Math. Struct. in Comp. Science (2003), *vol.* 13, *pp.* 799–855.
DOI: 10.1017/S0960129503003967 Printed in the United Kingdom

Obsessional experiments for linear logic proof-nets

LORENZO TORTORA de FALCO

Dipartimento di Filosofia Università Roma Tre,
Via Ostiense 234,
00146 Roma, Italy
Email: tortora@uniroma3.it

Received 10 January 2001; revised 20 August 2002

We address the question of injectivity of coherent semantics of linear logic proof-nets. Starting from Girard's definition of experiment, we introduce the key-notion of 'injective obsessional experiment', which allows us to give a positive answer to our question for certain fragments of linear logic, and to build counter-examples to the injectivity of coherent semantics in the general case.

1. Introduction

Denotational semantics associates with every proof in a given formal system a set in some space, which is usually called *the interpretation* or *the semantics* of the proof. This association can be seen as a way to define a (semantic) equivalence relation on proofs (of the same formula): roughly speaking, two proofs are equivalent when they have the same interpretation.

The cut-elimination procedure for the proofs of a given logical system can also be seen as a way to define a (syntactical) equivalence relation on proofs. If the cut-elimination procedure enjoys the confluence property, this relation can be (roughly) defined as: two proofs are equivalent when they have the same normal form.

Now, a *very* natural question arises:

Do these two equivalence relations (sometimes/always) coincide? (*)

Proofs of linear logic (LL) are represented as 'proof-nets', a graph-theoretic presentation (introduced in Girard (1987)) that gives a more geometric account of proofs. A unique proof-net may be associated with several sequent calculus proofs: this is reminiscent of the situation of natural deduction proofs (or λ-calculus terms) with respect to sequent calculus proofs, in the restricted framework of minimal logic. A net is both a canonical representative of a set of sequent calculus proofs and a computational object in itself (with a much better behaviour with respect to cut-elimination than sequential proofs).

In the present paper we ask question (*) for LL proof-nets. One thing is clear to begin with (by the definition of denotational semantics): two syntactically equivalent proof-nets are also semantically equivalent. If, for a given semantics s, the answer to our question is positive, we say that s is *injective*: two different proof-nets without cuts have different interpretations in the semantics s.

This kind of question has been studied for the pure and simply typed λ-calculus in several papers (see, for example, Statman (1983)). It is somewhat surprising that, while the question of 'surjectivity' of the semantics (also known as 'full completeness') has been extensively studied for LL, the question of injectivity had not been studied before Tortora de Falco (2000).

The technique we use to prove the injectivity property is 'to rebuild' a cut-free proof-net from its semantic interpretation. Notice that this allows us also to 'semantically compute' the normal form of a proof-net. Indeed, we can compute the semantics of a given proof-net (with cuts) and rebuild the unique cut-free proof-net with the given semantics: the net obtained in this way can only be the normal form of the proof-net we started from. This approach to computation is known, among researchers of the functional programming language community, as 'normalisation by evaluation' (see Berger *et al.* (1998) and Danvy *et al.* (2001)).

To ask about the injectivity of some semantics of LL is another way of asking whether it is possible to make 'more identifications' than proof-nets. In other words, it is a way of 'measuring' the quality of the representation of proofs as proof-nets. In Laurent and Tortora de Falco (2001), we show that some fragment F of LL containing the additive connectives enjoys the injectivity property, and we use this high-quality representation of the proofs of F to define a (sliced) cut-elimination procedure enjoying outstanding properties: this gives (for the fragment F) a rather convincing solution to the problem of normalisation in the presence of the additives.

We restrict our analysis to the coherent multiset-based semantics of LL, and extend the positive results of the paper to the relational semantics in Appendix B. Actually, the main tool of our approach (injective obsessional experiments) is fully relevant to the relational case (as shown in Tortora de Falco (2000)). As the question had not been studied before, there are a lot of (sometimes) simple properties to prove: we have tried to be convincing without being tedious, and some (easy) proofs are left to the reader, who can also refer to Tortora de Falco (2000).

Structure of paper

The rest of the paper is composed of four main sections:

2: The question of injectivity
3: Injectivity and obsessionality
4: Injective experiments for ($? \wp$)LL
5: Positive and negative results

and two appendices:

A: Proof-nets and coherent semantics
B: About injectivity for relational semantics.

In Section 2, we address the question of injectivity in a precise way, and give a positive answer in the multiplicative case.

We start in Section 2.1, by generalising to multiplicative and exponential LL the notion of experiment given in Girard (1987): an experiment associates with every edge a of a proof-net a multiset of elements of the web of the coherent space interpreting the

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

'type' of a (the formula associated with the edge a). The 'result' of an experiment of a proof-net R is the set of labels associated by the experiment with the conclusions of R. The interpretation of a proof-net is the set of all the results of the experiments of the proof-net (Definition 2.4). We end the section with some useful properties of experiments. In Section 2.2, we state our problem in mathematical terms and introduce the canonical representatives of syntactical equivalence classes: standard proof-nets. We (easily) prove in Section 2.3 that coherent semantics is injective for the multiplicative fragment of LL using the notion of 'injective experiment'.

In Section 3 we introduce and study 'obsessional experiments'. Using this new tool, we prove a sufficient condition for local injectivity: if, for a given (standard) proof-net, there exists a particular experiment, then this proof-net is the unique standard proof-net of its semantic equivalence class.

In Section 3.1 we define obsessional experiments and state some of their properties. In Section 3.2 we show the regularity of obsessional experiments: we prove (Propositions 3.14 and 3.15) that the obsessional feature of an experiment can be read in its result, using in a crucial way the uniformity of coherent semantics (Lemma 3.11). In Section 3.3, we show the power of obsessional experiments: we prove (Corollary 3.23) that they allow us to partially reconstruct a (standard) proof-net. Finally, in Section 3.4, we give the sufficient condition for local injectivity (Theorem 3.35): if there exists, for a (standard) proof-net R, an 'injective 1-experiment', then there are no other (standard) proof-nets with the same semantics as R.

In Section 4 we define the subsystem $(?\wp)$LL of multiplicative and exponential proof-nets, and we prove that every (standard) proof-net of $(?\wp)$LL satisfies the sufficient condition of local injectivity given in Section 3.

In Section 4.1, we define two operations on (standard) proof-nets: linearisation and par-mutilation. We then show that by first linearising and then par-mutilating a (standard) $(?\wp)$LL proof-net, we obtain a proof-net without boxes, without $\wp$ links, and with only terminal contraction links (Remark 4.5). Section 4.2 is devoted to proving that for such a proof-net there always exists an injective 1-experiment (Proposition 4.9). We conclude Section 4 by proving in Section 4.3 that the existence of such an experiment is preserved when we 'come back' from the linearised and par-mutilated proof-net to the original one (Lemma 4.24 and Proposition 4.26).

In Section 5 we prove the positive and negative results of the paper, which are summed up in Section 5.3. Even though the existence of an injective 1-experiment of a (standard) proof-net *is not* necessary for the proof-net to be alone in its semantic equivalence class (as shown in Remark 3.37), it certainly is a crucial property. Indeed, the existence of an injective 1-experiment for every (standard) $(?\wp)$LL proof-net allows us to prove (in Section 5.1) injectivity for $(?\wp)$LL, and the fact that such an experiment does not exist for every (standard) multiplicative and exponential proof-net allows us to build a counter-example to the injectivity of coherent semantics in the general case (in Section 5.2). Among the notable subsystems of $(?\wp)$LL (for which injectivity is proved), we can mention the 'weakly polarised' fragment of LL, which contains the simply typed λ-calculus (Theorem 5.5, Corollary 5.6, Theorem 5.7).

Appendix A is devoted to defining the (well-)known notions of coherent space and proof-net. However, in contrast with coherent spaces, the notion of proof-net is not canonical: in order to answer the question addressed in this paper, we need to refer to a precise definition (see Definition A.6 and Remark A.7).

Appendix B extends the positive results of the paper to relational semantics.

2. The question of injectivity

This section is mainly devoted to setting the stage and formulating the question of injectivity in precise mathematical terms. We first introduce (Section 2.1) the notion of experiment and mention some of its properties. We then turn to syntax (Section 2.2), by introducing the canonical representatives of 'syntactical equivalence classes': standard proof-nets. This allows us to state our problem precisely (Problem 2.16). Finally, we solve (in Section 2.3) the problem in the usual 'perfect' fragment of LL: the multiplicative fragment. The (very easy) proof has the virtue of indicating the path to follow in order to attack the problem in the more difficult (and interesting) multiplicative and exponential fragment.

We use the (well-known) notions of proof-net and coherent space. We refer to Appendix A, where all the main definitions are given, as well as a description of several conventions and the notation used in the paper.

2.1. *Experiments: the semantics of linear logic proof-nets*

We come now to the crucial notion of experiment, which was introduced in Girard (1987).

Experiments have hardly been used since Girard's first paper on LL, and the only other works dealing with this notion are (at least as far as we know) Duquesne and Van de Wiele (1994), Retoré (1997), and, more recently, Tortora de Falco (2000) and Barcaglioni (2001). We give here a precise definition of experiment for the multiplicative and exponential fragment of LL, and recall the main results of Girard (1987).

The following definition is the extension to multiplicative and exponential LL of the definition of Girard (1987): the main difference is due to the presence of exponential boxes. This means that an experiment is no longer a simple labelling of the edges of a proof-net. While in the absence of exponential boxes an experiment e associates with every edge a of type A of a proof-net a unique element of $|\mathscr{A}|$, this is not the case in general: it might associate with a an element of $|\mathscr{A}|$, several elements of $|\mathscr{A}|$, or the empty set. We will then say that e associates with a *a multiset* of elements of $|\mathscr{A}|$.

Definition 2.1 (Experiment). We define the notion of an experiment of a proof-structure S by induction on the depth p of S.

Let e be an application that associates with every edge a of type A of S a multiset $e(a)$ of elements of $|\mathscr{A}|$ in such a way that when a has depth 0 the multiset $e(a)$ contains exactly one element. The application e is an **experiment** of S when the following conditions hold:

— If $p = 0$, then:

- If $a = a_1$ is the conclusion of an axiom link with conclusions the edges a_1 and a_2 of type A and $A^{\perp}$, then $e(a_1) = e(a_2)$.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

- If a is the premise of a cut link with premises a and b, then $e(a) = e(b)$.
- If a is the conclusion of a $\wp$ (or $\otimes$) link with left premise a_1 and right premise a_2, then $e(a) = \{(x_1, x_2)\}$, where $e(a_1) = \{x_1\}$ and $e(a_2) = \{x_2\}$.
- If a is the conclusion of a dereliction link with premise a_1, then $e(a) = \{\{x_1\}\}$, where $e(a_1) = \{x_1\}$.
- If a is the conclusion of a weakening link, then $e(a) = \{\emptyset\}$.
- If a is the conclusion of a contraction link of arity k ($k \geqslant 2$), with premises $a_1, \ldots, a_k$, then $e(a) = \{x_1 \cup \ldots \cup x_k\}$, where $e(a_i) = \{x_i\}$ (for every $i \in \{1, \ldots, k\}$).

— If $p > 0$, then e has to satisfy the same conditions as when $p = 0$. Moreover, for every box $B_n^!$ with depth 0 in S whose front door n has conclusion c of type $!C$ and whose auxiliary doors have conclusions $a_1, \ldots, a_m$ ($m \geqslant 0$) of type $?A_1, \ldots, ?A_m$, respectively, let $S_n = S_{B_n^!}$ be the biggest subproof-structure of S contained in $B_n^!$. Let c' be the premise of the !-link n and (for every $i \in \{1, \ldots, m\}$) let a_i' be the premise of the pax link of $B_n^!$ having a_i as conclusion. Clearly, c' and $a_1', \ldots, a_m'$ are the conclusions of the proof-structure S_n.
In order for the application e to be an experiment of S, for every such box $B_n^!$ there has to exist a *unique* multiset† $\{e_1, \ldots, e_{k_n}\}$ ($k_n \geqslant 0$) of experiments of S_n satisfying the following conditions:
  - for every edge a of S_n, $e(a) = e_1(a) \cup \ldots \cup e_{k_n}(a)$,
  - $e(c) = \{\{x_1, \ldots, x_{k_n}\}\}$, where $e_j(c') = \{x_j\}$ ($\forall j \in \{1, \ldots, k_n\}$),
  - $\forall i \in \{1, \ldots, m\}$ we have $e(a_i) = \{x_1^i \cup \ldots \cup x_{k_n}^i\}$, where $\forall j \in \{1, \ldots, k_n\}$ we have $e_j(a_i') = \{x_j^i\}$.

If the conclusions of S are the edges $a_1, \ldots, a_l$ of type $A_1, \ldots, A_l$, respectively, and e is an experiment of S such that $\forall i \in \{1, \ldots, l\}$ $e(a_i) = \{x_i\}$, then we shall say that $(x_1, \ldots, x_l) \in |\mathscr{A}_1 \wp \ldots \wp \mathscr{A}_l|$ is the *conclusion* or the *result* of the experiment e of S. We shall also denote it by $x_1, \ldots, x_l$.

Remark 2.2. Let a be an edge of the proof-structure S and e be an experiment of S. We will often refer to the elements of $e(a)$ as 'the labels' associated by the experiment e with the edge a.

It is *crucial* to notice that the previous definition implies that the following conditions are fulfilled (inductively, with respect to the depth):

— The label $x_1 \cup \ldots \cup x_k$ of the conclusion a of type $?C$ of a contraction link with depth 0 satisfies $x_1 \cup \ldots \cup x_k \in |?\mathscr{C}|$.
— The label $\{x_1, \ldots, x_{k_n}\}$ of the conclusion a of type $!C$ of an of course link with depth 0 satisfies $\{x_1, \ldots, x_{k_n}\} \in |!\mathscr{C}|$.
— The label $x_1^i \cup \ldots \cup x_{k_n}^i$ of the conclusion a_i of type $?A_i$ of a pax link with depth 0 satisfies $x_1^i \cup \ldots \cup x_{k_n}^i \in |?\mathscr{A}_i|$.

† This simply means that for a given experiment e the multiset is unique, but (at least for the moment) it might be the case that there exists *another* experiment e' with the same result as e (see the end of the definition) and with a different multiset of experiments associated with the box $B_n^!$.

If R is a cut-free proof-net of multiplicative LL, a 'correct' assignment of labels to the conclusions of the axiom links of R always induces an experiment of R. The fact that an experiment of a cut-free proof-net R' of multiplicative and exponential LL has to fulfill the previous conditions implies that a 'correct' assignment of labels to the conclusions of the axiom links of R' *does not* necessarily induce an experiment of R'.

Remark 2.3. Let $\Gamma = A_1, \dots, A_l$ be the sequent conclusion of the two proof-nets R and R'. Let e (respectively, e') be an experiment of R (respectively, R') with result γ (respectively, γ'). By definition, there exists a permutation σ (respectively, σ') of $\{1, \dots, l\}$ such that $\gamma = (x_1, \dots, x_l) \in |\mathscr{A}_{\sigma(1)} \wp \dots \wp \mathscr{A}_{\sigma(l)}|$ (respectively, $\gamma' = (x'_1, \dots, x'_l) \in |\mathscr{A}_{\sigma'(1)} \wp \dots \wp \mathscr{A}_{\sigma'(l)}|$).

We shall write $\gamma = \gamma'$ when for every $i \in \{1, \dots, l\}$ we have $x_i = x'_i$ and $A_{\sigma(i)} = A_{\sigma'(i)}$. This means that the equality $\gamma = \gamma'$ induces a bijection between the conclusion edges of R and the conclusion edges of R': the one that associates with the conclusion $a_{\sigma(i)}$ of type $A_{\sigma(i)}$ of R the conclusion $a_{\sigma'(i)}$ of type $A_{\sigma'(i)}$ of R'.

We shall say that a given conclusion a of R 'corresponds' to a given conclusion a' of R' (or that a' is 'the corresponding edge' of R'), when a' is the image of a through the bijection.

The following definition is the one given in Girard (1987), which is extended here to the case of multiset-based coherent semantics. Notice that the semantics of a proof-net R depends on the choice of the coherent spaces associated with the atomic subformulas of R's conclusions (see Definition A.1).

Definition 2.4. Let R be a proof-net with conclusion Γ.

$$[\![R]\!] := \{\gamma \in |\wp\Gamma| : \text{there exists an experiment } e \text{ of } R \text{ with conclusion } \gamma\}.$$

$[\![R]\!]$ is said to be *the interpretation* or *the semantics* of R.

Theorem 2.5. If R is a proof-net with conclusion Γ, then $[\![R]\!] \in \wp\Gamma$ (here $\wp\,\Gamma$ is the space interpreting the formula $\wp\,\Gamma$).

Proof. The theorem is proved in Girard (1987) for the coherent (set-based) semantics. In the multiset case, we have to extend the previous result, which Barcaglioni (2001) does for multiplicative and exponential LL. □

The interpretation defined in Definition 2.4 yields a denotational semantics of proof-nets for multiplicative and exponential LL, as stated by the following theorem.

Theorem 2.6. Let R be a proof-net and let R' be a proof-net obtained from R by applying one step of cut-elimination. Then $[\![R]\!] = [\![R']\!]$.

Proof. The proof is given in Girard (1987) for the coherent (set-based) semantics, and can be straightforwardly extended to the multiset case. □

Notation. The set of proof-nets (defined in Appendix A) only makes use of the formulas of multiplicative and exponential LL, and will be denoted in this paper by *MELL*.

We now state some more properties of the notion of experiment that will be used later.

Definition 2.7 (Restriction of an experiment to a subproof-net). Let R be a proof-net, let e be an experiment of R with conclusion γ, and let R_1 be a subproof-net of R. We are going to define the multiset $e|_{R_1}$, whose elements are experiments of R_1, by induction on the depth p of the conclusions of R_1 in R.

If $p = 0$, then $e|_{R_1} = \{e_1\}$, where e_1 is the experiment of R_1 defined by: for every edge a of R_1, $e_1(a) = e(a)$.

Let $p+1$ be the depth of the conclusions of R_1 in R, let B be the box of R with depth 0 containing R_1 and let R_B be the biggest subproof-net of R contained in B. Let n be the cardinality of the unique label associated by e with the conclusion of the pal door of B and let $e_1, \ldots, e_n$ be the n ($n \geqslant 0$) experiments of R_B from which the unique experiment of the multiset $e|_B$ is built (following Definition 2.1)†. We define: $e|_{R_1} = e_1|_{R_1} \cup \ldots \cup e_n|_{R_1}$ (if $n = 0$, we have $e|_{R_1} = \emptyset$).

If $e|_{R_1} = \{e_1, \ldots, e_l\}$, we will use $\gamma|_{R_1}$ to denote the multiset $\{\gamma_1, \ldots, \gamma_l\}$ where γ_i is the conclusion of the experiment e_i of R_1, $\forall i \in \{1, \ldots, l\}$.

We now state some useful lemmas, the proofs of which are left to the reader.

Lemma 2.8. Let e be an experiment of R and R_1 be a subproof-net of R, and let $e|_{R_1} = \{e_1, \ldots, e_n\}$. For every edge a of R_1, we have $e(a) = e_1(a) \cup \ldots \cup e_n(a)$.

Lemma 2.9. Let e be an experiment of the (non-empty) proof-net R, and let a be an edge of R. We have that $e(a) = \emptyset$ iff there exists a box B of R containing a such that if we call c the conclusion of the pal door of B, then we have that $e(c) = \{n[\emptyset]\}$ for some integer n different from zero.

Lemma 2.10. Let a be an edge of the proof-net R with depth p, and let e be an experiment of R. Let $c_1, \ldots, c_p$ of type $!C_1, \ldots, !C_p$, respectively, be the conclusions of the p pal doors of the boxes of R containing a.

If $p \geqslant 1$ and $\forall i \in \{1, \ldots, p\}$ the cardinality of all the elements of $e(c_i)$ is equal to n_i (respectively, if $p = 0$), then $e(a)$ is a multiset of cardinality $n_1 \cdot \ldots \cdot n_p$ (respectively, of cardinality 1).

Definition 2.11. Let a be an edge of type A of the proof-net R and let e be an experiment of R. Let $x \in e(a)$ (that is, let $x \in |\mathscr{A}|$ be *one* of the labels associated by e with the edge a). For every **occurrence** of subformula C of A, we define the multiset 'multiset projection of x on C', denoted by $|x|_C$, by induction on the (logical) complexity of the formula $A \backslash C$:

— If $C = A$, then $|x|_C = \{x\}$.

— If $E = C \otimes D$ or $E = C \wp D$ (respectively, $E = D \otimes C$ or $E = D \wp C$) is an occurrence of subformula of A, then $|x|_C = \{y \in |\mathscr{C}| : \exists z \in |\mathscr{D}| \text{ s.t. } (y, z) \in |x|_E\}$ (respectively, $|x|_C = \{y \in |\mathscr{C}| : \exists z \in |\mathscr{D}| \text{ s.t. } (z, y) \in |x|_E\}$)

— If $D = ?C$ or $D = !C$ where D is an occurrence of subformula of A, then $|x|_C = \{y \in |\mathscr{C}| : \exists z \in |\mathscr{D}| \text{ s.t. } y \in z \in |x|_D\}$.

† We use here the unicity, for a given experiment e, of the multiset $\{e_1, \ldots, e_{k_n}\}$ of Definition 2.1.

Remark 2.12.

(i) Let e (respectively, e') be an experiment of the proof-net R, let a be an edge of type A of R, let C be an occurrence of a subformula of D, and D be an occurrence of a subformula of A. Let x (respectively, x') be a label associated by e (respectively, e') with the edge a. If $|x|_D = |x'|_D$, then $|x|_C = |x'|_C$.

(ii) Let a be an edge of type A of the proof-net R and e be an experiment of R. Let $x \in |\mathscr{A}|$ be *one* of the labels associated with a by e. We can extend the previous definition to the case where C is not an occurrence of a subformula of A by defining in that case $|x|_C = \emptyset$.

(iii) With the notation of (ii), if $|x|_C \neq \emptyset$, then C is an occurrence of a subformula of A. Observe that the converse is wrong: we might have $|x|_C = \emptyset$ with C an occurrence of a subformula of A.

Remark 2.13. In Girard (1987), there is a very nice proof of the following property for the multiplicative fragment of LL:

'An experiment of a given proof-net is uniquely determined by its result.'

This may be restated as follows:

Let e (respectively, e') be an experiment with result γ (respectively, γ') of the proof-net R. If $\gamma = \gamma'$, then $e = e'$.

In his proof, Girard uses the longtrip correctness criterion. E. Duquesne and J. Van de Wiele extended this result to pure LL proof-nets (Duquesne and Van de Wiele 1994), and Barcaglioni (2001) gives a similar proof in the *MELL* case for the set-based and multiset-based coherent semantics.

This last result will be used later but not in an essential way: it just simplifies our statements. All the results presented in the paper can be proved without using it.

Notice that this property entails the unicity of the multiset $\{e_1, \ldots, e_{k_n}\}$ of Definition 2.1 once the *result* of e is known: when we are dealing with a proof-net, the footnote of Definition 2.1 cannot hold (this is not true for general proof-structures).

2.2. *Standard proof-nets*

Before we can state our problem in a correct way, we need to make some preliminary remarks.

Let c be a binary connective of LL, and let $c^\perp$ be the dual connective. The semantics that we consider will identify the axiom with conclusions $AcB, A^\perp c^\perp B^\perp$ and the canonical proof of $AcB, A^\perp c^\perp B^\perp$ (obtained from the two axiom links with conclusions $A, A^\perp$ and $B, B^\perp$). We will often refer to this last proof(-net) as to the η**-expansion** of the axiom (terminology that obviously comes from the λ-calculus). A similar remark holds for unary connectives (the exponentials '!' and '?'). If R is obtained by applying to the proof-net R' some η-expansions, there is clearly no hope for our semantics to distinguish R from R'.

