{M}^\vee\otimes\mathcal{N}) \stackrel{1-F}{\to} H^0_{\mathrm{conv}}(X/K,\mathcal{M}^\vee\otimes\mathcal{N}) \\ \to \mathrm{Ext}^1_{F\text{-Isoc}}(\mathcal{M},\mathcal{N}) &\to H^1_{\mathrm{conv}}(X/K,\mathcal{M}^\vee\otimes\mathcal{N}) \stackrel{1-F}{\to} H^1_{\mathrm{conv}}(X/K,\mathcal{M}^\vee\otimes\mathcal{N}). \end{aligned}$$

Here $\mathcal{M}^\vee$ is the dual of $\mathcal{M}$. If furthermore the residue field k is algebraically closed, then the homomorphism $H^0_{\mathrm{conv}}(X/K,\mathcal{M}^\vee\otimes\mathcal{N}) \stackrel{1-F}{\to} H^0_{\mathrm{conv}}(X/K,\mathcal{M}^\vee\otimes\mathcal{N})$ is surjective.

The author does not know whether $\pi_1^{\mathrm{et}}(X_{\overline{k}})=\{1\}$ implies $H^1_{\mathrm{conv}}(X/K)_{\mathrm{fin}}=0$ or not in general because the convergent cohomology (not the rigid cohomology!) is huge in general. Esnault and Shiho proved the following theorem by a comparison with ℓ-adic cohomology theory. Hence $H^1_{\mathrm{conv}}(X/K)_{\mathrm{fin}}=H^1_{\mathrm{rig}}(X/K)=0$ at least when X is proper and $\pi_1^{\mathrm{et}}(X_{\overline{k}})=\{1\}$ holds.

Theorem 2.16 [20, Theorem 5.1]. *Let X be a smooth geometrically connected scheme over* $\mathrm{Spec}\,k$*. Suppose that either X is proper over* $\mathrm{Spec}\,k$ *or $p\geqslant 3$. Then the triviality $\pi_1^{\mathrm{et,ab}}(X_{\overline{k}})=\{1\}$ implies the vanishing $H^1_{\mathrm{rig}}(X/K)=0$. Here $\pi_1^{\mathrm{et,ab}}(X_{\overline{k}})$ is the maximal Abelian quotient of geometric etale fundamental group $\pi_1^{\mathrm{et}}(X_{\overline{k}})$.*

Corollary 2.17. *Let X be a smooth geometrically connected scheme over* $\mathrm{Spec}\,k$*. Suppose that X is proper over* $\mathrm{Spec}\,k$*. Then the equivalence $F\text{-Isoc}(X/K)^{\mathrm{CST}}=F\text{-Isoc}(X/K)^{\mathrm{CNP}}$ of categories holds if and only if $\pi_1^{\mathrm{et}}(X_{\overline{k}})=\{1\}$.*

3. Constancy of Newton polygons on Abelian varieties

In this section we prove any F-isocrystal on an Abelian variety over a finite field has constant Newton polygons.

3.1. An estimate of number of jumping points

Suppose k is a finite field of q elements in this section, and denote an algebraic closure of k by $\overline{k}$. In this case our q-Frobenius σ is the identity map on K. When we replace k by a finite extension k_f of q^f elements, then we also change the Frobenius σ by the identity map σ^f on the unramified extension K_f of K with the residue field k_f.

Let C be a projective smooth and geometrically connected curve of genus g over $\mathrm{Spec}\,k$ with p-rank e, i.e.,

$$e=\mathrm{rank}_{\mathbb{F}_p}J(C)[p](\overline{k})=\dim_K H^1_{\mathrm{rig}}(C/K)^0.$$

Here $J(C)$ is the Jacobian variety of C, $J(C)[p]$ is the subgroup scheme of $J(C)$ which is the kernel of the multiplication by p, $H^i_{\mathrm{rig}}(C/K)$ is the ith rigid cohomology of C/K, and $H^i_{\mathrm{rig}}(C/K)^0$ is the unit-root subspace of the F-space $H^1_{\mathrm{rig}}(C/K)$. Then $0\leqslant e\leqslant g$.

Proposition 3.1. *With the notation as above, suppose $g\geqslant 1$. Let $\mathcal{M}$ be a convergent F-isocrystal on C/K which satisfies the following conditions:*

(i) *The initial slope of $\mathcal{M}$ at the generic point of C is 0 of rank 1;*

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(ii) *Let U be the open subscheme $\{x \in C \mid$ the initial slope of $\mathcal{M}$ at x is $0\}$ of C with the open immersion $j_U : U \to X$, $\mathcal{L}$ the rank 1 unit-root convergent sub F-isocrystal of $j_U^*\mathcal{M}$ on U/K by Theorem 2.5, and $\rho : \pi_1^{\mathrm{et}}(U) \to R^\times$ the p-adic representation corresponding to $\mathcal{L}$ by Katz–Crew's equivalence such that the representation ρ satisfies*

$$\rho \equiv 1 \ (\mathrm{mod}\, \mathbf{m}R).$$

If $Z = C \setminus U$ is a reduced divisor of C over $\mathrm{Spec}\, k$*, then we have an inequality*

$$e + \deg Z \leqslant 1 + 2(g-1)\mathrm{rank}\mathcal{M}.$$

Proof. If $U = C$, then there is nothing to prove. So we suppose $U \neq C$ and hence the rank of $\mathcal{M}$ is greater than or equal to 2. Let us calculate the L-function

$$L(C/k, \mathcal{M}; t) = \prod_{x:\text{closed points of } C} \det(1 - F_x^{\deg(x)} t^{\deg(x)}; i_x^*\mathcal{M})^{-1} \in R[[t]]$$

of $\mathcal{M}$ modulo $\mathbf{m}$ by two ways. Here $\deg(x)$ is the degree of the function field $k(x)$ of x over k and $i_x : x \to C$ be the canonical morphism. At a closed point x of U all the p-adic valuations of Frobenius eigenvalues of $i_x^*\mathcal{M}$ except one eigenvalue which is 1 modulo $\mathbf{m}$ are positive. At a closed point x in Z all the p-adic valuations of Frobenius eigenvalues of $i_x^*\mathcal{M}$ are positive. Hence we have

$$(*) \qquad \begin{aligned} L(C/k, \mathcal{M}; t) &\equiv \prod_{x:\text{closed points of } U} (1 - t^{\deg(x)})^{-1} \\ &\equiv \mathrm{Zeta}(C/k; t)\mathrm{Zeta}(Z/k; t)^{-1} \\ &\equiv \frac{\det(1 - Ft; H^1_{\mathrm{rig}}(C/K))\mathrm{Zeta}(Z/k; t)^{-1}}{1-t} \ (\mathrm{mod}\ \mathbf{m}R[[t]]) \end{aligned}$$

in $k[[t]]$, where $\mathrm{Zeta}(C/k; t)$ (respectively $\mathrm{Zeta}(Z/k; t)$) is the zeta function of C (respectively Z) over k. Since

$$\begin{aligned} &\deg(\det(1 - Ft; H^1_{\mathrm{rig}}(C/K)) \ (\mathrm{mod}\, \mathbf{m}R[[t]])) = e \\ &\deg(\mathrm{Zeta}(Z/k; t)^{-1} \ (\mathrm{mod}\, \mathbf{m}R[[t]])) = \deg(Z) \end{aligned}$$

in $k[t]$, we have

$$\deg((1-t)L(C/k, \mathcal{M}; t) \ (\mathrm{mod}\ \mathbf{m}R[[t]])) = e + \deg(Z).$$

On the other hand, let us calculate the L-function of $\mathcal{M}$ modulo $\mathbf{m}$ using Lefschetz trace formula

$$L(C/k, \mathcal{M}; t) = \prod_i \det(1 - Ft; H^i_{\mathrm{rig}}(C/K, \mathcal{M}))^{(-1)^{i+1}}$$

for rigid cohomology [21, Théorème 6.3]. Since the characteristic polynomial of Frobenius in each degree belongs to $1 + t\mathcal{O}_K[t]$ by Remark 3.2, it makes sense to take the reduction modulo $\mathbf{m}$.

Let $\mathcal{N}$ be an irreducible subquotient of $\mathcal{M}$ as a convergent F-isocrystal on C/K such that $\mathcal{N}$ includes the generic slope 0 part. Since any slope of Frobenius F_x on $i_x^*\mathcal{N}$ at any point x of $Z \neq \emptyset$ is positive, $\mathcal{N}$ is not constant and is of rank $\geqslant 2$. If $\mathcal{N}^\vee$ denotes the dual of $\mathcal{N}$, then $H^0_{\mathrm{rig}}(C/K, \mathcal{N}) = H^0_{\mathrm{rig}}(C/K, \mathcal{N}^\vee) = 0$ by the irreducibility. Hence, Poincaré duality [31, Theorem 1.2.3] implies $H^2_{\mathrm{rig}}(C/K, \mathcal{N}) = 0$ and then $\dim_K H^1_{\mathrm{rig}}(C/K, \mathcal{N}) = 2(g-1)\mathrm{rank}\,\mathcal{N}$ by Euler–Poincaré formula of rigid cohomology of curves [9, Corollaire 5.0-12]. If $\mathcal{N}' \neq \mathcal{N}$ is another subquotient of $\mathcal{M}$, then any slope of Frobenius F_x on $i_x^*\mathcal{N}'$ at any point x is positive since the generic slope 0 subpart is of rank 1 and is included in $\mathcal{N}$. It implies that any slope of Frobenius F on the rigid cohomology $H^i_{\mathrm{rig}}(C/K, \mathcal{N}')$ is positive for $i = 0, 1, 2$ by Remark 3.2 if it does not vanish. Hence, we have a congruence

$$L(C/k, \mathcal{M}; t) \equiv \det(1 - Ft; H^1_{\mathrm{rig}}(C/K, \mathcal{N})) \pmod{\mathbf{m}R[[t]]}.$$

Since $\dim_K H^1_{\mathrm{rig}}(C/K, \mathcal{N}) = 2(g-1)\mathrm{rank}\,\mathcal{N} \leqslant 2(g-1)\mathrm{rank}\,\mathcal{M}$, we have the desired inequality. □

Remark 3.2. Let $\mathcal{M}$ be an overconvergent F-isocrystal $\mathcal{M}$ on a smooth scheme X separated of finite type over $\mathrm{Spec}\,k$ such that all slopes of $\mathcal{M}$ at generic points are nonnegative.

(1) All slopes of $H^i_{\mathrm{rig}}(X/K, \mathcal{M})$ are nonnegative for $i = 0, 1$. Indeed, when $i = 0$, the overconvergent F-isocrystal $H^0_{\mathrm{rig}}(X/K, \mathcal{M}) \otimes_K j^\dagger\mathcal{O}_{]X[}$ is regarded as a subobject of $\mathcal{M}$. In the case where $i = 1$ the nontrivial 1-cocycle in $H^1_{\mathrm{rig}}(X/K, \mathcal{M})$ of slope μ determines an F-space L over K with purely of slope μ and a nontrivial extension

$$0 \to \mathcal{M} \to \mathcal{E} \to L \otimes_K j^\dagger\mathcal{O}_{]X[} \to 0$$

of overconvergent F-isocrystals on X/K by Lemma 2.15. This extension is nontrivial on any open dense subscheme U of X as overconvergent F-isocrystals [32, Theorem 5.2.1] and is also nontrivial in the category of convergent F-isocrystals on U/K [30, Theorem 1.1]. However, if $\mu < 0$, then the extension above must be split. Indeed, there exists an open dense subscheme U of X with the open immersion $j_U : U \to X$ such that the Newton polygons of $j_U^*\mathcal{M}$ are constant on U by Proposition 2.2(2). Then there is a slope filtration $\{S_\lambda\}$ of $j_U^*\mathcal{M}$ by Corollary 2.6 and we have an isomorphism $S_\mu \cong L \otimes_K \mathcal{O}_{]U[}$ as convergent F-isocrystals. This contradicts the nontriviality of the extension on U. Hence μ is nonnegative. The same holds for $H^i_{\mathrm{conv}}(X/K, \mathcal{M})_{\mathrm{fin}}$ for $i = 0, 1$ by using the full faithfulness in [32, Theorem 5.2.1] [33, Theorem 4.2.1].

(2) In the case where X is a curve all slopes of $H^2_{\mathrm{rig}}(X/K, \mathcal{M})$ are greater than or equal to 1 by Poincaré duality [31, Theorem 1.2.3].

(3) Suppose X is purely of dimension d. In the case $\mathcal{M} = j^\dagger\mathcal{O}_{]X[}$ all slopes of $H^i_{\mathrm{rig}}(X/K)$ are in the interval $[\max\{0, i-d\}, \min\{i, d\}]$ [6, Theorem 3.1.2].

