





# Equilibrium states on higher-rank Toeplitz non-commutative solenoids

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*Abstract.* We consider a family of higher-dimensional non-commutative tori, which are twisted analogues of the algebras of continuous functions on ordinary tori and their Toeplitz extensions. Just as solenoids are inverse limits of tori, our Toeplitz non-commutative solenoids are direct limits of the Toeplitz extensions of non-commutative tori. We consider natural dynamics on these Toeplitz algebras, and we compute the equilibrium states for these dynamics. We find a large simplex of equilibrium states at each positive inverse temperature, parametrized by the probability measures on an (ordinary) solenoid.

Key words: KMS states,  $C^*$ -algebras, direct limit, Toeplitz algebra

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## 1. Introduction

Classical solenoids are inverse limits of tori. There are non-commutative analogues of tori, which are the twisted group algebras  $C^*(\mathbb{Z}^n, \sigma)$  of the abelian group  $\mathbb{Z}^n$ . For  $n = 2$ , these are the rotation algebras  $A_\theta$  generated by two unitaries  $U, V$  satisfying the commutation relation  $UV = e^{2\pi i\theta} VU$ . When  $\theta$  is irrational, these are simple  $C^*$ -algebras, and they have been extensively studied (see, for example, [10, Ch. VI]). For  $\theta = 0$ , we recover the commutative algebra  $C(\mathbb{T}^2)$ , and hence the  $A_\theta$  are also known as ‘non-commutative tori.’ In [24], Latrémolière and Packer studied a family of non-commutative solenoids that are direct limits of non-commutative tori. (The connection is that the commutative algebra of continuous functions on a solenoid is the direct limit of the algebras of continuous functions on the approximating tori.)

Following surprising results about phase transitions for the Kubo–Martin–Schwinger (KMS) states of the Toeplitz algebras of the  $ax + b$ -semigroup of the natural numbers [19, 21], many authors have studied the KMS structure of Toeplitz extensions in other settings. Typically, these Toeplitz extensions exhibit more interesting KMS structure. This recent work has covered Toeplitz algebras of directed graphs and their higher-rank analogues [7, 8, 13, 15, 16] (after earlier work in [11]), Toeplitz algebras arising in number theory [9], the Nica–Toeplitz extensions of Cuntz–Pimsner algebras [1, 4, 17–19] and Toeplitz algebras associated to self-similar actions [22, 23]. In [6], Brownlowe, Hawkins and Sims described Toeplitz extensions of the non-commutative solenoids from [24], and they considered a natural dynamics on this extension. They showed that, for each inverse temperature  $\beta > 0$ , the  $\text{KMS}_\beta$  states are parametrized by the probability measures on a commutative solenoid which is the inverse limit of one-dimensional tori [6, Theorem 6.6].

Here we consider a family of higher-rank non-commutative solenoids and their Toeplitz extensions. As for the algebras of higher-rank graphs [16], there is an obvious gauge action of a torus  $\mathbb{T}^d$  on these algebras, but to get a dynamics one has to choose an embedding of  $\mathbb{R}$  in the torus. We fix  $r \in [0, \infty)^d$ , giving an embedding  $t \mapsto e^{itr}$  of  $\mathbb{R}$  in  $\mathbb{T}^d$ , and we compose with the gauge action to get a dynamics  $\alpha^r$ .

The building blocks in [6] are Toeplitz non-commutative tori in which one generator  $U$  is unitary and the other  $V$  is an isometry; the relation is still given by  $UV = e^{2\pi i\theta} VU$ , and the dynamics fixes  $U$ . Here we fix  $d, k \in \mathbb{N}$ . Our blocks  $B_\theta$  are Toeplitz non-commutative tori generated by a unitary representation  $U$  of  $\mathbb{Z}^d$  and a Nica-covariant isometric representation  $V$  of  $\mathbb{N}^k$ , and the commutation relation is given by  $U_n V_p = e^{2\pi i p^T \theta n} V_p U_n$  for a fixed  $k \times d$  matrix  $\theta$  with entries in  $[0, \infty)$ . Then the dynamics  $\alpha^r$  is given by a vector  $r \in (0, \infty)^k$ ; it fixes the unitaries  $U_n$  and multiplies  $V_p$  by  $e^{itp^T r}$ .

We begin by describing the direct system of Toeplitz non-commutative tori whose limit is the Toeplitz non-commutative solenoid of the title. Everything is defined in terms of presentations of the blocks: building the connecting maps is, in particular, quite complicated, and it requires us to be careful with the notation, which we try to keep consistent throughout the paper. We then discuss the dynamics, which is again defined using actions on the individual blocks. Then, remarkably, we have a presentation of the direct limit which allows us to state our main result as Theorem 2.8. This gives a satisfyingly explicit description of the  $\text{KMS}_\beta$  states in terms of measures on a commutative solenoid of the form  $\varprojlim \mathbb{T}^d$ . This concrete description is new even in the case  $k = d = 1$  studied in [6].

The first step in the proof of our theorem is an analysis of the KMS states of a building block  $B_\theta$ , which we do in §3. The description in Proposition 3.7 looks rather like the descriptions of KMS states on graph algebras in [15, Theorem 3.1] and [16, Theorem 6.1] and on algebras associated to local homeomorphisms in [2, Theorem 5.1]: we find a subinvariance relation which identifies the measures on the torus associated to KMS states, and then we describe the solutions of that relation in terms of a concrete simplex of measures.

In the next section (§4), we show how the subinvariance relations for the building blocks combine to give one continuously parametrized subinvariance relation for the direct limit (Theorem 4.1). We then describe the solutions to this new subinvariance relation in

Theorem 5.1, which is the key technical result in the paper. This solution is very concrete, involving a formula which is reminiscent of a multi-variable Laplace transform, and it is much more direct than the *ad hoc* approach used in [6].

In the last section, we give a concrete description of the isomorphism  $\mu \mapsto \psi_\mu$  of the simplex  $P(\varprojlim \mathbb{T}^d)$  of probability measures on the solenoid onto the simplex of  $\text{KMS}_\beta$  states on the Toeplitz non-commutative torus. Then, by evaluating these KMS states on generators, we arrive at the explicit values described in Theorem 2.8.

2. Toeplitz non-commutative solenoids

We define a Toeplitz non-commutative solenoid as the direct limit of a sequence of blocks, which we call Toeplitz non-commutative tori. So we begin by looking at these blocks. In the course of this section, we will introduce notation which will be used throughout the paper.

First, we fix positive integers  $d$  and  $k$ . We write  $A^T$  for the transpose of a matrix  $A$ . We view elements of  $\mathbb{R}^k$  as column vectors, and we write the inner product of  $n, p \in \mathbb{R}^k$  in matrix notation as  $p^T n$ . We use similar conventions for  $\mathbb{R}^d$ .

The pair  $(\mathbb{Z}^k, \mathbb{N}^k)$  is a quasi-lattice-ordered group in the sense of Nica [25]. Indeed, for every  $p, q \in \mathbb{N}^k$ , the element  $p \vee q$  defined pointwise by

$$(p \vee q)_j = \max\{p_j, q_j\} \quad \text{for } 1 \leq j \leq k$$

is a least upper bound for  $p$  and  $q$ , so it is lattice ordered. An isometric representation  $V : \mathbb{N}^k \rightarrow B(H)$  is *Nica-covariant* if it satisfies

$$V_p V_p^* V_q V_q^* = V_{p \vee q} V_{p \vee q}^* \quad \text{for all } p, q \in \mathbb{N}^k,$$

or, equivalently [20, (1.4)], if

$$V_p^* V_q = V_{(p \vee q) - p} V_{(p \vee q) - q}^* \quad \text{for all } p, q \in \mathbb{N}^k.$$

For  $\theta \in M_{k,d}([0, \infty))$ , we consider the universal  $C^*$ -algebra  $B_\theta$  generated by a unitary representation  $U$  of  $\mathbb{Z}^d$  and a Nica-covariant isometric representation  $V$  of  $\mathbb{N}^k$  such that

$$U_n V_p = e^{2\pi i p^T \theta n} V_p U_n \quad \text{for } p, q \in \mathbb{N}^k \text{ and } n \in \mathbb{Z}^d. \tag{2.1}$$

We then have also

$$U_n V_p^* = (V_p U_{-n})^* = (e^{-2\pi i p^T \theta (-n)} U_{-n} V_p)^* = e^{-2\pi i p^T \theta n} V_p^* U_n. \tag{2.2}$$

Direct calculation shows that, for  $p, q, p', q' \in \mathbb{N}^k$  and  $n, n' \in \mathbb{Z}^d$ ,

$$\begin{aligned} V_p U_n V_q^* V_{p'} U_{n'} V_{q'}^* &= V_p U_n V_{(q \vee p') - q} V_{(q \vee p') - p'}^* U_{n'} V_{q'}^* \\ &= e^{2\pi i ((q \vee p') - q)^T \theta n + (q \vee p' - p')^T \theta n'} V_{p + (q \vee p') - q} U_{n+n'} V_{q' + (q \vee p') - p'}^*, \end{aligned}$$

and we deduce that

$$B_\theta = \overline{\text{span}}\{V_p U_n V_q^* : n \in \mathbb{Z}^d \text{ and } p, q \in \mathbb{N}^k\}.$$

We call  $B_\theta$  a *Toeplitz non-commutative torus*.

Now we move on to non-commutative solenoids. First, we need some more conventions. We write  $\mathbb{S}^d$  for the compact quotient space  $\mathbb{R}^d / \mathbb{Z}^d$ , and we view functions  $f \in C(\mathbb{S}^d)$  as

$\mathbb{Z}^d$ -periodic continuous functions  $f : \mathbb{R}^d \rightarrow \mathbb{C}$ . We write  $M(\mathbb{S}^d)$  for the set of positive measures on  $\mathbb{S}^d$ , and we view measures  $\mu \in M(\mathbb{S}^d)$  as positive functionals  $f \mapsto \int_0^1 f d\mu$  on  $C(\mathbb{S}^d)$ . Then  $\|\mu\| := \mu(\mathbb{S}^d)$  is the norm of the corresponding functional, and  $P(\mathbb{S}^d) := \{\mu \in M(\mathbb{S}^d) : \|\mu\| = 1\}$  is the set of probability measures.

We consider three sequences of matrices  $\{\theta_m\} \subset M_{k,d}([0, \infty))$ ,  $\{D_m\} \subset M_k(\mathbb{N})$  and  $\{E_m\} \subset M_d(\mathbb{N})$  such that each  $D_m$  is diagonal with entries larger than one, each  $E_m$  has  $\det E_m > 1$  and

$$D_m \theta_{m+1} E_m = \theta_m \quad \text{for } m \geq 1. \tag{2.3}$$

We choose a sequence  $\{r^m\} = \{(r_j^m)\}$  of vectors in  $(0, \infty)^k$  satisfying

$$r^{m+1} = D_m^{-1} r^m \quad \text{for } m \geq 1. \tag{2.4}$$

Notice that both sequences are determined by the first terms  $\theta_1 \in M_{k,d}([0, \infty))$  and  $r^1 \in [0, \infty)^k$ .

*Example 2.1.* We fix  $N \geq 2$ , and we set  $d = k = 1$ ,  $D_m = E_m = N$  for  $m \geq 2$ ,  $\theta_1 \in (0, \infty)$  and  $\theta_m = N^{-2(m-1)}\theta_1$ . Taking the equivalence classes of the  $\theta_m$  in  $\mathbb{S} = \mathbb{R}/\mathbb{Z}$  yields an example of the set-up of [6] except that we are insisting that  $\theta_m = N^2\theta_{m+1}$  as real numbers, not just as elements of  $\mathbb{S}$ . This has the consequence that  $\theta_m \rightarrow 0$  as  $m \rightarrow \infty$ , which need not happen in the situation of [6]; but see Remark 2.2 below.

*Remark 2.2.* Our hypothesis that  $D_m \theta_{m+1} E_m = \theta_m$  exactly, and not just modulo  $\mathbb{Z}^d$ , seems to be crucial in our arguments. Specifically, to assemble the sequences of KMS states that we will construct on the approximating subalgebras  $B_m$  into a KMS state on  $B_\infty$ , we will need to show that the associated probability measures  $\nu_m$  (see Proposition 3.7(a)) intertwine through the maps induced by the  $E_m^T$ . We do this in Lemma 6.2, and we indicate there the step in the first displayed calculation where it is critical that  $D_m \theta_{m+1} E_m = \theta_m$  exactly. This prompted us to review carefully the arguments of [6] and we believe that those arguments also require that  $N^2\theta_{m+1} = \theta_m$  exactly. Specifically, the calculation at the end of the proof of [6, Theorem 6.9] implicitly treats  $\theta_j$  as an element of  $\mathbb{R}$  (there are many solutions to  $N^k\gamma = \theta_j$  in  $\mathbb{S}$ ). Similarly, the formulas in [6, §8] that involve setting  $r_j = \beta/(N^j\theta_j)$  only make sense if  $\theta_j$  is an element of  $\mathbb{R}$ . In particular, in the final displayed calculation in the proof of [6, Lemma 8.1], it is critical that  $N^2\theta_{j+1} = \theta_j$  exactly.