It is also easy to see that the considered semantics cannot 'see' whether the conclusion of a weakening or a contraction link is the premise of a pax link. And a contraction or a weakening link whose conclusion is the premise of a contraction link is also 'semantically invisible'.

In order to state the problem, we then have to define more precisely the two equivalence relations we want to compare.

Definition 2.14. Let R be a proof-net of *MELL*. We shall say that R is **standard** when:

— R is cut-free.
— Every conclusion of an axiom link of R is of atomic type.
— If a is the conclusion of a $?w$ or $?co$ link of R, then a is neither a premise of a pax nor the permise of a $?co$ link.

It is fairly obvious that by performing the necessary η-expansions, by (possibly) erasing some $?w$ and $?co$ links, and by 'pushing' (when this is possible) the links $?w$ and $?co$ out of the boxes, we can associate with every cut-free proof-net of *MELL* a unique standard proof-net.

Notice, by the way, that (except for the η-expansions) a standard proof-net is nothing but a proof-net of the 'nouvelle syntaxe' defined by V. Danos and L. Regnier (for example in Regnier (1992)).

We now have to be precise in what we mean when we say 'R and R' are semantically equivalent' (in a given semantics), where R and R' are two proof-nets with the same conclusions. Indeed, we cannot simply say that this means $[\![R]\!] = [\![R']\!]$ (referring to Definition 2.4), because this equality depends on the interpretation of atomic formulas. We will say that two proof-nets with the same conclusions are semantically equivalent when they have the same semantics *for every* possible interpretation of the atoms of the types of their conclusions.

We use in the following definition the confluence property of (*MELL*) proof-nets, which is proved in Danos (1990).

Definition 2.15. Let R and R' be two proof-nets with the same conclusions, and let s be a semantics of *MELL*. We shall denote by $[\![T]\!]_s$ the element of $\wp\Gamma$ associated by the semantics s with the proof-net T with conclusion Γ.

Let R_0 (respectively, R'_0) be the unique standard proof-net associated with the unique normal form of R (respectively, R').

We shall say that R and R' are **semantically equivalent** when $[\![R]\!]_s = [\![R']\!]_s$ (for every interpretation of the atoms of the types of the conclusions of R). We shall simply write $[\![R]\!]_s = [\![R']\!]_s$ (or $[\![R]\!] = [\![R']\!]$ when there is no ambiguity).

We shall say that R and R' are **syntactically equivalent** or **$\beta\eta$-equivalent** when $R_0 = R'_0$. We will then write $R \simeq_{\beta\eta} R'$.

Now we can finally state the problem.

Problem 2.16. Let R and R' be two proof-nets with the same conclusions, let s be the coherent multiset-based semantics of *MELL*. If $R \simeq_{\beta\eta} R'$, then $[\![R]\!]_s = [\![R']\!]_s$.

Do we have 'if $[\![R]\!]_s = [\![R']\!]_s$, then $R \simeq_{\beta\eta} R'$'? In other words: 'is the semantics s injective?'

Convention. From now on, all the proof-nets considered in this paper will be standard *MELL* proof-nets.

We can then re-state the problem as follows:

'Let R and R' be two proof-nets with the same conclusions. If R and R' are semantically equivalent, do we have $R = R'$?'

We now introduce some notions that will be used later (mainly in Section 3).

Definition 2.17. With every proof-net R are naturally associated the following graphs:

— the open proof-structure, denoted by $OPS(R)$, that is obtained from R by erasing all the axiom links of R;

— the linear proof-structure (respectively, the open linear proof-structure), denoted by $LPS(R)$ (respectively, denoted by $OLPS(R)$), that is obtained from R (respectively, from $OPS(R)$) by erasing the connections between the different doors of a given box (the 'rectangular frame' of Definition A.2).

Remark 2.18. Let us be more precise in what we mean by 'erasing the connections between the different doors of a given box'. In $LPS(R)$ and $OLPS(R)$, we keep the pax and of course links of R. This means (in particular) that the notion of depth is still meaningful in $LPS(R)$ and $OLPS(R)$: an edge/a link α has depth p when the path with starting edge/link α and terminal edge a conclusion of R crosses exactly p of course or pax links (different from α).

The only information that is lost in $LPS(R)$ (respectively, $OLPS(R)$) is that we no longer know whether two given pax links (or a pax and an of course link) with the same depth are two doors of the same box.

Remark 2.19.

(i) Let a be the conclusion of m, which is a pax or a dereliction link of the proof-net R. Because the conclusion of every contraction and weakening link is not a premise of a pax link (and because every conclusion of an axiom link has an atomic type), there exists a unique dereliction link n with conclusion the edge a' of R and such that the path with starting edge a' and terminal edge a conclusion of R crosses a after a certain number of pax doors. We will say that n is **the dereliction link above** a or **above** m. (Of course we might have $n = m$).

(ii) Let a (respectively, $a_1, \ldots, a_k$) be the conclusion (respectively, the premises) of a contraction link m with arity k. Because of the position of the structural rules in a standard proof-net, $a_1, \ldots, a_k$ are conclusions of pax or dereliction links. Thanks to (i), we can then associate with the link m k dereliction links $n_1, \ldots, n_k$ (where n_i is the dereliction link above a_i), which will be called **the k dereliction links above** m.

The following definition of 'graph of an edge in a proof-net' can be given for every (not necessarily standard) cut-free proof-net of $MELL$.

Let R be a proof-net and a an edge of R. The graph of a in R is, essentially, the subgraph of R whose links are the elements of the following set of links: $\{n : n$ link of R such that there exists a path ϕ_a^n of R with starting link an axiom or a weakening and terminal edge a, which contains $n\}$.

However, this definition is not precise enough (especially with respect to the axiom links). This is why we give the inductive 'construction' of this graph.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

Definition 2.20. Let R be a proof-net and let a be an edge of R. Fix a 'graphical representation' $R^\star$ of R: by which we mean fix the order of the contraction links of R (see Remark A.3). Let k_a be the cardinality of the set $\{n : n$ link of $R^\star$ such that there exists a path ϕ_a^n of $R^\star$ with starting link an axiom or a weakening and terminal edge a, which contains $n\}$.

The tree of a in $R^\star$, denoted by $G_a^{R^\star}$, is an oriented tree, which we define by induction on k_a:

— If $k_a = 1$, then a is a conclusion of an axiom link or the conclusion of a weakening link (these are the only links with no premises) of $R^\star$. If a is a conclusion of an axiom link, then $G_a^{R^\star}$ is the edge a, and we will write $G_a^{R^\star} = \{a\}$. If a is the conclusion of a weakening link, then $G_a^{R^\star}$ is the weakening link and its conclusion, and we will write $G_a^{R^\star} = \{n\} \cup \{a\}$.

— Otherwise, let n be the link of $R^\star$ with conclusion a:

- If n is a logical link with two premises and a conclusion (that is, if $n = \otimes, \wp$), then let a_1 be the left premise and a_2 the right premise of n in $R^\star$. We have $k_{a_1} < k_a$ and $k_{a_2} < k_a$. We can then define $G_a^{R^\star}$ as the graph obtained by connecting the two graphs $G_{a_1}^{R^\star}$ and $G_{a_2}^{R^\star}$ by means of the link n with left premise a_1 and right premise a_2, and with conclusion a. We will write $G_a^{R^\star} = G_{a_1}^{R^\star} \cup G_{a_2}^{R^\star} \cup \{n\} \cup \{a\}$.
- If n is a link with one premise and one conclusion (that is, if $n = !, ?de, pax$), then let a' be the premise of n. We have $k_{a'} < k_a$. We can then define $G_a^{R^\star}$ as the graph obtained from $G_{a'}^{R^\star}$ by adding the link n with premise a' and with conclusion a. We will write $G_a^{R^\star} = G_{a'}^{R^\star} \cup \{n\} \cup \{a\}$.
- If n is a $?co$ link with $h \geqslant 2$ premises, then let (from left to right in the given representation $R^\star$ of R) $a_1, \dots, a_h$ be the premises of n in $R^\star$. We have $k_{a_i} < k_a$ (for $i \in \{1, \dots, h\}$). We can then define $G_a^{R^\star}$ as the graph obtained by connecting the h graphs $G_{a_1}^{R^\star}, \dots, G_{a_h}^{R^\star}$ by means of the link n with premises (from left to right) $a_1, \dots, a_h$, and with conclusion a.

It is rather obvious that the relation $\sim$ defined by $G_a^{R^\star} \sim G_a^{R^\bullet}$ iff $R^\star$ and $R^\bullet$ are two graphical representation of the proof-net R, is an equivalence relation.

The **graph of** a **in** R, denoted by G_a^R, or by G_a if there is no ambiguity, is the equivalence class of $G_a^{R^\star}$ with respect to $\sim$.

Remark 2.21 (and Definition).

(i) The graph G_a^R of the previous definition is an equivalence class of trees that can only differ in the order of the premises of the contraction links.

(ii) If a is an edge of the proof-net R and if e is an experiment of R, we will use $e|_{G_a^R}$ to denote the restriction of e to the edges of G_a^R.

Warning! If a' is an edge of the proof-net R' and if e' is an experiment of R', we will write $e|_{G_a^R} = e'|_{G_{a'}^{R'}}$ when *there exists* a tree $\mathcal{T}$ (respectively, $\mathcal{T}'$) of the equivalence class G_a^R (respectively, $G_{a'}^{R'}$) such that ($\mathcal{T} = \mathcal{T}'$ and) $e|_{\mathcal{T}} = e'|_{\mathcal{T}'}$.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

Remark 2.22.

(i) Let R be a proof-net, a an edge of type A of R and c an edge of type C of G_a^R. Then C is an occurrence of a subformula of A.
(ii) Let R be a proof-net, and a and c be two edges of R. If $a \in G_c^R$, then G_a^R is a subgraph of G_c^R.
(iii) Let R be a proof-net, and $a_1, \ldots, a_n$ be the conclusions of R. The graph $OLPS(R)$ is nothing but the juxtaposition of the graphs $G_{a_1}^R, \ldots, G_{a_n}^R$.
(iv) Let a and a' be two edges of the proof-net R. Then $G_a \cap G_{a'} = \emptyset$ if and only if $a \notin G_{a'}$ and $a' \notin G_a$.

2.3. *The case of MLL*

The first idea is to try to prove that the syntactical and semantic relations mentioned in the introduction (and precisely defined in the previous section) do coincide. We are going to see that this is indeed the case for the multiplicative fragment of LL. This is easy to show, and the reason we call this result a theorem (Theorem 2.26) is that its proof suggests the approach to the much more complex *MELL* case (which will be developed later).

We use *MLL* to denote the subset of *MELL* proof-nets containing just axiom, cut, $\otimes$ and $\wp$ links.

Observe that every experiment of an *MLL* proof-net R associates a unique label with every edge: in the multiplicative case, an experiment of R is a labelling of the edges of R.

It is also immediate that, if R is an *MLL* proof-net, then $OPS(R) = OLPS(R)$.

The following lemma is obvious: if we know the type of all the conclusions of the *MLL* proof-net R, we know the proof-net 'up to the axiom links'.

Lemma 2.23. If R and R' are two *MLL* proof-nets with the same conclusions, then $OPS(R) = OPS(R')$.

Lemma 2.23 stresses the fact that the unique information that we have to extract from the interpretation of an *MLL* proof-net R in order to be able to say that the semantics is injective, is the set of the pairs of edges of atomic type that are conclusions of the same axiom link of R.

Lemma 2.24. Let R (respectively, R') be an *MLL* proof-net with conclusion Γ, and let e (respectively, e') be an experiment of R (respectively, R') with result γ (respectively, γ'). If $\gamma = \gamma'$, then ($OPS(R) = OPS(R')$ and) $e|_{OPS(R)} = e'|_{OPS(R')}$.

Here is the only point requiring a bit of attention. We introduce the notion of 'injective experiment', which will be extended to *MELL* later.

Definition 2.25. Let R be an *MLL* proof-net and let e be an experiment of R. We will say that e is an **injective experiment** when $\forall a, a'$ edges of the same atomic type of R such that $a \neq a'$, we have $e(a) \neq e(a')$.

We can now prove the injectivity 'theorem' for *MLL*.

Theorem 2.26. (Injectivity for MLL). Let R and R' be two proof-nets with the same conclusions. If $[\![R]\!] = [\![R']\!]$, then $R = R'$.

Proof. Let e be an injective experiment of R with result γ.

Notice that such an experiment always exits, because there is no condition on the labels of the edges of a standard MLL proof-net: it contains no of course, *cut*, $?co$ or pax link (remember Remark 2.2). We only have to choose an interpretation of R's atomic formulas containing a sufficient number of elements.

Since $[\![R]\!] = [\![R']\!]$, there exists an experiment e' of R' with result γ. From Lemma 2.24, ($OPS(R) = OPS(R')$ and) $e|_{OPS(R)} = e'|_{OPS(R')}$. Because e is injective, for every atomic edge a_1 of type X of $OPS(R) = OPS(R')$, there exists a unique edge a_2 of $OPS(R) = OPS(R')$ of type $X^\perp$ such that $e(a_1) = e'(a_1) = e(a_2) = e'(a_2)$. Then $R = R'$. □

Remark 2.27. This proof shows that every injective experiment of an MLL proof-net R contains *all the informations* contained in R.

You might then wonder 'what are the other experiments for?'

We should not forget that denotational semantics does not yield a *static* representation of a given proof π: it mainly describes the possible interactions of π with the other proofs. Notice, by the way, that the interpretation of a proof-net by means of the results of its injective experiments is not correct: it does not yield an invariant of cut-elimination (just imagine two axiom links connected by a cut link).

3. Injectivity and obsessionality

The first idea is to apply the method used to prove Theorem 2.26 to $MELL$, that is, to prove the analogue of Lemma 2.24 and to extend the notion of injective experiment to $MELL$ proof-nets.

But we immediately stumble on several problems. The first is that the type A of a conclusion a of an $MELL$ proof-net R is not enough to know G_a^R (due to the presence of pax, $?w$ and $?co$ links). Worse, suppose you know all the labels that the experiments of R associate with an edge a that is the conclusion of a link n, suppose that this is enough to 'guess' which kind of link n is, and that n is a $?co$ link with arity k. How can you 'guess', for a given label of a, how to split it into the k labels of the premises of a?

We have discovered that the interpretation of a proof-net always contains the results of a very specific kind of experiment, *obsessional experiments*, which turned out to be 'at the heart of semantics'.

We introduce obsessional experiments in Section 3.1. These experiments have a particular nature: they are both regular and powerful. Their regularity is explored in Sections 3.1 and 3.2: we prove that coherent semantics 'can read the obsessional feature of an experiment in its result' (Propositions 3.14 and 3.15). In Section 3.3 we show the power of our new tool by proving the analogue of Lemma 2.24 (Corollary 3.23): as a consequence, we show that an obsessional experiment of a proof-net R allows us to determine R 'up to the axiom links and the boxes' (Theorem 3.27). Finally, in Section 3.4 we give a sufficient condition for 'local injectivity' (Theorem 3.35). This last result is the starting point for Sections 4 and 5: we try to fulfill the sufficient condition of Theorem 3.35 in order to answer our original question (Problem 2.16).

3.1. *Obsessional experiments*

We now introduce the main tool of our analysis: obsessional experiments. These experiments are very regular, as witnessed by the properties stated in this section.

Definition 3.1 (n-obsessional experiment). Let R be a proof-net, let e be an experiment of R, and let $n \geqslant 1$ be an integer.

We will say that e is an n-obsessional experiment of R iff:

(1) For every edge a of R of atomic type X, for every $x, y \in |\mathscr{X}|$, if $x \in e(a)$ and $y \in e(a)$, then $x = y$.

(2) For every edge c of type $!C$ of R, $e(c) \neq \emptyset$ and for every $y \in e(c)$, $card(y) = n$.

When $n = 1$, we will also say that e is a 1-experiment.

Remark 3.2.

(i) Let e be an n-obsessional experiment of the proof-net R and let a be an edge of R of atomic type with depth p_a in R. The experiment e satisfies the hypothesis of Lemma 2.10 where $\forall i \in \{1, \ldots, p_a\}$ $n_i = n$. Then $e(a)$ is a multiset of cardinality n^{p_a}, containing n^{p_a} occurrences of the same element of $|\mathscr{X}|$.

(ii) For 1-experiments, Conditions (1) and (2) of the definition above are equivalent to Condition (2).
Notice also that every experiment of a proof-net without boxes is always an n-obsessional experiment (for every $n \geqslant 1$): for these proof-nets, the notion of an n-obsessional experiment does not make much sense.

(iii) To understand what an n-obsessional experiment is, you might visualise it as follows: first fix a proof-net R and start from the subproof-nets of R that contain no box. Consider an experiment for each of these subproof-nets. So long as we meet a door of some box, the top-down propagation of the labels of the edges of R is completely deterministic. Either it fails or succeeds (remember Remark 2.2), but we have no choice. When we meet a door of a box, we stop and wait for friends (that is, we wait until every premise of every door of the box has a label). When everybody has arrived, we have an experiment e of the content R_B of every box B of R that contains no other box. We then take n copies of that same experiment e: we get an experiment of B (the condition on the labels of the conclusions of the pax and pal doors are necessarily satisfied). We then start the game again, following the same rules. This construction (when it succeeds) yields an n-obsessional experiment of R; and the results of the present section entail that every n-obsessional experiment of R can be built in such a way.

By now you should certainly agree with the terminology chosen for this kind of experiment.

Notice also, that the notion of obsessional experiment has a true meaning only in a multiset framework.

Lemmas 3.3 and 3.4, and Proposition 3.6 are rather intuitive, and their proofs are left to the reader.

Lemma 3.3. Let e be an n-obsessional experiment of the proof-net R and let R_1 be a subproof-net of R. If $e|_{R_1} = \{e_1, \ldots, e_l\}$, then $\forall i \in \{1, \ldots, l\}$ e_i is an n-obsessional experiment of R_1.

Lemma 3.4. Let e and e' be two n-obsessional experiments of the proof-net R. If for every edge a of atomic type of R $e(a) = e'(a')$, then $e = e'$.

Remark 3.5. The previous lemma is wrong if e and e' are not (both) obsessional: consider the proof-net obtained from an axiom link with conclusions a_1 of type X and a_2 of type $X^\perp$ by adding what is needed (that is, two ! links, a $?de$ link and two pax links) to obtain the proof-net with conclusions $!!X, ?X^\perp$. The edges a_1 and a_2 are contained in two boxes: we will call the smaller one B_1 and the bigger one B_2. Let e be the experiment of the subproof-net of R containing only the axiom link (and its conclusions) such that $e(a_1) = e(a_2) = \{x\}$, where $x \in |\mathscr{X}|$. Let e_1 (respectively, e_2) be the experiment of R obtained from e by taking 2 (respectively, 4) copies of e to exit from B_1 and 8 (respectively, 4) copies of the experiment thus obtained to exit from B_2. As $4 \times 4 = 2 \times 8$, we will indeed have that $e_1(a_1) = e_2(a_1)$ and $e_1(a_2) = e_2(a_2)$, despite the fact that $e_1 \neq e_2$.

Proposition 3.6. Let e be an n-obsessional experiment of the proof-net R' and let R be a subproof-net of R'. If the depth of the conclusions of R in R' is p, then $e|_R = \{e_1, \ldots, e_{n^p}\}$, where $e_1 = \ldots = e_{n^p}$ is an n-obsessional experiment of R.

In particular, for every edge a of type A of R' with depth p, $e(a)$ is a multiset of cardinality n^p containing n^p occurrences of the same element of $|\mathscr{A}|$.

In the rest of the paper we will constantly use the following remark (especially part (ii)).

Remark 3.7.

(i) Let e be an n-obsessional experiment of the proof-net R, c be an edge of type $!C$ of R and c' be the premise of the pal door whose conclusion is c. Let y be the unique element of $e(c)$. By definition of experiment, there exist $z_1, \ldots, z_n \in e(c')$ such that $y = \{z_1, \ldots, z_n\}$. We know from the previous proposition that $e(c')$ contains a unique element $z \in |\mathscr{C}|$: then $z_1 = \ldots = z_n = z$.

(ii) Let e (respectively, e') be an n-obsessional experiment of the proof-net R (respectively, of the proof-net R') and let a (respectively, a') be an edge of depth p in R (respectively, of depth p' in R'). If the unique element of $e(a)$ is equal to the unique element of $e'(a')$ and if $p = p'$, then $e(a) = e'(a')$.

Remark 3.8. Let R be a proof-net without $?co$ links. If we associate ('correctly') with every edge of atomic type α of R an element x_α of the web of the coherent space $\mathscr{A}$, then there exists a (unique) n-obsessional experiment e_n of R such that the unique element of $e_n(\alpha)$ is x_α (for every edge α of atomic type). Indeed, the fact that e_n is obsessional guarantees that the compatibility conditions of Remark 2.2 (required by Definition 2.1) when passing through the pax and pal doors are always satisfied. Notice that this is not the case for the $?co$ links, and this will be discussed in all details later in the paper.

3.2. *Obsessional results*

In the present section, we answer the following question: is it possible, from the result of an experiment, to deduce whether or not it is an obsessional experiment?

We first prove some general lemmas, and then show that the answer to this question is positive (Proposition 3.14), thanks to the uniformity property of coherent semantics (Lemma 3.11).

The following lemma is a consequence of the definitions. Its proof is rather long and not very interesting: we therefore omit it, and leave it as an exercise for the reader.

Lemma 3.9. Let R be a proof-net, c be an edge of R of type C and e be an experiment of R. For every x, the following two statements are equivalent:

(1) There exists an edge d of type A of G_c^R, with $A \neq ?F$ (for every formula F), such that $x \in e(d)$.
(2) There exists $y \in e(c)$ such that $x \in |y|_A$, with $A \neq ?F$ (for every formula F).

Proposition 3.10. Let R be a proof-net, c be a conclusion of R of type C, and e be an experiment of R. Let A be an occurrence of subformula of C such that $A \neq ?F$ (for every formula F), and $d_1, \dots, d_k$ be the k edges ($k \geqslant 0$) of type A such that $\forall i \in \{1, \dots, k\}$ $d_i \in G_c^R$.

If $e(c) = \{y\}$, then $|y|_A = e(d_1) \cup \dots \cup e(d_k)$.

Proof. The statement is a straightforward consequence of the previous lemma. □

Up to this point, we have not used the fact that the labels of the edges of type $!A$ (respectively, $?A$) are *cliques* of the coherent space $\mathscr{A}$ (respectively, $\mathscr{A}^\perp$). (By the way, this means that everything we have written up to now, is also true for the relational semantics of Appendix B.) This fact will now be used to prove Proposition 3.14, which states that coherent semantics ‘can read the obsessional feature of an experiment in its result’. In the proof, we will make an essential use of the following lemma (which can be seen as a way of expressing the ‘uniformity’ of coherent semantics).