Let C be a projective smooth and geometrically connected curve over $\mathrm{Spec}\,k$ with the generic point η of C, and $\mathcal{M}$ a convergent F-isocrystal on C/K. We study the closed subscheme

$$D_{\mathcal{M}} = \{x \in C \mid \mathrm{NP}(\mathcal{M}, x) \neq \mathrm{NP}(\mathcal{M}, \eta)\}$$

which is regarded as a reduced divisor.

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Theorem 3.3. *There exists a constant B depending on a positive power q of p, a nonnegative integer g, and a positive integer r such that the inequality*

$$\deg(D_{\mathcal{M}}) \leqslant B$$

holds for any projective smooth and geometrically connected curve C of genus g over the spectrum $\mathrm{Spec}\, k$ *of the field k of q elements and any convergent F-isocrystal $\mathcal{M}$ on C/K of rank r with respect to q-Frobenius σ.*

In order to apply Proposition 3.1 we prepare the lemma below.

Lemma 3.4. *Let Y be an irreducible scheme separated of finite type over* $\mathrm{Spec}\, k$ *with the generic point η, and $\mathcal{N}$ a convergent F-isocrystal on Y/K. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_m$ be the generic slopes of $\mathcal{N}$ of ranks $r_1, r_2, \ldots, r_m$.*

(1) *For $l = 1, \ldots, m$, the exterior power $\wedge^{r_1+\cdots+r_l}\mathcal{N}$ of $\mathcal{N}$ has an initial slope of $r_1\gamma_1 + \cdots + r_l\gamma_l$ of rank 1.*

(2) *For a point $y \in Y$, $\mathrm{NP}(\mathcal{N}, y) \neq \mathrm{NP}(\mathcal{N}, \eta)$ if and only if there is an integer l with $1 \leqslant l \leqslant m$ such that the initial slope of $\wedge^{r_1+\cdots+r_l}\mathcal{N}$ at y is greater than $r_1\gamma_1 + \cdots + r_l\gamma_l$.*

Proof. The assertion immediately follows from the properties of slopes under exterior powers. □

Proof of Theorem 3.3. Suppose $\mathcal{M}$ is a convergent F-isocrystal on C/K of rank r with the generic slopes $\gamma_1 < \gamma_2 < \cdots < \gamma_m$ of ranks $r_1, r_2, \ldots, r_m$. If we put $\mathcal{M}_l = \wedge^{r_1+\cdots+r_l}\mathcal{M}$, then

$$D_{\mathcal{M}} = \bigcup_l Z_l, \quad Z_l = Z_{\mathcal{M}_l} = \left\{ x \in C \,\middle|\, \begin{array}{l} \text{The initial slope of } \mathcal{M}_l \text{ at } x \\ \text{is greater than that at } \eta. \end{array} \right\}$$

by Lemma 3.4(2). Applying Lemma 3.4(1) to each $\mathcal{M}_l$, we may suppose the initial generic slope of $\mathcal{M}_l$ is 0 of rank 1 by a suitable twist of Frobenius. By taking a self tensor product, $\mathcal{M}_l^{\otimes(q-1)}$ satisfies the conditions (i) and (ii) of Proposition 3.1. Since $Z_l = Z_{\mathcal{M}_l^{\otimes(q-1)}}$ as closed subsets of C, $\deg(Z_l)$ is bounded by a constant depending only on g and $\mathrm{rank}\, \mathcal{M}_l^{\otimes(q-1)}$. Therefore, the bound B exists since $m \leqslant r$. □

Remark 3.5. (1) More precisely, the upper bound B in Theorem 3.3 depends on the genus g of curves, a p-power q of q-Frobenius, and the rank r of convergent F-isocrystals, but not on the cardinality of k. Indeed, if k is a field of q^f elements and σ is a q-Frobenius satisfying the conditions in § 2.1, then the $\pi_1^{\mathrm{et}}(U)$-representation of rank one arising from the generic slope 0 convergent sub F-isocrystal $\mathcal{M}$ is an R_σ-representation. Hence, after the $(q-1)$-st self tensor product, the representation modulo $\mathbf{m}_\sigma$ is equivalent to the trivial representation. Even if we replace q-Frobenius by q^f-Frobenius on K in order to apply Proposition 3.1, $\mathcal{M}^{\otimes(q-1)}$ satisfies the hypothesis of Proposition 3.1 and we do not need any further self tensor product to kill the representation modulo $\mathbf{m}_\sigma$.

(2) One can take an upper bound $B = r + 2^{1+(q-1)r}(g-1)$ in Theorem 3.3. Indeed, at the lth generic slope the number of jumping points is $\leqslant 1 + 2(g-1)$

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rank $(\wedge^{r_1+\cdots+r_l}\mathcal{M})^{\otimes(q-1)}$. Hence, we have an estimate

$$\begin{aligned} B &\leqslant \textstyle\sum_{l=1}^{m}(1+2(g-1)(\text{rank } \wedge^{r_1+\cdots+r_l}\mathcal{M})^{\otimes(q-1)}) \\ &\leqslant \textstyle\sum_{j=1}^{r}(1+2(g-1)(\text{rank } \wedge^{j}\mathcal{M})^{\otimes(q-1)}) \\ &\leqslant r+2(g-1)\times(2^r)^{q-1}. \end{aligned}$$

This upper bound is not sharp.

3.2. The case of elliptic curves

To clarify the idea of proof, we first prove the constancy theorem in the case of elliptic curves.

Theorem 3.6. *Let k be a finite field, and X an elliptic curve over* Spec k*. Then any convergent F-isocrystal $\mathcal{M}$ on X/K has constant Newton polygons.*

Proof. Suppose $\mathcal{M}$ is an F-isocrystal on X/K with nonconstant Newton polygons. Let $D_{\mathcal{M}} \neq \emptyset$ be the reduced divisor of X consisting of the points x such that

$$\mathrm{NP}(X,x) \neq \mathrm{NP}(X,\eta)$$

where η is the generic point of X. Let $[n] : X \to X$ be the morphism of multiplication n for a positive integer n. If n is prime to p, then $[n]$ is a finite etale morphism and we have

$$\deg(D_{[n]^*\mathcal{M}}) = n^2 \deg(D_{\mathcal{M}}).$$

It is a contradiction to the boundedness of $\deg(D_{[n]^*\mathcal{M}})$ in Theorem 3.3. Therefore, any F-isocrystal on X/K has constant Newton polygons. □

3.3. The case of Abelian varieties

Let us now prove our main theorem for general Abelian varieties.

Theorem 3.7. *Let k be a finite field, and X an Abelian variety over* Spec k *of dimension g. Any convergent F-isocrystal $\mathcal{M}$ on X/K has constant Newton polygons, i.e.,*

$$F\text{-Isoc}(X/K)^{\mathrm{CNP}} = F\text{-Isoc}(X/K).$$

Proof. Suppose $\mathcal{M}$ is a convergent F-isocrystal on X/K with nonconstant Newton polygons. Let $D_{\mathcal{M}} \neq \emptyset$ be the closed subscheme of X consisting of the points x such that

$$\mathrm{NP}(X,x) \neq \mathrm{NP}(X,\eta)$$

as in the proof of Theorem 3.6. Then $D_{\mathcal{M}}$ is purely of codimension 1 by de Jong–Oort's purity theorem (Theorem 2.3). After replacing k by a finite extension, we can find a projective smooth geometrically connected curve C over Spec k such that

(a) for any closed integral subscheme Z over Spec k of codimension 1 in X, the set-theoretical intersection $C \cap Z$ is nonempty;

(b) $O \in C$ and $O \notin D_{\mathcal{M}}$, where O is the origin of the Abelian variety X. Indeed, if we fix an embedding X into a projective space, then by Bertini's theorem and Moishezon–Nakai's criterion of ampleness [27, Appendix A, Theorem 5.1], one can obtain a projective smooth geometrically connected curve C in X as an intersection of different $g-1$ hyperplane sections such that the condition (a) holds after replacing k by a finite extension. Note that the genus of C is greater than or equal to g by weak Lefschetz theorem. Since $\mathcal{D}_{\mathcal{M}}$ is of codimension 1, one can choose C so that $C \not\subset D_{\mathcal{M}}$. Taking a translation by a k-rational point after replacing k by a finite extension if necessary, we may also assume the condition (b).

Lemma 3.8. (1) *For any positive integer n, we have $C \cap D_{[n]^*\mathcal{M}} \neq \emptyset$ and $C \not\subset D_{[n]^*\mathcal{M}}$.*

(2) *If we put $\Omega = \{x \in C(\bar{k}) \mid x \in C \cap D_{[n]^*\mathcal{M}}$ for some $n > 0\}$, then Ω is infinite.*

(3) *If $C \cap D_{[n]^*\mathcal{M}}$ denotes the set-theoretical intersection and $(C \cap D_{[n]^*\mathcal{M}})_{\mathrm{red}}$ denotes a reduced divisor of C, then*

$$\sup_n \deg((C \cap D_{[n]^*\mathcal{M}})_{\mathrm{red}}) = \infty.$$

Proof. First we remark $D_{[n]^*\mathcal{M}} = [n]^{-1}(D_{\mathcal{M}})$ set-theoretically.

(1) It follows from the condition (a) and $O \notin C \cap D_{[n]^*\mathcal{M}}$ for any n.

(2) Suppose that Ω is finite, namely, $\Omega = \{y_1, y_2, \ldots, y_s\}$. Let n be a positive integer such that n is a multiple of orders of all $y_1, y_2, \ldots, y_s$. Since $[n](y_i) = O$, $y_i \notin C \cap D_{[n]^*\mathcal{M}}$ for any i. This contradicts $C \cap D_{[n]^*\mathcal{M}} \neq \emptyset$.

(3) For a finite extension k_f of k of degree f, the cardinality of $X(k_f)$ is finite. Hence there is a properly infinitely increasing sequence $f_1 < f_2 < \cdots$ such that $x_{f_i} \in \Omega \cap X(k_{f_i})$ and $x_{f_i} \notin X(k')$ for any proper subfield k' of k_{f_i} by (2). Since C and $D_{[n]^*\mathcal{M}}$ are defined over $\mathrm{Spec}\, k$, the cardinality of the set-theoretical intersection $C \cap D_{[n]^*\mathcal{M}}$ is greater than or equal to f_i if $x_{f_i} \in C \cap D_{[n]^*\mathcal{M}}$ because of Galois conjugation. □

Now we return to the proof of Theorem 3.7. Since $\mathrm{rank}\, [n]^*\mathcal{M} = \mathrm{rank}\, \mathcal{M}$ for any positive integer n, the degree of $(C \cap D_{[n]^*\mathcal{M}})_{\mathrm{red}}$ in C is bounded by a constant independent of n by Theorem 3.3. The assertion of Lemma 3.8 (3) contradicts this boundedness. Therefore, any F-isocrystal on X/K has constant Newton polygons. □

4. Isotriviality of a family of curves

In this section we study the isotriviality of proper smooth families of connected curves in positive characteristic p. Let k be an algebraically closed field of characteristic p.

4.1. Isotriviality of families of ordinary Abelian varieties

We recall some definitions.

Definition 4.1. Let κ be a perfect field of characteristic p and $\bar{\kappa}$ an algebraic closure of κ.

(1) An Abelian variety S of dimension g over $\mathrm{Spec}\, \kappa$ is said to be ordinary if $\dim_{\mathbb{F}_p} S[p](\bar{\kappa}) = g$. Here $S[p]$ is the subgroup scheme defined by the kernel of multiplication with p.

(2) A projective smooth geometrically connected curve S over $\operatorname{Spec} \kappa$ is said to be ordinary if so is the Jacobian variety $J(S)$ of S.

The quantity $\dim_{\mathbb{F}_p} S[p](\overline{\kappa})$ is called p-rank of Abelian variety S and the equality $\dim_{\mathbb{F}_p} S[p](\overline{\kappa}) = \dim H^1_{\mathrm{rig}}(S/K)^0$ holds as well as in the case of curves (see the beginning of § 3.1).

Theorem 4.2. *Let S be a projective smooth connected scheme over* $\operatorname{Spec} k$ *such that any geometric convergent F-isocrystal on S/K has constant Newton polygons. Suppose $f : X \to S$ is a polarized Abelian scheme relatively of dimension g and of degree d^2. If there is a point $t \in S$ such that the fiber X_t of X at t is an ordinary Abelian variety, then any closed fiber of X is isomorphic to each other as polarized Abelian varieties. In particular, X is isotrivial over S. Moreover, for any closed point $s \in S$, there is an etale morphism $g : U \to S$ with $s \in g(U)$ such that $X \times_S U$ is a trivial deformation over U, that is, $X \times_S U \cong X_s \times_{\operatorname{Spec} k} U$ as U-schemes.*

Proof. Let us consider the relative first rigid cohomology $R^1 f_{\mathrm{rig}*}\mathcal{O}_{]X[}$ which is a convergent F-isocrystal on S/K (see Theorem 2.11). Since there is a point $t \in S$ such that X_t is ordinary, X is a family of ordinary Abelian varieties by our hypothesis on Newton polygons. Let $\mathcal{A}_{g,d,n/k}$ be the moduli space of g-dimensional polarized Abelian varieties of degree d^2 and with full level-n structure over $\operatorname{Spec} k$ (see [45] for the detail). Because the ordinary locus $\mathcal{A}^{\mathrm{ord}}_{g,d,1/k}$ of the coarse moduli space $\mathcal{A}_{g,d,1/k}$ is a quasi-affine scheme by [44, XI, Théorème 5.2], the canonical morphism $S \to \mathcal{A}^{\mathrm{ord}}_{g,d,1/k}$ is constant.