For each  $m$  there is a Toeplitz non-commutative torus  $B_m := B_{\theta_m}$  with generators  $U_{m,n}$  and  $V_{m,p}$  such that  $U : n \mapsto U_{m,n}$  is a unitary representation of  $\mathbb{Z}^d$ ,  $V : p \mapsto V_{m,p}$  is a Nica-covariant isometric representation of  $\mathbb{N}^k$  and the pair  $U, V$  satisfy the commutation relation (2.1) for the matrix  $\theta_m$ .

Next, we use the matrices  $D_m$  and  $E_m$  to build homomorphisms from  $B_m$  to  $B_{m+1}$ .

**PROPOSITION 2.3.** *Suppose that  $m$  is a positive integer. Then there is a homomorphism  $\pi_m : B_m \rightarrow B_{m+1}$  such that  $\pi_m(U_{m,n}) = U_{m+1, E_m n}$  and  $\pi_m(V_{m,p}) = V_{m+1, D_m p}$ .*

*Proof.* We define  $U : \mathbb{Z}^d \rightarrow B_{m+1}$  by  $U_n = U_{m+1, E_m n}$  and  $V : \mathbb{N}^k \rightarrow B_{m+1}$  by  $V_p = V_{m+1, D_m p}$ . Then, since  $D_m$  and  $E_m$  have entries in  $\mathbb{N}$ ,  $U$  is a unitary representation of  $\mathbb{Z}^d$  and  $V$  is an isometric representation of  $\mathbb{N}^k$ .

We claim that  $V$  is Nica-covariant. To see this, we take  $p, q \in \mathbb{N}^k$ . Then Nica-covariance of  $p \mapsto V_{m+1,p}$  implies that

$$\begin{aligned} V_p V_p^* V_q V_q^* &= V_{m+1,D_m p} V_{m+1,D_m p}^* V_{m+1,D_m q} V_{m+1,D_m q}^* \\ &= V_{m+1,(D_m p) \vee (D_m q)} V_{m+1,(D_m p) \vee (D_m q)}^* \end{aligned} \tag{2.5}$$

Now recall that  $D_m$  is diagonal with diagonal entries  $d_{m,j}$ , say. Then, for  $1 \leq j \leq k$ ,

$$\begin{aligned} ((D_m p) \vee (D_m q))_j &= \max\{(D_m p)_j, (D_m q)_j\} = \max\{d_{m,j} p_j, d_{m,j} q_j\} \\ &= d_{m,j} \max\{p_j, q_j\} = d_{m,j} (p \vee q)_j \\ &= (D_m (p \vee q))_j. \end{aligned}$$

Thus

$$V_{m+1,(D_m p) \vee (D_m q)} = V_{m+1,D_m (p \vee q)} = V_{p \vee q},$$

and (2.5) says that  $V$  is Nica-covariant.

We next claim that  $U$  and  $V$  satisfy the commutation relation (2.1). We take  $n \in \mathbb{Z}^d$  and  $p \in \mathbb{N}^k$ , and using the commutation relation in  $B_{m+1}$ , we compute

$$\begin{aligned} U_n V_p &= U_{m+1,E_m n} V_{m+1,D_m p} \\ &= e^{2\pi i (D_m p)^T \theta_{m_1} E_m n} V_{m+1,D_m p} U_{m+1,E_m n} \\ &= e^{2\pi i p^T (D_m \theta_{m_1} E_m)^n} V_{m+1,D_m p} U_{m+1,E_m n} \\ &= e^{2\pi i p^T \theta_m n} V_p U_n \quad \text{using (2.3)}. \end{aligned}$$

Now the universal property of  $B_m$  gives the desired homomorphism  $\pi_m$ . □

*Remark 2.4.* It was crucial in the proof of Proposition 2.3 that  $D$  is a diagonal matrix. To see this, consider, for example,

$$D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then  $De_1 = e_1$ ,  $De_2 = e_1 + e_2$ ,  $e_1 \vee e_2 = e_1 + e_2$ , and  $D(e_1 \vee e_2) = 2e_1 + e_2$  is not the same as  $(De_1) \vee (De_2) = e_1 + e_2$ .

*Remark 2.5.* Although we do not think that we use this anywhere, the homomorphisms  $\pi_m$  are, in fact, injective. One way to see this is to use the Nica-covariance of  $n \mapsto V_{m,n}$  to get a homomorphism  $\pi_{V_m} : \mathcal{T}(\mathbb{N}^k) \rightarrow B_{\theta^m}$  and to interpret (2.1) as saying that  $(\pi_{V_m}, U_m)$  is a covariant representation of a dynamical system  $(\mathcal{T}(\mathbb{N}^k), \mathbb{Z}^d, \gamma^m)$  in the algebra  $B_{\theta^m}$ . Then  $B_{\theta^m}$  has the universal property which characterizes the crossed product  $\mathcal{T}(\mathbb{N}^k) \rtimes_{\gamma^m} \mathbb{Z}^d$ , and we can deduce from the equivariant uniqueness theorem for the crossed product (for example, [26, Corollary 4.3]) that the representation

$$\pi_{D_m, E_m} := \pi_{V_{m+1} \circ D_m} \rtimes (U_{m+1} \circ E_m)$$

of  $\mathcal{T}^k(\mathbb{N}^k) \rtimes_{\gamma^m} \mathbb{Z}^d$  in  $\mathcal{T}^k(\mathbb{N}^k) \rtimes_{\gamma_{m+1}} \mathbb{Z}^d$  is faithful.

We now define our *higher-rank Toeplitz non-commutative solenoid* to be the direct limit

$$B_\infty := \varinjlim_{m \in \mathbb{N}} (B_m, \pi_m). \tag{2.6}$$

We write  $\pi_{m,\infty}$  for the canonical homomorphism of  $B_m$  into  $B_\infty$ . To ease notation, we also write  $U_{m,n}$  for the image  $\pi_{m,\infty}(U_{m,n})$  in  $B_\infty$ .

Now we use the vectors  $r^m \in (0, \infty)^k$  from our set-up to define the dynamics we propose to study.

PROPOSITION 2.6. *There is a dynamics  $\alpha : \mathbb{R} \rightarrow \text{Aut } B_\infty$  such that*

$$\alpha_t(V_{m,p}U_{m,n}V_{m,q}^*) = e^{it(p-q)^T r^m} V_{m,p}U_{m,n}V_{m,q}^*. \tag{2.7}$$

*Proof.* Since  $U_m$  and  $V'_m : p \mapsto e^{itp^T r^m} V_{m,p}$  satisfy the same relations in  $B_m$  as  $U_m$  and  $V_m$ , there is a dynamics  $\alpha^{r^m} : \mathbb{R} \rightarrow \text{Aut } B_m$  such that

$$\alpha^{r^m}(V_{m,p}U_{m,n}V_{m,q}^*) = e^{it(p-q)^T r^m} V_{m,p}U_{m,n}V_{m,q}^*.$$

We claim that  $\pi_m \circ \alpha_t^{r^{m+1}} = \alpha_t^{r^{m+1}} \circ \pi_m$ . To see this, we compute on generators. First, for  $n \in \mathbb{Z}^d$ ,

$$\begin{aligned} \alpha_t^{r^{m+1}}(\pi_m(U_{m,n})) &= \alpha_t^{r^{m+1}}(U_{m+1, E_m n}) = U_{m+1, E_m n} \\ &= \pi_m(U_{m,n}) = \pi_m(\alpha_t^{r^m}(U_{m,n})). \end{aligned}$$

Second, for  $p \in \mathbb{N}^k$ , and using the relation (2.4) at the crucial step to pass from  $r^{m+1}$  to  $r^m$ , we have

$$\begin{aligned} \alpha_t^{r^{m+1}}(\pi_m(V_{m,p})) &= \alpha_t^{r^{m+1}}(V_{m+1, D_m p}) = e^{it(D_m p)^T r^{m+1}} V_{m+1, D_m p} \\ &= e^{itp^T D_m r^{m+1}} \pi_m(V_{m,p}) = e^{itp^T r^m} \pi_m(V_{m,p}) \\ &= \pi_m(\alpha_t^{r^m}(V_{m,p})). \end{aligned}$$

Now the universal property of the direct limit implies that, for each  $t \in \mathbb{R}$ , there is an automorphism  $\alpha_t$  of  $B_\infty$  such that  $\alpha_t \circ \pi_{m,\infty} = \pi_{m,\infty} \circ \alpha_t^{r^m}$ . The formula (2.7) (which implicitly involves the homomorphisms  $\pi_{m,\infty}$ ) implies that  $t \mapsto \alpha_t$  is a strongly continuous action of  $\mathbb{R}$  on  $B_\infty$ . □

Our goal is to describe the KMS states of the dynamical system  $(B_\infty, \alpha)$ . But first we pause to establish some conventions about probability measures on inverse limits.

Remark 2.7. All measures in this paper are positive Borel measures. We view probability measures on a compact space  $X$  as states on the  $C^*$ -algebra  $C(X)$  of continuous functions. We write  $P(X)$  for the set of probability measures on  $X$ .

When  $\{h_m : X_{m+1} \rightarrow X_m : m \in \mathbb{N}\}$  is an inverse system of compact spaces with each  $h_m$  surjective, the inverse limit  $\varprojlim (X_m, h_m)$  is also a compact space. We write  $h_{m,\infty}$  for the canonical map of  $X_\infty := \varprojlim (X_m, h_m)$  onto  $X_m$ , so that we have  $h_{m,\infty} = h_m \circ h_{m+1,\infty}$  for all  $m \in \mathbb{N}$ . The maps  $h_{m,\infty}$  induce maps  $h_{m,\infty*}$  on measures: if  $\mu$  is a probability measure on  $X_\infty$ , then  $\mu_m := h_{m,\infty*}(\mu)$  is the measure on  $X_m$  such that

$$\int_{X_m} f \, d\mu_m = \int_{X_\infty} (f \circ h_{m,\infty}) \, d\mu \quad \text{for } f \in C(X_m).$$

Conversely, because each  $h_m$  is surjective, for any sequence of probability measures  $\{\mu_m \in P(X_m) : m \in \mathbb{N}\}$  such that  $\mu_m = h_{m*}(\mu_{m+1})$  for all  $m$ , there is a probability measure  $\mu \in P(X_\infty)$  such that  $\mu_m = h_{m,\infty*}(\mu)$  for all  $m$  (see [5, Lemma 6.1], for example). Thus the simplices  $P(\varprojlim X_m)$  and  $\varprojlim P(X_m)$  are canonically isomorphic.

To state our main result, we need to observe that, because the entries in the  $E_m$  are integers, multiplication by  $E_m^T$  on  $\mathbb{R}^d$  maps  $\mathbb{Z}^d$  into  $\mathbb{Z}^d$  and hence induces a homomorphism  $E_m^T$  of  $\mathbb{S}^d = \mathbb{R}^d/\mathbb{Z}^d$  onto itself. We show that the KMS states are parametrized by the probability measures on the inverse limit  $\varprojlim(\mathbb{S}^d, E_m^T)$ , which is an ordinary solenoid. We write  $E_{m,\infty}^T$  for the projection of  $\varprojlim(\mathbb{S}^d, E_m^T)$  on the  $m$ th copy of  $\mathbb{S}^d$ , so that

$$E_{m,\infty}^T = E_m^T \circ E_{m+1,\infty}^T \quad \text{for } m \in \mathbb{N}.$$

The main theorem of this paper is the following; we prove it at the end of the paper.

**THEOREM 2.8.** *Suppose that  $\mu \in P(\varprojlim(\mathbb{S}^d, E_m^T))$  and  $\beta > 0$ . Let  $\{\mu_m\}$  be the corresponding sequence of probability measures on  $\mathbb{S}^d$ . For  $m \in \mathbb{N}$  and  $n \in \mathbb{N}^d$ , we define the moment  $M_{m,n}(\mu)$  to be the number*

$$M_{m,n}(\mu) = \int_{\mathbb{S}^d} e^{2\pi i x^T n} d\mu_m(x) = \int_{\varprojlim(\mathbb{S}^d, E_m^T)} e^{2\pi i E_{m,\infty}^T(x)^T n} d\mu(x).$$

Then there is a  $KMS_\beta$  state  $\psi_\mu$  on  $(B_\infty, \alpha)$  such that

$$\psi_\mu(V_{m,p} U_{m,n} V_{m,q}^*) = \delta_{p,q} e^{-\beta p^T r^m} \prod_{j=1}^k \frac{\beta r_j^m}{\beta r_j^m - 2\pi i (\theta_m^T n)_j} M_{m,n}(\mu). \tag{2.8}$$

The map  $\mu \mapsto \psi_\mu$  is an affine homeomorphism of  $P(\varprojlim(\mathbb{S}^d, E_m^T))$  onto the simplex  $KMS_\beta(B_\infty, \alpha)$  of  $KMS_\beta$  states.