Lemma 3.11 (Uniformity property). Suppose that the proof-net R is a box B having as conclusions the edge d of type $!D$ and the edges $d_1, \dots, d_k$ of types $?D_1, \dots, ?D_k$, respectively. We will call d' (respectively, $d'_1, \dots, d'_k$) the premise of the pal door of B (respectively, the premises of the pax doors with conclusions $d_1, \dots, d_k$ of B): $d', d'_1, \dots, d'_k$ are the conclusions of R_B. Let e be an experiment of R such that $e(d) = \{\{y_1, \dots, y_n\}\}$, and $e_1, \dots, e_n$ be the n experiments of R_B from which e is built: we have $\forall i \in \{1, \dots, n\}$ $e_i(d') = \{y_i\}$. Finally, let $y_i, t_1^i, \dots, t_k^i$ be the result of the experiment e_i of R_B ($\forall i \in \{1, \dots, n\}$).

If, for some $i_1, i_2 \in \{1, \dots, n\}$ we have $y_{i_1} = y_{i_2}$, then $\forall j \in \{1, \dots, k\}$ we have $t_j^{i_1} = t_j^{i_2}$.

Proof. The results of the experiments $e_1, \dots, e_n$ of R_B all belong to the same clique of the coherent space $\mathscr{A} = \mathscr{D} \wp ?\mathscr{D}_1 \wp \dots \wp ?\mathscr{D}_k$ (from Theorem 2.5), so we have $(y_{i_1}, t_1^{i_1}, \dots, t_k^{i_1}) \frown\!\!\!\!\smile (y_{i_2}, t_1^{i_2}, \dots, t_k^{i_2})(\mathscr{A})$. Moreover, $\forall j \in \{1, \dots, k\}$ the label associated by e with the conclusion d_j of type $?D_j$ of R is $t_j^1 \cup \dots \cup t_j^n$, which is an element of $|?\mathscr{D}_j| = \mathscr{D}_j^\perp$. Then, in particular, $t_j^{i_1} \cup t_j^{i_2}$ is a clique of $\mathscr{D}_j^\perp$, that is, $t_j^{i_1} \asymp t_j^{i_2}(?\mathscr{D}_j)$. This means that, for $j \in \{1, \dots, k\}$ we never have $t_j^{i_1} \frown t_j^{i_2}(?\mathscr{D}_j)$. Because we also have $y_{i_1} = y_{i_2}$, we cannot have

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

$y_{i_1} \frown y_{i_2}$. The only remaining possibility to have $(y_{i_1}, t_1^{i_1}, \ldots, t_k^{i_1}) \frown\!\!\!\!\smile (y_{i_2}, t_1^{i_2}, \ldots, t_k^{i_2})(\mathscr{A})$ is then that $\forall j \in \{1, \ldots, k\}$ $t_j^{i_1} = t_j^{i_2}$. □

In the proof of the following lemma we use the result of Barcaglioni (2001) mentioned in Remark 2.13.

Lemma 3.12. Let n be a strictly positive integer. Let e be an experiment of the proof-net R such that for every edge c of type $!C$ (for some formula C) of R, if we call c' (of type C) the premise of the of course link having c as conclusion, we have that:

(1) $e(c) \neq \emptyset$.
(2) For every $y \in e(c)$, there exists $z \in e(c')$ such that $y = \{n[z]\}$.

Then e is an n-obsessional experiment of R.

Proof. We use induction on a sequentialisation of R.

If R is an axiom link, then the lemma holds.

Otherwise, let n be a terminal link of R, that is, such that there exists a sequentialisation of R whose last rule is the rule corresponding to n. If R is not a box, then $n = \otimes, \wp$, $?de$, $?co$, $?w$, and the result is a straightforward application of the induction hypothesis.

Now consider the case of a box B with conclusions the edge d of type $!D$ and the edges $d_1, \ldots, d_k$ of types $?D_1, \ldots, ?D_k$, respectively. We will call d' (respectively, $d'_1, \ldots, d'_k$) the premise of the pal door of B (respectively, the premises of the pax doors with conclusions $d_1, \ldots, d_k$ of B): $d', d'_1, \ldots, d'_k$ are the conclusions of R_B. By hypothesis, $e(d) = \{\{n[y]\}\}$. Thus, there exist n experiments $e_1, \ldots, e_n$ of R_B such that $\forall i \in \{1, \ldots, n\}$ $e_i(d') = \{y\}$. So, let $y, t_1^i, \ldots, t_k^i$ be the result of the experiment e_i of R_B ($\forall i \in \{1, \ldots, n\}$). We are going to prove that:

(a) $\forall i_1, i_2 \in \{1, \ldots, n\}$ $\forall j \in \{1, \ldots, k\}$, we have $t_j^{i_1} = t_j^{i_2}$.
(b) $\forall i \in \{1, \ldots, n\}$, we have e_i is an n-obsessional experiment of R_B.

Property (a) is a consequence of Lemma 3.11. To prove Property (b), by the induction hypothesis it will be enough to prove that $\forall i \in \{1, \ldots, n\}$ the experiment e_i of R_B satisfies hypotheses (1) and (2) of the lemma. If for some edge c of R_B $e_i(c) = \emptyset$, by Lemma 2.9, there necessarily exists an edge g of type $!G$ of R_B such that $\emptyset \in e_i(g)$. But by Lemma 2.8, we would then have $\emptyset \in e(g)$, which contradicts (2). Similarly, suppose there exists an edge c of type $!C$ (for some formula C) and $y \in e_i(c)$ such that $\forall z \in e_i(c')$ $y \neq \{n[z]\}$, where c' is the edge of R_B, which is the premise of the of course link having c as conclusion. In this case, either the cardinality of y (as a multiset) is different from n, or there exist $z_1, z_2 \in e_i(c')$ such that $z_1 \neq z_2$ and $z_1, z_2 \in y$. By Lemma 2.8, in both cases there exists $y \in e(c)$ such that $\forall z \in e(c')$ $y \neq \{n[z]\}$, thus contradicting the hypothesis of the lemma.

This entails that e_i is an n-obsessional experiment of R_B. Moreover, thanks to Property (a) above (and to Remark 2.13), we have $e_1 = \ldots = e_n$. The experiment e of R is then n-obsessional (remember Remark 3.2). □

Remark 3.13. Let e (respectively, e') be an n-obsessional experiment of the proof-net R (respectively, of the proof-net R') and let a (respectively, a') be an edge of R (respectively, of R') conclusion of the link m (respectively, m') whose premises are $a_1, \ldots, a_k$ (respectively, $a'_1, \ldots, a'_k$), with $k \geqslant 0$.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

If m and m' are two links of the same kind, and if $\forall i \in \{1, \dots, k\}$ $e(a_i) = e'(a_i)$, then $e(a) = e'(a')$.

Notice that this is wrong (in general) if e and e' are not n-obsessional.

To see why the property is wrong if we omit the hypothesis of n-obsessionality, take, for example, $k = 2$, m and m' of type $\otimes$, $e(a_1) = e'(a'_1) = \{x_1, x_2\}$ and $e(a_2) = e'(a'_2) = \{y_1, y_2\}$, with $x_1 \neq x_2$ and $y_1 \neq y_2$. Then we may very well have $e(a) = \{(x_1, y_1), (x_2, y_2)\}$ and $e'(a') = \{(x_1, y_2), (x_2, y_1)\}$.

Proposition 3.14. Let R be a proof-net, $\gamma \in [\![R]\!]$ and n be a strictly positive integer. The following two statements are equivalent:

(i) For every edge a of type A conclusion of R and for every (occurrence of the) subformula $!C$ of A, if x is the label of γ associated with a, we have: if $z \in |x|_{!C}$, then there exists $t \in |\mathscr{C}|$ such that $z = \{n[t]\}$.

(ii) The experiment e of R with result γ is an n-obsessional experiment.

Proof. Let e be an n-obsessional experiment of R with result γ, and a be an edge of type A conclusion of R. Let $!C$ be (an occurrence of) a subformula of A and x be the label that e associates with a. Let k be the number of edges of G^R_a of type $!C$. If $k = 0$, then $|x|_{!C} = \varnothing$ and we are done. Otherwise, $k \geqslant 1$, and by Proposition 3.10, $|x|_{!C} = e(c_1) \cup \ldots \cup e(c_k)$, where $c_1, \dots, c_k$ are the k edges of type $!C$ such that $\forall i \in \{1, \dots, k\}$ we have $c_i \in G^R_a$. If $z \in |x|_{!C}$, then there exists $i \in \{1, \dots, k\}$ such that $z \in e(c_i)$. By Remark 3.7(i), we then indeed have $z = \{n[t]\}$ for some $t \in |\mathscr{C}|$.

Conversely, let e be the experiment of R with result γ. We show that e satisfies the hypothesis of Lemma 3.12, by applying Lemma 3.9.

By contradiction, suppose that e does not satisfy the hypothesis of Lemma 3.12. There exists an edge c of type $!C$ conclusion of an of course link with premise c' and satisfying one of the following two conditions:

(1) $e(c) = \varnothing$

(2) $\exists y \in e(c)$ such that $\forall z \in e(c')$ $y \neq \{n[z]\}$.

In case (1), by Lemma 2.9, there exists a box B of R such that if we call d the conclusion of type $!D$ of the pal door of B, then $e(d) = \{m[\varnothing]\}$ for some integer m different from zero. Let a of type A be the conclusion of R such that $d \in G^R_a$ and let $x \in |\mathscr{A}|$ be such that $e(a) = \{x\}$. Lemma 3.9 applied to e gives $\varnothing \in |x|_{!D}$, which contradicts (i).

In case (2) we have two possibilities:

(2.1) $\exists y \in e(c)$ s.t. $card(y) \neq n$.

(2.2) $\exists y \in e(c)$ s.t. $card(y) = n$ and $\exists z_1, z_2 \in e(c')$, $z_1 \neq z_2$ s.t. $z_1, z_2 \in y$.

In both cases, let a of type A be the conclusion of R such that $c \in G^R_a$ and let $x \in |\mathscr{A}|$ be such that $e(a) = \{x\}$. Lemma 3.9 applied to e gives $y \in |x|_{!C}$, which again contradicts (i). □

The following proposition is a consequence of Proposition 3.14 and will be used later in the paper to answer the question of injectivity.

Proposition 3.15. Let R and R' be two proof-nets with the same conclusions. Let e (respectively, e') be an experiment of R (respectively, of R') with result γ (respectively, γ').

If e' is an n-obsessional experiment of R' and if $\gamma = \gamma'$, then e is an n-obsessional experiment of R.

Proof. We just have to show that the result γ of e satisfies (i) of Proposition 3.14. This is immediate, because e' is n-obsessional and its result $\gamma' = \gamma$ satisfies (i) of Proposition 3.14. □

3.3. *Recovering OLPS*

We show, in this section, that the coherent semantics of a proof-net R determines R 'up to the axiom links and the boxes' (Theorem 3.27).

The notion of n-obsessional experiment will allow us to prove Corollary 3.23, which is the analogue for *MELL* of Lemma 2.24. This is the second step allowing us to argue as for Theorem 2.26 (the first step being Proposition 3.15): suppose there exists an n-obsessional experiment e of R with result γ; because $[\![R]\!] = [\![R']\!]$, there exists an experiment e' of R' with result γ, and we know by Proposition 3.15 that e' is itself n-obsessional, which allows us to apply Corollary 3.23. To conclude, we prove that for every proof-net R and for every integer $n \geqslant 1$, there exists an n-obsessional experiment of R.

The obsessional feature of our obsessional experiments is crucial here.

We first have to introduce one more notion: the structural graph of a link.

Definition 3.16. Let R be a proof-net and n be a contraction link of arity k, a pax link, or a dereliction link. Let a be the conclusion of n. Let $a_1, \dots, a_k$ (respectively, $a'_1, \dots, a'_k$) be the conclusions (respectively, the premises) of the k dereliction links above a ($k \geqslant 1$).

The **structural graph** of n in R, denoted by SG^R_a, is the subgraph of G^R_a obtained by erasing $G^R_{a'_i}$ for every $i \in \{1, \dots, k\}$.

We will say that the edge a (respectively, the edges $a_1, \dots, a_k$) is (respectively, are) the conclusion (respectively, the hypothesis) of SG^R_a. For every edge c of SG^R_a, we will say that the number p of the pax links crossed by the path of SG^R_a with starting edge c and terminal edge a is the depth of c in SG^R_a.

This definition will mainly (but not only) be used when a is the conclusion of a contraction link.

Remark 3.17. Let a be an edge of the proof-net R that is the conclusion of a contraction link n of arity k, and let $a_1, \dots, a_k$ be the conclusions of the k dereliction links $m_1, \dots, m_k$ above a. For every edge c (respectively, for every link m) of SG^R_a such that $c \notin \{a, a_1, \dots, a_k\}$ (respectively, $m \notin \{n, m_1, \dots, m_k\}$), c is the conclusion of a pax link (respectively, m is a pax link).

The following lemma is the main ingredient of the main result of this section (Proposition 3.22). Every letter stands for a non-negative integer, and we deal (as usual) with *multisets*.

Lemma 3.18. Let $1 \leqslant l, m < n$. We have $\forall p_1, \dots, p_l, p'_1, \dots, p'_m$: the equality $n^{p_1} + \ldots + n^{p_l} = n^{p'_1} + \ldots + n^{p'_m}$ implies the equality $\{p_1, \dots, p_l\} = \{p'_1, \dots, p'_m\}$ (in particular, $l = m$).

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

Proof. This result is a straightforward consequence of the unicity of the decomposition of every integer in base n. □

The following definition associates with every proof-net R an integer (its $?co$-size), which depends on the arity of the contraction links of R. This 'size' is often used later in the paper, and allows us to simplify several statements.

Definition 3.19. Let k be the maximal arity of the contraction links of the proof-net R. The **$?co$-size** of R, denoted by $h(R)$, is the following integer:

— $h(R) = max(1, k)$, if there exists at least one box in R
— $h(R) = 0$ otherwise.

We shall often consider integers $n > h(R)$: when $h(R) = 0$, you can actually read $n = 1$.

We come now to a first application of the notion of obsessional experiment.

Lemma 3.20. Let $h(R)$ be the $?co$-size of the proof-net R, and let $n > h(R)$. Let e be an n-obsessional experiment of R. Let a be an edge of R of type $?A$ and let x be the unique element of $e(a)$ (following Proposition 3.6). Then:

(i) a is the conclusion of a weakening link iff $x = \emptyset$
(ii) a is the conclusion of a dereliction link iff x is a singleton
(iii) a is the conclusion of a pax link iff there exists an integer $p \geqslant 1$ such that $card(x) = n^p$
(iv) a is the conclusion of a contraction link with arity k iff there exist $p_1, \ldots, p_k$ non-negative integers such that $k \geqslant 2$ and $card(x) = n^{p_1} + \ldots + n^{p_k}$.

Proof. As an example, we will prove (iv) only.

— If a is the conclusion of a contraction link with arity k ($k \geqslant 2$), then let $a_1, \ldots, a_k$ be the edges conclusion of the k dereliction links above a, and let $p_1, \ldots, p_k$ be, respectively, their depths in SG_a^R. Let $\{y_i\}$, $\forall i \in \{1, \ldots, k\}$, be the unique element of $e(a_i)$. By definition of n-obsessional experiment, the unique element of $e(a)$ is the multiset of cardinality $n^{p_1} + \ldots + n^{p_k}$ containing n^{p_i} occurrences of y_i $\forall i \in \{1, \ldots, k\}$.
— Because $card(x) > 1$, by (i) and (ii), the edge a is the conclusion of a pax link or of a contraction link. If a were the conclusion of a pax link, then by (iii) there would exist $p \geqslant 1$ such that $card(x) = n^p$. But then we would have $n^p = n^{p_1} + \ldots + n^{p_k}$, thus contradicting Lemma 3.18 (because $k \geqslant 2$). This means that a is indeed the conclusion of a contraction link. Let l be the arity of this link. We have proved that in this case there exist $p'_1, \ldots, p'_l$ non-negative integers such that $card(x) = n^{p'_1} + \ldots + n^{p'_l}$. Then $n^{p_1} + \ldots + n^{p_k} = n^{p'_1} + \ldots + n^{p'_l}$. Because (when $h(R) \neq 0$) we have $n > h(R) \geqslant k, l \geqslant 1$, we can apply Lemma 3.18 to give $\{p'_1, \ldots, p'_l\} = \{p_1, \ldots, p_k\}$. In particular $l = k$, which means that a is the conclusion of a contraction link with arity k. □

Let e (respectively, e') be an n-obsessional experiment of the proof-net R (respectively, R') with $n > 1$. Let a (respectively, a') be an edge of R (respectively, of R') of type $?C$ conclusion of a contraction link with arity $k \geqslant 2$. Let $a_1, \ldots, a_k$ (respectively, $a'_1, \ldots, a'_k$) be the hypothesis of SG_a^R (respectively, of $SG_{a'}^{R'}$) and $p_1, \ldots, p_k$ (respectively, $p'_1, \ldots, p'_k$), respectively, be their depths in SG_a^R (respectively, in $SG_{a'}^{R'}$).

Suppose that $e(a) = e'(a')$ and let $t \in |?\mathscr{C}|$ be the unique element of $e(a) = e'(a')$ (following Proposition 3.6). We define, $\forall x \in t$:

— $k_x := card(\{i \in \{1,\ldots,k\} : \{x\}$ is the unique element of $e(a_i)\})$ (respectively, $k'_x := card(\{i \in \{1,\ldots,k\} : \{x\}$ is the unique element of $e'(a'_i)\})$)
— the multiset $\{p^x_1,\ldots,p^x_{k_x}\} \subseteq \{p_1,\ldots,p_k\}$ (respectively, the multiset $\{p'^x_1,\ldots,p'^x_{k'_x}\} \subseteq \{p'_1,\ldots,p'_k\}$) of the depths in SG^R_a (respectively, in $SG^{R'}_{a'}$) of the edges $c \in \{a_1,\ldots,a_k\}$ (respectively, of the edges $c' \in \{a'_1,\ldots,a'_k\}$) such that $\{x\}$ is the unique element of $e(c)$ (respectively, of $e'(c')$).

Lemma 3.21. If $\forall x \in t$, $k_x = k'_x$ and $\{p^x_1,\ldots,p^x_{k_x}\} = \{p'^x_1,\ldots,p'^x_{k'_x}\}$, then $SG^R_a = SG^{R'}_{a'}$ and $e|_{SG^R_a} = e'|_{SG^{R'}_{a'}}$.

Proof. We leave readers to convince themselves that the following holds and is enough to conclude:

$\forall c_1, c_2 \in \{a_1,\ldots,a_k\}$, $c_1 \neq c_2$, c_1 (respectively, c_2) of depth h_1 (respectively, h_2) in SG^R_a and such that $\{y_1\}$ (respectively, $\{y_2\}$) is the unique element of $e(c_1)$ (respectively, of $e(c_2)$), there exists an edge $c'_1 \in \{a'_1,\ldots,a'_k\}$ (respectively, an edge $c'_2 \in \{a'_1,\ldots,a'_k\}$) of depth h_1 (respectively, h_2) in $SG^{R'}_{a'}$ such that $c'_1 \neq c'_2$ and $\{y_1\}$ (respectively, $\{y_2\}$) is the unique element of $e'(c'_1)$ (respectively, of $e'(c'_2)$). □

Proposition 3.22. Let a (respectively, a') be an edge of type A of the proof-net R (respectively, of the proof-net R'). Let $h(R)$ (respectively, $h(R')$) be the $?co$-size of R (respectively, R'), and let $n > max(h(R), h(R'))$.

Let e (respectively, e') be an n-obsessional experiment of R (respectively, of R'). If $e(a) = e'(a')$, then:

(i) $G^R_a = G^{R'}_{a'}$
(ii) $e|_{G^R_a} = e'|_{G^{R'}_{a'}}$.

Proof. We use induction on $s(G^R_a)$, the number of links of G^R_a.

If $s(G^R_a) = 0$, then a is an edge of R that is the conclusion of an axiom link, and then A is an atomic formula. This means that a' is also the conclusion of an axiom and $G^R_a = G^{R'}_{a'}$. Then (ii) is a straightforward consequence of the hypothesis $e(a) = e'(a')$.

Otherwise, let m be the link of R having a as conclusion. The non-trivial cases are the ones in which $A = ?D$ for some formula D, especially when a is the conclusion of a contraction link. We nevertheless give a complete proof, precisely so that the reader can feel the difference between the cases.

— If $m = \otimes$ or $m = \wp$, then the edge a' of R' is also the conclusion of a link $m' = \otimes$ or $m' = \wp$. Let a_1 and a_2 (respectively, a'_1 and a'_2) be the premises of m in R (respectively, of m' in R'). The edges a_i and a'_i $(i = 1, 2)$ have the same type, and, by definition, $G^R_a = G^R_{a_1} \cup G^R_{a_2} \cup \{m\} \cup \{a\}$ and $G^{R'}_{a'} = G^{R'}_{a'_1} \cup G^{R'}_{a'_2} \cup \{m'\} \cup \{a'\}$. We have that $s(G^R_{a_1}) < s(G^R_a)$, $s(G^R_{a_2}) < s(G^R_a)$. Let x be the unique element of $e(a) = e'(a')$ (following Proposition 3.6). By the definition of experiment, there exists $x_1 \in e(a_1)$ and $x_2 \in e(a_2)$ (respectively, $x'_1 \in e'(a'_1)$ and $x'_2 \in e'(a'_2)$) such that $x = (x_1, x_2) = (x'_1, x'_2)$. Then $x_1 = x'_1$ is the unique

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

element of $e(a_1)$ and of $e'(a'_1)$, and $x_2 = x'_2$ is the unique element of $e(a_2)$ and of $e'(a'_2)$. Because $e(a) = e'(a')$, the edges a and a' have the same depth, and thus a_1, a_2, a'_1 and a'_2 all have the same depth. This implies that $e(a_1) = e'(a'_1)$ and $e(a_2) = e'(a'_2)$. The induction hypothesis gives $G^R_{a_1} = G^{R'}_{a'_1}$, $G^R_{a_2} = G^{R'}_{a'_2}$ and $e|_{G^R_{a_1}} = e'|_{G^{R'}_{a'_1}}$, $e|_{G^R_{a_2}} = e'|_{G^{R'}_{a'_2}}$. The conclusion then follows immediately.

— If m is an of course link, the edge a' of R' is also the conclusion of an of course link m'. Let a_1 (respectively, a'_1) be the premise of m in R (respectively, of m' in R'). The edges a_1 and a'_1 have the same type, and, by definition, $G^R_a = G^R_{a_1} \cup \{m\} \cup \{a\}$ and $G^{R'}_{a'} = G^{R'}_{a'_1} \cup \{m'\} \cup \{a'\}$. We have that $s(G^R_{a_1}) < s(G^R_a)$. Let x be the unique element of $e(a) = e'(a')$ (following Proposition 3.6). By the definition of n-obsessional experiment, there exist $x_1, \ldots, x_n \in e(a_1)$ (respectively, $x'_1, \ldots, x'_n \in e'(a'_1)$) such that $x = \{x_1, \ldots, x_n\} = \{x'_1, \ldots, x'_n\}$. Proposition 3.6 then gives $x_1 = \ldots = x_n = x'_1 = \ldots = x'_n$ that is, the unique element of $e(a_1)$ is also the unique element of $e'(a'_1)$. Because $e(a) = e'(a')$, the edges a and a' have the same depth, so a_1 and a'_1 also have the same depth. Thus $e(a_1) = e'(a'_1)$. The induction hypothesis gives $G^R_{a_1} = G^{R'}_{a'_1}$ and $e|_{G^R_{a_1}} = e'|_{G^{R'}_{a'_1}}$, which allows us to conclude.