Fix an integer $n \geqslant 3$ which is relatively prime to p. If $[n] : X \to X$ is the morphism of multiplication n, then $\Sigma_n = \operatorname{Ker}([n])$ is a finite etale group scheme over S. For each closed point $s \in S$, one can take an etale morphism $g : U \to S$ of irreducible schemes with $s \in g(U)$ such that $\Sigma_n \times_S U \cong (\mathbb{Z}/n\mathbb{Z})^{2g}_U$ as group schemes on U. Then the polarized Abelian scheme $X \times_S U$ over U with the level structure $\Sigma_n \times_S U \cong (\mathbb{Z}/n\mathbb{Z})^{2g}_U$ induces a commutative diagram

$$\begin{array}{ccc} U & \stackrel{g}{\to} & S \\ \downarrow & & \downarrow \\ \mathcal{A}_{g,d,n/k} & \to & \mathcal{A}_{g,d,1/k}. \end{array}$$

Here the bottom arrow is a level forgetful map which is finite. Hence, the canonical morphism $U \to \mathcal{A}_{g,d,n/k}$ is constant. Since $\mathcal{A}_{g,d,n/k}$ is a base change of the fine moduli scheme with the universal family [45, Chapter 7], $X \times_S U$ is a trivial deformation of X_s over U. □

Corollary 4.3. *Let S be a projective smooth connected scheme of finite type over* $\operatorname{Spec} k$ *such that any geometric convergent F-isocrystal on S/K has constant Newton polygons. Suppose $X \to S$ is a proper smooth family of connected curves of genus g. If there is a point $s \in S$ such that the fiber X_s of X at s is ordinary, then X is isotrivial over S.*

Note that Raynaud and Szpiro studied the isotriviality for a family of ordinary curves over a curve by using intersection theory [53, Théorème 5].

Proof of Corollary 4.3. Consider the relative Jacobian variety $J(X)$ of X over S. Then $J(X)$ is a principally polarized Abelian scheme over S with an ordinary fiber $J(X)_s$. Hence the canonical morphism $S \to \mathcal{A}_{g,1,1/k}$ associated to the family is constant by Theorem 4.2. Now apply the Torelli theorem (see [43, Theorem 12.1], for example), and then the family X is isotrivial over S. □

Proposition 4.4. *Let S be an Abelian variety over* Spec k*. Then any geometric convergent F-isocrystal on S/K has constant Newton polygons. In particular, if X is a polarized Abelian scheme over S with an ordinary fiber X_s for a point $s \in S$, then the conclusions of Theorem 4.2 hold for the family X over S.*

Proof. Let $Y \to S$ be a proper smooth morphism. Then there exist a smooth integral scheme $\mathcal{T}$ separated of finite type over a spectrum Spec κ of a finite field κ such that $\kappa(\mathcal{T}) \subset k$, an Abelian scheme $\mathcal{S}$ over $\mathcal{T}$ and a proper smooth morphism $g : \mathcal{Y} \to \mathcal{S}$ such that the base change by the morphism Spec $k \to \mathcal{T}$ is the given morphism $Y \to S$. Here $\kappa(\mathcal{T})$ is the function field of $\mathcal{T}$. Then, for any closed point $t \in \mathcal{T}$, the relative rigid cohomology $\mathcal{M} = R^i g_{\mathrm{rig}*}\mathcal{O}_{]\mathcal{Y}[}$ has a constant Newton polygon on $\mathcal{S}_t$ by Theorem 3.7. Hence, any geometric convergent F-isocrystal on S/K has constant Newton polygons by Proposition 2.13. The rest follows from Theorem 4.2. □

4.2. Isotriviality of family of curves

We will prove the isotriviality of a family of curves (Theorem 1.6) in the rest of this section. When the genus g of fibers is $\leqslant 1$, the isotriviality holds without the hypothesis of constancy of Newton polygons. Indeed, the coarse moduli space is a point if $g = 0$. When $g = 1$, one may assume the family has a section and the isotriviality follows from the fact that the coarse moduli space of elliptic curves is an affine line.

Theorem 4.5. *Let S be a projective smooth and connected scheme over* Spec k*, and $f : X \to S$ a proper smooth family of connected curves of genus $g \geqslant 2$. Suppose that any geometric convergent F-isocrystal on S/K has constant Newton polygons. Then the family X over S is isotrivial.*

We give a proof of Theorem 4.5 in § 4.4. The next corollary is a consequence of Proposition 4.4.

Corollary 4.6. *Let S be an Abelian variety over* Spec k*. If $f : X \to S$ is a proper smooth family of connected curves, then the family X over S is isotrivial.*

4.3. Torelli theorem

In this subsection we recall Tamagawa's work in [54]. Let X be a projective smooth curve of genus $g \geqslant 2$ over Spec k.

Definition 4.7. The gonality of X is a minimum degree of nonconstant morphisms $X \to \mathbb{P}^1_k$.

By [54, Theorem 2.7, Proposition 2.14] we have the following theorem.

Theorem 4.8. *Suppose that the p-rank of X is neither 0 nor g. For any sufficient large integer n which is prime to p, there exists a finite etale morphism $Y \to X$ cyclic of degree n such that the gonality of Y is greater than or equal to 5.*

Remark 4.9. Tamagawa gave a positive lower bound of number of finite etale Galois coverings Y of X with a certain condition "strictly non-$\mathcal{P}_{1/1}$" under the assumption that X is non-almost elliptic, i.e., the Jacobian variety $J(X)$ is not isogenous to E^g for an elliptic curve E over $\operatorname{Spec} k$ in [54, Proposition 2.14]. He also studied lower bounds of gonalities of coverings of curves and proved the gonality of Y is greater than or equal to $\sqrt{n}$ under the condition above in [54, Theorem 2.7]. Theorem 4.8 states what we need for the proof of Theorem 4.5.

Definition 4.10. Let $f : Y \to X$ be a finite etale morphism of projective smooth and connected curves over $\operatorname{Spec} k$, $J(X) = \operatorname{Pic}^0(X)$ and $J(Y) = \operatorname{Pic}^0(Y)$ the Jacobian varieties of X and Y, respectively, and define the new part of Jacobian variety of Y relatively to X by the Abelian variety

$$J(Y, X) = J(Y)/f^* J(X)$$

with a polarization. The finite etale covering Y over X is said to be new-ordinary if $J(Y, X)$ is an ordinary Abelian variety.

Theorem 4.11 [48, Theorem 4.3.1], [54, Corollary 5.3]. *Let X be a projective smooth curve of genus $g \geqslant 2$. For any sufficiently large prime number l which is prime to p, there exists a nontrivial μ_l-torsor Y of X which is new-ordinary. Here μ_l is the locally constant etale sheaf consisting of lth roots of unity.*

Let $\mathcal{C}_k$ be a category of Artinian local k-algebras with residue field k. For a proper smooth connected scheme T_0 over $\operatorname{Spec} k$, we define a deformation functor

$$M_{T_0} : \mathcal{C}_k \to (\text{Sets}),$$

that is, for $R \in \mathcal{C}_k$, the set $M_{T_0}(R)$ is a set of isomorphism classes of pairs (T, φ) such that T is a proper smooth scheme over $\operatorname{Spec} R$ and φ is an isomorphism $T \times_{\operatorname{Spec} R} \operatorname{Spec} k \cong T_0$. If T_0 is a projective smooth connected curve or an Abelian variety, then M_{T_0} is pro-representable by the formal spectrum of a ring of formal power series over k.

Now let l be a prime number which is prime to p, X_0 a projective smooth connected curve of genus $\geqslant 2$ over $\operatorname{Spec} k$, Y_0 a μ_l-torsor of X_0 which is associated to a nontrivial element $\mathcal{L}_0 \in J(X_0)[l](k) \cong H^1_{\text{et}}(X_0, \mu_l)$, and $J(Y_0, X_0)$ the new part of Jacobian variety of Y_0 with respect to X_0. Then there is a canonical map

$$T_{\mathcal{L}_0}(R) : M_{X_0}(R) \to M_{J(Y_0, X_0)}(R)$$

as follows. For $X \in M_{X_0}(R)$, there exists a unique μ_l-torsor $f : Y \to X$ over $\operatorname{Spec} R$ which is a lift of the μ_l-torsor $f_0 : Y_0 \to X_0$ over $\operatorname{Spec} k$ by [26, I, Corollaire 8.4]. Then $T_{\mathcal{L}_0}(R)(X)$ is the new part $J(Y, X) = J(Y)/f^* J(X)$ of the relative Jacobian variety $J(Y)$

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of Y with respect to the relative Jacobian variety $J(X)$ of X with a natural isomorphism $J(Y,X) \times_{\operatorname{Spec} R} \operatorname{Spec} k \cong J(Y_0, X_0)$. Hence one has a Torelli morphism

$$T_{\mathcal{L}_0} : M_{X_0} \to M_{J(Y_0,X_0)}.$$

By the numerical estimates in [54, Corollaries 4.7, 5.3] one has the following theorem.

Theorem 4.12 [49, Theorem 3.3]. *With the notation as above, suppose that the gonality of X_0 is greater than or equal to 5. Then, for any sufficiently large prime number l which is prime to p, there exists a nontrivial element $\mathcal{L}_0 \in J(X_0)[l](k)$ such that the corresponding μ_l-torsor Y_0 of X_0 is new-ordinary and the Torelli morphism $T_{\mathcal{L}_0}$ is an immersion.*

4.4. Proof of Theorem 4.5.

First we prepare several assertions. Lemmas 4.13 and 4.16 are higher dimensional generalizations of Lemma/Definition 4.7 and Lemma 2 in [49], respectively.

Lemma 4.13. *Let $f : X \to S$ be a proper smooth morphism of smooth schemes separated of finite type over $\operatorname{Spec} k$ such that S is connected and each fiber of f is a connected curve, $t \in X(k(S)^{\mathrm{sep}})$ a point of degree d over $k(S)$ where $k(S)^{\mathrm{sep}}$ is a separable closure of the function field $k(S)$ of S, and n a positive integer which is prime to p. Let s be a closed point of S, and Y_s a μ_n-torsor of the fiber X_s of f at s. If n is prime to d, then there exist a finite etale morphism $S' \to S$ with a base change morphism $f' : X' = X \times_S S' \to S'$ and a μ_n-torsor $Y' \to X'$ such that, for $s' \in S'$ which goes to s in S, the μ_n-torsor $Y'_{s'}$ of $X'_{s'}$ is naturally isomorphic to the μ_n-torsor Y_s of X_s.*

Proof. Since we may identify μ_n with the constant etale sheaf $\mathbb{Z}/n\mathbb{Z}$, $R^1 f_{\mathrm{et}*}\mu_n$ is locally constant. Moreover, we can take a finite etale cover S' of S such that $R^1 f_{\mathrm{et}*}\mu_n$ is constant. Take a point $s' \in S'$ with a closed immersion $i_{s'} : s' \to S'$ such that s' is above s. Then we have an isomorphism $i_{s'}^* R^1 f'_{\mathrm{et}*}\mu_n \cong H^1_{\mathrm{et}}(X_{s'}, \mu_n)$ by proper base change theorem of etale cohomology. Since $R^1 f'_{\mathrm{et}*}\mu_n$ is constant, the natural homomorphism

$$H^0_{\mathrm{et}}(S', R^1 f'_{\mathrm{et}*}\mu_n) \to H^1_{\mathrm{et}}(X'_{s'}, \mu_n) \cong H^1_{\mathrm{et}}(X_s, \mu_n)$$

is bijective. By composing with the exact sequence

$$0 \to H^1_{\mathrm{et}}(S', \mu_n) \to H^1_{\mathrm{et}}(X', \mu_n) \to H^0_{\mathrm{et}}(S', R^1 f'_{\mathrm{et}*}\mu_n) \to H^2_{\mathrm{et}}(S', \mu_n) \to H^2_{\mathrm{et}}(X', \mu_n)$$

arising from Leray spectral sequence, we have only to prove that the natural homomorphism $H^2_{\mathrm{et}}(S', \mu_n) \to H^2_{\mathrm{et}}(X', \mu_n)$ is injective. Indeed, if this is proved, the homomorphism

$$H^1_{\mathrm{et}}(X', \mu_n) \to H^1_{\mathrm{et}}(X_s, \mu_n)$$

is surjective, and hence there exists a μ_n-torsor $Y' \to X'$ whose fiber at s' is isomorphic to the given μ_n-torsor $Y_s \to X_s$.