*Remark 2.9.* As a reality check, we take  $p = q = 0$  and  $n = 0$ . Then  $V_{m,p} U_{m,n} V_{m,q}^*$  is the identity  $1_{B_m} = 1_{B_\infty}$ , and our formula collapses to  $\psi_\mu(1) = 1$ .

*Remark 2.10.* It is interesting to set  $d = k = 1$  and compare the formula (2.8) with the formula (6.4) in [6, Theorem 6.9], which, on the face of it, looks different. The point is that the integral on the right-hand side of [6, (6.4)] is with respect to the subinvariant measure associated to the probability measure  $\mu$ , which in our notation would be  $\nu_{\mu_m}$ . There is no specific description for this measure in [6]: they get an isomorphism of the simplex  $P(\varprojlim \mathbb{S})$  onto the simplex of subinvariant measures by specifying it on the extreme points (see [6, Lemma 8.2]). We reconcile the two approaches in Remark 5.3.

### 3. Equilibrium states on a Toeplitz non-commutative torus

In this section, we fix  $\theta \in M_{k,d}([0, \infty))$ , and we investigate the KMS states on the Toeplitz non-commutative torus  $B_\theta$ .

For  $n \in \mathbb{Z}^d$ , we write  $g_n$  for the character on  $\mathbb{S}^d$  given by  $g_n(x) = e^{2\pi i x^T n}$ , and we write  $\iota : C(\mathbb{S}^d) \rightarrow C^*(\mathbb{Z}^d) \subset B_\theta$  for the isomorphism such that  $\iota(g_n) = U_n$ . Then

$$B_\theta = \overline{\text{span}}\{V_p \iota(f) V_q^* : f \in C(\mathbb{S}^d), p, q \in \mathbb{N}^k\}.$$

For  $y \in \mathbb{R}^d$ , we define  $R_y : \mathbb{S}^d \rightarrow \mathbb{S}^d$  by  $R_y(x) = x + y$ . Later, we will also write  $R_y^*$  for the automorphism of  $C(\mathbb{S}^d)$  given by  $R_y^* f = f \circ R_y$ , and we write  $R_{y*}$  for the dual map on measures defined by

$$\int_{\mathbb{S}^d} f dR_{y*}(\mu) = \int_{\mathbb{S}^d} R_y^*(f) d\mu = \int_{\mathbb{S}^d} f \circ R_y d\mu.$$

The assignment  $y \mapsto R_y^*$  is a strongly continuous action  $R$  of  $\mathbb{R}^d$  on  $C(\mathbb{S}^d)$ , and each  $R_{y*}$  is norm-preserving.

LEMMA 3.1. For  $f \in C(\mathbb{S}^d)$  and  $p \in \mathbb{N}^k$ ,

$$V_p \iota(f) = \iota(f \circ R_{-\theta^T p}) V_p \quad \text{and} \quad V_p^* \iota(f) = \iota(f \circ R_{\theta^T p}) V_p^*. \tag{3.1}$$

*Proof.* Since  $C(\mathbb{S}^d) = \overline{\text{span}}\{g_n : x \mapsto e^{2\pi i x^T n} : n \in \mathbb{Z}^d\}$ , it suffices to check (3.1) for  $f = g_n$ . Let  $n \in \mathbb{Z}^d$ . Then (2.1) gives

$$V_p \iota(g_n) = V_p U_n = e^{-2\pi i p^T \theta n} U_n V_p = e^{-2\pi i p^T \theta n} \iota(g_n) V_p.$$

Since

$$e^{-2\pi i p^T \theta n} g_n(x) = e^{-2\pi i p^T \theta n} e^{2\pi i x^T n} = g_n(x - \theta^T p) = (g_n \circ R_{-\theta^T p})(x),$$

the first equality follows. The second follows from a similar computation using (2.2).  $\square$

*Remark 3.2.* The minus sign in the first identity in (3.1) is crucial. As a reality check, notice that the signs in the two formulas have to be different, because  $V_p^* V_p = 1$  means the  $\pm \theta^T p$  have to cancel. As a corollary, note that  $V_p V_p^*$ , which is a proper projection, commutes with the  $\iota(f)$ . (To see that  $V_p V_p^* \neq 1$ , we can use the specific representation of  $B_\theta$  constructed in the proof of Proposition 3.7(b).)

We now fix  $r \in (0, \infty)^k$ . The universal property of  $B_\theta$  gives a dynamics  $\alpha^r : \mathbb{R} \rightarrow \text{Aut } B_\theta$  such that

$$\alpha_t^r(U_n) = U_n \quad \text{and} \quad \alpha_t^r(V_p) = e^{itp^T r} V_p \quad \text{for } n \in \mathbb{Z}^d, p \in \mathbb{N}^k, t \in \mathbb{R}. \tag{3.2}$$

Then  $\alpha_t^r(V_p U_n V_q^*) = e^{it(p-q)^T r} V_p U_n V_q^*$ , and hence

$$\{V_p U_n V_q^* : n \in \mathbb{Z}^d, p, q \in \mathbb{N}^k\}$$

is a set of  $\alpha^r$ -analytic elements spanning an  $\alpha^r$ -invariant dense subset of  $B_\theta$ .

To describe the  $\text{KMS}_\beta$  states of  $(B_\theta, \alpha^r)$ , it was tempting to apply [3, Theorem 6.1] to the Toeplitz algebra of the commuting homeomorphisms  $h_j : x \mapsto x + \theta_j$  associated to the rows  $\theta_j$  of  $\theta$ . That result is, in several ways, more general than we need, but it has an unfortunate hypothesis of rational independence on the set  $\{r_j\}$  which we prefer to avoid.

PROPOSITION 3.3. Suppose that  $\beta > 0$  and  $\phi$  is a state of  $(B_\theta, \alpha^r)$ . Then  $\phi$  is a  $\text{KMS}_\beta$  state of  $(B_\theta, \alpha^r)$  if and only if

$$\phi(V_p U_n V_q^*) = \delta_{p,q} e^{-\beta p^T r} \phi(U_n) \quad \text{for } n \in \mathbb{Z}^d \text{ and } p, q \in \mathbb{N}^k. \tag{3.3}$$

To prove Proposition 3.3, we need two lemmas. The arguments are based on [16, proofs of Lemmas 5.2 and 5.3].

LEMMA 3.4. Suppose that  $\beta > 0$  and  $\phi$  is a  $\text{KMS}_\beta$  state of  $(B_\theta, \alpha^r)$ . If  $p, q \in \mathbb{N}^k$  satisfy  $p^T r = q^T r$ , then:

- (a)  $\phi(V_p U_n V_p^*) = \phi(V_q U_n V_q^*)$  for  $n \in \mathbb{Z}^d$ ; and
- (b)  $|\phi(V_p \iota(f) V_q^*)| \leq \phi(V_p \iota(f) V_p^*)$  for positive  $f \in C(\mathbb{S}^d)$ .



*Proof.* For (a), since  $V_q$  is an isometry,

$$\phi(V_p U_n V_p^*) = \phi(V_p U_n (V_q^* V_q) V_p^*) = \phi((V_p U_n V_q^*)(V_q V_p^*)),$$

and since  $p^T r = q^T r$ , the KMS condition gives

$$\phi(V_p U_n V_p^*) = e^{-\beta(p-q)^T r} \phi(V_q V_p^* (V_p U_n V_q^*)) = \phi(V_q U_n V_q^*).$$

For (b), we take a positive function  $f$  in  $C(\mathbb{S}^d)$ . By linearity and continuity, part (a) implies that  $\phi(V_q \iota(f) V_q^*) = \phi(V_p \iota(f) V_p^*)$ . Using the Cauchy–Schwarz inequality at the second step, we calculate

$$\begin{aligned} |\phi(V_p \iota(f) V_q^*)|^2 &= |\phi((V_p \iota(\sqrt{f})) (V_q \iota(\sqrt{f}))^*)|^2 \\ &\leq \phi(V_p \iota(f) V_p^*) \phi(V_q \iota(f) V_q^*) \\ &= \phi(V_p \iota(f) V_p^*)^2. \end{aligned}$$

Since both sides are the squares of non-negative numbers, we can take square roots, and we retrieve (b). □

LEMMA 3.5. *Suppose that  $\beta > 0$  and  $\phi$  is a  $KMS_\beta$  state of  $(B_\theta, \alpha^r)$ . Suppose that  $p, q \in \mathbb{N}^k$  satisfy  $p^T r = q^T r$  and that  $f \in C(\mathbb{S}^d)$ . Write  $P := (p \vee q) - p$ . Then*

$$\phi(V_p \iota(f) V_q^*) = \phi(V_{p+lP} \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*) \quad \text{for all } l \in \mathbb{N}. \tag{3.4}$$

If  $p \neq q$ , then  $\phi(V_p \iota(f) V_q^*) = 0$ .

*Proof.* We prove (3.4) by induction on  $l$ . The base case  $l = 0$  is trivial. Now suppose that (3.4) holds for  $l \geq 0$ . The inductive hypothesis gives

$$\begin{aligned} \phi(V_p \iota(f) V_q^*) &= \phi(V_{p+lP} \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*) \\ &= \phi(V_{p+lP} \iota(f \circ R_{l\theta^T P}) V_{q+lP}^* V_{q+lP} V_{q+lP}^*). \end{aligned}$$

Since the dynamics  $\alpha^r$  fixes the element  $V_{q+lP} V_{q+lP}^*$ , the KMS condition implies that

$$\phi(V_p \iota(f) V_q^*) = \phi(V_{q+lP} V_{q+lP}^* V_{p+lP} \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*),$$

and Nica-covariance gives

$$\begin{aligned} \phi(V_p \iota(f) V_q^*) &= \phi(V_{q+lP} V_{((q+lP) \vee (p+lP)) - (q+lP)} V_{((q+lP) \vee (p+lP)) - (p+lP)}^* \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*). \end{aligned}$$

For  $c \in \mathbb{N}^k$ , we have  $(p + c) \vee (q + c) = (p \vee q) + c$ . Thus

$$\begin{aligned} \phi(V_p \iota(f) V_q^*) &= \phi(V_{q+lP} V_{(p \vee q) - q} V_{(p \vee q) - p}^* \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*) \\ &= \phi(V_{(p \vee q) + lP} V_p^* \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*) \\ &= \phi(V_{(p \vee q) + lP} \iota(f \circ R_{l\theta^T P} \circ R_{\theta^T P}) V_{q+(l+1)P}^*) \quad \text{by Lemma 3.1} \\ &= \phi(V_{p+(l+1)P} \iota(f \circ R_{(l+1)\theta^T P}) V_{q+(l+1)P}^*) \end{aligned}$$

because  $(p \vee q) + lP = p + (l + 1)P$ . This completes the inductive step, and hence the proof of (3.4).

Now suppose that  $p \neq q$ . Then at least one of  $P$  and  $(p \vee q) - q$  is non-zero. We argue the case where  $P \neq 0$ , and the other case follows by taking adjoints. For  $l \in \mathbb{N}$ ,

$$\begin{aligned} |\phi(V_p \iota(f) V_q^*)| &= |\phi(V_{p+lP} \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*)| \\ &\leq \phi(V_{p+lP} \iota(f \circ R_{l\theta^T P}) V_{p+lP}^*) \text{ by Lemma 3.4(b)} \\ &= e^{-\beta(p+lP)T r} \phi(V_{p+lP}^* V_{p+lP} \iota(f \circ R_{l\theta^T P})) \\ &= e^{-\beta(p+lP)T r} \phi(\iota(f \circ R_{l\theta^T P})) \\ &\leq e^{-\beta(p+lP)T r} \|f\|_\infty. \end{aligned}$$

Since  $P > 0$  and  $r \in (0, \infty)^k$ , we have  $(p + lP)^T r \rightarrow \infty$  as  $l \rightarrow \infty$ , and hence  $e^{-\beta(p+lP)T r} \|f\|_\infty \rightarrow 0$  as  $l \rightarrow \infty$ . Thus  $\phi(V_p \iota(f) V_q^*) = 0$ .  $\square$

*Proof of Proposition 3.3.* First, suppose that  $\phi$  is a  $\text{KMS}_\beta$  state for  $(B_\theta, \alpha^r)$ . For  $n \in \mathbb{Z}^d$  and  $p, q \in \mathbb{N}^k$ , two applications of the KMS condition give

$$\phi(V_p U_n V_q^*) = e^{-\beta p^T r} \phi(U_n V_q^* V_p) = e^{-\beta(p-q)^T r} \phi(V_p U_n V_q^*). \tag{3.5}$$

It follows immediately that if  $(p - q)^T r \neq 0$ , then  $\phi(V_p U_n V_q^*) = 0$ . If  $(p - q)^T r = 0$  but  $p \neq q$ , then Lemma 3.5 gives  $\phi(V_p U_n V_q^*) = 0$ . This, combined with the first equality in (3.5), gives

$$\phi(V_p U_n V_q^*) = \delta_{p,q} e^{-\beta p^T r} \phi(U_n V_q^* V_p) = \delta_{p,q} e^{-\beta p^T r} \phi(U_n)$$

because  $V_p$  is an isometry. This is the desired formula (3.3).