— If m is a dereliction link of R, then $A = ?C$, and, by Lemma 3.20, the unique element of $e(a) = e'(a')$ is a singleton. By that same lemma, a' is also the conclusion of a dereliction link m'. Let a_1 (respectively, a'_1) be the premise of m in R (respectively, of m' in R'). The edges a_1 and a'_1 have the same type, and, by definition, $G^R_a = G^R_{a_1} \cup \{m\} \cup \{a\}$ and $G^{R'}_{a'} = G^{R'}_{a'_1} \cup \{m'\} \cup \{a'\}$. We have that $s(G^R_{a_1}) < s(G^R_a)$. Let x be the unique element of $e(a) = e'(a')$ (following Proposition 3.6). By the definition of experiment, there exists $y_1 \in e(a_1)$ (respectively, $y'_1 \in e'(a'_1)$) such that $x = \{y_1\} = \{y'_1\}$. Then $y_1 = y'_1$, that is, the unique element of $e(a_1)$ is also the unique element of $e'(a'_1)$. Because $e(a) = e'(a')$, the edges a and a' have the same depth, so a_1 and a'_1 also have the same depth. Thus $e(a_1) = e'(a'_1)$. The induction hypothesis gives $G^R_{a_1} = G^{R'}_{a'_1}$ and $e|_{G^R_{a_1}} = e'|_{G^{R'}_{a'_1}}$, which allows us to conclude.

— If m is a pax link of R, let x be the unique element of $e(a)$ (following Proposition 3.6). By Lemma 3.20, there exists $p \geqslant 1$ such that $card(x) = n^p$. By the same lemma, because $e(a) = e'(a')$, a' is necessarily the conclusion of a pax link m' of R'. Let a_1 (respectively, a'_1) be the premise of m in R (respectively, of m' in R'). The edges a_1 and a'_1 have the same type, and, by definition, $G^R_a = G^R_{a_1} \cup \{m\} \cup \{a\}$ and $G^{R'}_{a'} = G^{R'}_{a'_1} \cup \{m'\} \cup \{a'\}$. We have that $s(G^R_{a_1}) < s(G^R_a)$. By the definition of an n-obsessional experiment, there exists $x_1, \ldots, x_n \in e(a_1)$ (respectively, $x'_1, \ldots, x'_n \in e'(a'_1)$) such that $x = x_1 \cup \ldots \cup x_n = x'_1 \cup \ldots \cup x'_n$. Proposition 3.6 then gives $x_1 = \ldots = x_n = x'_1 = \ldots = x'_n$, that is, the unique element of $e(a_1)$ is also the unique element of $e'(a'_1)$. Because $e(a) = e'(a')$, the edges a and a' have the same depth, so a_1 and a'_1 also have the same depth. Thus $e(a_1) = e'(a'_1)$. The induction hypothesis gives $G^R_{a_1} = G^{R'}_{a'_1}$ and $e|_{G^R_{a_1}} = e'|_{G^{R'}_{a'_1}}$, which allows us to conclude.

— If m is a weakening link, then, by Lemma 3.20, the unique element of $e(a) = e'(a')$ is $\emptyset$. By the same lemma, because $e(a) = e'(a')$, a' is necessarily the conclusion of a weakening link, and we are done.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

— If m is a contraction link with arity $k \geqslant 2$, let t be the unique element of $e(a) = e'(a')$. By lemma 3.20, there exist $p_1, \ldots, p_k$ non-negative integers such that $card(t) = n^{p_1} + \ldots + n^{p_k}$. By the same lemma, because $e(a) = e'(a')$, a' is necessarily the conclusion of a contraction link m' with arity k.

Let $a_1, \ldots, a_k$ (respectively, $a'_1, \ldots, a'_k$) be the hypothesis of SG^R_a (respectively, of $SG^{R'}_{a'}$) and let $p_1, \ldots, p_k$ (respectively, $p'_1, \ldots, p'_k$) be, respectively, their depths in SG^R_a (respectively, in $SG^{R'}_{a'}$). With this notation (which is the same as in Lemma 3.21), we are going to prove that the hypotheses of Lemma 3.21 are satisfied. Let $x \in t$ and $\{p^x_1, \ldots, p^x_{k_x}\} \subseteq \{p_1, \ldots, p_k\}$ (respectively, $\{p'^x_1, \ldots, p'^x_{k'_x}\} \subseteq \{p'_1, \ldots, p'_k\}$) be the multiset of the depths in SG^R_a (respectively, in $SG^{R'}_{a'}$) of the edges $c \in \{a_1, \ldots, a_k\}$ (respectively, of the edges $c' \in \{a'_1, \ldots, a'_k\}$) such that $\{x\}$ is the unique element of $e(c)$ (respectively, of $e'(c')$). Let α be the cardinality of t and let α_x be the number of occurrences of x in t. Because e and e' are two n-obsessional experiments, we have that $\alpha_x = n^{p^x_1} + \ldots + n^{p^x_{k_x}} = n^{p'^x_1} + \ldots + n^{p'^x_{k'_x}}$. Because (when $h(R) \neq 0$ or $h(R') \neq 0$) we have $n > h(R), h(R') \geqslant k_x, k'_x \geqslant 1$, we can apply Lemma 3.18, and obtain $\{p^x_1, \ldots, p^x_{k_x}\} = \{p'^x_1, \ldots, p'^x_{k'_x}\}$. The hypotheses of Lemma 3.21 are then satisfied, and we have that $SG^R_a = SG^{R'}_{a'}$ and $e|_{SG^R_a} = e'|_{SG^{R'}_{a'}}$. In particular, *possibly after a renaming of the hypotheses of* SG^R_a *and of* $SG^{R'}_{a'}$ (remember that G^R_a and $G^{R'}_{a'}$ are equivalence classes of trees), we have that $\forall i \in \{1, \ldots, k\}$ $e(a_i) = e'(a'_i)$, and $s(G^R_{a_i}) < s(G^R_a)$. The induction hypothesis then gives $G^R_{a_i} = G^{R'}_{a'_i}$ and $e|_{G^R_{a_i}} = e'|_{G^{R'}_{a'_i}}$, $\forall i \in \{1, \ldots, k\}$, and we are done. □

Corollary 3.23. Let R and R' be two proof-nets with conclusion Γ, let $h(R)$ (respectively, $h(R')$) be the $?co$-size of R (respectively, R'), and let $n > max(h(R), h(R'))$. Let e (respectively, e') be an n-obsessional experiment of R (respectively, of R') with conclusion γ (respectively, γ'). If $\gamma = \gamma'$, then $OLPS(R) = OLPS(R')$ and $e|_{OLPS(R)} = e'|_{OLPS(R')}$.

We have just proved the analogue (for $MELL$) of Lemma 2.24. So let us try to conclude that when $[\![R]\!] = [\![R']\!]$, we have $OLPS(R) = OLPS(R')$: let e be an n-obsessional experiment of R (**Question: does it exist?**) with result γ. Because $[\![R]\!] = [\![R']\!]$, there exists an experiment e' of R' with result γ, which, by Proposition 3.15, is n-obsessional. If the answer to the question is positive, we can apply Corollary 3.23: $OLPS(R) = OLPS(R')$.

In order to answer this question, we introduce the notion of 'simple experiment'. We also generalise the notion of injective experiment to $MELL$ (which will be used later in the paper).

Definition 3.24. Let R be a proof-net and let e be an n-obsessional experiment of R. We will say that e is **injective** (respectively, **simple**) when $\forall a, a'$ edges of the same atomic type of R such that $a \neq a'$, the unique element of $e(a)$ is different from (respectively, equal to) the unique element of $e(a')$.

The existence of an n-obsessional experiment has to be proved. Indeed, even for n-obsessional experiments, even for standard proof-nets, we cannot say '*a priori*' that the labels of the premises of a contraction link satisfy the condition of Remark 2.2 (which is required by Definition 2.1), as already mentioned in Remark 3.8.

We are going to prove that a simple n-obsessional experiment does always exist, for every (standard) $MELL$ proof-net (Proposition 3.26).

Notation. From now on, if e is an n-obsessional experiment of the proof-net R and a is an edge of R, we will use $|e(a)|$ to denote the unique element of $e(a)$. We have not introduced this notation before, because of the possible confusion with the notion of projection (Definition 2.11), which will only be used rarely later in the paper (and it will be clear from the context which of the two notions the notation refers to).

Lemma 3.25. Let R be a proof-net, let a and a' be two edges of R of type A. Let e be an n-obsessional experiment of R ($n \geqslant 1$).
If $\forall \alpha \in G_a$, and $\forall \alpha' \in G_{a'}$, with α and α' of the same atomic type, we have $|e(\alpha)| = |e(\alpha')|$, then we have also $|e(a)| \smile\!\!\!\!\frown |e(a')|(\mathscr{A})$.

Proof. We use induction on $p = s(G_a) + s(G_{a'})$, where for every edge b of R, $s(G_b) =$ number of links of G_b. The details are left to the reader. □

Proposition 3.26. Let R be a proof-net. If n is a strictly positive integer, there exists a simple n-obsessional experiment of R.

Proof. Let $\mathscr{X}$ be a coherent space and $x \in |\mathscr{X}|$. If we interpret every atomic formula of R by the coherent space $\mathscr{X}$ and associate with every axiom link of R the label x, the previous lemma shows that we can always perform a contraction between two different edges of the same type.

More formally, we can proceed by induction on a sequentialisation of R, the only significant case being the one of the terminal contraction link, for which we make use of the previous lemma. □

Theorem 3.27. Let R and R' be two proof-nets with the same conclusions. If $[\![R]\!] = [\![R']\!]$, then $OLPS(R) = OLPS(R')$.

Proof. Let $h(R)$ (respectively, $h(R')$) be the $?co$-size of R (respectively, R'), and let $n > max(h(R), h(R'))$. Let e_n be a simple n-obsessional experiment of R (which exists by Proposition 3.26). Because $[\![R]\!] = [\![R']\!]$, there exists an experiment e'_n of R' with the same result as e_n. By Proposition 3.15, e'_n is an n-obsessional experiment of R'. From Corollary 3.23, $OLPS(R) = OLPS(R')$. □

3.4. *Local injectivity*

We give, in this section, a sufficient condition of 'local injectivity' (Theorem 3.35): if there exists an injective 1-experiment of a given proof-net, it is 'alone' in its (semantic) equivalence class.

We begin by trying to push further the similarity between the $MELL$ case and the MLL one, and try to argue in the same way as for Theorem 2.26: let $h(R)$ (respectively, $h(R')$) be the $?co$-size of R (respectively, R'), and let $n > max(h(R), h(R'))$; let e be an injective n-obsessional experiment of R (**Question: does it exist?**) with result γ. Because

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

$[\![R]\!] = [\![R']\!]$, there exists an experiment e' of R' with result γ, which, by Proposition 3.15, is n-obsessional.

Provided the answer to the question is positive, we can apply Corollary 3.23 to get $OLPS(R) = OLPS(R')$ and $e|_{OLPS(R)} = e'|_{OLPS(R')}$. The injectivity of e (then of e') allows us to conclude in that case that $LPS(R) = LPS(R')$, and we shall see that *in the absence of weakenings* this implies that $R = R'$.

Let R be a proof-net. If we associate with every axiom link l with conclusions α_l and $\alpha_l^\perp$ of R an element x_l of the web of the coherent space $\mathscr{A}$ in such a way that if $l \neq l'$, then $x_l \neq x_{l'}$, does there exist an n-obsessional experiment e_n of R such that for every axiom link l we have $|e_n(\alpha_l)| = |e_n(\alpha_l^\perp)| = x_l$? The definition of experiment (Definition 2.1, see also Remark 2.2) clearly shows that, in the absence of cut links, the constrained labels are the ones of the conclusions of the of course, pax and $?co$ links.

We show that if e_n exists for $n = 1$ (that is, if there exists an injective 1-experiment of the proof-net R), then e_n exists for every $n > h(R)$ (that is, there also exists an injective n-obsessional experiment of R, for every 'big enough' n): e_n is the n-obsessional experiment 'induced' by the 1-experiment e_1 (Proposition 3.33). The existence of such an experiment e_1 of R is then proved to be enough to conclude that R is 'alone' in its (semantic) equivalence class (Theorem 3.35).

In the rest of the paper our key question will be whether or not, for a given proof-net (or a given set of proof-nets), there exists an injective 1-experiment. Sections 4 and 5 give the answer for some particular fragments of $MELL$: when it is positive, we obtain a positive answer to our original problem (Problem 2.16), when it is negative, we can build counter-examples, thus answering negatively to Problem 2.16.

Convention. Let e be a 1-experiment of the proof-net R. For every edge a of type A of R, $e(a) = \{x\}$ for some $x \in |\mathscr{A}|$ (from Proposition 3.6).

In the rest of the paper, we will identify the multiset (of cardinality 1) $e(a)$ with its unique element x.

Definition 3.28. Let R be a proof-net and e_1 be an injective 1-experiment of R. We say that e_n is an n-obsessional experiment induced by e_1, if e_n is an n-obsessional experiment of R such that for every atomic edge a of R $|e_n(a)| = e_1(a)$.

Remark 3.29.

(i) *A priori* the existence of an n-obsessional experiment induced by a given injective 1-experiment is not obvious. But if it exists, it is unique (by Lemma 3.4 and Proposition 3.6) and obviously injective.

(ii) You may be tempted to think that if e_n is the n-obsessional experiment induced by the injective 1-experiment e_1 of R, then for every edge a, we have $|e_n(a)| = e_1(a)$. However, you should resist the temptation, because this is wrong (in general): suppose that $|e_n(c')| = e_1(c')$, where c' is the premise of an of course link of R with conclusion c; we have $|e_n(c)| = \{n[|e_n(c')|]\}$, and $e_1(c) = \{e_1(c')\}$.

The proofs of the following lemmas are simple applications of the definition of coherence in the spaces interpreting LL formulas (see Definition A.1). We give an example by proving the first, and leave the rest as an exercise for the reader.

Lemma 3.30. Let R be a proof-net containing at least one box, let $h(R)$ be the $?co$-size of R, and let $n > h(R)$. Let a and a' be two different edges of R of the same type A. Let e_1 be an injective 1-experiment of R and let e_n be the n-obsessional experiment of R induced par e_1. Then we have:

(1) If there exists an edge α of atomic type such that $\alpha \in G_a$ or $\alpha \in G_{a'}$, then $|e_n(a)| \neq |e_n(a')|$

(2) Otherwise, either $G_a = G_{a'}$ and then $|e_n(a)| = |e_n(a')|$, or $G_a \neq G_{a'}$ and then $|e_n(a)| \smile |e_n(a')|$.

Proof.

(1) Let $x = |e_n(a)|$ (respectively, $x' = |e_n(a')|$), and suppose, for example, that $\alpha \in G_a$ is of type X.
If $a' \in G_a$ or $a \in G_{a'}$, then, necessarily, $A = ?B$ (remember that a and a' have the same type), and because $n > 1$, we have $x \neq x'$.
Otherwise, $G_a \cap G_{a'} = \varnothing$, and thus $\alpha \notin G_{a'}$. For every atomic edge $\beta \in G_{a'}$ of type X $|e_n(\beta)| \neq |e_n(\alpha)|$ (by definition of injective experiment). By Lemma 3.9, we then have $|e_n(\alpha)| \in |x|_X$ and $|e_n(\alpha)| \notin |x'|_X$. Thus $x \neq x'$.

(2) If such an edge α does not exist, none of the edges of G_a or $G_{a'}$ is an edge of atomic type. Therefore, in G_a, and also in $G_{a'}$, there is at least one weakening link. And every 'leaf' of G_a and $G_{a'}$ is the conclusion of a weakening link.
If $G_a = G_{a'}$, then (because e_n is n-obsessional) $e_n(a) = e_n(a')$ (and, in particular, $|e_n(a)| = |e_n(a')|$).
If $G_a \neq G_{a'}$, we proceed by induction on $k = s(G_a) + s(G_{a'})$, the sum of the number of links of G_a and of $G_{a'}$.
If $k = 2$, then a and a' are conclusions of two weakening links and $G_a = G_{a'}$.
Otherwise, if $A = B \otimes C$ (respectively, $A = B \wp C$), let b and b' be the premises of type B and let c and c' be the premises of type C of the $\otimes$ (respectively, $\wp$) links having a and a' as conclusions. By definition, $|e_n(a)| = (|e_n(b)|, |e_n(c)|)$ and $|e_n(a')| = (|e_n(b')|, |e_n(c')|)$. We have $G_b \neq G_{b'}$ and/or $G_c \neq G_{c'}$. Suppose, for example, that $G_b \neq G_{b'}$: by the induction hypothesis, we then have $|e_n(b)| \smile |e_n(b')|$. In the $\otimes$ case this is enough to conclude that $|e_n(a)| \smile |e_n(a')|$. In the $\wp$ case, observe that if $G_c = G_{c'}$, we have already proved that $|e_n(c)| = |e_n(c')|$, while if $G_c \neq G_{c'}$, by the induction hypothesis, $|e_n(c)| \smile |e_n(c')|$: whatever happens we indeed have that $|e_n(a)| \smile |e_n(a')|$.
If $A = !B$, the conclusion is a straightforward application of the induction hypothesis.
If $A = ?B$, let m (respectively, m') be the link having a (respectively, a') as conclusion. Suppose, for example, that $s(G_a) \geqslant s(G_{a'})$, and consider first the case where $a' \in G_a$. We have $|e_n(a')| \subseteq |e_n(a)|$, so $|e_n(a')| \cup |e_n(a)|$ is a clique of $|?\mathscr{B}| = \mathscr{B}^{\perp}$: thus $|e_n(a')| \asymp |e_n(a)| (?\mathscr{B})$. Because $a \neq a'$ and $n > 1$, we have $|e_n(a)| \neq |e_n(a')|$. Thus $|e_n(a')| \smile |e_n(a)|$.
We can now turn our attention to the case $a' \notin G_a$, that is, $G_a \cap G_{a'} = \varnothing$. If m or m' (but not both!) is a weakening link, then $|e_n(a')| \smile \varnothing$ or $|e_n(a)| \smile \varnothing$: we can then suppose that m and m' are $?de$, pax or $?co$ links.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

Now let $a_1, \ldots, a_l$ (respectively, $a'_1, \ldots, a'_q$) be the premises of the l (respectively, q) dereliction links above a (respectively, a'). Observe that because $G_a \cap G_{a'} = \emptyset$, we have $a_i \neq a'_j$ $\forall i \in \{1, \ldots, l\}$ and $\forall j \in \{1, \ldots, q\}$.

There is no edge of atomic type in either G_{a_i} or $G_{a'_j}$, $\forall i \in \{1, \ldots, l\}$ and $\forall j \in \{1, \ldots, q\}$ (because there is no such edge in either G_a or $G_{a'}$). $\forall i \in \{1, \ldots, l\}$ and $\forall j \in \{1, \ldots, q\}$ we then have two possibilities for G_{a_i} and $G_{a'_j}$: either $G_{a_i} = G_{a'_j}$, and thus $|e_n(a_i)| = |e_n(a'_j)|$, or $G_{a_i} \neq G_{a'_j}$, and thus (by the induction hypothesis) $|e_n(a_i)| \smile |e_n(a'_j)|$. In any case, we will then have $|e_n(a')| \stackrel{\smile}{\frown} |e_n(a)|$.

If $SG_a = SG_{a'}$ (see Definition 3.16 for the notation), then (because $G_a \neq G_{a'}$) there necessarily exists $i \in \{1, \ldots, l\}$ such that $\forall j \in \{1, \ldots, q\}$ $G_{a_i} \neq G_{a'_j}$, or there exists $j \in \{1, \ldots, q\}$ such that $\forall i \in \{1, \ldots, l\}$ $G_{a_i} \neq G_{a'_j}$: in both cases the induction hypothesis gives $|e_n(a)| \neq |e_n(a')|$.

If $SG_a \neq SG_{a'}$, by Lemmas 3.20 and 3.18, we have $card(|e_n(a)|) \neq card(|e_n(a')|)$, and thus $|e_n(a)| \neq |e_n(a')|$. (Notice that our use of the hypothesis $n > h(R)$ is crucial here: if we omit it, the lemma is wrong). □

Lemma 3.31. Let a and a' be two different edges of the same type A of the proof-net R, and let e_1 be an injective 1-experiment of R.

(1) If $a \in G_{a'}$ or $a' \in G_a$, either one of a and a' is the conclusion of a $?co$ link and then $e_1(a) \neq e_1(a')$, or it is not the case and then $e(a) = e(a')$.

(2) If $G_a \cap G_{a'} = \emptyset$ and there exists an edge α of atomic type such that $\alpha \in G_a$ or $\alpha \in G_{a'}$, then $e_1(a) \neq e_1(a')$.

(3) If $G_a \cap G_{a'} = \emptyset$ and there exists no edge of atomic type α such that $\alpha \in G_a$ or $\alpha \in G_{a'}$, then $e_1(a) \stackrel{\smile}{\frown} e_1(a')$.

Lemma 3.32. Let R be a proof-net containing at least one box, let $h(R)$ be the $?co$-size of R, and let $n > h(R)$. Let a and a' be two different edges of R of the same type A. Let e_1 be an injective 1-experiment of R and let e_n be the n-obsessional experiment of R induced by e_1.

If $e_1(a) \smile e_1(a')$, then $|e_n(a)| \smile |e_n(a')|$.

Proof. The proof is by induction on $k = s(G_a) + s(G_{a'})$, the sum of the number of links of G_a and of $G_{a'}$. □

Proposition 3.33. Let R be a proof-net, let $h(R)$ be the $?co$-size of R, and let $n > h(R)$.

If there exists an injective 1-experiment of R, there also exists an injective n-obsessional experiment of R: it is the (unique) n-obsessional experiment of R induced by e_1.

Proof. If R contains no boxes, $h(R) = 0$ and (remember the convention of Definition 3.19) $n = 1$: in this case the result is obvious with $e_n = e_1$.

Otherwise, we proceed by induction on a sequentialisation π of R. Of course, we consider a slightly modified sequent calculus, the contraction rule of which is a k-ary rule (with $k \geqslant 2$ active formulas in the sequent premise of the rule).

The most significant case is when the last rule r_m of π is a contraction with arity k. In this case, let π_1 be the subproof of π obtained by erasing r_m, and its conclusion, and let R_1 be the subproof-net of R obtained by erasing the last contraction link m of arity k

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

corresponding to r_m. The proof π_1 is a sequentialisation of R_1, and the restriction e_1^1 of the injective 1-experiment e_1 of R to R_1 is an injective 1-experiment of R_1. Let $a_1, \ldots, a_k$ be the premises of m and a be its conclusion. We have $e_1^1(a_i) \asymp e_1^1(a_j)$ $\forall i, j \in \{1, \ldots, k\}$. Let e_n^1 be the n-obsessional experiment of R_1 given by the induction hypothesis. Notice that because $\forall i \in \{1, \ldots, k\}$ a_i has depth 0 in R_1, $e_n^1(a_i)$ is a singleton.

If for some $i, j \in \{1, \ldots, k\}$ we have $e_1^1(a_i) = e_1^1(a_j)$, then (because a_i and a_j are premises of the same $?co$ link of R, we have $G_{a_i} \cap G_{a_j} = \varnothing$), by Lemma 3.31, there is no atomic edge in either G_{a_i} or G_{a_j}: in this case Lemma 3.30 allows us to conclude that $|e_n^1(a_i)| \asymp |e_n^1(a_j)|$.