Let T be the normalization of S in the function field $k(S)(t)$ of t. Then the restriction $g = f|_T : T \to S$ is generically etale and finite of degree d. Since X is proper over S, there exists a morphism $h : T \to X$ over S except a closed subscheme of codimension $\geqslant 2$ in

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T. If we put $T' = T \times_S S'$ with a finite morphism $g' : T' \to S'$, then there exists a closed subscheme E' of codimension $\geqslant 2$ in S' such that h induces a commutative diagram

$$\begin{array}{ccccc} T' \setminus g'^{-1}(E') & \xrightarrow{h'} & X' \setminus f'^{-1}(E') & \subset & X' \\ & g'' \searrow & \downarrow f'' & & \downarrow f' \\ & & S' \setminus E' & \subset & S', \end{array}$$

where g'' is the restriction of g' to $T' \setminus g'^{-1}(E')$. Hence we have a commutative diagram of etale cohomology groups:

$$\begin{array}{ccc} H^2_{\text{et}}(S', \mu_n) & \xrightarrow{(f')^*} & H^2_{\text{et}}(X', \mu_n) \\ \cong\downarrow & & \downarrow\cong \\ H^2_{\text{et}}(S' \setminus E', \mu_n) & \xrightarrow{(f'')^*} & H^2_{\text{et}}(X' \setminus f'^{-1}(E'), \mu_n) \\ & (g'')^* \searrow & \downarrow (h')^* \\ & & H^2_{\text{et}}(T' \setminus g'^{-1}(E'), \mu_n). \end{array}$$

Indeed, both upper vertical maps are isomorphisms by the cohomological purity since E' (respectively $f'(E')$) is of codimension $\geqslant 2$ in the smooth scheme S' (respectively X') [42, VI, Theorem 5.1]. The injectivity of $(f')^*$ follows from that of $(g'')^*$ which will be proved in Proposition 4.14. □

Proposition 4.14. *Let S be a smooth connected scheme separated of finite type over* Spec k *of an algebraic closed field k of characteristic $p \geqslant 0$, and $g : T \to S$ a generically etale and finite morphism of degree d such that T is normal. Suppose n is a positive integer which is prime to dp (respectively d) if $p > 0$ (respectively $p = 0$). Then the homomorphism*

$$g^* : H^2_{\text{et}}(S, \mu_n) \to H^2_{\text{et}}(T, \mu_n)$$

is injective.

Proof. Let U be an open dense subscheme of S such that the inverse image $g^{-1}(U)$ is regular and the complement E of U in S is of codimension $\geqslant 2$. Such a U exists since T is normal and g is finite. Then we have a natural commutative diagram

$$\begin{array}{ccc} H^2_{\text{et}}(S, \mu_n) & \xrightarrow{g^*} & H^2_{\text{et}}(T, \mu_n) \\ \downarrow & & \downarrow \\ H^2_{\text{et}}(U, \mu_n) & \xrightarrow[g^*]{} & H^2_{\text{et}}(g^{-1}(U), \mu_n). \end{array}$$

Since the left vertical homomorphism is injective by the cohomological purity theorem, we may assume that T is smooth over Spec k.

Let us take a largest open dense subscheme V of S such that the inverse image $W = g^{-1}(V)$ is etale over S, and E (respectively F) is the complement of V (respectively W) in S (respectively T). Let us now consider the commutative diagram

$$\begin{array}{ccccc} H^2_{\text{et},E}(S, \mu_n) & \to & H^2_{\text{et}}(S, \mu_n) & \to & H^2_{\text{et}}(V, \mu_n) \\ g^* \downarrow & & g^* \downarrow & & \downarrow g^* \\ H^2_{\text{et},F}(T, \mu_n) & \to & H^2_{\text{et}}(T, \mu_n) & \to & H^2_{\text{et}}(W, \mu_n) \\ g_* \downarrow & & g_* \downarrow & & \downarrow g_* \\ H^2_{\text{et},E}(S, \mu_n) & \to & H^2_{\text{et}}(S, \mu_n) & \to & H^2_{\text{et}}(V, \mu_n), \end{array}$$

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where the left items are etale cohomology groups with supports, each sequence of horizontal homomorphisms is a localization sequence. The homomorphisms $g_* : H^2_{\mathrm{et}}(T, \mu_n) \to H^2_{\mathrm{et}}(S, \mu_n)$ are defined by Poincaré duality

$$\langle g_*(a), b\rangle_S = \langle a, g^*(b)\rangle_T$$

for $a \in H^2_{\mathrm{et}}(T, \mu_n)$ and $b \in H^{2\dim S-2}_{\mathrm{et},c}(S, \mu_n^{\otimes \dim S-1})$ where $g^* : H^{2\dim S-2}_{\mathrm{et},c}(S, \mu_n^{\otimes \dim S-1}) \to H^{2\dim T-2}_{\mathrm{et},c}(T, \mu_n^{\otimes \dim S-1})$ is a pullback of etale cohomology with compact supports and $\mu_n^{\otimes l}$ is a tensor product of l copies of μ_n. Note that Poincaré duality is applicable since $g : T \to S$ is a finite morphism of smooth schemes [42, VI, Remark 11.6]. The same works for the etale cohomology with supports. Then the composite g_*g^* in the left (respectively right) vertical homomorphisms is the map of multiplication with d by Lemma 4.15 (respectively the finite etaleness of W over V). Hence, the middle g_*g^* is surjective. Indeed, for any $a \in H^2_{\mathrm{et}}(S, \mu_n)$, $g_*g^*(a) - da$ is included in the image of $H^2_{\mathrm{et},E}(S, \mu_n)$. Hence there exists an element $b \in H^2_{\mathrm{et}}(S, \mu_n)$ coming from $H^2_{\mathrm{et},E}(S, \mu_n)$ such that $g_*g^*(a + b) = da$. Therefore the finiteness of $H^2_{\mathrm{et}}(S, \mu_n)$ implies that the homomorphism $g^* : H^2_{\mathrm{et}}(S, \mu_n) \to H^2_{\mathrm{et}}(T, \mu_n)$ is injective. □

Lemma 4.15. *With the notation in Proposition 4.14, the following hold.*

(1) *Any generic point of complement of V in S is pure of codimension 1. The same holds for the complement of W in T.*

(2) *Let $E_1, \dots, E_r$ be reduced irreducible components of the complement E of V in S, and $F_{i,1}, \dots, F_{i,s_i}$ the reduced irreducible components of the inverse image F_i of E_i in T with multiplicity $e_{i,j}$ of $F_{i,j}$ in $T \times_S E_i$ and the degree $f_{i,j}$ of $F_{i,j}$ over E_i (here we consider the ramification index and the residual degree for the extension $\mathcal{O}_{T,F_{i,j}}/\mathcal{O}_{S,E_i}$ of discrete valuation rings, respectively). Then $\sum_j e_{i,j} f_{i,j} = d$ for any i.*

(3) *The homomorphism $g^* : H^2_{\mathrm{et},E_i}(S, \mu_n) \to H^2_{\mathrm{et},F_i}(T, \mu_n)$ is given by the homomorphism*
$$g^* : H^0_{\mathrm{et}}(E_i, \mu_n) \to \oplus_j H^0_{\mathrm{et}}(F_{i,j}, \mu_n)$$
under the isomorphism (and the same for $F_{i,j}$) induced by the bottom horizontal Gysin isomorphism [42, VI, Theorem 5.1]:
$$\begin{array}{cccc} \mathbb{Z}/n\mathbb{Z} = & H^0_{\mathrm{et}}(E_i, \mu_n)(-1) & \to & H^2_{\mathrm{et},E_i}(S, \mu_n) \\ & \cong\downarrow & & \downarrow\cong \\ & H^0_{\mathrm{et}}(E_i^{\mathrm{sm}}, \mu_n)(-1) & \overset{\cong}{\to} & H^2_{\mathrm{et},E_i^{\mathrm{sm}}}(S \setminus E_i^{\mathrm{sing}}, \mu_n), \end{array}$$
where E_i^{sing} is a singular locus of E_i (note that E_i^{sing} is of codimension $\geqslant 2$ in S), $E_i^{\mathrm{sm}} = E_i \setminus E_i^{\mathrm{sing}}$, and (-1) means the (-1)-st Tate twist.

(4) *The homomorphism $g_* : H^2_{\mathrm{et},F_i}(T, \mu_n) \to H^2_{\mathrm{et},E_i}(S, \mu_n)$ is given by the homomorphism*
$$\oplus_j H^0_{\mathrm{et}}(F_{i,j}, \mu_n) \to H^0_{\mathrm{et}}(E_i, \mu_n) \qquad (a_j) \mapsto \sum_j a_j$$
under the isomorphisms as in (3).

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(5) *The composite* $g_*g^* : H^2_{\mathrm{et},E}(S, \mu_n) \to H^2_{\mathrm{et},E}(S, \mu_n)$ *is the map of multiplication with* d*.*

Proof. (1) Since V is the largest and T is finite over S, the assertion follows from Zariski–Nagata purity theorem [24, X, Théorème 3.4].

(2) is standard. (3) follows from the functoriality of Gysin morphisms.

(4) The homomorphism is induced by the natural homomorphism

$$g^* : H^{2\dim E_i}_{\mathrm{et},c}(E_i, \mu_n) \to H^{2\dim F_i}_{\mathrm{et},c}(F_i, \mu_n) \cong \oplus_j H^{2\dim F_{i,j}}_{\mathrm{et},c}(F_{i,j}, \mu_n).$$

(5) Since $H^2_{\mathrm{et},E}(S, \mu_n) \cong \bigoplus_i, H^2_{\mathrm{et},E_i}(S, \mu_n)$, the assertion follows from (2), (3), and (4). □

Lemma 4.16. *Let* S *be a projective smooth and connected scheme over* Spec k*, and* X *a proper smooth family of connected curves of genus* $g \geqslant 2$*. Let* $S' \to S$ *be a finite etale morphism and* $Y' \to X' = X \times_S S'$ *a finite etale morphism. If the family* Y' *over* S' *is isotrivial, then so is the family* X *over* S*.*

Proof. When S is a curve, the lemma is just Lemma 2 in [49, p.435]. Let T be a projective smooth curve in S, and T', X_T, X'_T, Y'_T base changes of S', X, X', Y' by $T \to S$, respectively. Then the family X_T over T is isotrivial. By varying projective smooth curves in S, there exists an open dense subscheme U of S such that $X_U = X \times_S U$ over U is isotrivial by Bertini's theorem. Hence, the family X over S is isotrivial because the canonical morphism $S \to \mathfrak{M}_g$ is constant, where $\mathfrak{M}_g$ is the coarse moduli space of projective smooth curves of genus g. □

Now let us prove Theorem 4.5. The following arguments are essentially due to the proof of [49, Theomre 4.6]. Let us fix a closed point s in S. If the fiber X_s of X at s is ordinary, then the assertion follows from Corollary 4.3. Hence we may suppose the fiber X_s is not ordinary. Then there exist

1° (only in the case where the p-rank of X_s is 0; if not, then $S_0 = S$ and $s_0 = s$) a point t_0 of $X(k(S)^{\mathrm{sep}})$ of degree d_0, a finite etale morphism $S_0 \to S$ of connected schemes and a finite etale morphism $Y_0 \to X_0 = X \times_S S_0$ such that, for a closed point s_0 of S_0 which goes to s in S, $Y_0 \to X_0$ is a μ_{l_0}-torsor for a prime number l_0 which is prime to pd_0 and that the p-rank of the fiber Y_{0,s_0} of Y_0 at s_0 is neither 0 nor g;

2° a point t_1 of $Y_0(k(S_0)^{\mathrm{sep}})$ of degree d_1, a finite etale morphism $S_1 \to S_0$ of connected schemes and a finite etale morphism $Y_1 \to X_1 = Y_0 \times_{S_0} S_1$ such that, for a closed point s_1 of S_1 which goes to s_0 in S_0, $Y_1 \to X_1$ is a μ_{l_1}-torsor for a prime number l_1 which is prime to pd_1 and that the gonality of the fiber Y_{1,s_1} of Y_1 at s_1 is greater than or equal to 5;

3° a point t_2 of $Y_1(k(S_1)^{\mathrm{sep}})$ of degree d_2, a finite etale morphism $S_2 \to S_1$ of connected schemes, a prime number l_2 which is prime to pd_2, and a nontrivial μ_{l_2}-torsor $Y_2 \to X_2 = Y_1 \times_{S_1} S_2$ such that, if s_2 is a closed point of S_2 which goes to s_1 in S_1 and $\mathcal{L} \in J(X_{2,s_2})[l_2](k)$ corresponds to the μ_{l_2}-torsor $Y_{2,s_2} \to X_{2,s_2}$, then (i) $Y_{2,s_2} \to X_{2,s_2}$ is new-ordinary and (ii) the Torelli morphism $T_{\mathcal{L}} : M_{X_{2,s_2}} \to M_{J(Y_{2,s_2}, X_{2,s_2})}$ is an immersion.