Now suppose that  $\phi$  is a state satisfying (3.3). Since the  $V_p U_n V_q^*$  are analytic elements spanning a dense  $\alpha^r$ -invariant subspace of  $B_\theta$ , it suffices to fix  $p, q, b, c \in \mathbb{N}^k$  and  $n, n' \in \mathbb{Z}^d$  and to show that

$$\phi(V_p U_n V_q^* V_b U_{n'} V_c^*) = e^{-\beta(p-q)^T r} \phi(V_b U_{n'} V_c^* V_p U_n V_q^*). \tag{3.6}$$

Let  $P := (q \vee b) - b$  and  $Q := (q \vee b) - q$ . Then  $P, Q \in \mathbb{N}^k$  are the unique elements such that  $P \wedge Q = 0$  and  $P + b = Q + q$ , and Nica-covariance says that  $V_q^* V_b = V_Q V_P^*$ . Now, using first the identities (2.1) and (2.2) and then (at the last step) the assumption (3.3), we calculate

$$\begin{aligned} \phi(V_p U_n V_q^* V_b U_{n'} V_c^*) &= \phi(V_p U_n V_Q V_P^* U_{n'} V_c^*) \\ &= e^{2\pi i Q^T \theta n} \phi(V_p (V_Q U_n) V_P^* U_{n'} V_c^*) \\ &= e^{2\pi i (Q^T \theta n + P^T \theta n')} \phi(V_{Q+P} U_{n+n'} V_{P+c}^*) \\ &= \delta_{Q+P, P+c} e^{-\beta(Q+p)^T r} e^{2\pi i (Q^T \theta n + P^T \theta n')} \phi(U_{n+n'}). \end{aligned} \tag{3.7}$$

Similarly, let  $M := (c \vee p) - p$  and  $N := (c \vee p) - c$ . Then  $M, N \in \mathbb{N}^k$  are the unique elements such that  $M \wedge N = 0$  and  $M + p = N + c$ , and the right-hand side of (3.6) is

$$\begin{aligned} e^{-\beta(p-q)^T r} \phi(V_b U_{n'} V_N V_M^* U_n V_q^*) &= e^{-\beta(p-q)^T r} e^{2\pi i (N^T \theta n' + M^T \theta n)} \phi(V_{b+N} U_{n+n'} V_{q+M}^*) \\ &= \delta_{N+b, M+q} e^{-\beta(p-q+b+N)^T r} e^{2\pi i (N^T \theta n' + M^T \theta n)} \phi(U_{n+n'}). \end{aligned} \tag{3.8}$$

To see that (3.7) is equal to (3.8), we first show that the two Kronecker deltas have the same value. For this, observe that, by definition of  $M, N, P, Q$ ,

$$(P + b) + (N + c) = (Q + q) + (M + p),$$

and, consequently,  $(N + b) - (M + q) = (Q + p) - (P + c)$ . Thus  $\delta_{Q+p, P+c} = 1$  if and only if  $\delta_{N+b, M+q} = 1$ . So it now suffices to prove that (3.7) equals (3.8) when  $Q + p = P + c$  and  $N + b = M + q$ .

We first claim that  $M = Q$  and  $N = P$ . By assumption, we have  $M + q = N + b$ , and we have  $P + b = Q + q$  by definition of  $P, Q$ . Subtracting these equations, we obtain  $M - Q = N - P$ , and rearranging gives  $M - N = Q - P$ . Since  $P \wedge Q = 0$  and  $M \wedge N = 0$ , we deduce that  $Q = (Q - P) \vee 0 = (M - N) \vee 0 = M$ , and then  $P = N$  too, as claimed.

We now have

$$e^{2\pi i(Q^T \theta_n + P^T \theta_{n'})} = e^{2\pi i(M^T \theta_n + N^T \theta_{n'})},$$

and so it remains to check that

$$e^{-\beta(p-q+b+N)^T r} = e^{-\beta(p+Q)^T r}.$$

For this, we apply  $N = P$ , from above, at the second equality and  $b + P = q + Q$ , by definition of  $Q, P$ , at the third to get

$$\begin{aligned} (p - q) + (b + N) &= p + (b + N - q) = p + (b + P - q) \\ &= p + (q + Q - q) = p + Q, \end{aligned}$$

which gives the result. Thus  $\phi$  is a  $\text{KMS}_\beta$  state. □

LEMMA 3.6. Write  $\theta_j$  for the  $j$ th row of  $\theta$ . Then the series

$$\sum_{p \in \mathbb{N}^k} e^{-\beta p^T r} R_{\theta^T p^*} \tag{3.9}$$

converges in the operator norm of  $B(C(\mathbb{S}^d))$  to an inverse for  $\prod_{j=1}^k (\text{id} - e^{-\beta r_j} R_{\theta_j^T *})$ .

*Proof.* We first need to understand the sum (3.9), which we want to calculate as an iterated sum. So we interpret (3.9) as a  $B(C(\mathbb{S}^d))$ -valued integral over  $\mathbb{N}^k$  with respect to counting measure  $\sigma$  (for which all functions on  $\mathbb{N}^k$  are measurable). Since each  $R_{\theta^T p}$  is norm-preserving,

$$\|e^{-\beta p^T r} R_{\theta^T p^*}\| = e^{-\beta p^T r} = \prod_{j=1}^k e^{-\beta p_j r_j}.$$

By Tonelli's theorem,

$$\begin{aligned} \sum_{p \in \mathbb{N}^k} \|e^{-\beta p^T r} R_{\theta^T p^*}\| &= \sum_{p_k=0}^\infty \cdots \sum_{p_1=0}^\infty \left( \prod_{j=1}^k e^{-\beta p_j r_j} \right) \\ &= \sum_{p_k=0}^\infty \cdots \sum_{p_2=0}^\infty \left( \prod_{j=2}^k e^{-\beta p_j r_j} \right) \left( \sum_{p_1=0}^\infty e^{-\beta p_1 r_1} \right) \\ &= \sum_{p_k=0}^\infty \cdots \sum_{p_2=0}^\infty \left( \prod_{j=2}^k e^{-\beta p_j r_j} \right) (1 - e^{-\beta r_1})^{-1}. \end{aligned}$$

Repeating this  $k - 1$  more times gives

$$\sum_{p \in \mathbb{N}^k} \|e^{-\beta p^T r} R_{\theta_j^T p^*}\| = \prod_{j=1}^k (1 - e^{-\beta r_j})^{-1}.$$

Thus the function  $p \mapsto e^{-\beta p^T r} R_{\theta_j^T p^*}$  is integrable with respect to  $\sigma$ , and Fubini’s theorem for functions with values in a Banach space (for example, [12, Theorem II.16.3]) implies that

$$\begin{aligned} \sum_{p \in \mathbb{N}^k} e^{-\beta p^T r} R_{\theta^T p^*} &= \sum_{p_k=0}^{\infty} \cdots \sum_{p_1=0}^{\infty} \left( \prod_{j=1}^k e^{-\beta p_j r_j} R_{p_j \theta_j^T} \right) \\ &= \prod_{j=1}^k \left( \sum_{p_j=0}^{\infty} (e^{-\beta r_j} R_{\theta_j^T})^{p_j} \right). \end{aligned}$$

Writing the infinite sum as a limit of partial sums shows that

$$\sum_{p_j=0}^{\infty} (e^{-\beta r_j} R_{\theta_j^T})^{p_j} (\text{id} - e^{-\beta r_j} R_{\theta_j^T}) = \text{id}. \tag{3.10}$$

To simplify the product

$$\left( \prod_{j=1}^k \left( \sum_{p_j=0}^{\infty} (e^{-\beta r_j} R_{\theta_j^T})^{p_j} \right) \right) \left( \prod_{j=1}^k (\text{id} - e^{-\beta r_j} R_{\theta_j^T}) \right),$$

we write the left-hand product from  $j = k$  to  $j = 1$  and the right-hand one from  $j = 1$  to  $j = k$ . Now  $k$  applications of (3.10) show that the product telescopes to the identity  $\text{id}$  of  $B(C(\mathbb{S}^d))$ . □

The next proposition is an analogue of [16, Theorem 6.1] and [3, Theorem 6.1].

PROPOSITION 3.7. Fix  $\beta \in (0, \infty)$ .

(a) Suppose that  $\phi$  is a  $KMS_{\beta}$  state for  $(B_{\theta}, \alpha^r)$ , and let  $\nu \in P(\mathbb{S}^d)$  be the measure such that

$$\phi(\iota(f)) = \int_{\mathbb{S}^d} f \, d\nu \quad \text{for } f \in C(\mathbb{S}^d).$$

Suppose that  $F \subset \mathbb{N}^k$  is a finite set such that  $p \neq q \in F$  implies  $p \wedge q = 0$ . Then the measure  $\nu$  satisfies the subinvariance relation

$$\prod_{p \in F} (\text{id} - e^{-\beta p^T r} R_{\theta^T p^*})(\nu) \geq 0. \tag{3.11}$$

(b) Define  $y_{\beta} := \sum_{p \in \mathbb{N}^k} e^{-\beta p^T r}$ , and suppose that  $\kappa$  is a positive measure on  $\mathbb{S}^d$  with total mass  $y_{\beta}^{-1}$ . Write  $\theta_j$  for the  $j$ th row of  $\theta$ . Then

$$\nu = \nu_{\kappa} := \prod_{j=1}^k (\text{id} - e^{-\beta r_j} R_{\theta_j^T})^{-1}(\kappa)$$

is a subinvariant probability measure, and there is a  $KMS_\beta$  state  $\phi_\nu$  of  $(B_\theta, \alpha^r)$  such that

$$\phi_\nu(V_p \iota(f) V_q^*) = \delta_{p,q} e^{-\beta p^T r} \int_{\mathbb{S}^d} f \, d\nu \quad \text{for } p, q \in \mathbb{N}^k \text{ and } f \in C(\mathbb{S}^d). \quad (3.12)$$

(c) The map  $\kappa \mapsto \phi_{\nu_\kappa}$  is an affine isomorphism of the simplex

$$\Sigma_{\beta,r} = \{\text{positive measures } \kappa : \|\kappa\| = y_\beta^{-1}\}$$

onto the simplex of  $KMS_\beta$  states of  $(B_\theta, \alpha^r)$ .

*Proof.* (a) We take a positive function  $f \in C(\mathbb{S}^d)_+$  and compute

$$\begin{aligned} \int_{\mathbb{S}^d} f \, d\left(\prod_{p \in F} (\text{id} - e^{-\beta p^T r} R_{\theta^T p^*})(\nu)\right) &= \int_{\mathbb{S}^d} f \circ \left(\prod_{p \in F} (\text{id} - e^{-\beta p^T r} R_{\theta^T p})\right) \, d\nu \\ &= \int_{\mathbb{S}^d} \sum_{S \subset F} (-1)^{|S|} \left(\prod_{p \in S} e^{-\beta p^T r}\right) \left(f \circ \prod_{p \in S} R_{\theta^T p}\right) \, d\nu. \end{aligned} \quad (3.13)$$

We write  $p_S := \sum_{p \in S} p$ , and we observe that  $\prod_{p \in S} e^{-\beta p^T r} = e^{-\beta p_S^T r}$  and  $\prod_{p \in S} R_{\theta^T p} = R_{\theta^T p_S}$ . Thus

$$\begin{aligned} (3.13) &= \int_{\mathbb{S}^d} \sum_{S \subset F} (-1)^{|S|} e^{-\beta p_S^T r} (f \circ R_{\theta^T p_S}) \, d\nu \\ &= \phi\left(\sum_{S \subset F} (-1)^{|S|} e^{-\beta p_S^T r} \iota(f \circ R_{\theta^T p_S}) V_{p_S}^* V_{p_S}\right) \quad \text{since } V_{p_S}^* V_{p_S} = 1 \\ &= \phi\left(\sum_{S \subset F} (-1)^{|S|} V_{p_S} \iota(f \circ R_{\theta^T p_S}) V_{p_S}^*\right) \quad \text{by the KMS condition} \\ &= \phi\left(\sum_{S \subset F} (-1)^{|S|} V_{p_S} V_{p_S}^* \iota(f)\right) \quad \text{by (3.1).} \end{aligned}$$

Because the set  $F$  has the property that  $p \wedge q = 0$  for  $p \neq q \in F$ , Nica-covariance gives  $V_p V_p^* V_q V_q^* = V_{p \vee q} V_{p \vee q}^* = V_{p+q} V_{p+q}^*$  for  $p \neq q \in F$ . Thus, for each  $S \subset F$ , we have  $V_{p_S} V_{p_S}^* = \prod_{p \in S} V_p V_p^*$ , and

$$\sum_{S \subset F} (-1)^{|S|} V_{p_S} V_{p_S}^* = \prod_{p \in F} (1 - V_p V_p^*).$$

The latter product is a projection, and it is fixed by the action  $\alpha$ . Hence another application of the KMS condition gives

$$\begin{aligned} \int_{\mathbb{S}^d} f \, d\left(\prod_{p \in F} (\text{id} - e^{-\beta p^T r} R_{\theta^T p^*})(\nu)\right) &= \phi\left(\prod_{p \in F} (1 - V_p V_p^*) \iota(f)\right) \\ &= \phi\left(\left(\prod_{p \in F} (1 - V_p V_p^*)\right)^2 \iota(f)\right) \\ &= \phi\left(\prod_{p \in F} (1 - V_p V_p^*) \iota(f) \prod_{p \in F} (1 - V_p V_p^*)\right). \end{aligned}$$

This last term is positive because the argument of  $\phi$  is a positive element of  $B_\theta$ , and this proves (a).