$\forall i, j \in \{1, \ldots, k\}$ we have two possibilities: either $e_1^1(a_i) \smile e_1^1(a_j)$, and thus, using Lemma 3.32, $|e_n^1(a_i)| \smile |e_n^1(a_j)|$, or $e_1^1(a_i) = e_1^1(a_j)$, and thus we have just proved that $|e_n^1(a_i)| \asymp |e_n^1(a_j)|$. In either case we have $|e_n^1(a_i)| \asymp |e_n^1(a_j)|$ $\forall i, j \in \{1, \ldots, k\}$. We can then define the experiment e_n of R such that $\forall a'$ edge of R, if $a' \neq a$, then $e_n(a') = e_n^1(a')$, and $e_n(a) = \{|e_n^1(a_1)| \cup \ldots \cup |e_n^1(a_k)|\}$.

Let us also mention the case of a last promotion rule for π. We argue as before: the experiment e_n is obtained from the experiment e_n^1 given by the induction hypothesis by 'repeating n times' e_n^1, following the definition of n-obsessional experiment (see Remark 3.2). □

We now prove the following (very simple) characterisation of exponential boxes in the absence of weakenings.

Proposition 3.34. Let R be a proof-net that contains no weakening links, and R' be a proof-net with the same conclusions as R.

If $LPS(R) = LPS(R')$, then $R = R'$.

Proof. We will use the following characterisation of boxes:

Let l (respectively, m) be an of course or a pax link with depth p in the proof-net T (T contains no weakening links). The links l and m are two doors of the same box if and only if there exists an oriented path Φ, which is *not necessarily straight* (see Definition A.9), having l as starting link and m as terminal link, and such that *every* link of Φ different from l and m has depth (strictly) greater than p.

The conclusion then follows from the fact that the paths of $LPS(T)$ are exactly the paths of T for every proof-net T. □

The following theorem states that a given proof-net (without weakenings) is 'alone' in its (semantic) equivalence class, provided there exists an injective 1-obsessional experiment for it. This can be seen as a condition of 'local injectivity'.

Theorem 3.35. Let R be a proof-net, and suppose there exists an injective 1-experiment of R.

Then, for every proof-net R' with the same conclusions as R, from $[\![R]\!] = [\![R']\!]$ it follows that $LPS(R) = LPS(R')$. Moreover, if R contains no weakening links, $R = R'$.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

Proof. Let R' be a proof-net with the same conclusions as R, let $h(R)$ (respectively, $h(R')$) be the $?co$-size of R (respectively, R'), and let $n > max(h(R), h(R'))$.

By Proposition 3.33, there exists an injective n-obsessional experiment of R. Let e_n be this experiment and γ be its result. Because $[\![R]\!] = [\![R']\!]$, by Proposition 3.15, there exists an n-obsessional experiment e'_n of R' with result γ. Corollary 3.23 then gives $OLPS(R) = OLPS(R')$ and $e_n|_{OLPS(R)} = e'_n|_{OLPS(R')}$. The injectivity of e_n then allows us to conclude that $LPS(R) = LPS(R')$. When R contains no weakening links, by Proposition 3.34, we have $R = R'$. □

Remark 3.36. The following (delicate) point may be of interest. One might think that if for a given fragment F of $MELL$ (without weakenings) the answer to our new question (does there exist an injective 1-experiment of every proof-net of F?) were positive, the analogue of Remark 2.27 would apply (which, in the presence of exponentials, would be more striking). But this is not precisely the case: indeed, suppose the answer is positive for F, let R be a proof-net of F, e_1 be an injective 1-experiment of R, $n > h(R)$, and e_n be the injective n-obsessional experiment of R induced by e_1.

If R' is a proof-net, different from R, with the same conclusions as R and such that $h(R') > n$, the experiment e_n might not be enough to distinguish R from R': we only know of the existence of $m > h(R')$ and of an injective m-obsessional experiment e_m of R such that the result of e_m is not an element of $[\![R']\!]$ (which means that e_m allows us to distinguish R and R').

Contrary to the MLL case, we cannot conclude that for every proof-net of F, there exists an experiment containing *all the information* contained in R.

Remark 3.37. It is rather natural to wonder whether the converse of Theorem 3.35 holds. We cannot exclude the possibility that 'some kind of converse' does, but, for sure, strictly speaking, the answer is negative, as we now show.

Consider the (standard) proof-net R associated with the following sequent calculus proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash X, X^\perp}{\vdash ?X, X^\perp}}{\vdash ?X, ?X^\perp}}{\vdash ?X, ?X^\perp, ?A}\,?W \qquad \dfrac{\dfrac{\dfrac{\vdash X, X^\perp}{\vdash ?X, X^\perp}}{\vdash ?X, ?X^\perp}}{\vdash ?X, ?X^\perp, ?B}\,?W}{\vdash ?X, ?X, ?A \otimes ?B, ?X^\perp, ?X^\perp}}{\vdash ?X, ?A \otimes ?B, ?X^\perp, ?X^\perp}\,?co}{\vdash ?X, ?A \otimes ?B, ?X^\perp}\,?co$$

There cannot exist any injective experiment of R: if x and y are the two elements of $|\mathscr{X}|$ that the experiment e of R associates with the two axiom links of R, we have $x \asymp y(\mathscr{X})$ and $x \asymp y(\mathscr{X}^\perp)$, that is $x = y$. This is due to the presence of the two contraction links of R. The same phenomenon will be used (in a more subtle way) in Section 5.2 to prove that coherent semantics is not injective for $MELL$.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

However, R is alone in its semantic equivalence class: if R' is a proof-net with the same conclusions as R and such that $[\![R]\!] = [\![R']\!]$, then, by Theorem 3.27, we have $OLPS(R) = OLPS(R')$. Well, in this very special case, this implies $R = R'$.

This example is clearly related to the fact, already mentioned several times, that a proof-net is not a graph but an equivalence class of graphs (Remark A.3).

4. Injective experiments for $(?\wp)$LL

A $(?\wp)$LL formula A is any formula built as follows:

$$A ::= X \mid ?A \wp A \mid A \wp ?A \mid A \wp A \mid A \otimes A \mid !A.$$

An *MELL* proof-net is a $(?\wp)$LL proof-net when the types of its conclusions are all subformulas of $(?\wp)$LL formulas. We use the same conventions for $(?\wp)$LL proof-nets as we have used for *MELL* ones: in particular, every $(?\wp)$LL proof-net will be standard (unless explicitly stated).

This section is devoted to proving that for every proof-net of $(?\wp)$LL there exists an injective 1-experiment (Proposition 4.4). This result will be used in Section 5: it allows us to prove injectivity of coherent semantics for $(?\wp)$LL and (then) for some remarkable fragments of *MELL*.

In outline the proof is as follows. With every $(?\wp)$LL proof-net R we can associate its 'linearised' $L(R)$ (which contains no boxes), and then the set of proof-nets $L(R)^{\wp}$ obtained from the proof-net without boxes $L(R)$ by 'removing' the $\wp$ links of $L(R)$. For every proof-net R of $(?\wp)$LL, there exists a proof-net of $L(R)^{\wp}$ containing only terminal contraction links. We prove that for such a proof-net of $L(R)^{\wp}$ there exists an injective experiment (Proposition 4.9). We then prove that this implies the existence of an injective experiment of $L(R)$ (Lemma 4.24), and that every injective experiment of $L(R)$ yields an injective 1-experiment of R (Proposition 4.26).

4.1. *Definitions and result*

We define two operations on proof-nets: linearisation and par-mutilation. The former associates with every proof-net a proof-net without boxes, and the latter associates with some proof-nets without boxes a set of proof-nets without boxes or $\wp$ links.

We are going to show how these two operations are used to prove the main result of Section 4 (Proposition 4.4).

4.1.1. *Linearisation.* With every proof-net R we can associate in a canonical way a (unique) proof-net without boxes, which will be called **the linearised** of R and will be denoted by $L(R)$: it is the proof-net obtained by erasing all the connections between the doors of the boxes of R and by erasing every of course and pax link of R.

The fact that $L(R)$ is a standard proof-net is obvious. We use L to denote the application that associates with every formula A the formula $L(A)$, which is obtained by erasing every occurrence of the connective '!' in A. Note that the types of the conclusions of R are not (in general) the types of the conclusions of $L(R)$: if the conclusions of R are of type Γ, then the conclusions of $L(R)$ will be of type $L(\Gamma)$ (with the usual convention for this kind of notation).

Remark 4.1.

(i) Let R be a proof-net. There exists a canonical application L that associates with every edge a of type A of R the edge $L(a)$ of type $L(A)$ of $L(R)$. Notice that L is far from being injective if there is at least one box in R.

(ii) Let R be a proof-net. If e_L is an injective experiment of $L(R)$, there exists at most one 1-experiment e of R such that for every atomic edge α of R (and thus of $L(R)$) $e(\alpha) = e_L(\alpha)$. We will say that e is the **delinearised** of e_L. Obviously, if e exists, it is injective.

We now turn our attention to the presence of $\wp$ links. We define a procedure of '$\wp$-mutilation', allowing us to associate with some proof-nets R, a set $L(R)^{\wp}$ (defined in Remark 4.2.(ii)) of proof-nets, obtained from the proof-net without boxes $L(R)$ by 'removing' the $\wp$ links of $L(R)$.

4.1.2. *The procedure of par-mutilation.* Recall that a $?co$ link of a proof-net is terminal when its conclusion is also a conclusion of the proof-net.

Let R be a proof-net without boxes, and let Γ be the sequent conclusion of R. Let $A \in \Gamma$ be such that $C \wp D$ is an *occurrence* of subformula of A, and let a of type A be a conclusion of R.

In order to be able to apply our procedure, we require that for every edge h of type H of G_a^R such that $C \wp D$ is a subformula of H, there exists an edge ξ_h of type $C \wp D$ of G_h^R. Intuitively, we are requiring that there is no weakening above a 'introducing' an occurrence of $C \wp D$: in the particular case of a proof-net R without weakenings, the procedure can always be applied.

Let $a_1, \ldots, a_k$ be the edges of G_a^R of type $C \wp D$ (we will always speak of the *occurrence* of subformula $C \wp D$ of A). Let $n_1, \ldots, n_k$ be the k $\wp$ links with conclusions $a_1, \ldots, a_k$, respectively, let $c_1, \ldots, c_k$ (respectively, $d_1, \ldots, d_k$) be the premises of type C (respectively, D) of $n_1, \ldots, n_k$.

We will use R' for the graph obtained from R as follows (for every $i \in \{1, \ldots, k\}$, we perform the first three operations, and we then apply the fourth to the graph thus obtained, in order to get the graph R'):

1 Erase the link n_i and its conclusion a_i.
2 If a_i is a premise of a link m_i, replace the premise a_i of m_i by d_i.
3 If the formula C is different from $?F$ (for every F), add a dereliction link with premise c_i and conclusion an edge g'_i of type $?C$ (otherwise $g'_i = c_i$ will be of type C).
4 If $k \geqslant 2$, add to the graph obtained after the application of the operations 1-3 to R, a contraction link with premises $g'_1, \ldots, g'_k$ and conclusion the edge g (of type $?C$ or C).

It is easy to see that the graph thus obtained is a standard proof-net whose sequent conclusion is $\Gamma \backslash A, A[D/(C \wp D)], ?C$ (or $\Gamma \backslash A, A[D/(C \wp D)], C$). You simply have to notice that if some edges among the c_i are conclusions of a $?co$ link, then the $?co$ link of R' with conclusion g will have more than k premises†.

By permuting the role played by the edges c_i and d_i, we clearly obtain a proof-net whose sequent conclusion is $\Gamma\backslash A, A[C/(C \wp D)], ?D$ (or $\Gamma\backslash A, A[C/(C \wp D)], D$).

We will say that the proof-net R' is obtained from R by mutilation of a $\wp$ formula.

Remark 4.2.

(i) The procedure of $\wp$-mutilation just described might not make sense in presence of weakening links. Indeed, if $C \wp D$ is an occurrence of the subformula of the type A of the conclusion a of R, if there is an edge $b \in G_a^R$ of type $?B$, such that $C \wp D$ is an occurrence of subformula of $?B$, and if b is the conclusion of a weakening link, then the procedure of $\wp$-mutilation cannot be applied. This is precisely what is avoided by the hypothesis 'for every edge h of type H of G_a^R such that $C \wp D$ is a subformula of H, there exists an edge ξ_h of type $C \wp D$ of G_h^R'.

(ii) By applying the procedure of $\wp$-mutilation several times (provided it is possible) to a proof-net $L(R)$, we might obtain a proof-net R^- containing no $\wp$ links: R^- will only contain atomic axiom links, $?de$, $?co$, $?w$, $\otimes$ links.
Of course (when it exists) R^- is not necessarily unique: we will use $L(R)^{\wp}$ to denote the set of proof-nets without $\wp$ links obtained from $L(R)$ by a sequence of $\wp$-mutilations, following the procedure previously described. If for a proof-net R there exists no sequence of $\wp$-mutilations starting from $L(R)$ and leading to a proof-net without $\wp$ links, then we will have $L(R)^{\wp} = \emptyset$.
It is important to notice that every $?co$ link of a proof-net of $L(R)^{\wp}$ that is not a link of R is a terminal link.

(iii) You may wonder why we distinguished (in the procedure of $\wp$-mutilation) the mutilation of a formula $C \wp D$ from that of a formula $?C \wp D$ (where $C \neq ?F$ for every formula F). This is in order to remain in the subsystem $(?\wp)$LL of *MELL* (more precisely so that we still satisfy the property (P) of Lemma 4.3): the formula $??C$ is not contained in this subsystem.

Convention. We will say from now on that an edge a is a premise of a link n 'up to a link m' when a is a premise of n or when a is a premise of m and the conclusion of m is a premise of n. We will use a similar convention for the terminal edges of a proof-net.

We now show how linearisation and $\wp$-mutilation are used to prove the main result of this section (Proposition 4.4).

Lemma 4.3. Let R be a proof-net without boxes. Consider the following property:

(P) Every edge of type $?F$ (for some formula F of LL) is (up to a $?co$ link) a premise of a $\wp$ link or a terminal edge, and every $\wp$ link has at least a premise that is not of type $?F$ (for every formula F of LL)

If R satisfies (P), there exists an injective experiment of R.

† To be precise, we should also mention that some edges among the c_i might be conclusions of $?w$ links. When $k \geqslant 2$, by the definition of standard proof-net, these links and edges disappear.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

Proof. We use induction on the number k of $\wp$ links of R.

— If $k = 0$, every edge of type $?F$ of R is terminal (up to a contraction link). Let R' be the proof-net obtained from R by removing all the (necessarily terminal) weakening links of R. Every contraction link of R' (if any) is terminal, and there are no $\wp$ links in R': by Proposition 4.9 (which will be proved in Section 4.2), there exists an injective experiment of R'. Then (obviously) there exists an injective experiment of R.

— If $k > 0$, let a be a conclusion of R such that there exists a $\wp$ link in G_a^R. And let n be a $\wp$ link of G_a^R that is (one among) the 'closest' to a. This clearly makes sense (it is easy to measure the distance from a in the tree G_a^R). We claim that for the $\wp$ link n, we can apply the procedure of $\wp$-mutilation described at the start of Section 4.1.2. Indeed, Remark 4.2.(i) does not apply in this case, because the conclusion of every non-terminal weakening link of R is the premise of a $\wp$ link, and n is among the closest to a. We now have to apply the procedure carefully: if (using the notation of Section 4.1.2) one of the two premises of the a_i (for example the c_i) is of type $C = ?E$, we apply the procedure in such a way as to obtain a proof-net with conclusion $\Gamma \backslash A, A[D/C \wp D], C$ (and not $\Gamma \backslash A, A[C/C \wp D], ?D$). To apply the induction hypothesis, we first check that the proof-net thus obtained (we will call it R'') still satisfies (P): this is true, because (by hypothesis) in any case $D \neq ?F$, for every formula F. We then obtain an injective experiment of R'', and by Lemma 4.24 (which will be proved in Section 4.3), this yields an injective experiment of R. □

Proposition 4.4. If R is a $(?\wp)$LL proof-net, there exists an injective 1-experiment of R.

Proof. By the definition of $(?\wp)$LL, the linearised $L(R)$ of R satisfies the hypotheses of Lemma 4.3: so there exists an injective experiment of $L(R)$. To conclude, we simply have to apply Proposition 4.26 (which will be proved in Section 4.3). □

Remark 4.5. Notice that we have implicitly proved (in the proof of Lemma 4.3) that if R is a $(?\wp)$LL proof-net, there always exists a proof-net of $L(R)^{\wp}$ whose $?co$ links are all terminal links.

The following two sections are devoted to proving the three results mentioned at the beginning of Section 4 (Proposition 4.9, Lemma 4.24 and Proposition 4.26), which are used to prove Proposition 4.4.

4.2. *The case of terminal contraction links*

We prove that for every proof-net without boxes, without weakenings, without $\wp$ links, and such that it contains only terminal contraction links (we obviously mean that all its $?co$ links are terminal links), there exists an injective experiment (Proposition 4.9).

You should note the role played in the proof by the connectivity of our proof-nets (see also Remark A.7).

In this section, a proof-net will always be a proof-net without boxes, without weakenings, and without $\wp$ links.

From now on, if α is an edge of atomic type conclusion of the axiom link n of the proof-net R, we will use $\alpha^\perp$ to denote the edge conclusion of n different from α.

Remark 4.6. Let a and a' be two (different) edges of the same type of the proof-net R. Because there are no weakening links in R, case (3) of Lemma 3.31 has to be excluded. Moreover, if $a \in G_{a'}$ or if $a' \in G_a$, then (because there are no boxes in R) necessarily a or a' is the conclusion of a $?co$ link. Lemma 3.31 tells us that for every injective experiment e of R, $e(a) \neq e(a')$. This means that we have only two possibilities for the coherence relation between $e(a)$ and $e(a')$: namely, $e(a) \smile e(a')$ or $e(a) \frown e(a')$.

An injective experiment of a proof-net R then associates with a pair of different edges of the same type one of the two elements of the set $\{\frown, \smile\}$. We will use $e(a, a')$ to denote the value '$\frown$' or '$\smile$' of e on the pair of edges of the same type $\{a, a'\}$. By the definition of experiment (Definition 2.1), the value of the injective experiment e on a pair $\{a, a'\}$ of edges depends only on the value of e on the pairs of *atomic* edges of the same type (actually on the pairs of 'similar' atomic edges, see Definition 4.14) $\{\alpha, \alpha'\}$, where $\alpha \in G_a$ and $\alpha' \in G_{a'}$. The data of an injective experiment of a proof-net R can be seen as the data of a function that associates with every pair of edges of the same atomic type one of the two elements of the set $\{\frown, \smile\}$, with the *only* constraint that if α and α' are two edges of the same atomic type, the value of the function on the pair $\{\alpha, \alpha'\}$ is different from the value of the function on the pair $\{\alpha^\perp, \alpha'^\perp\}$.

Notice also that we can define a would-be experiment e' from an experiment e by modifying the coherence relation between the edges of the same atomic type α and α', with the effect only that we also modify the coherence relation between the edges $\alpha^\perp$ and $\alpha'^\perp$.

Remark 4.7. From now on, we will use the term 'pair' to mean 'ordered pair' (unless we write 'unordered pair'). Actually, the use of unordered pairs would be more suitable for our purposes, but it turns out that handling ordered pairs is easier.

Remark 4.8. Let R be a proof-net. With every axiom link l with conclusions α_l and $\alpha_l^\perp$ we associate an element x_l of the web of the coherent space $\mathscr{A}$ in such a way that if $l \neq l'$, then $x_l \neq x_{l'}$.

We want to know whether there exists an experiment e of R such that for every axiom link l we have $e(\alpha_l) = e(\alpha_l^\perp) = x_l$. In other words, we wonder whether or not the labelling of R's edges induced by the previous assignment of labels to the atomic edges of R is an experiment.

We only have to beware of $?co$ links: if we use e for this labelling of R, e is an experiment of R *if and only if* for every $?co$ link of R with premises $a_1, \ldots, a_k$ we have $e(a_i) \smile e(a_j)$, $\forall i, j \in \{1, \ldots, k\}$.

We want to prove the following proposition.

Proposition 4.9. If R is a proof-net whose $?co$ links are all terminal links, then there exists an injective experiment of R.

Proof. The statement is a consequence of Proposition 4.10. □

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

The following proposition is only apparently stronger than the previous one: you can easily see that they are in fact equivalent. We will prove this second statement at the very end of this section (Section 4.2).

Proposition 4.10. Let R be a proof-net whose $?co$ links are all terminal links. There exists an injective experiment e of R such that, for every pair (a, a') of edges of the same type A and conclusions of R, we have $e(a) \smile e(a')(\mathscr{A})$.

Remark 4.11. If R is a proof-structure satisfying condition (AC) of Definition A.6 but not condition (ACC) of Remark A.7, the previous proposition is wrong (in general). A simple counter-example is given by the proof-structure consisting in two axiom links whose conclusions have the same types.

The failure of Proposition 4.10 in the absence of connectivity will be used in Section 5.2.

Convention. *In the rest of this section* (Section 4.2), every proof-net will contain no contraction links. In other words, our proof-nets will only contain axiom, dereliction and tensor links.

We start by providing a little familiarisation with the objects that are now our proof-nets. From now on, we will constantly switch from a proof-net to its 'tree-like representation', introduced in the following remark.

Remark 4.12 (Tree-like representation). A proof-net with conclusions the edges $a_1, \ldots, a_n$ is made of n 'blocks' (the graphs $G_{a_1}, \ldots, G_{a_n}$), connected by some axiom links (and their conclusions). This case looks like the purely multiplicative one of Section 2.3, except for the fact that in our proof-nets there is a unique correctness graph (see Definition A.5), which is connected: the proof-net itself.

This allows us to represent our proof-nets in an alternative way: every block is a node, and every axiom link with conclusions α and $\alpha^\perp$ is an edge connecting two nodes. If R is a proof-net, we will refer to this representation as the 'tree-like representation' of R, and we will denote it by $R^\star$. The motivation for this way of representing nets lies in the behaviour of the coherence relation with respect to the connectives $\otimes$ and $?$, as explained in Remark 4.16.

Clearly, for every proof-net R, the graph $R^\star$ is a tree (that is, it is acyclic and connected).

If a and b are two conclusion edges of R, if α and $\alpha^\perp$ are two edges of atomic type that are conclusions of the same axiom link such that $\alpha \in G_a$ and $\alpha^\perp \in G_b$, then we will still use a and b to denote the nodes of $R^\star$ corresponding to the blocks G_a and G_b, and we will use $\alpha\alpha^\perp$ (respectively, $\overrightarrow{\alpha\alpha^\perp}$) to denote the unoriented (respectively, oriented) edge of $R^\star$ connecting a and b (respectively, with source a and target b).

If Λ and Λ' are two oriented paths of $R^\star$ such that the last node of Λ is the first node of Λ', then we will use $\Lambda \star \Lambda'$ to denote the oriented path of $R^\star$ having as first (respectively, last) node the first (respectively, the last) node of Λ (respectively, Λ').

Definition 4.13 (Paths of $R^\star$). If a and b are two conclusion edges of R, then there exists a unique unoriented path (which will be denoted by $\Theta_{a,b}$) of $R^\star$ connecting a and b. We will use $\overrightarrow{\Theta_{a,b}}$ (respectively, $\overrightarrow{\Theta_{b,a}}$) to denote the oriented path of $R^\star$ having as first node a (respectively, b) and as last node b (respectively, a).

If a_1 and a_2 are two edges of a proof-net, G_{a_1} and G_{a_2} are now trees (instead of equivalence classes of trees). This implies that when a_1 and a_2 have the same type, there exists an obvious correspondence between G_{a_1} and G_{a_2}. Two edges 'corresponding to each other' are said to be similar. The following (horrible) definition makes this notion more precise.