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by Theorem 4.11 and Lemma 4.13 for 1°, by Theorem 4.8 and Lemma 4.13 for 2°, and by Theorem 4.12 and Lemma 4.13 for 3°. Note that the existence of t_0, t_1, t_2 in separable closures follows from [60, 0CD4, Lemma 50.13.9]. Hence the gonality of $X_{2,s_2} \cong Y_{1,s_1}$ is greater than or equal to 5 by 2°. In this situation we have only to prove the family X_2 over S_2 is isotrivial under the assumption that any geometric convergent F-isocrystal on S_2/K has constant Newton polygons. Indeed, one can replace X/S by X_0/S_0, X_0/S_0 by X_1/S_1, and X_1/S_1 by X_2/S_2 by Proposition 2.8(2) and Lemma 4.16.

Let $J(X_2)$ and $J(Y_2)$ be relative Jacobian varieties over S_2 and define the new part

$$J(Y_2, X_2) = J(Y_2)/f_2^* J(X_2)$$

where $f_2 : Y_2 \to X_2$ is the canonical morphism. Then $J(Y_2, X_2)$ is a principally polarized Abelian scheme over S_2. Since $J(Y_2, X_2)_{s_2} = J(Y_{2,s_2}, X_{2,s_2})$ is an ordinary Abelian variety, there exists an etale morphism $g : U \to S_2$ with $s_2 \in g(U)$ such that $J(Y_2, X_2) \times_{S_2} U$ is a trivial deformation of $J(Y_{2,s_2}, X_{2,s_2})$ over U by Theorem 4.2. Hence the formal deformation $J(Y_2, X_2) \times_{S_2} \mathrm{Spf}\, \widehat{\mathcal{O}}_{S_2,s_2}$ of $J(Y_{2,s_2}, X_{2,s_2})$ over $\mathrm{Spf}\, \widehat{\mathcal{O}}_{S_2,s_2}$ is trivial where $\widehat{\mathcal{O}}_{S_2,s_2}$ is the completion of $\mathcal{O}_{S_2}$ along s_2. Since the Torelli morphism $T_{\mathcal{L}}$ is an immersion by 3°, the formal deformation $X_2 \times_{S_2} \mathrm{Spf}\, \widehat{\mathcal{O}}_{S_2,s_2}$ of X_{2,s_2} over $\mathrm{Spf}\, \widehat{\mathcal{O}}_{S_2,s_2}$ is also trivial. It implies an existence of an etale morphism $h : V \to S_2$ with $s_2 \in h(V)$ such that X_2 is a trivial deformation of X_{2,s_2} over V by [50, Proposition 2.6.10]. Note that the projectivity X_2 over S_2 is required in the proposition of [50] and it follows from the fact that $J(X_2)$ is projective over S_2 (see [22, Proof of Theorem 1.9]). Therefore, X_2 is isotrivial over an open subscheme of S_2 including s_2 and then the canonical morphism $S_2 \to \mathfrak{M}_{g_2}$ is constant where g_2 is the genus of the geometric generic fiber of X_2 over S_2. This completes a proof. □

4.5. Existence of convergent F-isocrystals with nonconstant Newton polygons

In the study of nonconstant geometric etale fundamental groups on a family, M.Saïdi proved the following theorem.

Theorem 4.17 [49, Theorem 4.5]. *Let C be a projective smooth and connected curve over* $\mathrm{Spec}\, k$*, and $f : X \to C$ a proper smooth family of connected curves of genus $\geqslant 2$. If X is not isotrivial over C, then the p-ranks of fibers X_s are not constant on C.*

Since there always exists a projective smooth and connected curve C with a nonisotrivial family of curves (see [47, Theorem 3.1], [17]), we have an existence theorem below. Moreover, such a curve C and a convergent F-isocrystal are possibly defined over a finite base field in any characteristic p.

Corollary 4.18. *There exist a projective smooth and connected curve C over* $\mathrm{Spec}\, k$ *and a convergent F-isocrystal on C/K with nonconstant Newton polygons.*

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A. Proof of Theorem 2.5.

A.1. First reduction

In this appendix we will prove the following theorem.

Theorem A.1. *Let X be a smooth scheme separated of finite type over k, and $\mathcal{M}$ a convergent F-isocrystal on X/K. Suppose that*

(∗) *the initial slope of $\mathcal{M}$ at the generic point of X is greater than or equal to 0 and the rank of slope 0 of F-isocrystal $i_x^*\mathcal{M}$ is constant on points of X.*

Then there exists a unit-root convergent sub F-isocrystal $\mathcal{L}$ of $\mathcal{M}$ on X/K of rank r_0 (if $r_0 = 0$, then $\mathcal{L} = 0$), where r_0 is the rank of slope 0 at the generic point.

Theorem 2.5 follows from the theorem above. Indeed, by taking a ramified finite extension K' of K with an extension of Frobenius such that the valuation corresponding to the initial slope is contained in the valuation group of K', we can reduce the assertion to that in the case where the initial slope is 0 by Lemma A.2.

Lemma A.2. *Let K' be a finite extension of K with a residue field k' such that there exists a q-Frobenius σ' on K' satisfying $\sigma'|_K = \sigma$, and put $X' = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k$. Let $\mathcal{M}$ be a convergent F-isocrystal on X/K, and $\mathcal{M}'$ the inverse image of $\mathcal{M}$ on X'/K'. Suppose there exists a convergent sub F-isocrystal $\mathcal{L}'$ of $\mathcal{M}'$ such that all slopes of $i_x^*(\mathcal{M}'/\mathcal{L}')$ are greater than those of $i_x^*\mathcal{L}'$ for any point $i_x : x \to X$. Then there exists a convergent sub F-isocrystal $\mathcal{L}$ of $\mathcal{M}$ on X/K such that the inverse image of $\mathcal{L}$ on X'/K' is isomorphic to $\mathcal{L}'$.*

Proof. Let $g : X' \to X$ be a canonical morphism. If we put $\mathcal{L}$ to be the kernel of the natural homomorphism $\mathcal{M} \to g_*\mathcal{M}'/g_*\mathcal{L}'$, i.e.,

$$\mathcal{L} = \mathrm{Ker}(\mathcal{M} \to g_*\mathcal{M}'/g_*\mathcal{L}'),$$

then $\mathcal{L}$ is the desired convergent sub F-isocrystal of $\mathcal{M}$ by the hypothesis of slopes. □

In order to prove Theorem A.1 we may assume that the conditions (i) and (ii) of Frobenius σ in § 2.1 and X are geometrically connected by Lemma A.2. Our strategy of the proof of Theorem A.1 is as follows:

1° To construct a $\mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X))$-representation $V(\mathcal{M})$ over K_σ which corresponds to the unit-root sub of $\mathcal{M}$ at the generic fiber. Here $k(X)$ is the function field of X and $k(X)^{\mathrm{sep}}$ is a separable closure of $k(X)$.

2° To show $V(\mathcal{M})$ is unramified at each point of X of codimension 1. Then $V(\mathcal{M})$ is a representation of $\pi_1^{\mathrm{et}}(X)$ by Zariski–Nagata purity theorem [24, X, Theorem 3.4].

3° To take a unit-root convergent F-isocrystal $\mathcal{L}$ on X/K corresponding to $V(\mathcal{M})$ by Katz–Crew equivalence [11, Theorem 2.1].

4° To show $\mathcal{L}$ is a subobject of $\mathcal{M}$.

A.2. Unramified representations at a point of codimension 1

Let E be a complete discrete valuation field of mixed characteristic $(0, p)$, and $R_E, k_E, \mathbf{m}_E$ the ring of integers, the residue field (allowing a non-perfect field), and the maximal ideal of R_E, respectively. Let φ be a q-Frobenius on E. Suppose that (i) $\mathbb{F}_q \subset k_E$ and (ii) the absolute ramification index of E is equal to that of E_φ. Here E_φ is a φ-invariant subfield of E and it is a totally ramified finite extension of the field of fractions of $W(\mathbb{F}_q)$. Note that, for arbitrary E and φ, the hypotheses (i), (ii) hold after a finite unramified extension of E. Indeed, a φ-invariant uniformizer π exists in the extension $\widehat{E}^{\mathrm{ur,pf}}$ of E which is defined in the next paragraph and π is algebraic over $\mathbb{Q}_p$ by Remark 2.1.

Let k_E^{alg} (respectively k_E^{sep}) be an algebraic closure of k_E (respectively a separable closure of k_E in k_E^{alg}), $\widehat{E}^{\mathrm{ur}}$ the p-adic completion of a maximal unramified extension of E, and $\widehat{E}^{\mathrm{ur,pf}}$ the p-adic completion of the inductive limit of the inductive system

$$\widehat{E}^{\mathrm{ur}} \stackrel{\varphi}{\to} \widehat{E}^{\mathrm{ur}} \stackrel{\varphi}{\to} \widehat{E}^{\mathrm{ur}} \stackrel{\varphi}{\to} \cdots$$

such that $\widehat{E}^{\mathrm{ur}} \to \widehat{E}^{\mathrm{ur,pf}}$ is defined by the inclusion into the first component. $\widehat{E}^{\mathrm{ur}}$ (respectively $\widehat{E}^{\mathrm{ur,pf}}$) is a complete discrete valuation field of mixed characteristic $(0, p)$ with residue field k_E^{sep} (respectively k_E^{alg}). Then the q-Frobenius φ on E extends uniquely on $\widehat{E}^{\mathrm{ur}}$ and $\widehat{E}^{\mathrm{ur,pf}}$ where the Frobenius on $\widehat{E}^{\mathrm{ur,pf}}$ is induced by $(\varphi, \varphi, \varphi, \ldots)$ on the inductive system, and we also denote them by φ.

Lemma A.3. (1) *There exists a canonical isomorphism*

$$\widehat{E}^{\mathrm{ur,pf}} \cong E_\varphi \otimes_{W(\mathbb{F}_q)} W(k_E^{\mathrm{alg}})$$

such that the isomorphism commutes with Frobenius and the actions of $\mathrm{Gal}(k_E^{\mathrm{sep}}/k_E)$*. Here* Fr_q *is the canonical* q*-Frobenius on* $W(k_E^{\mathrm{alg}})$*, the Frobenius* $\widetilde{\varphi}$ *on* $E_\varphi \otimes_{W(\mathbb{F}_q)} W(k_E^{\mathrm{alg}})$ *is defined by* $\mathrm{id}_{E_\varphi} \otimes \mathrm{Fr}_q$*,* $\mathrm{Gal}(k_E^{\mathrm{sep}}/k_E)$ *acts on* $W(k_E^{\mathrm{alg}})$ *by each entry of Witt vectors and on* $E_\varphi \otimes_{W(\mathbb{F}_q)} W(k_E^{\mathrm{alg}})$ *by* $1 \otimes \tau$ *for* $\tau \in \mathrm{Gal}(k_E^{\mathrm{sep}}/k_E)$*.*

(2) $(\widehat{E}^{\mathrm{ur,pf}})_\varphi = (\widehat{E}^{\mathrm{ur}})_\varphi = E_\varphi$*.*

Proof. It follows from the universal property of Witt vector rings. □

Even in the case where k_E is an arbitrary field of characteristic p, one can also define F-spaces over E, slopes of Frobenius, and slope filtrations of F-spaces as in § 2.2 and 2.3. We replace the notation of Frobenius F of F-spaces by Φ in this Appendix. An F-space M over E is unit-root, i.e., all slopes are 0, if and only if there exists a finitely generated Φ-stable R_E-submodule L in M such that $L \otimes_{R_E} E \cong M$ and $\Phi(L)$ generates L over R_E. Such an L is called a lattice of M. Indeed, when the residue field k_E is infinite, there exists a cyclic vector $v \in M$, namely $\{v, \Phi(v), \ldots, \Phi^{r-1}(v)\}$ forms a basis of M (see [55, Lemma 3.1.4]). If $\Phi^r(v) + a_1\Phi^{r-1}(v) + \cdots + a_r v = 0$ for some $a_1, \ldots, a_r \in E$ $(a_r \neq 0)$, then the

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unit-rootness of M is equivalent to the conditions $a_1, \dots, a_r \in R_E$ and $\mathrm{ord}_p(a_r) = 0$. Hence a unit-root F-space over E admits a lattice L. When k_E is finite, $L_{\widehat{E}^{\mathrm{ur}}} \cap M$ is a lattice of M where $L_{\widehat{E}^{\mathrm{ur}}}$ is a lattice of $M \otimes_E \widehat{E}^{\mathrm{ur}}$. For the converse, we may assume the residue field k_E is algebraically closed and then it follows from Dieudonné–Manin classification theorem.