(b) We have

$$\prod_{j=1} (\text{id} - e^{-\beta r_j} R_{\theta_j^*})(v) = \kappa \geq 0,$$

so  $\nu$  is subinvariant. By Lemma 3.6,

$$\begin{aligned} \int_{\mathbb{S}^d} 1 \, d\nu &= \int_{\mathbb{S}^d} 1 \, d\left(\sum_{p \in \mathbb{N}^k} e^{-\beta p^T} R_{\theta^T p^*}(\kappa)\right) \\ &= \sum_{p \in \mathbb{N}^k} e^{-\beta p^T r} \int_{\mathbb{S}^d} 1 \circ R_{\theta^T p} \, d\kappa \\ &= \sum_{p \in \mathbb{N}^k} e^{-\beta p^T r} \|\kappa\| = y_\beta \|\kappa\| = 1, \end{aligned} \tag{3.14}$$

and hence  $\nu$  is a probability measure.

We will build a  $\text{KMS}_\beta$  state using a representation of  $B_\theta$  on  $\ell^2(\mathbb{N}^k) \otimes L^2(\mathbb{S}^d, \kappa)$ . Recall that we write  $g_n$  for the trigonometric polynomial  $g_n(x) = e^{2\pi i x^T n}$ . Then the formula  $W_n f := g_n f$  defines a unitary representation  $W$  of  $\mathbb{Z}^d$  on  $L^2(\mathbb{S}^d, \kappa)$ . Write  $\{\delta_p : p \in \mathbb{N}^k\}$  for the orthonormal basis of point masses for  $\ell^2(\mathbb{N}^k)$ , and let  $D_n$  be the bounded operator such that  $D_n \delta_p := e^{2\pi i p^T \theta n} \delta_p$ . Then  $D$  is a unitary representation of  $\mathbb{Z}^d$  on  $\ell^2(\mathbb{N}^k)$ , and hence  $D \otimes W$  is a unitary representation of  $\mathbb{Z}^d$  on  $\ell^2(\mathbb{N}^k) \otimes L^2(\mathbb{S}^d, \kappa)$ .

Let  $T$  be the usual Toeplitz representation of  $\mathbb{N}^k$  by isometries on  $\ell^2(\mathbb{N}^k)$ . Then

$$(T_p \otimes 1)(D_n \otimes W_n)(\delta_q \otimes f) = e^{2\pi i q^T \theta n} (\delta_{p+q} \otimes W_n f),$$

and

$$\begin{aligned} (D_n \otimes W_n)(T_p \otimes 1)(\delta_q \otimes f) &= e^{2\pi i (p+q)^T \theta n} (\delta_{p+q} \otimes W_n f) \\ &= e^{2\pi i p^T \theta n} (T_p \otimes 1)(D_n \otimes W_n). \end{aligned}$$

Hence the universal property of  $B_\theta$  gives a representation

$$\pi : B_\theta \rightarrow B(\ell^2(\mathbb{N}^k) \otimes L^2(\mathbb{S}^d, \kappa))$$

such that  $\pi(U_n) = (D_n \otimes W_n)$  and  $\pi(V_p) = T_p \otimes 1$ .

Since  $\sum_{p \in \mathbb{N}^k} e^{-\beta p^T r}$  is convergent, there is a positive linear functional  $\phi_\nu : B_\theta \rightarrow \mathbb{C}$  such that

$$\phi_\nu(a) := \sum_{p \in \mathbb{N}^k} e^{-\beta p^T r} (\pi(a)(\delta_p \otimes 1) \mid \delta_p \otimes 1).$$

Then (3.14) implies that  $\phi_\nu(1) = 1$ , and  $\phi_\nu$  is a state. To see that  $\phi_\nu$  is a  $\text{KMS}_\beta$  state, we take  $p, q \in \mathbb{N}^k, n \in \mathbb{Z}^d$  and calculate

$$\begin{aligned} \phi_\nu(V_p \iota(g_n) V_q^*) &= \phi_\nu(V_p U_n V_q^*) \\ &= \sum_{b \in \mathbb{N}^k} e^{-\beta b^T r} ((D_n \otimes W_n)(T_q^* \delta_b \otimes 1) \mid T_p^* \delta_b \otimes 1) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{b \geq p \vee q} e^{-\beta b^T r} (e^{2\pi i(b-q)^T \theta n} \delta_{b-q} \otimes g_n \mid \delta_{b-p} \otimes 1) \\
 &= \delta_{p,q} \sum_{b \geq p} e^{-\beta b^T r} e^{2\pi i(b-p)^T \theta n} (g_n \mid 1) \\
 &= \delta_{p,q} \left( \sum_{b \in \mathbb{N}^k} e^{-\beta(b+p)^T r} e^{2\pi i b^T \theta n} \right) \int_{\mathbb{S}^d} g_n \, d\kappa. \tag{3.15}
 \end{aligned}$$

In particular,

$$\phi_\nu(\iota(g_n)) = \phi_\nu(U_n) = \left( \sum_{b \in \mathbb{N}^k} e^{-\beta b^T r} e^{2\pi i b^T \theta n} \right) \int_{\mathbb{S}^d} g_n \, d\kappa. \tag{3.16}$$

Thus

$$\phi_\nu(V_p U_n V_q^*) = \delta_{p,q} e^{-\beta p^T r} \phi_\nu(U_n),$$

and  $\phi_\nu$  is a  $\text{KMS}_\beta$  state by Proposition 3.3.

From (3.15),

$$\begin{aligned}
 \phi_\nu(U_n) &= \sum_{b \in \mathbb{N}^k} e^{-\beta b^T r} \int_{\mathbb{S}^d} e^{2\pi i(x+b^T \theta)^T n} \, d\kappa(x) \\
 &= \sum_{b \in \mathbb{N}^k} e^{-\beta b^T r} \int_{\mathbb{S}^d} g_n \circ R_{\theta^T b} \, d\kappa \\
 &= \int_{\mathbb{S}^d} g_n \, d \left( \sum_{b \in \mathbb{N}^k} e^{-\beta b^T r} R_{\theta^T b*}(\kappa) \right),
 \end{aligned}$$

which, by Lemma 3.6, is  $\int_{\mathbb{S}^d} g_n \, d\nu$ . Thus

$$\phi_\nu(V_p \iota(g_n) V_q^*) = \delta_{p,q} e^{-\beta p^T r} \phi_\nu(\iota(g_n)) = \delta_{p,q} e^{-\beta p^T r} \int_{\mathbb{S}^d} g_n \, d\nu.$$

Since  $C(\mathbb{S}^d) = \overline{\text{span}}\{g_n : n \in \mathbb{Z}^d\}$ , (3.12) follows from (3.16) and the linearity and continuity of  $\phi_\nu$ .

(c) We first observe that both maps  $\kappa \mapsto \nu_\kappa$  and  $\nu \mapsto \phi_\nu$  are affine, and hence so is the composition. To see that the composition is surjective, we take a  $\text{KMS}_\beta$  state  $\phi$ , restrict it to the range of  $\iota$  to get a measure  $\nu$  and take

$$\kappa = \prod_{j=1}^k (\text{id} - e^{-\beta r_j} R_{\theta_j*})(\nu).$$

Then the formula (3.3) implies that  $\phi$  and  $\phi_{\nu_\kappa}$  agree on the elements  $V_p \iota(f) V_q^*$ , and hence by linearity and continuity on all of  $B_\theta$ . Thus  $\phi = \phi_{\nu_\kappa}$ . The procedure that sends  $\phi$  to  $\kappa$  is weak\* continuous and inverts  $\kappa \mapsto \phi_{\nu_\kappa}$ . Thus it is a continuous bijection of one compact Hausdorff space onto another, and it is therefore a homeomorphism. Thus so is the inverse  $\kappa \mapsto \phi_{\nu_\kappa}$ . □

4. The subinvariance relation for the direct limit

We now return to the set-up in which the dynamics  $\alpha$  on the direct limit  $B_\infty$  is given by a sequence  $\{r^m\}$ .

Suppose that  $\phi$  is a  $\text{KMS}_\beta$  state of  $(B_\infty, \alpha)$  and  $\nu_m$  are the measures on  $\mathbb{S}^d$  that implement the restrictions of  $\phi \circ \pi_{m,\infty}$  to  $C(\mathbb{S}^d) \subset B_m$ . Since the embeddings  $\pi_m$  are all unital, so are the  $\pi_{m,\infty}$ . Thus, for each  $m$ , the restriction  $\phi \circ \pi_{m,\infty}$  is a  $\text{KMS}_1$  state of  $(B_m, \alpha^{r^m})$ , and hence is given by a probability measure  $\nu_m$  that satisfies the subinvariance relations for  $\theta = \theta_m$  in (3.11) parametrized by subsets  $F$  of  $\{1, \dots, k\}$ . But here, since  $\phi \circ \pi_{m,\infty} = \phi \circ \pi_{m+l,\infty} \circ \pi_{m,m+l}$  for  $l \in \mathbb{N}$ , the measure  $\nu_m$  satisfies a sequence of subinvariance relations parametrized by  $l$  as well as  $F$ . Our first main result says that these can be combined into one master subinvariance relation with real parameters  $s \in [0, \infty)^k$ .

We now describe our continuously parametrized subinvariance relation. For  $k = 1$ , this follows from [6, Definition 6.7 and Theorem 6.9].

**THEOREM 4.1.** *Suppose that  $\phi$  is a  $\text{KMS}_\beta$  state on  $(B_\infty, \alpha)$  and  $m \in \mathbb{N}$ . We write  $\iota_m$  for the inclusion of  $C(\mathbb{S}^d)$  in  $B_m$ , and then*

$$\iota_m(C(\mathbb{S}^d)) = \overline{\text{span}}\{U_{m,n} : n \in \mathbb{N}^d\}.$$

Let  $\nu_m$  be the probability measure on  $\mathbb{S}^d$  such that

$$\phi \circ \pi_{m,\infty}(\iota_m(f)) = \int_{\mathbb{S}^d} f d\nu_m \quad \text{for } f \in C(\mathbb{S}^d). \tag{4.1}$$

Write  $\theta_{m,j}$  for the  $j$ th row of the matrix  $\theta_m$ . Then, for every  $s \in [0, \infty)^k$ ,

$$\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j^m} R_{s_j \theta_{m,j}^T, *})(\nu_m) \geq 0. \tag{4.2}$$

We prove Theorem 4.1 at the end of this section. We first need two preliminary results.