Definition 4.14 (Similar edges). Let A be a formula containing just the connectives '?' and '$\otimes$', and let A' be any formula obtained from A by changing the name of its propositional variables in such a way that every propositional variable of the language occurs at most once in A'.

Let a_1 and a_2 be two conclusions of type A of the proof-net R. Let G'_{a_1} be the tree obtained from G_{a_1} by changing the types of the atomic edges of G_{a_1} in such a way that now the edge a_1 is of type A'. We consider here that the edges of G'_{a_1} are the same as the ones of G_{a_1}, but with different types. We define G'_{a_2} in the same way.

Let c_1 (respectively, c_2) be an edge of G_{a_1} (respectively, of G_{a_2}). We will say that c_1 and c_2 are similar when, as edges of G'_{a_1} and G'_{a_2}, c_1 and c_2 have the same type.

Remark 4.15. Let a_1 and a_2 be two conclusion edges of the proof-net R. If c_1 (respectively, c_2) is an edge of G_{a_1} (respectively, of G_{a_2}), and if c_1 and c_2 are similar, then a_1 and a_2 are also similar (that is, a_1 and a_2 are two conclusions of the same type).

Remark 4.16. By the definition of the coherence relation in the spaces $\mathscr{A} \otimes \mathscr{B}$ and $?\mathscr{B}$, for every proof-net R and for every injective experiment e of R, we have:

> For every pair (a, a') of conclusions of the same type of R, $e(a) \smile e(a')$ iff there exists a pair $(\alpha, \alpha') \in G_a \times G_{a'}$ of similar edges of atomic type such that $e(\alpha) \smile e(\alpha')$.

Definition 4.17 ((C)-pair). Let R be a proof-net and let a and a' be two conclusions of R of the same type.

We will say that the pair $(\alpha, \alpha') \in G_a \times G_{a'}$ of similar edges of atomic type is a (C)-pair for (a, a') when the path $\Theta_{a,a'}$ of $R^\star$ connecting a and a' contains the edge $\alpha\alpha^\perp$ and/or the edge $\alpha'\alpha'^\perp$.

Remark 4.18. For every pair (a, a') of conclusions of the same type of a proof-net R, there always exists a (C)-pair for (a, a'), and there are at most two.

Notice that when the type of a and a' contains no occurrence of the connective $\otimes$, there will be a unique (C)-pair for (a, a').

Notice also that if (α, α') is the unique (C)-pair for (a, a'), then $(\alpha^\perp, \alpha'^\perp)$ cannot be a (C)-pair (for any pair of conclusions of R). This remark is important: it will be used in the proof of Proposition 4.22.

Definition 4.19 (Pair of similar paths). Let R be a proof-net and let a, a' be two edges of the same type and conclusions of R. Let n be a strictly positive integer. Let Φ

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

(respectively, $\overrightarrow{\Phi'}$) be an oriented path of $\overrightarrow{R^\star}$ with starting node a (respectively, a') whose edges are $\alpha_1\alpha_1^\perp, \ldots, \alpha_n\alpha_n^\perp$ (respectively, $\alpha'_1\alpha'^\perp_1, \ldots, \alpha'_n\alpha'^\perp_n$).

We will say that (Φ, Φ') is a pair of similar paths starting from (a, a') when:

— The edges α_i and α'_i are similar $\forall i \in \{1, \ldots, n\}$.
— The edges $\alpha_i^\perp$ and $\alpha'^\perp_i$ are similar $\forall i \in \{1, \ldots, n-1\}$.

We will say that the pair of similar paths (Φ, Φ') is maximal when the edges $\alpha_n^\perp$ and $\alpha'^\perp_n$ are not similar.

The following proposition is a consequence of the connectivity of the tree $R^\star$, and it is the essential ingredient of the proof of Proposition 4.22.

Proposition 4.20 (Maximal pair of similar paths). Let a and a' be two conclusions of the proof-net R of the same type, and suppose that there exist for (a, a') two (C)-pairs.

There then exists a maximal pair of similar paths (Φ, Φ') starting from (a, a') and such that Φ is a prefix of the oriented path $\overrightarrow{\Theta_{a,a'}} \star \Phi'$.

Proof. Let (α, α') be a (C)-pair for (a, a') and suppose that $\overrightarrow{\alpha\alpha^\perp}$ is the first edge of $\overrightarrow{\Theta_{a,a'}}$. We then call $\overrightarrow{\alpha_n\alpha_n^\perp}$ the last edge of $\overrightarrow{\Theta_{a,a'}}$: because there are two (C)-pairs for (a, a'), we have $\alpha' \neq \alpha_n^\perp$.

Let b (respectively, b') be the edge conclusion of R such that $\alpha^\perp \in G_b$ (respectively, $\alpha'^\perp \in G_{b'}$).

If $\alpha^\perp$ and $\alpha'^\perp$ are not similar, we are done. If they are similar, then b and b' are two conclusions of the same type of R (by Remark 4.15). Because $\alpha' \neq \alpha_n^\perp$, the node b' is not a node of $\overrightarrow{\Theta_{a,a'}}$ and the edge $\alpha'\alpha'^\perp$ is not an edge of $\overrightarrow{\Theta_{a,a'}}$. More precisely, we have $\overrightarrow{\Theta_{a,a'}} \star \overrightarrow{\alpha'\alpha'^\perp} = \overrightarrow{\alpha\alpha^\perp} \star \overrightarrow{\Theta_{b,b'}}$.

Let $\overrightarrow{\beta\beta^\perp}$ be the first edge of $\overrightarrow{\Theta_{b,b'}}$: we have $\beta \neq \alpha^\perp$. Let $\beta' \in G_{b'}$ be the edge similar to β: we have $\beta' \neq \alpha'^\perp$. The pair (β, β') is a (C)-pair for (b, b'), and there exist two (C)-pairs for (b, b'): (β, β') and $(\alpha^\perp, \alpha'^\perp)$. We are then back to the original situation, this time with the pair (b, b'). The reason why we will have to stop one day, is that $R^\star$ is a tree.

The situation is represented in Figure 1, where the nodes of the tree $R^\star$ $(a, a', b, b', \ldots)$ are (in general) connected with several nodes, but we have only drawn the ones we were concerned with.

It is (very) easy to give a formal version of the previous proof: we can, for example, define a size $|| \cdot ||$ on the oriented paths of $R^\star$ and argue by induction on this size, proving that $|| \overrightarrow{\Theta_{a,a'}} || > || \overrightarrow{\Theta_{b,b'}} ||$. □

We will use the following remark in the proof of the following proposition.

Remark 4.21. We use the notation of the previous proposition. Let c and c' be two conclusions of the same type of R. If $(\gamma, \gamma') \in G_c \times G_{c'}$ is a pair of similar edges of atomic type such that $\overrightarrow{\gamma\gamma^\perp}$ (respectively, $\overrightarrow{\gamma'\gamma'^\perp}$) is an edge of Φ (respectively, Φ'), then (γ, γ') is a (C)-pair for (c, c').

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

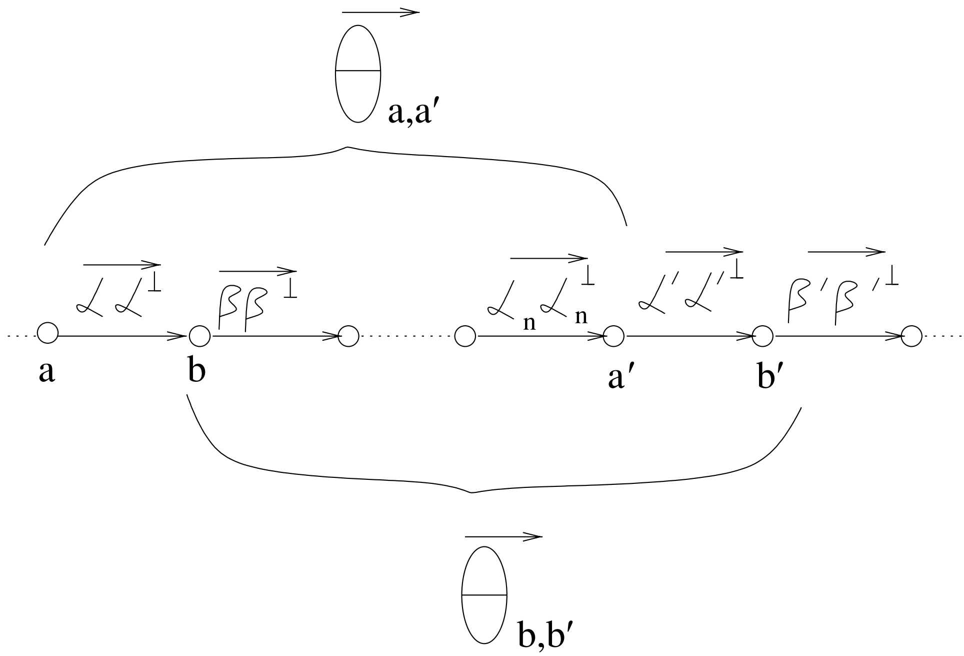

Fig. 1. Construction of the maximal pair of similar paths starting from (a, a').

The following proposition entails Proposition 4.10 (and then Proposition 4.9, which we want to prove). It shows that the solution to our problem for a pair of conclusions of the same type (a, a') is given *precisely* by the pairs of edges 'participating in the connection between a and a'': the (C)-pairs.

Proposition 4.22. Let R be a proof-net. There exists an injective experiment e of R such that:

> For every pair (a, a') of edges of the same type and conclusions of R, there exists a (C)-pair (α, α') for (a, a') satisfying $e(\alpha) \smile e(\alpha')$ (and then $e(a) \smile e(a')$ by Remark 4.16).

Proof. Let h be the number of unordered pairs of edges of the same type and conclusions of the proof-net R. Let e be any injective experiment of R (which obviously exists because there are no $?co$ links in R). Let k_e be the number of unordered pairs of conclusion edges of the same type $\{c, c'\}$ of R such that there exists a (C)-pair $(\alpha_c, \alpha'_c) \in G_c \times G_{c'}$ such that $e(\alpha_c) \smile e(\alpha'_{c'})$.

We prove, by induction on $h - k_e$, that there exists an injective experiment e' of R satisfying the conclusion of the proposition.

If $h - k_e = 0$, then e is the injective experiment of R we are looking for.

Otherwise, there exists a pair (a, a') of conclusions of the same type of R such that for the (C)-pair(s) $(\alpha_a, \alpha_{a'})$ for (a, a'), we have $e(\alpha_a) \frown e(\alpha'_{a'})$. We now build, starting from e, an injective experiment e' of R such that $k_{e'} > k_e$. We will then conclude by applying the induction hypothesis.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

Fix a (C)-pair (α, α') for (a, a'). If (α, α') is the unique (C)-pair for (a, a'), then (see Remark 4.18) $(\alpha^{\perp}, \alpha'^{\perp})$ cannot be a (C)-pair. In this case it is enough to define the experiment e' as the experiment e except on $\{\alpha, \alpha'\}$ and on $\{\alpha^{\perp}, \alpha'^{\perp}\}$: we define $e'(\alpha) \smile e'(\alpha')$ (and then $e'(\alpha^{\perp}) \frown e'(\alpha'^{\perp})$). We indeed have $k_{e'} = k_e + 1$.

If (α, α') is not the unique (C)-pair for (a, a'), we can apply Proposition 4.20: let (Φ, Φ') be the maximal pair of similar paths starting from (a, a') such that Φ is a prefix of $\overrightarrow{\Theta_{a,a'}} \star \Phi'$.

Suppose that Φ and Φ' contain $k+1$ different edges ($k \geqslant 0$). We call $(b_1, b'_1), \ldots, (b_k, b'_k)$ the pairs of conclusion edges of R and $(\beta_1, \beta'_1), \ldots, (\beta_k, \beta'_k)$ the pairs of atomic edges of R such that:

- The nodes crossed by Φ' are, successively, $a', b'_1, \ldots, b'_k, b'_{k+1}$.
- The nodes crossed by Φ are, successively, $a, b_1, \ldots, b_k, b_{k+1}$.
- $\beta'_i \in G_{b'_i}$ (respectively, $\beta_i \in G_{b_i}$) and $\beta'^{\perp}_i \in G_{b'_{i+1}}$ (respectively, $\beta^{\perp}_i \in G_{b_{i+1}}$), for $i \in \{1, \ldots, k\}$.
- $\forall i \in \{1, \ldots, k\}$ (respectively, $\forall i \in \{1, \ldots, k-1\}$), β'_i and β_i (respectively, $\beta'^{\perp}_i$ and $\beta^{\perp}_i$) are similar, while $\beta'^{\perp}_k$ and $\beta^{\perp}_k$ are not similar.
- $\overrightarrow{\alpha'\alpha'^{\perp}}$ (respectively, $\overrightarrow{\alpha\alpha^{\perp}}$) is the edge of Φ' (respectively, of Φ) connecting a' to b'_1 (respectively, a to b_1).
- $\forall i \in \{1, \ldots, k\}$, $\overrightarrow{\beta'_i\beta'^{\perp}_i}$ (respectively, $\overrightarrow{\beta_i\beta^{\perp}_i}$) is the edge of Φ' (respectively, of Φ) connecting b'_i and b'_{i+1} (respectively, b_i and b_{i+1}).

We then define e' on every unordered pair of atomic edges of the same type $\{\delta, \delta'\}$ of R as follows:

- If $\{\delta, \delta'\} \notin \{\{\alpha, \alpha'\}, \{\alpha^{\perp}, \alpha'^{\perp}\}\} \cup \{\{\beta_i, \beta'_i\}, \{\beta^{\perp}_i, \beta'^{\perp}_i\} : i \in \{1, \ldots, k\}\}$, then $e'(\delta, \delta') = e(\delta, \delta')$.
- $e'(\alpha) \smile e'(\alpha')$ (then $e'(\alpha^{\perp}) \frown e(\alpha'^{\perp})$).
- $\forall i \in \{1, \ldots, k\}$, $e'(\beta_i) \smile e'(\beta'_i)$ (then $e'(\beta^{\perp}_i) \frown e'(\beta'^{\perp}_i)$).

We will now show that $k_{e'} > k_e$.

If $\{c, c'\} \notin \{\{a, a'\}, \{b_1, b'_1\}, \ldots, \{b_k, b'_k\}\}$, then $\forall \delta \in G_c$ and $\forall \delta' \in G_{c'}$, where δ and δ' are similar atomic edges, we have $e'(\delta, \delta') = e(\delta, \delta')$ (remember that $\beta'^{\perp}_k \in G_{b'_{k+1}}$ and $\beta^{\perp}_k \in G_{b_{k+1}}$ are not similar). This means that if (δ, δ') is a (C)-pair for (c, c') such that $e(\delta) \smile e(\delta')$, it will still be the case for e'.

If $\{c, c'\} \in \{\{b_1, b'_1\}, \ldots, \{b_k, b'_k\}\}$, by Remark 4.21, we know that $\forall i \in \{1, \ldots, k\}$ the pair (β_i, β'_i) is a (C)-pair for (b_i, b'_i), and by the definition of e' we have $e'(\beta_i) \smile e'(\beta'_i)$.

If $(c, c') = (a, a')$, we have by the definition of e' that $e(\alpha) \smile e(\alpha')$ (where (α, α') is a (C)-pair).

In all cases, we clearly have $k_{e'} \geqslant k_e + 1$. □

Proof of Proposition 4.10. Let R be a proof-net whose $?co$ links are all terminal links, and let R' be the subproof-net of R obtained from R by erasing all the (terminal) $?co$ links of R and their conclusions.

By Proposition 4.22, there exists an injective experiment e' of R' satisfying the conclusion of Proposition 4.10. This experiment can be extended straightforwardly into an experiment e of R satisfying the conclusion of Proposition 4.10. □

4.3. *Adding par links and boxes*

We proved in Section 4.1 (Remark 4.5) that we can associate with every proof-net R of $(?\wp)$LL a proof-net of $L(R)^{\wp}$ containing only terminal $?co$ links. The previous section (Section 4.2, in particular, Proposition 4.9) allows us to conclude that for such a net there exists an injective experiment. The point now is to follow the path leading from R to $L(R)^{\wp}$ in the opposite direction, and prove that along this 'reverted' path the property we are interested in (the existence of an injective experiment) is preserved.

In other words, we fill the last two holes in the proof of Proposition 4.4: we show that if there exists an injective experiment of an element of $L(R)^{\wp}$, there exists an injective experiment of $L(R)$ (Lemma 4.24), and that in this last case there also exists an injective 1-experiment of R (Proposition 4.26).

Let R be a proof-net and $L(R)$ its linearised. Remember that because $L(R)$ is without boxes, an experiment of $L(R)$ associates with every edge of $L(R)$ a unique label: it is a labelling of the edges of $L(R)$ (as in the multiplicative case).

Remark 4.23. If R is a proof-net without boxes, we can associate with every axiom link l with conclusions α_l and $\alpha_l^{\perp}$ an element x_l of the web of the coherent space $\mathscr{A}$ in such a way that if $l \neq l'$, then $x_l \neq x_{l'}$.

We can extend Remark 4.8, but not straightforwardly, due to the possible presence in R of some weakening links. The labelling e of R's edges induced by the previous assignment of labels to the atomic edges of R is an experiment of R *if and only if* for every $?co$ link of R with premises $b_1, \dots, b_h$, we have $e(b_i) \asymp e(b_j)$, $\forall i, j \in \{1, \dots, h\}$.

Lemma 4.24. Let R be a proof-net without boxes, and let R' be a proof-net obtained from R by mutilation of a $\wp$ formula.

If there exists an injective experiment e' of R', there exists an injective experiment e of R.

Proof. We use the same notation in the proof as in the definition of the $\wp$-mutilation procedure (described at the start of Section 4.1.2).

In particular, we will suppose that the sequent conclusion of R (respectively, R') is Γ (respectively, $\Gamma\backslash A, A[D/(C \wp D)], ?C$, or $\Gamma\backslash A, A[D/(C \wp D)], C$). We will also assume that none of the c_i is the conclusion of a $?w$ link (remember the footnote in the definition of the procedure of $\wp$-mutilation), and leave it to the reader to extend the proof to that case.

Every edge b of R different from $a_1, \dots, a_k$ is an edge of R', which will be denoted by b'. Notice that the edges b and b' are not necessarily of the same type: if B is the type of the edge b' of R', then the edge b of R is either of type B or of type $B[(C \wp D)/D]$.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

We are going to show that the experiment e of R that we are looking for is nothing but 'the experiment e' defined on the edges of R in the only possible way'.

To define correctly the experiment e, we introduce some terminology that will only be used in the present proof. We will say that an edge b of R such that $b \notin \{a_1, \ldots, a_k\}$ is 'special', when there exists a path (of course, a straight path following Definition A.9) containing b, starting from an edge among the a_i ($i \in \{1, \ldots, k\}$) and going downwards to a conclusion of R. (Actually, this conclusion is always the edge a of the procedure described in Section 4.1.2).

Notice that every special edge b of R is also an edge b' of R', but having a type different from the type of b.

For every edge b of R:

— If b is not special and $b \notin \{a_1, \ldots, a_k\}$, we define $e(b) = e'(b')$.
— If $b = a_i$ for some $i \in \{1, \ldots, k\}$, we define $e(a_i) = (e'(c'_i), e'(d'_i))$.
— If b is a special edge, the definition of $e(b)$ follows from the previous ones (by Definition 2.1).

What we have to prove is that the labelling e of R thus defined is indeed an experiment of R. By Remark 4.23, the only possibility for our labelling not to be an experiment is the presence of a contraction link m of R having as conclusion the edge q and as premises the edges $q_1, \ldots, q_h$, such that $e(q_l) \frown e(q_s)$ for some $l, s \in \{1, \ldots, h\}$. If none of the edges $q_1, \ldots, q_h$ is special, then, because for every $l, s \in \{1, \ldots, h\}$ we have $e'(q'_l) \asymp e'(q'_s)$, we also have (by the definition of e), $e(q_l) \asymp e(q_s)$. Otherwise, *all* the edges $q_1, \ldots, q_h$ (so as q) are special: we then simply have to show that in this case $\forall l, s \in \{1, \ldots, h\}$ we have $e(q_l) \asymp e(q_s)$. Notice that the edges $q'_1, \ldots, q'_h$ are premises of a $?co$ link of R'.

We are going to prove that $\forall i, j \in \{1, \ldots, k\}$, we have the following property:

($\star$.1) If $e'(d'_i) \smile e'(d'_j)$, then $e(a_i) \smile e(a_j)$.
($\star$.2) If $e'(d'_i) = e'(d'_j)$, then $e(a_i) \asymp e(a_j)$.

We first show that this implies that $\forall l, s \in \{1, \ldots, h\}$ we have $e(q_l) \asymp e(q_s)$ (and thus we are done, because the only problematic case, mentioned above, cannot occur).

We fix two distinct edges t_1 and t_2 of the same type T, both special, and such that $G_{t_1} \cap G_{t_2} = \emptyset$. Intuitively, ($\star$) says that if $e'(t'_1) \asymp e'(t'_2)$ and the coherence relation between $e'(d'_i)$ and $e'(d'_j)$ has anything to do with the fact that $e'(t'_1) \asymp e'(t'_2)$, then the coherence relation between $e(a_i)$ and $e(a_j)$ will play the same role, and we will then be able to claim that $e(t_1) \asymp e(t_2)$. More precisely, we prove that:

— If $e'(t'_1) \smile e'(t'_2)$, then $e(t_1) \smile e(t_2)$.
— If $e'(t'_1) = e'(t'_2)$, then $e(t_1) \asymp e(t_2)$.

From $e'(q'_l) \asymp e'(q'_s)$, we will then be able to deduce that $e(q_l) \asymp e(q_s)$.

What we want to prove is rather clear, but the only way we see to give a convincing proof is to argue by induction. For every edge b of R, let $\sharp\Phi_b$ be the number of edges of the path having b as first edge and a conclusion of R as terminal edge. We make a proof by induction on $p = \Sigma_{i=1}^{k} \sharp\Phi_{a_i} - (\sharp\Phi_{t_1} + \sharp\Phi_{t_2})$.

The case $p = 0$ has to be excluded. So let $p > 0$.

We have to check all the possible cases for T, but here we will only consider, as an example, the case $T = U \wp S$. Let u_1 and u_2 be the premises of type U and let s_1 and

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

s_2 be the premises of type S of the links $\wp$ whose conclusions are t_1 and t_2, respectively. By definition, $e(t_1) = (e(u_1), e(s_1))$ and $e(t_2) = (e(u_2), e(s_2))$. Because t_1 and t_2 are special, exactly one of u_1 and s_1 and one of u_2 and s_2 is a special edge of R. Suppose, for example, that u_1 (and then u_2) are special edges. Then s_1 and s_2 are not, and we have that $e(s_1) = e'(s'_1)$ and $e(s_2) = e'(s'_2)$. If u_1 is not one of $a_1, \ldots, a_k$, then neither is u_2. And the result is a straightforward application of the induction hypothesis. If u_1 is one of $a_1, \ldots, a_k$, then u_2 is also one of $a_1, \ldots, a_k$ (and this is the interesting case). In this case, one of the two premises of each of the two $\wp$ links of R' having t'_1 and t'_2 as conclusions is then one of the edges d'_i ($i \in \{1, \ldots, k\}$): the premises of these links are the edges d'_{i_1} and s'_1 (for t'_1) and d'_{i_2} and s'_2 (for t'_2), where, of course, $i_1, i_2 \in \{1, \ldots, k\}$. The coherence relation ($\frown, \smile, =$) between $e(s_1)$ and $e(s_2)$ is then the same as the one between $e'(s'_1)$ and $e'(s'_2)$. The property ($\star$) then allows us to conclude. Indeed, if $e'(t'_1) \smile e'(t'_2)$, then either $e'(d'_{i_1}) \smile e'(d'_{i_2})$ and $e'(s'_1) \asymp e'(s'_2)$, or $e'(d'_{i_1}) \asymp e'(d'_{i_2})$ and $e'(s'_1) \smile e'(s'_2)$: in both cases we have by ($\star$) that $e(t_1) \smile e(t_2)$. While, if $e'(t'_1) = e'(t'_2)$, we have that $e'(d'_{i_1}) = e'(d'_{i_2})$ and $e'(s'_1) = e'(s'_2)$, and then by ($\star$), we have $e(a_{i_1}) \asymp e(a_{i_2})$ and $e(s_1) = e(s_2)$: thus $e(t_1) \asymp e(t_2)$.