For an F-space M over E, we put

$$V_E(M) := \mathrm{Ker}(1 - \Phi \otimes \varphi; M \otimes_E \widehat{E}^{\mathrm{ur}}) = \{v \in M \otimes_E \widehat{E}^{\mathrm{ur}} \mid (\Phi \otimes \varphi)(v) = v\}.$$

Then $V_E(M)$ is an E_φ-space of dimension $\leqslant \dim_E M$ since the canonical map $V_E(M) \otimes_{E_\varphi} E \to M$ is injective by Lemma A.3(2). The K_σ-space $V_E(M)$ is furnished with a $\mathrm{Gal}(k_E^{\mathrm{sep}}/k_E)$-action defined by $1 \otimes \tau$ ($\tau \in \mathrm{Gal}(k_E^{\mathrm{sep}}/k_E)$).

Proposition A.4. *Let M be an F-space over E.*

(1) *There is a slope filtration $\{S_\lambda M\}$ of M as F-spaces over E.*

(2) *Suppose $S_\lambda = 0$ for $\lambda < 0$. If L is a lattice of $S_0 M$, then*

$$V_E(M) \cong V_E(S_0 M) \cong \mathrm{Ker}(1 - \Phi \otimes \varphi; L \otimes_{R_E} R_{\widehat{E}^{\mathrm{ur}}}) \otimes_{(R_E)_\varphi} E_\varphi$$

and it is of dimension $r_0 = \dim_E S_0 M$. Here $R_{\widehat{E}^{\mathrm{ur}}}$ is the integer ring of $\widehat{E}^{\mathrm{ur}}$.

(3) *$V_E(M) \cong \mathrm{Ker}(1 - \Phi \otimes \varphi; M \otimes_E \widehat{E}^{\mathrm{ur,pf}})$ by the natural inclusion.*

Proof. (1) See [34, Remark 1.7.8] and [8, Theorems 2.4].

(2) The first equality is trivial by slopes under the hypothesis. Let $(e_1, \dots, e_{r_0})$ be a basis of $S_0 M$ and put $\Phi(e_1, \dots, e_{r_0}) = (e_1, \dots, e_{r_0})F$ for an $F \in \mathrm{GL}_{r_0}(E)$. Then

$$V_E(M) \cong \{\mathbf{y} \in (\widehat{E}^{\mathrm{ur}})^{r_0} \mid F\varphi(\mathbf{y}) = \mathbf{y}\}.$$

If one takes a basis of M in L, then $F \in \mathrm{GL}_{r_0}(R_E)$ and it implies that the second equality of the assertion. The Frobenius equation has enough solutions by [28, Proposition 4.1.1] ([23, A1.2] or by direct calculations of the equations modulo $\mathbf{m}^n_{\widehat{E}^{\mathrm{ur}}}$ step by step and completeness of $R_{\widehat{E}^{\mathrm{ur}}}$). Hence $\dim_{E_\varphi} V_E(M) = r_0$.

(3) It follows from Lemma A.3(2). □

Let us put $E[\![t]\!]_0 = R_E[\![t]\!][1/p]$ (respectively $\mathcal{E} = \widehat{E[\![t]\!]_0[1/t]}$ to be the p-adic completion of $E[\![t]\!]_0[1/t]$), and φ a q-Frobenius on $E[\![t]\!]_0$ (respectively the unique extension to $\mathcal{E}$) with respect to the Frobenius φ on E, that is, $\varphi(a) \equiv a^q \pmod{\mathbf{m} R_E[\![t]\!]}$ for $a \in R_E[\![t]\!]$. We define the Gauss norm on $E[\![t]\!]_0$ (respectively $\mathcal{E}$) by

$$\left| \sum_n a_n t^n \right|_{\mathrm{Gauss}} = \sup_n |a_n|_p,$$

where $|a|_p = p^{-\mathrm{ord}_p(a)}$ is the p-adic norm. Note that $E[\![t]\!]_0$ is a principal ideal domain by Weierstrass preparation theorem such that it is complete in (t, p)-adic topology, and $\mathcal{E}$ is a complete discrete valuation field under Gauss norm. The φ-invariant subfield $\mathcal{E}_\varphi$ of $\mathcal{E}$ coincides with E_φ, and hence the conditions (i), (ii) at the beginning of A.2 hold for $\mathcal{E}$ and φ.

For either $B = E[\![t]\!]_0$ or $\mathcal{E}$, a (φ, ∇)-module (M, ∇, Φ) over B is a free B-module of finite rank with a E-connection $\nabla : M \to M \otimes_B \Omega^1_{B/E}$, $(\Omega^1_{B/E} = Bdt)$ and a Frobenius $\Phi : \varphi^* M \stackrel{\cong}{\to} M$ such that Φ is a horizontal isomorphism with respect to connections. Then the category of (φ, ∇)-modules over B is Abelian and it is independent of the choice of φ up to canonical equivalences [56, Proposition 3.4.9].

A generic slope (respectively a special slope) of a (φ, ∇)-module M over $E[\![t]\!]_0$ is a slope of the (φ, ∇)-module $M \otimes_{E[\![t]\!]_0} \mathcal{E}$ over $\mathcal{E}$ (respectively the φ-module $M \otimes_{E[\![t]\!]_0} E$ over E, where the specialization map $E[\![t]\!]_0 \to E$ is defined by $\sum_n a_n t^n \mapsto a_0$). We define generic (respectively special) Newton polygon of M by Newton polygon of $M \otimes_{E[\![t]\!]_0} \mathcal{E}$ (respectively $M \otimes_{E[\![t]\!]_0} E$). A (φ, ∇)-module M over $E[\![t]\!]_0$ is said to be unit-root if all slopes both at the generic point and at the special point are 0.

Theorem A.5. *Let M be a (φ, ∇)-module M over $E[\![t]\!]_0$.*

(1) ([29, Corollary 2.6.2] if k_E is perfect, [8, Theorems 6.21] in general) *If a (φ, ∇)-module M over $E[\![t]\!]_0$ has constant Newton polygons (i.e., both generic and special Newton polygons are same), then M admits a slope filtration $\{S_\lambda M\}_\lambda$ as (φ, ∇)-modules over $E[\![t]\!]_0$. If furthermore that M is unit-root and the residue field k_E of E is separably closed, then M is isomorphic to a direct sum of copies of the trivial (φ, ∇)-module $E[\![t]\!]_0$.*

(2) *Suppose M admits a slope filtration $\{S_\lambda M\}_\lambda$ such that $S_\lambda M = 0$ for $\lambda < 0$. Then there are natural isomorphisms*

$$V_{\mathcal{E}}(M \otimes_{E[\![t]\!]_0} \mathcal{E}) \stackrel{\cong}{\leftarrow} \mathrm{Ker}(1 - \Phi \otimes \varphi; M \otimes_{E[\![t]\!]_0} \widehat{E}^{\mathrm{ur}}[\![t]\!]_0) \stackrel{\cong}{\to} V_E(M \otimes_{E[\![t]\!]_0} E)$$

which are compatible with the actions of $\mathrm{Gal}(k_E((t))^{\mathrm{sep}}/k_E((t)))$. In particular, the representation $V_{\mathcal{E}}(M \otimes_{E[\![t]\!]_0} \mathcal{E})$ is unramified, that is, the Galois group acts via the quotient $\mathrm{Gal}(k_E((t))^{\mathrm{ur}}/k_E((t))) \cong \mathrm{Gal}(k_E^{\mathrm{sep}}/k_E)$.

Proof. (2) If we put $r_0 = \dim_E S_0 M$, then $V_{\mathcal{E}}(M \otimes_{E[\![t]\!]_0} \mathcal{E})$ (respectively the middle term, respectively $V_E(M)$) is of dimension r_0 by Proposition A.4(2) (respectively by (1) for the middle term). Hence the natural maps are isomorphisms. By the construction they are compatible with the Galois actions, and it implies that the representation $V_{\mathcal{E}}(M \otimes_{E[\![t]\!]_0} \mathcal{E})$ is unramified. □

A.3. Construction of a functor V

Now we return to the situation of Theorem A.1. We will define a functor

$$V : F\text{-Isoc}(X/K)^{(*)} \to \mathrm{Rep}_{K_\sigma}(\pi_1^{\mathrm{et}}(X))$$

and show several properties of V, where $F\text{-Isoc}(X/K)^{(*)}$ is the full subcategory of $F\text{-Isoc}(X/K)$ consisting of objects which satisfy the hypothesis $(*)$ in Theorem A.1 and $\mathrm{Rep}_{K_\sigma}(\pi_1^{\mathrm{et}}(X))$ is the category of continuous finite dimensional K_σ-representations of $\pi_1^{\mathrm{et}}(X)$.

Let X be a smooth geometrically connected scheme separated of finite type over $\mathrm{Spec}\, k$. We may assume that X is affine by gluing after Lemma A.7. Then there exists a smooth affine formal scheme $\mathrm{Spf}\, A$ topologically of finite type over $\mathrm{Spf}\, R$ such that $X = \mathrm{Spec}\, A \otimes_R k$ and A is furnished with a q-Frobenius endomorphism φ_A compatible with the q-Frobenius σ on K. Indeed, such a lift A (respectively a Frobenius φ_A on A) exists by [18, Théorème 6] (respectively by formal smoothness).

Let E_A be the field of fractions of the p-adic completion of the localization $A_{\mathbf{m}}$ of A along $\mathbf{m}A$, and $\widehat{E}_A^{\mathrm{ur}}$ the p-adic completion of the maximal unramified extension of E_A, and $\widehat{E}_A^{\mathrm{ur,pf}}$ as before. Then

$$K \subset E_A \subset \widehat{E}_A^{\mathrm{ur}} \subset \widehat{E}_A^{\mathrm{ur,pf}}$$

is a sequence of extensions of discrete valuation fields with residue fields $k, k(X), k(X)^{\mathrm{sep}}$ and $k(X)^{\mathrm{alg}}$ respectively such that they have Frobenius φ_A which are unique extensions of that of A. The φ_A-invariant subfields of them coincide with K_σ by the hypotheses (i), (ii) in § 2.1, the formal smoothness of A over R and Lemma A.3. So the conditions (i), (ii) at the beginning of A.2 hold. Moreover, the absolute Galois group $\mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X))$ of the function field $k(X)$ of X acts continuously on the above sequence.

Let $\mathrm{Spf}\, B$ be another formally smooth lift of X over $\mathrm{Spf}\, R$, and take $E_B \subset \widehat{E}_B^{\mathrm{ur}} \subset \widehat{E}_B^{\mathrm{ur,pf}}$ and φ_B as before. Let $A\widehat{\otimes}_R B$ be the tensor product of A and B over R in the category of formal R-algebras, and $p_A : \mathrm{Spf}\, A\widehat{\otimes}_R B \to \mathrm{Spf}\, A$ and $p_B : \mathrm{Spf}\, A\widehat{\otimes}_R B \to \mathrm{Spf}\, B$ two projections.