The homomorphism  $\pi_m : B_m \rightarrow B_{m+1}$  maps  $\iota_m(C(\mathbb{S}^d))$  into  $\iota_{m+1}(C(\mathbb{S}^d))$ . When we view  $\iota_m(C(\mathbb{S}^d))$  as  $\overline{\text{span}}\{U_{m,n}\}$ , the homomorphism  $\pi_m$  is characterized by

$$\pi_m(U_{m,n}) = U_{m+1, E_m n},$$

and when we view  $\iota_m(C(\mathbb{S}^d))$  as  $\{\iota_m(f) : f \in C(\mathbb{S}^d)\}$ ,  $\pi_m$  is induced by the covering map  $E_m^T : \mathbb{S}^d \rightarrow \mathbb{S}^d$ . Hence we have  $\pi_m(\iota_m(f)) = \iota_{m+1}(f \circ E_m^T)$ . In particular,  $\pi_m|_{C(\mathbb{S}^d)}$  is  $(E_m^T)^* : f \mapsto f \circ E_m^T$ . The corresponding map on measures is given by  $E_{m*}^T$ : i.e.,

$$\int_{\mathbb{S}^d} f d\pi_m(\nu) = \int_{\mathbb{S}^d} f dE_{m*}^T(\nu) = \int_{\mathbb{S}^d} (f \circ E_m^T) d\nu.$$

**LEMMA 4.2.** *Suppose that  $\phi$  is a  $\text{KMS}_\beta$  state on  $(B_\infty, \alpha)$ . For  $m \in \mathbb{N}$ , let  $\nu_m$  be the probability measure on  $\mathbb{S}^d$  satisfying (4.1). Then, for every finite subset  $F$  of  $\mathbb{N}^k$  such that  $p \wedge q = 0$  for all  $p \neq q \in F$ ,*

$$\prod_{p \in F} (\text{id} - e^{-\beta(D_m^{-1} p)^T r^m} R_{\theta_m^T D_m^{-1} p, *})(\nu_m) \geq 0.$$



*Proof.* We apply Proposition 3.7(a) to the state  $\phi \circ \pi_{m+1, \infty}$  of  $(B_{m+1}, \alpha^{r^{m+1}})$ . We deduce that

$$\prod_{p \in F} (\text{id} - e^{-\beta p^T r^{m+1}} R_{\theta_{m+1}^T p^*})(v_{m+1}) \geq 0. \tag{4.3}$$

To convert this to a statement about  $v_m$ , we want to apply  $E_{m^*}^T$  to the left-hand side. We first observe that

$$\begin{aligned} E_m^T \circ R_{\theta_{m+1}^T p}(x) &= E_m^T x - E_m^T \theta_{m+1}^T p \\ &= E_m^T x - \theta_m^T D_m^{-1} p \quad \text{using (2.3)} \\ &= R_{\theta_m^T D_m^{-1} p} \circ E_m^T(x). \end{aligned} \tag{4.4}$$

Since  $E_{m^*}^T$  preserves positivity and  $h \mapsto h_*$  is covariant with respect to composition, (4.3) implies that

$$\begin{aligned} 0 &\leq E_{m^*}^T \left( \prod_{p \in F} (\text{id} - e^{-\beta p^T r^{m+1}} R_{\theta_{m+1}^T p^*})(v_{m+1}) \right) \\ &= \left( \prod_{p \in F} (\text{id} - e^{-\beta p^T r^{m+1}} R_{\theta_m^T D_m^{-1} p^*}) \right) \circ E_{m^*}^T(v_{m+1}) \quad \text{using (4.4)} \\ &= \prod_{p \in F} (\text{id} - e^{-\beta p^T (D_m)^{-1} r^m} R_{\theta_m^T D_m^{-1} p^*})(v_m) \quad \text{using (2.4)} \\ &= \prod_{p \in F} (\text{id} - e^{-\beta (D_m^{-1} p)^T r^m} R_{\theta_m^T D_m^{-1} p^*})(v_m). \end{aligned} \quad \square$$

For a positive integer  $l$ , we can apply the argument of Lemma 4.2 to the embedding  $\pi_{m, m+l}$  of  $B_m$  in  $B_{m+l}$ . This amounts to replacing the matrix  $D_m$  with  $D_{m, m+l} := D_{m+l-1} D_{m+l-2} \cdots D_{m+1} D_m$ ,  $E_m$  with a similarly defined  $E_{m, m+l}$ ,  $\theta_{m+1}$  with  $\theta_{m+l}$ , and  $r^{m+1}$  with  $r^{m+l}$ . We obtain the following corollary.

**COROLLARY 4.3.** *Suppose that  $\phi$  is a  $KMS_\beta$  state on  $(B_\infty, \alpha)$  and  $v_m$  is the probability measure satisfying (4.1). Then, for every positive integer  $l$  and for every finite subset  $F$  of  $\mathbb{N}^k$  such that  $p \wedge q = 0$  for all  $p \neq q \in F$ ,*

$$\prod_{p \in F} (\text{id} - e^{-\beta (D_{m, m+l}^{-1} p)^T r^m} R_{\theta_m^T D_{m, m+l}^{-1} p^*})(v_m) \geq 0.$$

*Proof of Theorem 4.1.* For each  $l \geq 0$  and  $p \in \mathbb{N}^k$ , we can apply Corollary 4.3 to the finite subset  $F_p := \{p_j e_j : 1 \leq j \leq k\}$  of  $\mathbb{N}^k$ . This gives us the subinvariance relation

$$\prod_{j=1}^k (\text{id} - e^{-\beta (D_{m, m+l}^{-1} p_j e_j)^T r^m} R_{\theta_m^T D_{m, m+l}^{-1} p_j e_j^*})(v_m) \geq 0. \tag{4.5}$$

Each factor in the left-hand side  $L$  of (4.5) has the form  $\text{id} - e^{-s} R_{v^*}$ . Since  $(e^{-s} R_{v^*})(e^{-t} R_{w^*}) = e^{-(s+t)} R_{(v+w)^*}$ , the product  $(\text{id} - e^{-s} R_{v^*})(\text{id} - e^{-t} R_{w^*})$  of two such terms collapses to

$$\text{id} - e^{-s} R_{v^*} - e^{-t} R_{w^*} + e^{-(s+t)} R_{(v+w)^*}.$$

Thus we can expand

$$L = \text{id} + \sum_{\emptyset \neq G \subset \{1, \dots, k\}} (-1)^{|G|} e^{-\beta(D_{m,m+l}^{-1} p_G)^T r^m} R_{\theta_m^T D_{m,m+l}^{-1} p_G^*}(\nu_m),$$

where  $p_G := \sum_{j \in G} p_j e_j$ .

For each fixed  $f \in C(\mathbb{S}^d)$  and  $\nu \in P(\mathbb{S}^d)$ , the function  $s \mapsto \int R_s(f) d\nu$  on  $\mathbb{R}^k$  is continuous, being the composition of the norm-continuous map  $s \mapsto R_s(f)$  and the bounded functional given by integration against  $\nu$ . We now consider a positive function  $f$  in  $C(\mathbb{S}^d)$ : we write  $f \in C(\mathbb{S}^d)_+$ . For  $s \in [0, \infty)^k$  and  $G \subset \{1, \dots, k\}$ , we write  $s_G = \sum_{j \in G} s_j e_j$ . Then

$$g_G : s \mapsto \int f d(e^{-\beta s_G^T r^m} R_{\theta_m^T s_G^*})(\nu_m)$$

is continuous, and so is the linear combination

$$L(s) := \int f d\left(\sum_{G \subset \{1, \dots, k\}} (-1)^{|G|} e^{-\beta s_G^T r^m} R_{\theta_m^T s_G^*}\right)(\nu_m).$$

The subinvariance relation (4.5) says that  $L(s) \geq 0$  for all  $s$  of the form  $D_{m,m+l}^{-1} p$  for  $l \geq 0$  and  $p \in \mathbb{N}^k$ .

Since each of the matrices  $D_m$  is diagonal with entries  $d_{m,j}$ , say, at least two,

$$D_{m,m+l}^{-1} p_j e_j = \left(\prod_{n=0}^{l-1} d_{m+n,j}^{-1}\right) p_j e_j.$$

Since  $d_{n,j} \geq 2$  for all  $n$  and  $j$ , the rational numbers of the form  $(\prod_{n=0}^{l-1} d_{m+n,j}^{-1}) p_j$  are dense in  $[0, \infty)$ . Thus the vectors  $s$  for which  $L(s) \geq 0$  form a dense subset of  $[0, \infty)^k$ , and the continuity of  $L$  implies that  $L(s) \geq 0$  for all  $s \in [0, \infty)^k$ . A measure  $\nu$  which has  $\int f d\nu \geq 0$  for all  $f \in C(\mathbb{S}^d)_+$  is a positive measure, and this is what we had to prove.  $\square$

5. The solution of the subinvariance relation

We now describe the solutions to the subinvariance relation (4.2). We observe that the formula on the right of (5.1) below is the Laplace transform of a periodic function, and as such is given by an integral over a finite rectangle. This observation motivated our calculations, but in the end we found it easier to work with the trigonometric polynomials  $x \mapsto e^{2\pi i n x}$ .

**THEOREM 5.1.** *Let  $\theta \in M_{k,d}(\mathbb{S}^d)$ ,  $\beta \in (0, \infty)$  and  $r = (r_j) \in (0, \infty)^k$ . Denote the  $j$ th row of  $\theta$  by  $\theta_j$ .*

(a) *For each  $\mu \in M(\mathbb{S}^d)$ , there is a non-negative measure  $\nu_\mu \in M(\mathbb{S}^d)$  such that*

$$\int_{\mathbb{S}^d} f d\nu_\mu = \int_{[0, \infty)^k} e^{-\beta w^T r} \int_{\mathbb{S}^d} f(x + \theta^T w) d\mu(x) dw \quad \text{for } f \in C(\mathbb{S}^d), \quad (5.1)$$

*and  $\nu_\mu$  has total mass  $\|\mu\| \prod_{j=1}^k (\beta r_j)^{-1}$ . The measure  $\nu = \nu_\mu$  satisfies the subinvariance relation*

$$\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})(\nu) \geq 0 \quad \text{for } s \in [0, \infty)^k. \quad (5.2)$$

- (b) For each  $\nu$  satisfying the subinvariance relation (5.2), there is a measure  $\mu_\nu \in M(\mathbb{S}^d)$  such that

$$\int_{\mathbb{S}^d} f d\mu_\nu = \lim_{s_k \rightarrow 0^+} \cdots \lim_{s_1 \rightarrow 0^+} \frac{1}{s_k \cdots s_1} \int_{\mathbb{S}^d} f d\left(\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^*})\right)(\nu) \tag{5.3}$$

for  $f \in C(\mathbb{S}^d)$ , and  $\mu_\nu$  has total mass  $\|\nu\| \prod_{j=1}^k (\beta r_j)$ .

- (c) The map  $\mu \mapsto \nu_\mu$  is an affine homeomorphism of  $M(\mathbb{S}^d)$  onto the simplex of measures satisfying the subinvariance relation (5.2), and the inverse takes  $\nu$  to  $\mu_\nu$ .

*Remark 5.2.* A measure  $\nu$  that satisfies the subinvariance relation (5.2) also satisfies the analogous relation involving  $\prod_{j \in J} (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^*})$  for any subset  $J$  of  $\{1, \dots, k\}$ . To see this, observe that, for any vector  $y \in [0, \infty)^d$ ,  $R_y$  is an isometric positivity-preserving linear operator on  $C(\mathbb{S}^d)$ . Hence so are  $R_{y^*}$  and  $e^{-\beta s_j r_j} R_{y^*}$ . Since the numbers  $-\beta r_j$  are negative, the series  $\sum_{n=0}^\infty e^{-\beta s_j r_j n} R_{y^*}^n$  converges in norm in the Banach space of bounded linear operators on  $M(\mathbb{S}^d)$  to an inverse for  $\text{id} - e^{-\beta s_j r_j} R_{y^*}$ . Hence applying this inverse allows us to remove factors from the subinvariance relation without losing positivity.

*Remark 5.3.* (Reality check) We reassure ourselves that the description of subinvariant measures in Theorem 5.1 is consistent with the description in [6, Theorem 7.1]. There  $d = k = 1$ , and they describe the simplex of subinvariant probability measures by specifying the extreme points of the simplex.

We recall that the matrices  $D_m \in M_1(\mathbb{N}) = \mathbb{N}$  and  $E_m \in M_1(\mathbb{N})$  are all the same integer  $N \geq 2$ , and the sequence  $\theta_m$  then satisfies  $N^2 \theta_{m+1} = \theta_m$ . In terms of our generators, the dynamics  $\alpha : \mathbb{R} \rightarrow \text{Aut } B_m$  in [6] is given by

$$\alpha_t(V_{m,p} U_{m,n} V_{m,q}^*) = e^{it(p-q)N^{-m}} V_{m,p} U_{m,n} V_{m,q}^*$$

(see [6, Proposition 6.3]), which is our  $\alpha^{r^m}$  with  $r^m = N^{-m}$ . We are interested in KMS $_\beta$  states, so the subinvariant probability measures for  $(B_m, \alpha)$  are those in the set denoted by  $\Omega_{\text{sub}}^r$  for  $r = \beta N^{-m} \theta_m^{-1} = \beta r^m \theta_m^{-1}$  (see [6, Notation 6.8]; the displayed equation there is meant to say this, as opposed to  $r = \beta N^{-m} \theta_m$ , which is the way we first read it).