To conclude, it is therefore enough to prove ($\star$). Remember that $\forall i \in \{1, \ldots, k\}$ the edges c_i and d_i (premises of the link n_i with conclusion a_i) are not special, which means that $e(c_i) = e'(c'_i)$ and $e(d_i) = e'(d'_i)$.

The property ($\star$) is actually a consequence of the fact that $\forall i, j \in \{1, \ldots, k\}$ the edges c'_i and c'_j of R' are premises (up to a $?de$ link) of a $?co$ link: we have $e'(c'_i) \asymp e'(c'_j)$ and then $e(c_i) \asymp e(c_j)$. Thus, let $i, j \in \{1, \ldots, k\}$. By definition of the coherence relation in the space $\mathscr{C} \wp \mathscr{D}$, we have:

($\star$.1) If $e'(d'_i) \smile e'(d'_j)$, then $e(d_i) \smile e(d_j)$ and $e(c_i) \asymp e(c_j)$: thus $e(a_i) \smile e(a_j)$.
($\star$.2) If $e'(d'_i) = e'(d'_j)$, then $e(d_i) = e(d_j)$ and $e(c_i) \asymp e(c_j)$: thus $e(a_i) \asymp e(a_j)$. □

We have proved that 'reverting' the operation of $\wp$ mutilation preserves the existence of injective experiments. We are going to proceed in a similar way with the operation of linearisation (defined in Section 4.1.1).

Lemma 4.25. Let R be a proof-net, and let a and a' be two different edges of the same type A of R. Let e_L be an injective experiment of $L(R)$ and e the delinearised of e_L (this means that we are supposing the existence of e).

If $e_L(L(a)) \smile e_L(L(a'))(\mathscr{L}(\mathscr{A}))$, then $e(a) \smile e(a')(\mathscr{A})$.

Proof. As in the case of the lemmas of Section 3.4, the proof is a simple application of the definition of coherence in the spaces interpreting LL formulas. We argue, as usual, by induction on $s(G^R_a) + s(G^R_{a'})$, and use Lemma 3.31. The details are left to the reader. □

Proposition 4.26. Let R be a proof-net. If there exists an injective experiment e_L of $L(R)$, there exists an injective 1-experiment e of R: it is the delinearised of e_L.

Proof. As in the proof of Proposition 3.33, we argue by induction on a sequentialisation π of R. The most significant case will again be when the last rule of π is a contraction rule, and we will apply Lemma 4.25 (and Lemma 3.31) in this case. □

5. Positive and negative results

At the beginning of Section 3.4, we replaced our original problem (Problem 2.16) by the question concerning the existence of a 1-injective experiment for a given set of proof-nets.

We gave a positive answer to this question for $(?\wp)$LL in the previous section. We now prove that this yields a positive answer to Problem 2.16 for $(?\wp)$LL (Section 5.1). In Section 5.2, we give a negative answer to the question for *MELL*, and we show how this yields a counter-example allowing us to answer negatively to Problem 2.16.

We end the paper by giving a table summing up the results obtained and some (seemingly) interesting conjectures (Section 5.3).

5.1. *Fragments of injectivity*

We now plug together the results of the previous sections and present the positive outcomes of the paper: the injectivity of multiset-based coherent semantics is proved for $(?\wp)$LL (Theorem 5.2) and for the 'weakly polarised' fragment of LL (Theorem 5.5). This last result yields a proof of the injectivity of the coherent model of the simply typed λ-calculus, with a bound on the cardinality of the model that separates two λ-terms that are not $\beta\eta$-equivalent (Theorem 5.7).

Appendix B guarantees that all the injectivity results just mentioned for multiset-based coherent semantics also hold in the relational case.

For the present section (only) we will forget the conventions we have used up to now for proof-nets.

Let R be an *MELL* proof-net. We use R_0 to denote the standard proof-net associated with the normal form of R (Definition 2.14). We use the notation $\simeq_{\beta\eta}$ introduced in Problem 2.16.

Theorem 5.1. Let R be an *MELL* proof-net such that R_0 contains no weakening links and such that there exists an element of $L(R_0)^{\wp}$ whose $?co$ links are all terminal links.

If R' is a proof-net with the same conclusions as R and if $[\![R]\!] = [\![R']\!]$, then $R \simeq_{\beta\eta} R'$.

Proof. Let $L(R_0)^-$ be the element of $L(R_0)^{\wp}$ that contains only terminal contraction links. By Proposition 4.9, there exists an injective experiment of $L(R_0)^-$. By Lemma 4.24, there exists an injective experiment of $L(R_0)$. By Proposition 4.26, there exists an injective 1-experiment of R_0. Let R'_0 be the standard proof-net associated with R'. By Theorem 3.35, we have $R_0 = R'_0$, that is, $R \simeq_{\beta\eta} R'$. □

Theorem 5.2. The multiset-based coherent semantics is injective for $(?\wp)$LL proof-nets.

Proof. Let R and R' be two semantically equivalent $(?\wp)$LL proof-nets. We know by Proposition 4.4 that there exists an injective 1-experiment of R_0. By Theorem 3.35, we have $LPS(R_0) = LPS(R'_0)$.

Notice now that because R_0 and R'_0 are $(?\wp)$LL (and standard) proof-nets, every $?w$ link of R_0 or R'_0 that is not terminal is the premise of a $\wp$ link; and every $\wp$ link of these two proof-nets has at least one premise that is not the conclusion of a $?w$ link.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

Well, *in this very particular case*, even in the presence of weakenings, the characterisation of boxes given by Proposition 3.34 is still valid for R_0 and R'_0.

We can then deduce $R_0 = R'_0$, that is, $R \simeq_{\beta\eta} R'$. □

We now define the notion of a 'weakly polarised formula', which is related to the idea of a 'polarised formula', which has been widely studied in the last 10 years: see Girard (1991), Danos *et al.* (1997), Quatrini and Tortora de Falco (1996), Tortora de Falco (1997), Laurent (1999), Laurent *et al.* (2000)...

Definition 5.3 (Weakly polarised formulas). A propositional formula P (respectively, N) of LL is weakly positive (respectively, weakly negative) when it is built as follows (where X is an atomic formula):

$$\begin{array}{rcl} P & ::= & X \mid P \otimes P \mid P \otimes !N \mid !N \otimes P \\ N & ::= & X \mid N \wp N \mid ?P \wp N \mid N \wp ?P \end{array}$$

We will say that a formula is weakly polarised when it is weakly positive or weakly negative.

A proof-net of *MELL* is weakly polarised when the types of its conclusions are all subformulas of weakly polarised formulas.

Remark 5.4. The difference between the weakly polarised formulas and the (strongly) polarised ones (coming from Girard (1991)) is that every atomic formula is both weakly positive and weakly negative: we do not assume anything about the atoms for weakly polarised formulas.

In particular, a weakly polarised formula A is not equivalent to either $?A$ or $!A$ (contrary to polarised formulas, see Danos *et al.* (1997) or Tortora de Falco (2000)).

Theorem 5.5. The multiset-based coherent semantics is injective for the set of weakly polarised proof-nets.

Proof. Simply notice that a weakly polarised proof-net is a $(?\wp)$LL proof-net. □

Corollary 5.6. The coherent multiset-based semantics is injective for the intuitionistic fragment ILU of Girard's unified logic (Girard 1993).

Proof. The system ILU is actually the t-fragment of LK^η (which is defined in Danos *et al.* (1997)), and Danos, Joinet and Schellinx proved that for this fragment Girard's translation $A \to B = !A \multimap B$ yields a denotational semantics for ILU. It then suffices to note that this translation uses only weakly polarised formulas, and to apply Theorem 5.5. □

Theorem 5.7. Let t_1 and t_2 be two terms of the simply typed λ-calculus, and let R_0^1 (respectively, R_0^2) be the proof-net associated with t_1 (respectively, t_2) by Girard's translation ($A \to B = !A \multimap B$). Let k_1 (respectively, k_2) be the number of axiom links of R_0^1 (respectively, R_0^2), and $k \geqslant max(k_1, k_2)$.

If t_1 and t_2 are not $\beta\eta$-equivalent, there exists a coherent space $\mathscr{X}$ such that $card(|\mathscr{X}|) = k$, and the model of the simply typed λ-calculus obtained by interpreting every atomic type by the space $\mathscr{X}$ distinguishes t_1 and t_2.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

Proof. The standard proof-net R_0^1 (respectively, R_0^2) is weakly polarised, and thus it is also a $(?\wp)$LL proof-net. By Proposition 4.4, there exists an injective experiment e_1^1 of R_0^1. Now choose $\mathscr{X}$ in such a way that the k elements of the web $|\mathscr{X}|$ satisfy the coherence relations required by the existence of e_1^1, and let $n > max(h(R_0^1), h(R_0^2))$ (remember that $h(R)$ is the $?co$-size of the proof-net R, defined in Definition 3.19). We can then apply Proposition 3.33: there exists (with the chosen interpretation for the atoms) an injective n-obsessional experiment e_n^1 of R_0^1 (the one induced by e_1^1).

Because t_1 and t_2 are not $\beta\eta$-equivalent, we have $R_0^1 \neq R_0^2$, so there cannot be in the model an experiment of R_0^2 with the same result as e_n^1. □

5.2. *Counter-examples*

We prove that there does not exist (in general) an injective experiment for a given proof-net without boxes. This leads immediately to a negative answer to our original question (Problem 2.16), thus corroborating the pertinence of our approach.

We then give another counter-example to the injectivity of coherent semantics for *MELL*, of a (slightly) different nature.

Both our counter-examples also hold in a coherent set-based framework.

5.2.1. *The first counter-example.* In order to prove injectivity for *MELL*, Sections 2–4 (and especially Theorem 3.35 and Proposition 4.26) suggest we try to show the existence of an injective experiment for every *MELL* (standard) proof-net without boxes.

But for both the standard proof-nets of Figure 2, such an experiment does not exit.

The digits that we have associated with the different edges of the proof-nets indicate which are the requests of the contraction links: if with the two edges a and a' of R (respectively, R') is associated the same integer, then every experiment e of R (respectively, R') must satisfy $e(a) \asymp e(a')$.

Let $x, y \in |\mathscr{X}|$ be the labels that the experiment e of, say, R associates with the conclusions of the two axiom links of R. We see very well that we must have on the one hand $x \asymp y(\mathscr{X})$ (by request of the square $?co$ link in the figure) and on the other hand $x \asymp y(\mathscr{X}^{\perp})$ (by request of the triangular $?co$ link in the figure), that is $x = y$. This precisely means that there exists no injective experiment of R.

You have probably noticed that we could find a simpler example of proof-net for which there is no injective experiment (as shown in Remark 3.37), but the reason we chose this one appears clearly if we cross the conclusion edges of the two axiom links: because denotational semantics is unable to distinguish between the two axiom links, by crossing the edges we get a proof-net R', with the same semantics as R but different from R. The previous proof-net was chosen in order to obtain $R \neq R'$ (with R and R' both standard).

To convince yourself that $R \neq R'$, note (for example) that there exists a subproof-net of R that is not a subproof-net of R'. The fact that R and R' are semantically equivalent, both for the set-based and for the multiset-based coherent semantics, is an

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

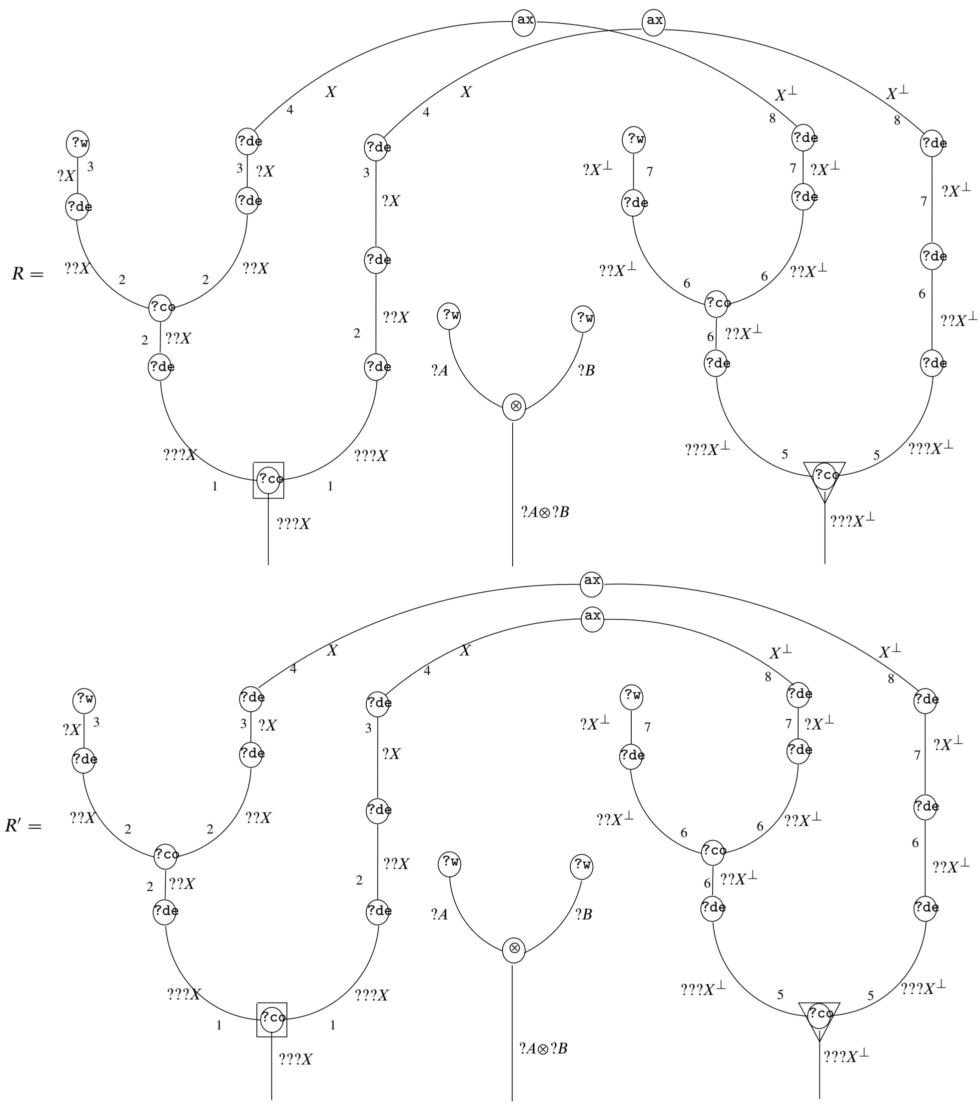

Fig. 2. Counter-example 1: for the proof-nets R and R' above, $[\![R]\!] = [\![R']\!]$ and $R \neq R'$.

immediate consequence of the absence of an injective experiment. Notice that (of course) $OLPS(R) = OLPS(R')$, following Theorem 3.27.

It is interesting to notice that if we 'open' the two contraction links of R and R' (that is, if we erase the $?co$ links and their conclusions), we get two proof-nets R_1 and R'_1 with the same conclusions, which are subproof-nets of R and R', respectively. The proof-nets R_1 and R'_1 *do not* have the same semantics: the (uniformity) condition imposed by the erased contraction links has disappeared, and we can now find injective experiments that distinguish (semantically speaking) the two proof-nets.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

5.2.2. *The second counter-example.* We are now going to show another phenomenon (still due to the 'uniformity of coherent semantics'), which also leads us to answer negatively to the question in Problem 2.16.

In the presence of weakenings, the characterisation of the boxes given by Proposition 3.34 is obviously wrong (in general). We show that the semantics cannot uniquely determine the connections between the different doors of the boxes of a proof-net. In other words, we give two semantically equivalent proof-nets R and R' satisfying $LPS(R) = LPS(R')$, and such that $R \neq R'$.

This counter-example also shows that even for the system ELL (which was defined in Girard (1995) and simplified in Danos and Joinet (2001)) both the coherent set-based semantics and the coherent multiset-based semantics are not injective. To be precise, we should slightly modify it (see Tortora de Falco (2000) for a more precise discussion).

The two proof-nets R and R' of Figure 3 are (clearly) different and they have the same coherent (set-based and multiset-based) semantics. We now show that R and R' are semantically equivalent. We have associated with some edges a digit, following the same convention as in the previous counter-example: if with the two edges a_1 and a_2 of R (respectively, of R') is associated the same integer, every experiment e of R (respectively, of R') must satisfy $e(a_1) \asymp e(a_2)$. Notice that this notation is meaningful, because the edges with which we have associated a digit all have depth zero, and, following Definition 2.1, every experiment associates a unique label with these edges. We identify here, as we did for 1-experiments, $e(a)$ with the unique element of $e(a)$, for every experiment e of R (respectively, R') and for every edge *a with depth zero* in R (respectively, R').

Let c_1 and c_2 (respectively, c'_1 and c'_2) be the conclusions of the two pal doors with depth zero in R (respectively, in R'). The uniformity condition coming from the $?co$ link with depth zero in R (respectively, in R') requires that for every experiment e of R (respectively, e' of R') $e(c_1) \asymp e(c_2)$ (respectively, $e'(c'_1) \asymp e'(c'_2)$). But the element of $e(c_i)$ (respectively, of $e'(c'_i)$), for $i \in \{1, 2\}$, is of the form $\{n_i[\varnothing]\}$ (respectively, $\{n'_i[\varnothing]\}$): the only possibility is then that $n_1 = n_2$ (respectively, $n'_1 = n'_2$). In the set-based case we have $n_1 = n_2 = n'_1 = n'_2 = 1$.

Observe now that the unique way for the semantics to distinguish between R and R' is to be able to express the fact that the subgraph T of R and R' is in a given box and not in the other one: this is precisely what the presence of the $?co$ link forbids. The semantics cannot then tell us which box T is in, and this allows us to conclude: with every experiment of R we can associate an experiment of R' with the same result (and conversely, of course).

We have not mentioned the neutral elements of LL in this paper. We simply point out that the previous counter-example is also a counter-example to the injectivity of coherent multiset-based semantics for the multiplicative and exponential fragment of LL, without axiom links but with the links introducing the multiplicative constants 1 and $\perp$. In the set-based case, the non-injectivity is obvious (see Tortora de Falco (2000) for more details).

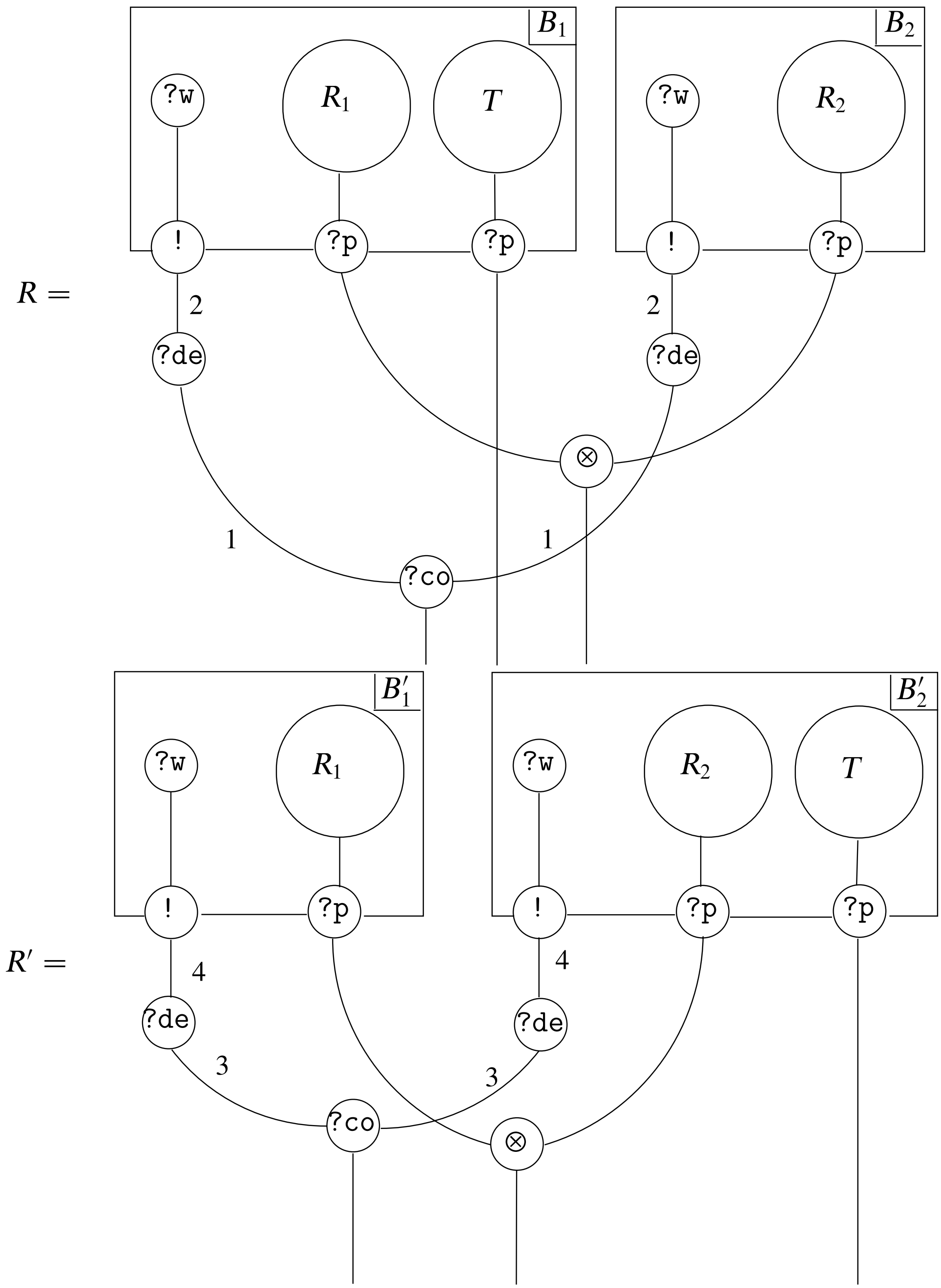

Fig. 3. Counter-example 2: for the proof-nets R and R' above, $[\![R]\!] = [\![R']\!]$ and $R \neq R'$.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

Remark 5.8. Our counter-examples also show that the result of Statman (1983) on the maximality of the $\beta\eta$-equivalence for the simply typed λ-calculus does not extend to *MELL* (see Tortora de Falco (2000) for more details).

We conclude the present section by observing that in both our counter-examples the presence of weakenings is crucial, and that none of our proof-nets is *polarised*.

5.3. *Summing up*

We use the content of Appendix B in this section.