Lemma A.6. *Let $]X[_{\mathrm{Spf}\, A\widehat{\otimes}_R B}$ be the tubular neighborhood of X in $\mathrm{Spf}\, A\widehat{\otimes}_R B$ (see [3] for rigid analytic spaces and the definition of tubular neighborhoods), and $f_1, \ldots, f_s \in A\widehat{\otimes}_R B$ lifts of generators of the kernel of the multiplication map $A/\mathbf{m}A \otimes_k B/\mathbf{m}B \to \Gamma(X, \mathcal{O}_X)$.*

(1) *Let π be a generator of the maximal ideal $\mathbf{m}_\sigma$ of R_σ (note that it is also a generator of $\mathbf{m}$). If U_n denotes an affinoid subspace of the quasi-Stein space $]X[_{\mathrm{Spf}\, A\widehat{\otimes}_R B}$ which is defined by $|f_i^{q^n}| \leqslant |\pi|$ for $n \geqslant 1$, then*

$$C := \Gamma(]X[_{\mathrm{Spf}\, A\widehat{\otimes}_R B}, \mathcal{O}_{]X[_{\mathrm{Spf}\, A\widehat{\otimes}_R B}}) = \varprojlim_n C_n$$

$$\text{where} \begin{cases} C_n := \Gamma(U_n, \mathcal{O}_{]X[_{\mathrm{Spf}\, A\widehat{\otimes}_R B}}) = \dfrac{(A\widehat{\otimes}_R B)[t_{n,1}, \ldots, t_{n,s}]\widehat{\ }[1/p]}{(\pi t_{n,i} - (\varphi_A\widehat{\otimes}\varphi_B)^n(f_i); i = 1, \ldots, s)} \\ C_{n+1} \to C_n \quad t_{n+1,i} \mapsto \pi^{q-1} t_{n,i}^q - (\varphi_A\widehat{\otimes}\varphi_B)^n(f_i^q)/\pi + (\varphi_A\widehat{\otimes}\varphi_B)^{n+1}(f_i)/\pi, \end{cases}$$

and $\widehat{\ }$ means the p-adic completion. Moreover, the Frobenius $\varphi_A\widehat{\otimes}\varphi_B$ on $A\widehat{\otimes}_R B$ induces a Frobenius $\varphi_C : C \to C$ which is given by

$$C_n \to C_{n+1} \quad t_{n,i} \mapsto t_{n+1,i}.$$

(2) *Let $A[1/p] \to K_\sigma \otimes_{W(\mathbb{F}_q)} W(k(X)^{\mathrm{alg}})$ be a K-algebra homomorphism which is a composite of $A[1/p] \to \widehat{E}_A^{\mathrm{ur,pf}}$ and the canonical isomorphism $\widehat{E}_A^{\mathrm{ur,pf}} \cong K_\sigma \otimes_{W(\mathbb{F}_q)} W(k(X)^{\mathrm{alg}})$ in Lemma A.3(1), and the same for B. Then they induce a K-algebra homomorphism*

$$\mu : C \to K_\sigma \otimes_{W(\mathbb{F}_q)} W(k(X)^{\mathrm{alg}})$$

such that μ commutes with Frobenius φ_C and $\widetilde{\varphi} = \mathrm{id}_{K_\sigma} \otimes \mathrm{Fr}_q$, respectively.

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(3) *If Q is the topological closure of the field of fractions of $\mu(C)$ under the p-adic norm, then we have a commutative diagram*

$$\begin{array}{ccccccc} & & A[1/p] & \to E_A & \to \widehat{E}_A^{\mathrm{ur}} & \to & \widehat{E}_A^{\mathrm{ur,pf}} \\ & \nearrow & \downarrow & \downarrow & & & \uparrow\cong \\ K & \to & C & \to Q & \subset & & K_\sigma \otimes_{W(\mathbb{F}_q)} W(k(X)^{\mathrm{alg}}) \\ & \searrow & \uparrow & \uparrow & & & \downarrow\cong \\ & & B[1/p] & \to E_B & \to \widehat{E}_B^{\mathrm{ur}} & \to & \widehat{E}_B^{\mathrm{ur,pf}} \end{array}$$

of topological K-algebras such that all morphisms commute with Frobenius and the actions of $\mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X))$.

Proof. (1) Since $(\varphi_A\widehat{\otimes}\varphi_B)(f_i) \equiv f_i^q\ (\mathrm{mod}\,\mathbf{m}(\varphi_A\widehat{\otimes}\varphi_B))$, the condition $|f_i^{q^n}| \leqslant |\pi|$ is equivalent to $|\varphi^n(f_i)| \leqslant |\pi|$ on $]X[_{\mathrm{Spf}\,A\widehat{\otimes}_R B}$. Since $|\varphi^n(f_i)/\pi| \leqslant 1$ on U_n, the K-algebra homomorphism $C_{n+1} \to C_n$ is well defined.

(2) Put $h : (A\widehat{\otimes}_R B)[1/p] \to K_\sigma \otimes_{W(\mathbb{F}_q)} W(k(X)^{\mathrm{alg}})$ to be the induced morphism. Since $f_i = 0$ in $\Gamma(X, \mathcal{O}_X)$, the inequality $|h(f_i)| \leqslant |\pi|$ holds in $K_\sigma \otimes_{W(\mathbb{F}_q)} W(k(X)^{\mathrm{alg}})$ and $h((\varphi_A\widehat{\otimes}\varphi_B)^n(f_i)/\pi)$ exists in $R_\sigma \otimes_{W(\mathbb{F}_q)} W(k(X)^{\mathrm{alg}})$. Hence the K-algebra homomorphism μ exists. Lemma A.3(1) and the universal property of tensor products induce the compatibility of Frobenius and h, so that μ is compatible with Frobenius.

(3) It follows from (1), (2) and Lemma A.3(1). □

Now we define the functor V. Let $\mathcal{M}$ be a convergent F-isocrystal on X/K which satisfies the hypothesis $(*)$ of Theorem A.1. Put $M = \Gamma(]X[_{\mathrm{Spf}\,A}, \mathcal{M})$. Then M is a projective $A[1/p]$-module of finite type which is furnished with an integrable connection $\nabla_M : M \to M \otimes_A \Omega^1_{\mathrm{Spf}\,A/\mathrm{Spf}\,R}$ and a Frobenius $\Phi_M : \varphi^* M \xrightarrow{\cong} M$ such that Φ_M is horizontal with respect to integrable connections. We define a K_σ-space $V(\mathcal{M})$ by

$$V(\mathcal{M}) = \mathrm{Ker}(1 - \Phi_M \otimes \varphi_A;\ M \otimes_{A[1/p]} \widehat{E}_A^{\mathrm{ur}}).$$

Since $M \otimes_{A[1/p]} E_A$ admits a slope filtration $\{S_\lambda\}$ with $S_\lambda = 0$ for $\lambda < 0$ by Theorem A.5, there is an equality $V(\mathcal{M}) = \mathrm{Ker}(1 - \Phi_M \otimes \varphi_A;\ S_0 \otimes_{E_A} \widehat{E}_A^{\mathrm{ur}})$ by the inclusion $S_0 \subset M \otimes_{A[1/p]} E_A$. Hence $V(\mathcal{M})$ is a K_σ-space of rank $r_0(= \dim_E S_0)$ with a continuous action of the Galois group $\mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X))$ by Proposition A.4. Moreover, there is a canonical isomorphism

$$V(\mathcal{M}) \cong \mathrm{Ker}(1 - \Phi_M \otimes \varphi_A;\ M \otimes_{A[1/p]} \widehat{E}_A^{\mathrm{ur,pf}}).$$

of K_σ-representations of $\mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X))$ by Lemma A.3(2) and Proposition A.4.

Lemma A.7. *The K_σ-representation $V(\mathcal{M})$ of* $\mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X))$ *does not depend on the choices of the lift* $\mathrm{Spf}\,A$ *of X over* $\mathrm{Spf}\,R$ *and the Frobenius φ_A on* $\mathrm{Spf}\,A$ *up to canonical isomorphisms.*

Proof. Keep the notation in Lemma A.6. Let $\mathrm{Spf}\,B$ be another formally smooth lift of X over $\mathrm{Spf}\,R$. We fix the notation as in Lemma A.6. If $\mathcal{N}$ is a convergent F-isocrystal on X/K with respect to the frame $(X, \mathrm{Spf}\,B)$ which corresponds to $\mathcal{M}$, then there is a canonical isomorphism

$$\epsilon : p_B^*\mathcal{N} \xrightarrow{\cong} p_A^*\mathcal{M}$$

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of convergent F-isocrystals on X/K with respect to the frame $(X, \operatorname{Spf} A\widehat{\otimes}_R B)$ and a commutative diagram

$$\begin{array}{ccc} \varphi_C^*(p_B^*\mathcal{N}) & \overset{\varphi_C^*\epsilon}{\underset{\cong}{\to}} & \varphi_C^*(p_A^*\mathcal{M}) \\ 1\otimes\Phi_N \downarrow & & \downarrow \Phi_M\otimes 1 \\ p_B^*\mathcal{N} & \overset{\cong}{\underset{\epsilon}{\to}} & p_A^*\mathcal{M} \end{array}$$

of Frobenius.

Let $\mathbf{e}_M$ (respectively $\mathbf{e}_N$) be a basis of $M\otimes_{A[1/p]} E_A$ (respectively $N\otimes_{B[1/p]} E_B$ for $N = \Gamma(]X[_{\operatorname{Spf} B}, \mathcal{N}))$. Then there exist matrices $F_M \in \mathrm{GL}_r(E_A)$, $F_N \in \mathrm{GL}_r(E_B)$, and $H \in \mathrm{GL}_r(C)$ such that

$$\Phi_M(1\otimes\mathbf{e}_M) = \mathbf{e}_M F_M, \quad \Phi_N(1\otimes\mathbf{e}_N) = \mathbf{e}_N F_N, \quad \epsilon(1\otimes\mathbf{e}_N) = (\mathbf{e}_M\otimes 1)H.$$

Then the above commutativity on Frobenius induces the identity

$$F_M\widetilde{\varphi}(\mu(H)) = \mu(H)F_N$$

in $\mathrm{GL}_r(Q) \subset \mathrm{GL}_r(K_\sigma \otimes_{W(\mathbb{F}_q)} W(k(X)^{\mathrm{alg}}))$. Let us consider a K_σ-space

$$\begin{aligned} &\mathrm{Ker}(1-\Phi_M\otimes\widetilde{\varphi}; M\otimes_{A[1/p]}(K_\sigma\otimes_{W(\mathbb{F}_q)} W(k(X)^{\mathrm{alg}}))) \\ &\qquad\cong \{\mathbf{y}\in (K_\sigma\otimes_{W(\mathbb{F}_q)} W(k(X)^{\mathrm{alg}}))^r \mid F_M\widetilde{\varphi}(\mathbf{y}) = \mathbf{y}\} \end{aligned}$$

for M and the same for N. Then we have an equivalence

$$F_M\widetilde{\varphi}(\mathbf{y}) = \mathbf{y} \Leftrightarrow F_N\widetilde{\varphi}(\mu(H)^{-1}\mathbf{y}) = \mu(H)^{-1}\mathbf{y}.$$

Since H is fixed by the action of $\mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X))$, we have isomorphisms

$$\begin{aligned} V(\mathcal{M}) &= \mathrm{Ker}(1-\Phi_M\otimes\varphi_A; M\otimes_{A[1/p]}\widehat{E}_A^{\mathrm{ur,pr}}) \\ &\cong \mathrm{Ker}(1-\Phi_N\otimes\varphi_B; N\otimes_{B[1/p]}\widehat{E}_B^{\mathrm{ur,pf}}) = V(\mathcal{N}). \end{aligned}$$

as K_σ-representations of $\mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X))$. □

Applying Theorem A.5 at each point of X of codimension 1, we have the following proposition.

Proposition A.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be convergent F-isocrystals on X/K which satisfy the hypothesis* (∗) *in Theorem A.1.*

(1) $\mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X))$ *acts on $V(\mathcal{M})$ via the etale fundamental group $\pi_1^{\mathrm{et}}(X)$ of X.*
(2) *The functor $V : F\text{-}\mathrm{Isoc}(X/K)^{(*)} \to \mathrm{Rep}_{K_\sigma}(\pi_1^{\mathrm{et}}(X))$ is exact and satisfies*

$$V(\mathcal{M}\otimes_{\mathcal{O}_{]X[}}\mathcal{N}) \cong V(\mathcal{M})\otimes_{K_\sigma} V(\mathcal{N}).$$

Proof. (1) Let x be a point of X of codimension 1, and $\mathbf{p}_x$ the inverse image of the prime ideal associated to x by the natural surjection $A \to A/\mathbf{m}A$. Let A_x be the p-adic completion of the localization of A along $\mathbf{p}_x$, $\widehat{A}_x$ the completion of A_x along $\mathbf{p}_x A_x$, $\mathbf{p}_x\widehat{A}_x = (\mathbf{m}, t_x)\widehat{A}_x$ for $t_x \in \mathbf{p}_x A_x$ such that $t_x \pmod{\mathbf{m}A_x}$ is a local parameter at the point

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x in X, and E_x the field of fractions of A_x/t_xA_x. Such t_x can be taken since A_x is a regular local domain of dimension 2 with the maximal ideal $\mathbf{p}_xA_x$. The ring $\widehat{A}_x$ is an integral domain and we have an isomorphism $\widehat{A}_x[1/p] \cong E_x[\![t_x]\!]_0$. We put $\mathcal{E}_x$ to be the p-adic completion of $E_x[\![t_x]\!]_0[1/t_x]$ and we regard E_A as a subfield of $\mathcal{E}_x$ by the natural injection $A \to \mathcal{E}_x$. If we denote the field of fractions of $\widehat{A}_x/\mathbf{m}\widehat{A}_x$ by $k(X)_x$, then the residue extension of $E_A \to \mathcal{E}_x$ is a natural embedding $k(X) \subset k(X)_x$. Let us fix an embedding $k(X)^{\mathrm{sep}} \subset k(X)_x^{\mathrm{sep}}$. Then the embedding induces a continuous K-algebra homomorphism $\widehat{E}_A^{\mathrm{ur}} \to \widehat{\mathcal{E}}_x^{\mathrm{ur}}$ which is compatible with Frobenius by the uniqueness of extension and with the actions of $\mathrm{Gal}(k(X)_x^{\mathrm{sep}}/k(X)_x)$ via the induced homomorphism $\mathrm{Gal}(k(X)_x^{\mathrm{sep}}/k(X)_x) \to \mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X))$.