Since the calculation in [6] is about extreme points, we start with a point mass  $\delta_y \in P(\varprojlim \mathbb{S})$ . Then  $(E_m^T)_* \delta_y$  is the point mass  $\mu_m = \delta_{y_m}$ , where  $y_m$  is obtained by realizing  $y$  as a sequence  $\{y_m\}$  satisfying  $N y_{m+1} = y_m$ . Then the measure  $\nu_{\mu_m}$  in Theorem 5.1(a) is given by

$$\begin{aligned} \int f d\nu_{\mu_m} &= \int_0^\infty e^{-\beta w r^m} \int_0^1 f(x + \theta_m w) d\mu_m(x) dw \\ &= \int_0^\infty e^{-\beta w r^m} f(y_m + \theta_m w) dw. \end{aligned}$$

For  $f(x) = e^{2\pi i n x}$ , we get

$$\int f d\nu_{\mu_m} = e^{2\pi i n y_m} \int_0^\infty e^{-\beta w r^m} e^{2\pi i n \theta_m w} dw,$$

and a change of variables gives

$$\begin{aligned} \int f \, d\nu_{\mu_m} &= e^{2\pi i n y_m} \int_0^\infty e^{-\beta \theta_m^{-1} v r^m} e^{2\pi i n v} \theta_m^{-1} \, dv \\ &= e^{2\pi i n y_m} \theta_m^{-1} \int_0^\infty e^{-(\beta r^m \theta_m^{-1})v} e^{2\pi i n v} \, dv. \end{aligned}$$

Now we recognize the integral as the Laplace transform of the periodic function  $x \mapsto e^{2\pi i n x}$ , and hence

$$\int f \, d\nu_{\mu_m} = e^{2\pi i n y_m} \theta_m^{-1} \frac{1}{1 - e^{-\beta r^m \theta_m^{-1}}} \int_0^1 e^{-(\beta r^m \theta_m^{-1})v} e^{2\pi i n v} \, dv. \tag{5.4}$$

In the notation of [6], we set  $r := \beta r^m \theta_m^{-1}$  and rewrite (5.4) as

$$\begin{aligned} \int f \, d\nu_{\mu_m} &= e^{2\pi i n y_m} \beta^{-1} N^m \int_0^1 \frac{r}{1 - e^{-r}} e^{-rv} e^{2\pi i n v} \, dv \\ &= \beta^{-1} N^m \int_0^1 e^{2\pi i n v} \, d(R_{y_m})_*(m_r)(v). \end{aligned}$$

This shows that the measure  $\nu_{\mu_m}$  is a multiple of the measure  $(R_{y_m})_*(m_r)$  appearing in [6, Theorem 7.1]. We are off by the scalar  $\beta^{-1} N^m$  because that theorem is about the simplex of subinvariant probability measures, and the measures  $\nu_\mu$  in Theorem 5.1 have total mass  $(\beta r^m)^{-1} = \beta^{-1} N^m$ .

For the proof of Theorem 5.1(a), we need the following lemma, which is known to probabilists as the *inclusion–exclusion principle*. We could not find a good reference for this measure-theoretic version, but fortunately it is relatively easy to prove by induction on the number  $k$  of subsets.

LEMMA 5.4. *Suppose that  $\lambda$  is a finite measure on a space  $X$  and  $\{S_j : 1 \leq j \leq k\}$  is a finite collection of measurable subsets of  $X$ . For each subset  $G$  of  $\{1, \dots, k\}$ , we set  $S_G := \bigcap_{j \in G} S_j$ . Then*

$$\lambda\left(\bigcup_{j=1}^k S_j\right) = \sum_{\emptyset \neq G \subset \{1, \dots, k\}} (-1)^{|G|-1} \lambda(S_G).$$

*Proof of Theorem 5.1(a).* We first claim that there is a positive functional  $I$  on  $C(\mathbb{S}^d)$  such that  $I(f)$  is given by the right-hand side of (5.1). Indeed, the estimate

$$\left| \int_{[0, \infty)^k} e^{-\beta w^T r} \int_{\mathbb{S}^d} f(x + \theta^T w) \, d\mu(x) \, dw \right| \leq \int_{[0, \infty)^k} e^{-\beta w^T r} \int_{\mathbb{S}^d} \|f\|_\infty \, d\mu(x) \, dw,$$

shows that the right-hand side of (5.1) determines a bounded function  $I : C(\mathbb{S}^d) \rightarrow \mathbb{C}$ . This function  $I$  is linear because the integral is linear, and  $f \geq 0$  implies that  $I(f) \geq 0$  because all the integrands in (5.1) are non-negative. Thus there is a finite non-negative measure  $\nu_\mu$  satisfying (5.1). The norm of the integral is given by the total mass of the measure  $\nu_\mu$ , which is

$$\int_{\mathbb{S}^d} 1 \, d\nu_\mu = \int_{[0, \infty)^k} e^{-\beta w^T r} \|\mu\| \, dw.$$

To compute the exact value of the integral, observe that

$$e^{-\beta w^T r} = e^{-\beta \sum_{j=1}^k w_j r_j} = \prod_{j=1}^k e^{-\beta w_j r_j}.$$

Thus

$$\|v_\mu\| = \|\mu\| \int_{[0, \infty)^k} \prod_{j=1}^k e^{-\beta w_j r_j} dw = \|\mu\| \prod_{j=1}^k \int_0^\infty e^{-\beta w_j r_j} dw_j = \|\mu\| \prod_{j=1}^k (\beta r_j)^{-1}.$$

This proves the assertions in the first sentence of part (a).

For the subinvariance relation, we fix  $f \in C(\mathbb{S}^d)_+$ , and we aim to prove that

$$\int_{\mathbb{S}^d} f d\left(\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^*}) (v_\mu)\right) \geq 0.$$

As in the proof of Theorem 4.1, we write

$$\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^*}) = \text{id} + \sum_{\emptyset \neq G \subset \{1 \leq j \leq k\}} (-1)^{|G|} e^{-\beta s_G^T r} R_{\theta^T s_G},$$

with  $s_G := \sum_{j \in G} s_j e_j$ . For  $j \leq k$ , we define  $S_j = \{v \in [0, \infty) : v_j \geq s_j\}$  and  $S_G := \bigcap_{j \in G} S_j$ . Then

$$\begin{aligned} \int_{\mathbb{S}^d} f d(e^{-\beta s_G^T r} R_{\theta^T s_G}) (v_\mu) &= \int_{\mathbb{S}^d} e^{-\beta s_G^T r} (f \circ R_{\theta^T s_G}) dv_\mu \\ &= \int_{[0, \infty)^k} e^{-\beta w^T r} e^{-\beta s_G^T r} \int_{\mathbb{S}^d} f(x + \theta^T w + \theta^T s_G) d\mu(x) dw \\ &= \int_{S_G} e^{-\beta v^T r} \int_{\mathbb{S}^d} f(x + \theta^T v) d\mu(x) dv. \end{aligned} \tag{5.5}$$

Since  $f$  is fixed, we can define a measure  $m$  on  $[0, \infty)^k$  by

$$\int_{[0, \infty)^k} g dm = \int_{[0, \infty)^k} g(v) e^{-\beta v^T r} \int_{\mathbb{S}^d} f(x + \theta^T v) d\mu(x) dv.$$

Now (5.5) says that

$$\int_{\mathbb{S}^d} f d(e^{-\beta s_G^T r} R_{\theta^T s_G}) (v_\mu) = m(S_G).$$

Thus

$$\int_{\mathbb{S}^d} f d\left(\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^*}) (v_\mu)\right) = m([0, \infty)^k) + \sum_{\emptyset \neq G \subset \{1 \leq j \leq k\}} (-1)^{|G|} m(S_G).$$

By the inclusion–exclusion principle, this is

$$m([0, \infty)^k) - m\left(\bigcup_{j=1}^k S_j\right) = m\left(\prod_{j=1}^k [0, s_j]\right) \geq 0. \quad \square$$

We now move towards a proof of part (b), and for that the first problem is to prove that the iterated limit in (5.3) exists. We will work with  $l$  satisfying  $1 \leq l \leq k$  and show by induction on  $l$  that the iterated limit

$$\lim_{s_l \rightarrow 0^+} \cdots \lim_{s_1 \rightarrow 0^+}$$

exists. We will be doing some calculus, so we often assume that our test functions  $f$  belong to the dense subalgebra  $C^\infty(\mathbb{S}^d)$  of  $C(\mathbb{S}^d)$  consisting of smooth functions, all of whose derivatives are also periodic.

We begin by establishing that, even after dividing by the numbers which are going to zero, the norms of the measures remain uniformly bounded.

LEMMA 5.5. *Suppose that  $\nu$  is a finite positive measure on  $\mathbb{S}^d$  satisfying the subinvariance relation (5.2). Then, for each  $s \in (0, \infty)^k$ ,*

$$\lambda_s := \frac{1}{s_k s_{k-1} \cdots s_1} \prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})(\nu)$$

is a positive measure with total mass

$$\|\lambda_s\| \leq \left( \prod_{j=1}^k (\beta r_j) \right) \|\nu\|. \tag{5.6}$$

*Proof.* The subinvariance relation implies that the measure is positive. For the estimate on the total mass of  $\lambda_s$ , we deal with the variables  $s_i$  separately. So, for  $1 \leq l \leq k$ , we set

$$\sigma_l := \prod_{j=l}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})(\nu),$$

which, by Remark 5.2, are all positive measures. We have

$$\begin{aligned} \int_{\mathbb{S}^d} 1 d \left( \prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})(\nu) \right) &= \int_{\mathbb{S}^d} 1 d\sigma_1 \\ &= \int_{\mathbb{S}^d} 1 d((\text{id} - e^{-\beta s_1 r_1} R_{s_1 \theta_1^T *})(\sigma_2)) \\ &= \int_{\mathbb{S}^d} 1 \circ (\text{id} - e^{-\beta s_1 r_1} R_{s_1 \theta_1^T}) d\sigma_2 \\ &= \int_{\mathbb{S}^d} (1 - e^{-\beta s_1 r_1}) d\sigma_2. \end{aligned}$$

So for all  $s_1 > 0$ ,

$$\frac{1}{s_1} \int_{\mathbb{S}^d} 1 d \left( \prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})(\nu) \right) = \int_{\mathbb{S}^d} \frac{1 - e^{-\beta s_1 r_1}}{s_1} d\sigma_2. \tag{5.7}$$

The integrand here is

$$\frac{1 - e^{-\beta s_1 r_1}}{s_1} = \frac{f(0) - f(s_1)}{s_1} \quad \text{for } f(s_1) := e^{-\beta s_1 r_1}.$$

Hence, for each fixed  $s_1 > 0$ , the mean value theorem implies that there exists  $c \in (0, s_1)$  such that

$$\frac{1 - e^{-\beta s_1 r_1}}{s_1} = -f'(c) = -(-\beta r_1 e^{-\beta c r_1}),$$

which is a positive number less than  $\beta r_1$ . Thus (5.7) is at most  $\beta r_1 \|\sigma_2\|$ .

Now we repeat this argument. We first see that

$$\frac{1}{s_2} \int_{\mathbb{S}^d} 1 d(\text{id} - e^{-\beta s_2 r_2} R_{s_2 \theta_2^T *})(\sigma_3)$$

has mass at most  $\beta r_2 \|\sigma_3\|$ . After  $k - 2$  more steps, we arrive at the estimate (5.6). □

LEMMA 5.6. *Suppose that  $1 \leq j \leq k$  and that  $\lambda \in M(\mathbb{S}^d)$  satisfies*

$$(\text{id} - e^{-\beta s r_j} R_{s \theta_j^T *})(\lambda) \geq 0 \quad \text{for all } s > 0.$$

Then, for all  $f \in C^\infty(\mathbb{S}^d)$ ,

$$\lim_{s \rightarrow 0^+} \left( \frac{1}{s} \int_{\mathbb{S}^d} f d(\text{id} - e^{-\beta s r_j} R_{s \theta_j^T *})(\lambda) \right) = \beta r_j \int_{\mathbb{S}^d} f d\lambda - \int_{\mathbb{S}^d} \theta_j^T (\nabla f) d\lambda. \quad (5.8)$$

*Proof.* Let  $g : \mathbb{S}^d \times \mathbb{R} \rightarrow \mathbb{C}$  be the function  $g(x, s) = e^{-\beta s r_j} f(x + s \theta_j^T)$ . The term on the left of (5.8) can be rewritten as

$$\frac{1}{s} \int_{\mathbb{S}^d} (f(x) - e^{-\beta s r_j} f(x + s \theta_j^T)) d\lambda(x) = \frac{1}{s} \int_{\mathbb{S}^d} (g(x, 0) - g(x, s)) d\lambda(x).$$

So we want to show that the function  $G$  defined by  $G(s) := \int_{\mathbb{S}^d} g(x, s) d\lambda(x)$  is differentiable at zero with  $-G'(0)$  equal to the right-hand side of (5.8).