We use:

— $MELL\backslash\{?W\}$, to denote the subsystem of *MELL* containing all the proof-nets of *MELL* whose normal forms do not contain any weakening link.
— LL_{pol} to denote the system of polarised proof-nets: an *MELL* proof-net is polarised when the types of its conclusions are all subformulas of a positive (P) or of a negative (N) formula, where:
$P ::= \ !X \mid P \otimes P \mid !N$
$N ::= \ ?X^{\perp} \mid N \wp N \mid ?P$
— $[\![\ ]\!]_{cohs}$ to denote the set-based coherent semantics.
— $[\![\ ]\!]_{cohm}$ to denote the multiset-based coherent semantics.
— $[\![\ ]\!]_{rel}$ to denote the relational semantics.

The following table sums up the state of the art on the question of injectivity: the answers that we do have (the results of the present paper) are written in capital letters, and the conjectures (which are actually open problems) in small letters.

	$[\![\ ]\!]_{cohs}$	$[\![\ ]\!]_{cohm}$	$[\![\ ]\!]_{rel}$
MELL	NO	NO	? (yes)
$MELL\backslash\{?W\}$	? (yes)	? (yes)	? (yes)
LL_{pol}	? (yes)	? (yes)	? (yes)
$(?\wp)$LL	? (yes)	YES	YES

Let us comment a bit on this table. The choice of the subsystems of *MELL* mentioned above is easy to justify: we will not comment on $(?\wp)$LL (for obvious reasons!), $MELL\backslash\{?W\}$ seems interesting because a positive answer for this subsystem (in the coherent case) would probably help us to understand more precisely the relation between connectivity and coherence, and LL_{pol} is certainly interesting, because it allows us to encode classical logic (Girard 1991; Danos *et al.* 1997; Laurent *et al.* 2000).

Notice that (thanks to Appendix B) any positive answer in the coherent (multiset-based) case immediately gives a positive answer in the relational case. We can also mention the paper Barreiro and Ehrhard (1997), which relates the semantic equivalence relation induced by the set-based coherent semantics and the one induced by the multiset-based coherent semantics. In particular, they show that two equivalent proofs with respect to the multiset-based coherent semantics are always equivalent with respect to the set-based coherent semantics. And the converse does not hold as soon as we add the constants 1 and $\perp$.

Appendix A. Proof-nets and coherent semantics

In this paper, we deal with both syntax and semantics, and more precisely with proof-nets and their semantics. To make the paper more self-contained, we recall here the notions of proof-net and coherent space. Notice that while the latter is standard, this is not the true for the former: there are several variants of LL proof-nets in the literature. Tortora de Falco (2000) (see also Tortora de Falco (2003)) gives a detailed description of proof-nets and their normalisation for second order LL. We refer to this notion of proof-net, and recall it here for the multiplicative and exponential fragment of LL.

A.1. *Coherent spaces*

We give the definition of coherent space (see, for example, Girard (1987)), and of the *multiset* based interpretation of the exponential connectives (Girard 1991).

Definition A.1 (Coherent space). A **coherent space** $\mathscr{A}$ is the data of a set $|\mathscr{A}|$ (the **web** of $\mathscr{A}$) and of a binary reflexive and symmetric relation denoted by $\mathrel{\substack{\frown\\\smile}}$ (the **coherence** relation on $|\mathscr{A}|$). If $x, y \in |\mathscr{A}|$ and if (x, y) is an element of the relation $\mathrel{\substack{\frown\\\smile}}$, we say that x and y are coherent and write $x \mathrel{\substack{\frown\\\smile}} y(\mathscr{A})$. We often use the following notation: $x \frown y(\mathscr{A})$ (when $x \mathrel{\substack{\frown\\\smile}} y(\mathscr{A})$ and $x \neq y$), $x \smile y(\mathscr{A})$ (when x and y are not coherent) and $x \mathrel{\substack{\smile\\\frown}} y(\mathscr{A})$ (when $x \smile y(\mathscr{A})$ or $x = y$). The elements of $\mathscr{A}$, called **cliques**, are the multisets of elements of $|\mathscr{A}|$ that are pairwise coherent. In the original set-based definition (see, for example, Girard (1987)), the cliques are the sets of elements of $|\mathscr{A}|$ that are pairwise coherent.

The interpretation of LL formulas is defined by induction on their complexity. We associate some arbitrary coherent spaces with atomic formulas (which means that we get a different interpretation for every such choice). Then the coherent spaces associated with compound formulas are defined as follows:

— $|\mathscr{A}^\perp| = |\mathscr{A}|$, and for every $x, y \in |\mathscr{A}|$, we have $x \mathrel{\substack{\frown\\\smile}} y(\mathscr{A}^\perp)$ iff $x \mathrel{\substack{\smile\\\frown}} y(\mathscr{A})$.

— $|\mathscr{A} \otimes \mathscr{B}| = |\mathscr{A}| \times |\mathscr{B}|$, and for every $x, x' \in |\mathscr{A}|$ and $y, y' \in |\mathscr{B}|$, we have $(x, y) \mathrel{\substack{\frown\\\smile}} (x', y')$ $(\mathscr{A} \otimes \mathscr{B})$ iff $x \mathrel{\substack{\frown\\\smile}} x'(\mathscr{A})$ and $y \mathrel{\substack{\frown\\\smile}} y'(\mathscr{B})$.

— $|!\mathscr{A}| = \mathscr{A}_f$, whose elements are the finite elements of $\mathscr{A}$ (notice that here the set-based and the multiset-based webs are different), and for every $x, y \in |!\mathscr{A}|$ we have $x \mathrel{\substack{\frown\\\smile}} y(!\mathscr{A})$ iff $x \cup y \in \mathscr{A}$ (that is, $x \cup y \in \mathscr{A}_f$).

A.2. *Proof-nets*

Definition A.2. A **proof-structure** is an oriented graph whose nodes are called links, and whose edges are labelled by formulas of LL. When drawing a proof-structure, we represent edges oriented up-down so that we may speak of moving upwards or downwards in the graph. Links are defined together with an arity and a coarity, that is, a given number of incident edges, called the premises of the link, and a given number of emergent edges, called the conclusions of the link.

— A **Hypothesis** or H link has $n \geqslant 1$ conclusions, each of them labelled by a formula, and no premise.
— An **axiom** link has no premise and two conclusions labelled by dual formulas.
— A **cut** link has two premises labelled by dual formulas (which are also called the active formulas of the cut link) and no conclusion.
— A **par** or $\wp$ (respectively, **times** or $\otimes$) link has two premises and one conclusion. If the left premise is labelled by the formula A and the right premise is labelled by the formula B, then the conclusion is labelled by the formula $A \wp B$ (respectively, $A \otimes B$).
— An **of course** link has one premise and one conclusion labelled by the **of course** of the premise.
— A **dereliction** or $?de$ link has one premise and one conclusion labelled by the **why not** of the premise.
— A **weakening** or $?w$ link has no premise and one conclusion labelled by $?A$ for some formula A.
— A **contraction** or $?co$ link has $k \geqslant 2$ premises and one conclusion, all labelled by $?A$ for some formula A.
— A **pax** link has one premise and one conclusion, both labelled by $?A$ for some formula A.

Let G be a set of links such that:

(α) Every edge of G is the conclusion of a unique link
(β) Every edge of G is the premise of at most one link.

We say that the edges that are not a premise of a link are the conclusions of G.

We will say that such a graph is a proof-structure if the two following conditions are satisfied:

(1) !-box condition:

— With each of course link n is associated a (unique) sub-graph $B^!$ of G (satisfying (α) and (β)), such that one of the conclusions of $B^!$ is the conclusion of n and every other conclusion of $B^!$ (there may be no other conclusions) is the conclusion of a pax link. $B^!$ is called an exponential box and it is represented by a rectangular frame, and n is called the front door or the pal door of $B^!$.
— With each pax link n is associated an exponential box $B^!$ of G, such that one of the conclusions of $B^!$ is the conclusion of n. The link n is called a pax door of $B^!$.

(2) Nesting condition:

— Two boxes are either disjoint or included one in the other.

We will often speak of a box, a link or an edge of a proof-structure R contained in a box B of R. In the case of links, we will not consider the doors of B as links contained in B. We will also speak of 'a link l (respectively, an edge a) of a box B' of a given proof-structure, meaning that l (respectively, a) is contained in B or it is a door (respectively, a conclusion) of B. If B is a box of a proof-structure R, the biggest (respectively, the smallest) box of R containing B is clearly well-defined, thanks to the nesting condition of Definition A.2.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

We shall say that a link or an edge of a given proof-structure R has **depth** n in R if it is contained in exactly n boxes of R. For a box B, we shall say that B has **depth** n in R if it is contained in exactly n boxes of R, all different from B. When R is a proof-net, the same definition will extend to the case of a subproof-net of R (as defined in Definition A.8). Clearly, we can define in the same way the depth of a subproof-structure S (a subgraph which is itself a proof-structure) of the proof-structure R.

The depth of a proof-structure is the maximal depth of its links.

Remark A.3. Notice that (contrary to the premises of a $\otimes$ or of a $\wp$ link) the premises of a contraction link *are not* ordered. This means that a proof-structure is defined up to the order of the premises of the $?co$ links: we are actually dealing with an equivalence class of graphs rather than with a graph.

Definition A.4 (Graph with pairs). We will say that two edges of an oriented graph are coincident when they have the same target. The couple $(G, App(G))$ is called a **graph with pairs** when G is an oriented graph and $App(G)$ is a set of n-tuples ($n \geqslant 2$) of coincident edges.

Let R be a proof-structure and let $B_1, \dots, B_k$ be the boxes of R with depth zero. We are going to associate with R a set $App(R)$ and a graph with pairs $R_{ap} = (G_R, App(R))$.

The graph G_R is obtained from R as follows:

— substitute for each box B_i with p_i conclusions ($i \in \{1, \dots, k\}$), a link H with p_i conclusions

The set $App(R)$ contains the following (and only the following) m-tuples:

— The couples of premises of every $\wp$ link of G_R with depth zero
— The p-tuples of premises of every $?co$ link of G_R with depth zero.

Definition A.5 (Correctness graph). Let R be a proof-structure and let $B_1, \dots, B_k$ be the boxes with depth zero in R. Let $R_{ap} = (G_R, App(R))$ be the graph with pairs associated with R by Definition A.4.

A **switching S** of R is the choice of an edge for every n-tuple of $App(R)$.

With each switching S is associated an unoriented graph $S(R)$, called a **correctness graph**: for every n-tuple of $App(R)$, erase the edges of G_R that are not selected by S, and then forget the labels and the orientation of the edges of the graph. The correctness graph of R associated with S will be denoted by $S(R)$.

Definition A.6 (Proof-net). Let R be a proof-structure that contains no occurrences of the link H, and let $B_1, \dots, B_k$ be the boxes with depth zero in R. We say that R is a **proof-net** when the following conditions are satisfied:

— R satisfies (AC): for every switching S of R, the correctness graph $S(R)$ is acyclic (there is no cycle in $S(R)$).
— For every box $B_i \in \{B_1, \dots, B_k\}$, the proof-structure R_i contained in B_i is a proof-net.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

This definition of proof-net corresponds to the standard notion of sequent calculus proof, *provided* we add to the multiplicative and exponential linear sequent calculus the two following rules:

1 The so-called 'mix' rule:

$$(\text{mix})\ \frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}$$

2 The following 'proof' of $?A$ for every formula A:

$$(?\ \text{Hyp})\ \frac{}{\vdash ?A}$$

Remark A.7. In Tortora de Falco (2003), we used the notion of 'jump' and modified the previous definition in order to get the correspondence between proof-nets and the usual multiplicative and exponential linear sequent calculus without the mix rule and without the (? Hyp) rule. The notion of proof-net thus obtained is also proved to be stable with respect to the usual cut-elimination steps.

We will not develop such a notion here: for more information, refer to Tortora de Falco (2003) (and also to Tortora de Falco (2000) for a more detailed version). For this paper, you can simply refer to Definition A.6, keeping in mind that all the proof-nets considered in the present paper satisfy (AC) and can be sequentialised in the usual multiplicative and exponential sequent calculus (without adding any rule).

It is, however, important to stress the fact that, in the absence of weakenings, a proof-net R can be sequentialised in the usual multiplicative and exponential linear sequent calculus (without either mix or (? Hyp)) iff R satisfies Definition A.6, where the (AC) condition is substituted by:

(ACC) for every switching S of R, the correctness graph $S(R)$ is acyclic and connected (that is, $S(R)$ is a tree).

Definition A.8. Let T be a proof-net. A subgraph R of T is a **subproof-net** of T, when R is a proof-net.

A.2.1. *Cut-elimination for LL Proof-nets.* We are only concerned with the multiplicative and exponential fragment of LL proof-nets, for which the cut-elimination procedure is standard. We will not define here the elementary reduction steps (see Girard (1987), Danos (1990) and Tortora de Falco (2000; 2003)).

A.3. *Conventions and notation*

A.3.1. *Some conventions.* In this paper (except for Section A.2) we restrict the term 'label' to the elements of the web of some space associated with an edge of a proof-net by an experiment (see Definition 2.1). The formula (which is called 'label' in Section A.2) associated with a given edge of a proof-net is said to be **the type** of the edge.

There is the obvious remark to make concerning formulas: when we say ‘a formula’ we may mean ‘a formula’ or ‘ an occurrence of a formula’. Which of these is meant is generally clear from the context, but sometimes when it is absolutely crucial, we say explicitly that we mean occurrences of formulas (and not formulas).

A sequent is (as usual) a multiset of formulas, sometimes prefixed by the symbol $\vdash$.

A last link of a proof-structure S is a link whose conclusion(s) is (are) a conclusion(s) of S. A cut link of S is a last link if it has depth 0. We say that n is a terminal link of a proof-net when there exists a sequentialisation having as last rule the one with which is associated the link n. Of course, there are *some* links for which the two notions coincide.

If A is a formula and E is an occurrence of a subformula of A, we sometimes speak of ‘the complexity of $A \backslash E$’: we mean the integer $c_A - c_E$, where c_A (respectively, c_E) is the number of occurrences of connectives of A (respectively, E).

We always work with **multisets** (unless explicitly mentioned), for which we use the same notation as for sets.

The notion of ‘path’ used in the paper (unless explicitly mentioned) is the notion of ‘straight path’ introduced in Danos and Regnier (1995): a straight path is an oriented path changing direction only when crossing a cut link or an axiom link. Throughout the paper, we simply write ‘path’, but always mean ‘straight path’.

Definition A.9 (Danos and Regnier 1995). Let R be a proof-net. A path of R is a sequence of edges or reverted edges (that is, a path may take an edge from its goal to its source). We sometimes make the abuse of considering links as parts of paths (the idea is that the node associated with the link is crossed by the path). We use $\alpha, \beta, \ldots$ to denote the edges of a proof-net (oriented as usual, following Definition A.2) and $\alpha^*, \beta^*, \ldots$ to denote the previous edges ‘reverted’ (that is, oriented now in the opposite direction). Now let a (respectively, b) be an edge or a reverted edge whose goal (respectively, source) is the link n; we use ab to denote the path consisting of the edge a (followed by the link n, itself) followed by the edge b.
We say that the path Φ of R is a **straight path** if:

(i) Φ does not contain any $\alpha^*\alpha$ nor any $\alpha\alpha^*$.

(ii) If α and β are two distinct premises (and then, according to our notations, two edges) of a same link n, and if $\alpha\beta^*$ is a subpath of Φ, then n is a cut link.

A.3.2. *Notation.* In the paper we use:

— a, b, c, … to denote the edges of a proof-net.
— A, B, C, … to denote the types of these edges.
— $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, … to denote the structures interpreting these types.
— $|\mathscr{A}|$, $|\mathscr{B}|$, $|\mathscr{C}|$, … to denote the webs of these structures.
— α, β, α', β', … to denote the edges of atomic type of a proof-net.
— e, e', e_i, … to denote the experiments.
— γ, γ', δ, δ',… to denote the results of some experiments.
— Γ, Γ', Δ, Δ',… to denote multisets of formulas.
— $\wp\,\Gamma$ to denote the formula obtained by performing the $\wp$ between the formulas of Γ. For simplicity, we still use $\wp\,\Gamma$ to denote the structure interpreting the formula $\wp\,\Gamma$.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

— R_B to denote the biggest subproof-net of the proof-net R that is contained in the box B of R (when R is a proof-structure, R_B is the biggest subproof-structure contained in B).
— $card(y)$ to denote the cardinality of the multiset y.
— $n[z]$ to denote the repetition n times of the element z: for example $\{n[z]\}$ is the multiset containing n occurrences of the element z.

We use the standard notation for coherence and incoherence (strict and strict or equal), introduced by Definition A.1. When the context makes this unambiguous, we simply write $x \asymp y$, instead of $x \asymp y(\mathscr{A})$, for $x, y \in |\mathscr{A}|$.

Appendix B. About injectivity for relational semantics

Several results presented in the paper are also valid for the relational semantics, which can be roughly defined as 'coherent semantics without coherence'.

We define the relational interpretation of LL formulas, and state a result (Proposition B.2), whose immediate consequence is the fact that when the multiset-based coherent semantics is injective, the relational semantics is also (Remark B.3).

Definition B.1. Let $|\ |$ be a function that associates a set with every propositional variable. We define the extension of $|\ |$ to LL formulas as follows:

— $|\mathscr{A}^\perp| = |\mathscr{A}|$.
— $|\mathscr{A} \otimes \mathscr{B}| = |\mathscr{A}| \times |\mathscr{B}|$.
— $|!\mathscr{A}| = M_f(|\mathscr{A}|)$, where $M(|\mathscr{A}|)$ is the free commutative monoid generated by $|\mathscr{A}|$ and $M_f(|\mathscr{A}|)$ is the set of the finite multisets of elements of $|\mathscr{A}|$.

For every LL formula A, the elements of the space $\mathscr{A}$ associated with A by the relational semantics are the multisets of elements of $|\mathscr{A}|$.

Definitions 2.1 and 2.4 straightforwardly extend to relational semantics, and the analogue of Theorem 2.6 can be proved. The same holds for the notion of obsessional experiment, and a study of these experiments in the relational framework has been undertaken in Tortora de Falco (2000).

Nevertheless, several crucial results do not extend to relational semantics: contrary to the coherent case, there exist several experiments with the same result, and Proposition 3.15 does not hold in the relational case.

However, there is a very natural and useful property allowing us to extend all our injectivity results to the relational case. The statement of the following proposition (proved in Tortora de Falco (2000)) was suggested to us by Thomas Ehrhard.

In the following, we will use $|\mathscr{A}|_{cohm}$ to denote the multiset-based web of the coherent space $\mathscr{A}$, and $[\![R]\!]_{cohm}$ (respectively, $[\![R]\!]_{rel}$) to denote the multiset-based coherent semantics (respectively, the relational semantics) of the proof-net R.

We choose a relational and a coherent interpretation of the propositional variables of the language such that if $\mathscr{X}_{rel}$ (respectively, $\mathscr{X}_{cohm}$) is the space interpreting the propositional variable X, then $|\mathscr{X}|_{cohm} = |\mathscr{X}|_{rel}$.

https://doi.org/10.1017/S0960129503003967 Published online by Cambridge University Press

Proposition B.2. Let R be a proof-net with conclusion Γ.

$$[\![R]\!]_{cohm} = [\![R]\!]_{rel} \cap |\wp\, \Gamma|_{cohm}.$$

Remark B.3. Proposition B.2 implies that if R and R' are two proof-nets with the same conclusions, then, from $[\![R]\!]_{rel} = [\![R']\!]_{rel}$, we deduce that $[\![R]\!]_{cohm} = [\![R']\!]_{cohm}$. This means that for every fragment F of LL, if the coherent (multiset-based) semantics is injective for F, then the relational semantics is injective too.

Acknowledgments

The author thanks the anonymous referee for his careful reading and suggestions. His report was very helpful in improving the structure of the paper, and thus (hopefully!) its readability.

References

Barcaglioni, C. (2001) *Le dimostrazioni logiche come costruzioni: spazi coerenti ed esperienze*, Tesi di laurea in matematica, Università di Roma Tre.

Barreiro, N. and Ehrhard, T. (1997) Anatomy of an extensional collapse. (Private comunication, Available at `ftp://iml.univ-mrs.fr/pub/ehrhard/anat.ps.Z.`)

Berger, U., Eber, M. and Schwichtenberg, H. (1998) Normalization by evaluation. *Springer-Verlag Lecture Notes in Computer Science* **1546** 117–137.

Danos, V. (1990) *La Logique Linéaire appliquée à l'étude de divers processus de normalisation (principalement du λ-calcul)*, Thèse de doctorat, Université Paris VII.

Duquesne, E. and Van de Wiele, J. (1994) Modèle cohérent des réseaux de preuve. *Archive for Mathematical Logic* **33** 133–158.

Danos, V. and Joinet, J.-B. (2001) Linear logic and elementary time. *Information and Computation* (to appear).

Danos, V., Joinet, J.-B. and Schellinx, H. (1997) A new deconstructive logic: linear logic. *Journal of Symbolic Logic* **62** (3) 755–807.

Danos, V. and Regnier, L. (1995) Proof-nets and the Hilbert space. In: Girard, J.-Y., Lafont, Y. and Regnier, L. (eds.) Advances in Linear Logic. *London Mathematical Society Lecture Note Series* **222** 307–328.

Danvy, O., Rhiger, M. and Rose, K. H. (2001) Normalization by evaluation with typed abstract syntax. *J. Funct. Programming* **11** (6) 673–680.

Girard, J.-Y. (1987) Linear logic. *Theoretical Computer Science* **50** 1–102.

Girard, J.-Y. (1991) A new constructive logic: classical logic. *Mathematical Structures in Computer Science* **1** (3) 255–296.

Girard, J.-Y. (1993) On the unity of logic. *Annals of Pure and Applied Logic* **59** 201–217.

Girard, J.-Y. (1995) Light linear logic. *Information and Computation* **143** (2) 175–204.

Laurent, O. (1999) Polarized proof-nets: proof-nets for LC (extended abstract). In: Girard, J.-Y. (ed.) Typed Lambda Calculi and Applications '99. *Springer-Verlag Lecture Notes in Computer Science* **1581** 213–227.

Laurent, O., Quatrini, M. and Tortora de Falco, L. (2000) Polarized and focalized linear and classical proofs. *Annals of Pure and Applied Logic* (to appear).

Laurent, O. and Tortora de Falco, L. (2001) Slicing polarized additive normalization. *Quaderno I.A.C.-C.N.R.* **12**. (To appear in *Advances in Linear Logic*, Cambridge University Press.)

Quatrini, M. and Tortora de Falco, L. (1996) Polarisation des preuves classiques et renversement. *Compte-Rendu de l'Académie des Sciences de Paris* **323** 113–116.

Regnier, L. (1992) *Lambda-Calcul et Réseaux*, Thèse de doctorat, Université Paris VII.

Retoré, C. (1997) A semantic characterisation of the correctness of a proof net. *Mathematical Structures in Computer Science* **7** (5) 445–452.

Statman, R. (1983) λ-definable functionals and $\beta\eta$ conversion. *Arch. Math. Logik Grundlagenforsch* **23** 21–26.

Tortora de Falco, L. (1997) Denotational semantics for polarized (but non-constrained) *LK* by means of the additives. In: 5^{th} Kurt Goedel Colloquium KGC'97, computational logic and proof theory. *Springer-Verlag Lecture Notes in Computer Science* **1289** 290–304.

Tortora de Falco, L. (2003) Additives of linear logic and normalization – part I: a (restricted) Church–Rosser property. *Theoretical Computer Science* **294** (3) 489–524.

Tortora de Falco, L. (2000) *Réseaux, cohérence et expériences obsessionnelles*, Thèse de doctorat, Université Paris VII. (Available at `http://www.logique.jussieu.fr/www.tortora`.)