Let $\mathcal{M}$ be a convergent F-isocrystal on X/K satisfying the condition $(*)$, $x \in X$ a point of codimension 1, and use the notation as before. By comparing the dimensions over K_σ, the inclusion $\widehat{E}_A^{\mathrm{ur}} \subset \widehat{\mathcal{E}}_x^{\mathrm{ur}}$ provides an isomorphism

$$V(\mathcal{M}) \overset{\cong}{\to} V_{\mathcal{E}_x}(M \otimes_{A[1/p]} \mathcal{E}_x)$$

by Proposition A.4(2). Since $M \otimes_{A[1/p]} \mathcal{E}_x$ has a (φ, ∇)-submodule $M \otimes_{A[1/p]} E_x[\![t_x]\!]_0$ over $E_x[\![t_x]\!]_0$, the compatibility implies that $V(\mathcal{M})$ is unramified at x by Theorem A.5(2) and Lemma A.7. Here we may replace Frobenius φ_A by what induces Frobenius on $E_x[\![t_x]\!]_0$. Applying Zariski–Nagata purity theorem [24, X, Theorem 3.4], $V(\mathcal{M})$ is a representation of $\pi_1^{\mathrm{et}}(X)$.

(2) The exactness follows from the additivity of ranks of graduation of slope filtrations. □

Theorem A.9. *Let X be a smooth geometrically connected scheme of finite type over k. Then the functor V is compatible with the functor V^0 of Katz–Crew's equivalence:*

$$\begin{array}{ccc} F\text{-Isoc}(X/K)^{(*)} & \xrightarrow{V} & \mathrm{Rep}_{K_\sigma}(\pi_1^{\mathrm{et}}(X)) \\ \bigcup & \nearrow_{V^0} & \\ F\text{-Isoc}(X/K)^0 & & \end{array}$$

where F-Isoc$(X/K)^0$ is the full subcategory of F-Isoc(X/K) consisting of unit-root convergent F-isocrystals.

Proof. We recall a construction of the functor V^0 which is a quasi-inverse of the equivalence $G : \mathrm{Rep}_{K_\sigma}(\pi_1^{\mathrm{et}}(X)) \to F\text{-Isoc}(X/K)^0$ defined by Crew in [11, Section 2]. Let $\mathcal{M}$ be a unit-root convergent F-isocrystal on X/K of rank r_0. By patching technique, we may assume that X is affine, and hence we follow the notation at the beginning of this subsection. Let $\mathcal{M}$ be a unit-root convergent F-isocrystal on X/K. Since $\mathcal{M}$ is unit-root, there is a locally free $\mathcal{O}_{\mathrm{Spf}\,A}$-module $\mathcal{L}$ of finite rank with Frobenius $\Phi : \varphi_A^*\mathcal{L} \overset{\cong}{\to} \mathcal{L}$ such that the analytification of $\mathcal{L} \otimes_R K$ is $\mathcal{M}$. Such an $\mathcal{L}$ is called an F-lattice of $\mathcal{M}$ and it always exists by [11, Proposition 2.5]. Then there is a sequence

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$$

of finite etale Galois coverings of X with structure morphisms $\pi_n : X_n \to X$ and a sequence

$$\mathrm{Spf}\,A = \mathrm{Spf}\,A_0 \leftarrow \mathrm{Spf}\,A_1 \leftarrow \mathrm{Spf}\,A_2 \leftarrow \ldots,$$

of lifts of $X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$ over $\mathrm{Spf}\, R$ with a system of embeddings $A_n \to \widehat{E}_A^{\mathrm{ur}}$ and a system of Frobenius φ_n compatible with Frobenius φ_A on $\widehat{E}_R^{\mathrm{ur}}$ such that $\pi_n^*(\mathcal{L}/\mathbf{m}^n\mathcal{L})$ is a trivial φ_n-module over $A_n/\mathbf{m}^n A_n$. Then

$$\Gamma_n = \mathrm{Ker}(1 - \Phi \otimes \varphi_n; \Gamma(\mathrm{Spf}\, A_n, \pi_n^*(\mathcal{L}/\mathbf{m}^n\mathcal{L})))$$

is a free $R_\sigma/\mathbf{m}_\sigma^n$-module of rank r_0 with an action of $\mathrm{Gal}(X_n/X)$ and $\{\Gamma_n\}_n$ forms a projective system of $\pi_1^{\mathrm{et}}(X)$-representations by [28, Proposition 4.1.1]. Then V^0 is defined by

$$V^0(\mathcal{M}) := K_\sigma \otimes_{R_\sigma} \varprojlim_n \Gamma_n.$$

The fact that V^0 is a quasi-inverse of Crew's functor G follows from a projective system of isomorphisms $\Gamma_n \otimes_{R_\sigma[\mathrm{Gal}(X_n/X)]} \mathcal{O}_{\mathrm{Spf}\, A_n} \cong \mathcal{L}/\mathbf{m}^n\mathcal{L}$, where $\mathrm{Gal}(X_n/X)$ is the Galois group of the covering X_n over X. Hence we have isomorphisms

$$\begin{aligned} V^0(\mathcal{M}) &\cong K_\sigma \otimes_{R_\sigma} \varprojlim_n \mathrm{Ker}(1 - \Phi \otimes \varphi_A; \Gamma(\mathrm{Spf}\, A, \mathcal{L}/\mathbf{m}^n\mathcal{L}) \otimes_{A/\mathbf{m}^n A} R_{\widehat{E}^{\mathrm{ur}}}/\mathbf{m}^n_{\widehat{E}^{\mathrm{ur}}}) \\ &\cong K_\sigma \otimes_{R_\sigma} \mathrm{Ker}(1 - \Phi \otimes \varphi_A; \Gamma(\mathrm{Spf}\, A, \mathcal{L}) \otimes_A R_{\widehat{E}^{\mathrm{ur}}}) \\ &= V(\mathcal{M}) \end{aligned}$$

as K_σ-spaces with continuous $\pi_1^{\mathrm{et}}(X)$-actions by our construction of V in § A.3. Indeed, the first isomorphism follows from the fact that the φ-invariant subring of $R_{\widehat{E}_A^{\mathrm{ur}}}/\mathbf{m}^n_{\widehat{E}_A^{\mathrm{ur}}}$ coincides with $R_\sigma/\mathbf{m}_\sigma^n$, and the second isomorphism follows from p-adic completeness. Therefore our construction is compatible with Crew's one. □

A.4. End of the proof of Theorem A.1

Keep the notation as in the previous section.

Lemma A.10. *Let $\mathcal{M}$ be a convergent F-isocrystal on X/K which satisfies the hypothesis $(*)$ of Theorem A.1. Suppose that $V(\mathcal{M})$ contains a trivial representation $V_0(\cong K_\sigma)$ of rank 1. Then there exists a convergent sub F-isocrystal of $\mathcal{M}$ which is isomorphic to the unit-root trivial object $(\mathcal{O}_{]X[}, d, \varphi)$ of rank 1.*

Proof. We may suppose X is affine by the full faithfulness of restriction functors of F-isocrystals [32, Theorem 5.2.1] [33, Theorem 4.2.1] and assume the same geometric situation as in the proof of Proposition A.8 holds. Let us put $M = \Gamma(]X[_{\mathrm{Spf}\, A}, \mathcal{M})$. Then M is a projective $A[1/p]$-module of finite rank. Our claim is that $V_0 \subset M$ under the inclusion

$$V_0 \subset V(\mathcal{M}) = \mathrm{Ker}(1 - \Phi_M \otimes \varphi_A; M \otimes_{A[1/p]} \widehat{E}_A^{\mathrm{ur}}) \subset M \otimes_{A[1/p]} \widehat{E}_A^{\mathrm{ur}}.$$

Indeed, $\mathcal{O}_{]X[} = V_0 \otimes_{K_\sigma} \mathcal{O}_{]X[}$ is a unit convergent sub F-isocrystal of $\mathcal{M}$.

Since V_0 is $\mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X))$-invariant, V_0 is included in $M \otimes_{A[1/p]} E_A$. Let us take a point x of X of codimension 1, and keep the notation as in the proof of Proposition A.8. The identity $V_{\mathcal{E}_x}(M \otimes_{A[1/p]} \mathcal{E}_x) = \mathrm{Ker}(1 - \Phi_M \otimes \varphi_A; M \otimes_{A[1/p]} \widehat{E}_x^{\mathrm{ur}}[\![t_x]\!])$ (Theorem A.5(2)) implies that V_0 is included in $M \otimes_{A[1/p]} \widehat{A}_x[1/p]$. Moreover, since M is a direct summand of a free $A[1/p]$-module of finite type, the equality

$$M \otimes_{A[1/p]} A_x[1/p] = (M \otimes_{A[1/p]} E_A) \cap (M \otimes_{A[1/p]} \widehat{A}_x[1/p])$$

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holds in $M \otimes_{A[1/p]} \mathcal{E}_x$ by Lemma A.11(1). Hence V_0 is included in $M \otimes_{A[1/p]} A_x[1/p]$. Now our claim $V_0 \subset M$ follows from Lemma A.11(2). □

Lemma A.11. *With the notation above, we have*

(1) $A_x[1/p] = E \cap \widehat{A}_x[1/p]$ *in* $\mathcal{E}_x$.

(2) $A[1/p] = \bigcap_x A_x[1/p]$ *in* E_A, *where* x *runs through all points of* X *of codimension* 1.

Proof. Since $E_A \subset \mathcal{E}_x$ is an extension as discrete valuation rings, we have only to prove $A_x = \widehat{A}_{\mathbf{m}} \cap \widehat{A}_x$ and $A = \bigcap_x A_x$, where $\widehat{A}_{\mathbf{m}}$ is the p-adic completion of $A_{\mathbf{m}}$. Since the ideals generated by $\mathbf{m}$ are prime in A, A_x, $\widehat{A}_x$ and $A_{\mathbf{m}}$ and since all rings are complete and separated in p-adic topology, respectively, it is sufficient to prove $A_x/\mathbf{m}A_x = (A_{\mathbf{m}}/\mathbf{m}A_{\mathbf{m}}) \cap (\widehat{A}_x/\mathbf{m}\widehat{A}_x)$ and $A/\mathbf{m}A = \bigcap_x A_x/\mathbf{m}A_x$. Note that $K(X) = A_{\mathbf{m}}/\mathbf{m}A_{\mathbf{m}}$ is a field of fractions of $A/\mathbf{m}A$, $A_x/\mathbf{m}A_x$ is a localization of $A/\mathbf{m}A$ at the point x, and its completion by the ideal of definition of x is $\widehat{A}_x/\mathbf{m}\widehat{A}_x$. Hence (1) holds. Since $A/\mathbf{m}A$ is a Noetherian normal domain, it is an intersection of all localizations of height one prime ideals. Hence (2) holds. □

Now let us complete a proof of Theorem A.1. Let $\mathcal{M}$ be a convergent F-isocrystal on X/K which satisfies the hypothesis of Theorem A.1. We have only to prove that there exist a unit-root convergent F-isocrystal $\mathcal{N}$ on X/K with a nontrivial homomorphism $\mathcal{N} \to \mathcal{M}$. Indeed, when such an $\mathcal{N}$ exists, the quotient $\mathcal{M}/\mathrm{Im}(\mathcal{N} \to \mathcal{M})$ satisfies the hypothesis of Theorem A.1. Repeating this argument, one has the unit-root subobject $\mathcal{L}$ of $\mathcal{M}$ whose rank is r_0. Hence $\mathcal{L}$ is a desired unit-root convergent F-isocrystal on X/K.

Applying Katz–Crew's quasi-inverse G of V^0, we have a nontrivial unit-root convergent F-isocrystal $\mathcal{N} = G(V(\mathcal{M}))$ on X/K since $V(\mathcal{M})$ is a continuous K_σ-representation $V(\mathcal{M})$ of $\pi_1^{\mathrm{et}}(X)$ of dimension r_0 by Proposition A.8(1). Applying Proposition A.8(2) and Lemma A.10 to $\mathcal{N}^\vee \otimes_{\mathcal{O}_{]X[}} \mathcal{M}$, we have a unit-root trivial convergent subobject in $\mathcal{N}^\vee \otimes_{\mathcal{O}_{]X[}} \mathcal{M}$. Hence we have a nontrivial homomorphism $\mathcal{N} \to \mathcal{M}$. □

Remark A.12. If $\mathcal{L}$ is the maximal unit-root subobject of $\mathcal{M}$, then

$$\mathcal{L} \cong G(V^0(\mathcal{L})) \cong G(V(\mathcal{L})) \cong G(V(\mathcal{M}))$$

by Katz–Crew equivalence and Theorem A.9.

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