We compute

$$\frac{\partial g}{\partial s}(x, s) = -\beta r_j e^{-\beta s r_j} f(x + s \theta_j^T) + e^{-\beta s r_j} \theta_j^T \nabla f(x + s \theta_j^T).$$

The Cauchy–Schwarz inequality for the inner product  $\theta_j^T (\nabla f) = (\theta_j | \nabla f)$  then gives

$$\left| \frac{\partial g}{\partial s}(x, s) \right| \leq \beta r_j \|f\|_\infty + \|\theta_j^T\|_2 \|\nabla f(x + s \theta_j^T)\|_2. \quad (5.9)$$

The right-hand side is uniformly bounded on  $\mathbb{S}^d$ , and hence there is an integrable function on  $\mathbb{S}^d$  that dominates the right-hand side for all  $s \in [0, 1]$ , say. Thus we can differentiate under the integral sign, using [14, Theorem 2.27], for example. We deduce that  $G$  is differentiable on  $[0, 1]$  with derivative

$$G'(s) = \int_{\mathbb{S}^d} (-\beta r_j e^{-\beta s r_j} f(x + s \theta_j^T) + e^{-\beta s r_j} \theta_j^T \nabla f(x + s \theta_j^T)) d\lambda(x).$$

Taking  $s = 0$  gives the negative of the right-hand side of (5.8), as required. □

Our next step is the inductive argument, which is quite a complicated one. As a point of notation, for each tuple  $I = \{i_1, \dots, i_m\}$  with entries in  $\{1, 2, \dots, k\}$ , and for  $f \in C^\infty(\mathbb{S}^d)$ , we write  $|I| := m$  and  $D_I f$  for the partial derivative

$$D_I f := \frac{\partial^m f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_m}}.$$

LEMMA 5.7. Suppose that  $\nu$  is a positive measure on  $\mathbb{S}^d$  satisfying the subinvariance relation (5.2). Let  $1 \leq l \leq k$ .

(a) The iterated limit

$$\lim_{s_l \rightarrow 0^+} \cdots \lim_{s_1 \rightarrow 0^+} \frac{1}{s_l \cdots s_1} \int_{\mathbb{S}^d} f d \left( \prod_{j=1}^l (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *}) \right) (\nu)$$

exists for all  $f \in C(\mathbb{S}^d)$ .

(b) Write

$$\Sigma_l := \bigcup_{n=0}^l \{1, \dots, k\}^n.$$

Then there are real scalars  $\{K_I^l : I \in \Sigma_l\}$  such that  $K_\emptyset^l = \prod_{j=1}^l (\beta r_j)$  and, for every  $f \in C^\infty(\mathbb{S}^d)$  and for every measure  $\nu$  on  $\mathbb{S}^d$  satisfying the subinvariance relation (5.2), the limit in (a) is

$$\int_{\mathbb{S}^d} \left( \sum_{I \in \Sigma_l} K_I^l D_I f \right) d\nu. \tag{5.10}$$

*Proof.* We prove by induction on  $l$  that the limit in (a) exists for every  $f \in C^\infty(\mathbb{S}^d)$ , and that there exist the scalars  $K_I^l$ . Then, since we know from Lemma 5.5 that the measures  $\lambda_s$  are norm-bounded by  $(\prod_{j=1}^k (\beta r_j)) \|\nu\|$  and that  $C^\infty(\mathbb{S}^d)$  is norm dense in  $C(\mathbb{S}^d)$ , we get convergence in (a) also for  $f \in C(\mathbb{S}^d)$ .

When  $l = 1$ , the index set  $\Sigma_1$  consists of the empty set  $\emptyset$  and the one-point sets  $\{j\}$ . Lemma 5.6 implies that  $K_\emptyset^1 = \beta r_1$  and  $K_{\{j\}}^1 = \theta_j^T e_j = \theta_{1j}$ .

We fix  $l$  between 1 and  $k - 1$ , and suppose as our inductive hypothesis that, for every measure  $\lambda$  such that

$$\prod_{j=1}^l (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *}) (\lambda) \geq 0 \quad \text{for all } s \in [0, \infty)^k, \tag{5.11}$$

we have such scalars  $\{K_I^l\}$  parametrized by  $I \in \Sigma_l$ . We now have to start with a measure  $\kappa$  that satisfies

$$\prod_{j=1}^{l+1} (\text{id} - e^{-\beta s_j r_j} R_{\theta_j^T *}) (\kappa) \geq 0 \quad \text{for all } s \in [0, \infty)^k \tag{5.12}$$

and find suitable scalars  $K_I^{l+1}$ .

We define

$$\lambda := (\text{id} - e^{-\beta s_{l+1} r_{l+1}} R_{s_{l+1} \theta_{l+1}^T *}) (\kappa).$$

Remark 5.2 reassures us that  $\lambda$  is another positive measure, and (5.12) implies that it satisfies (5.11). The induction hypothesis gives

$$\begin{aligned} L(s_{l+1}) &:= \lim_{s_l \rightarrow 0^+} \cdots \lim_{s_1 \rightarrow 0^+} \frac{1}{s_{l+1} \cdots s_1} \int_{\mathbb{S}^d} f d \left( \prod_{j=1}^{l+1} (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *}) \right) (\kappa) \\ &= \frac{1}{s_{l+1}} \left( \int_{\mathbb{S}^d} \left( \sum_{I \in \Sigma_l} K_I^l D_I f \right) d\lambda \right). \end{aligned}$$



Lemma 5.6 implies that

$$\begin{aligned} \lim_{s_{l+1} \rightarrow 0^+} L(s_{l+1}) &= \beta r_{l+1} \int_{\mathbb{S}^d} \left( \sum_{I \in \Sigma_l} K_I^l D_I f \right) d\lambda - \int_{\mathbb{S}^d} \theta_{l+1}^T \nabla \left( \sum_{I \in \Sigma_l} K_I^l D_I f \right) d\lambda \\ &= \beta r_{l+1} \int_{\mathbb{S}^d} \left( \sum_{I \in \Sigma_l} K_I^l D_I f \right) d\lambda - \int_{\mathbb{S}^d} \left( \sum_{I \in \Sigma_l} \sum_{i=1}^d K_I^l \theta_{l+1,i} \frac{\partial D_I f}{\partial x_i} \right) d\lambda(x). \end{aligned}$$

To finish off the inductive step, we set  $K_{\emptyset}^{l+1} = \beta r_{l+1} K_{\emptyset}^l$  and, for  $I' = (I, i_{|I|+1})$ , we set

$$K_{I'}^{l+1} = \begin{cases} K_I^l \theta_{(l+1)i_{|I|+1}} & \text{if } |I| = l, \\ \beta r_{l+1} K_{I'}^l - K_I^l \theta_{l+1,i_{|I|+1}} & \text{if } |I| < l. \end{cases}$$

This completes the inductive step, and hence the proof. □

*Proof of Theorem 5.1(b).* Lemma 5.7 shows that the limit exists for all  $f \in C(\mathbb{S}^d)$  and for  $f \in C^\infty(\mathbb{S}^d)$  gives us a formula for the limit. The limit is linear in  $f$ , positive when  $f$  is, and is bounded by  $\|f\|_\infty (\prod_{j=1}^k (\beta r_j)) \|v\|$ . Thus it is given by a finite positive measure  $\mu_\nu$ . Since the total mass of the measure is integration against the constant function 1, and since 1 is smooth, the total mass is given by (5.10). But since all derivatives of 1 are zero, the only non-zero terms are the ones on which  $I = \emptyset$ . Now the formula for  $K_{\emptyset}^k$  implies that  $\|\mu_\nu\| = (\prod_{j=1}^k (\beta r_j)) \|v\|$ . □

We now work towards the proof of Theorem 5.1(c). To prove that  $N : \mu \mapsto \nu_\mu$  is a bijection of the measures arising from  $\text{KMS}_\beta$  states onto the subinvariant measures, we prove that  $N$  is one-to-one and that  $M : \nu \mapsto \mu_\nu$  satisfies  $N \circ M(\nu) = \nu$  for all subinvariant measures  $\nu$ . We then have  $N \circ (M \circ N)(\mu) = (N \circ M) \circ N(\mu) = N(\mu)$ , and injectivity of  $N$  implies that  $(M \circ N)(\mu) = \mu$ . Thus Theorem 5.1(c) follows from the following proposition.

**PROPOSITION 5.8.** *Suppose that  $\nu$  is a measure on  $\mathbb{S}^d$  satisfying the subinvariance relation (5.2) and with total mass  $\prod_{j=1}^k (\beta r_j)^{-1}$ . Then  $\nu = \nu_{\mu_\nu}$ .*

Suppose that  $\nu$  is a subinvariant measure and  $f \in C^\infty(\mathbb{S}^d)$ . We need to show that the functional defined by integrating against  $\nu_{\mu_\nu}$ , which is defined in parts (a) and (b) of the theorem as

$$\int_{[0, \infty)^k} e^{-\beta w^T r} \lim_{s \rightarrow 0^+} \frac{1}{s_k \cdots s_1} \int_{\mathbb{S}^d} f(x + \theta^T w) d\left( \prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *}) \right) (\nu)(x) dw, \tag{5.13}$$

is, in fact, implemented by  $\nu$ . We will do this by peeling off the iterated limit one variable at a time. For this, the next lemma is crucial.

**LEMMA 5.9.** *Consider a positive measure  $\lambda$  on  $\mathbb{S}^d$ ,  $b \in (0, \infty)$  and  $v \in \mathbb{S}^d$ . For  $f \in C^\infty(\mathbb{S}^d)$ ,*

$$\begin{aligned} &\int_0^\infty e^{-bt} \lim_{s \rightarrow 0^+} \frac{1}{s} \int_{\mathbb{S}^d} f(y + tv) d((\text{id} - e^{-bs} R_{sv*})\lambda)(y) dt \\ &= \lim_{s \rightarrow 0^+} \int_0^\infty \frac{e^{-bt}}{s} \int_{\mathbb{S}^d} f(y + tv) d((\text{id} - e^{-bs} R_{sv*})\lambda)(y) dt. \end{aligned}$$

*Proof.* For  $s > 0$ ,

$$\begin{aligned} & \frac{1}{s} \int_{\mathbb{S}^d} f(y + tv) d((\text{id} - e^{-bs} R_{sv*})\lambda)(y) \\ &= \frac{1}{s} \int_{\mathbb{S}^d} (f(y + tv) - e^{-bs} f(y + tv + sv)) d\lambda(y). \end{aligned}$$

We write this last integrand as

$$\begin{aligned} K(y, s, t) &= s^{-1} (f(y + tv) - e^{-bs} f(y + tv + sv)) \\ &= s^{-1} (f(y + tv) - e^{-bs} f(y + tv) + e^{-bs} f(y + tv) - e^{-bs} f(y + tv + sv)) \\ &= \frac{1 - e^{-bs}}{s} f(y + tv) + e^{-bs} \frac{f(y + tv) - f(y + tv + sv)}{s}. \end{aligned}$$

We estimate the first summand using the mean value theorem on  $e^{-bs}$  and the second summand using the same theorem on  $f$ , to find

$$|K(y, s, t)| \leq b\|f\|_\infty + \|v^T(\nabla f)\|_\infty.$$

Thus

$$\left| \frac{e^{-bt}}{s} \int_{\mathbb{S}^d} K(y, s, t) d\lambda(y) \right| \leq e^{-bt} (b\|f\|_\infty + \|v^T(\nabla f)\|_\infty) \|\lambda\|.$$

Now the result follows from the dominated convergence theorem for Lebesgue measure on  $[0, \infty)$  (modulo the trick of observing that it suffices to work with sequences  $s_n \rightarrow 0+$  — see the proof of [14, Theorem 2.27]).  $\square$

*Proof of Proposition 5.8.* As in the proof of Theorem 5.1(b), Lemma 5.7 implies that there is a positive measure  $\eta$  on  $\mathbb{S}^d$  such that, for  $g \in C^\infty(\mathbb{S}^d)$ ,

$$\int_{\mathbb{S}^d} g d\eta = \lim_{s_k, \dots, s_2 \rightarrow 0^+} \frac{1}{s_k \cdots s_2} \int_{\mathbb{S}^d} g d\left(\prod_{j=2}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})\right)(\nu).$$

Since the operators  $\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *}$  commute with each other,

$$\begin{aligned} & \int_{\mathbb{S}^d} g d(\text{id} - e^{-\beta s_1 r_1} R_{s_1 \theta_1^T *})(\eta) \\ &= \lim_{s_k, \dots, s_2 \rightarrow 0^+} \frac{1}{s_k \cdots s_2} \int_{\mathbb{S}^d} g d\left(\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})\right)(\nu). \end{aligned} \tag{5.14}$$

Now we need some complicated notation to implement the peeling process. First of all, we fix  $f \in C^\infty(\mathbb{S}^d)$ . In an attempt to avoid an overdose of subscripts, we write  $s = (s_1, \hat{s})$ ,  $w = (w_1, \hat{w})$  and  $r = (r_1, \hat{r})$ . We also write  $\hat{\theta}$  for the  $k - 1 \times d$  matrix obtained from  $\theta$  by deleting the first row: thus  $\theta^T$  has block form  $(\theta_1^T \hat{\theta}^T)$ . With the new notation, (5.13) becomes

$$\begin{aligned} & \int_{[0, \infty)^{k-1}} e^{-\beta \hat{w}^T \hat{r}} \int_0^\infty e^{-\beta r_1 w_1} \\ & \times \lim_{s_1 \rightarrow 0^+} \frac{1}{s_1} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) d(\text{id} - e^{-\beta r_1 w_1} R_{s_1 \theta_1^T *})(\eta)(x) dw_1 d\hat{w}. \end{aligned}$$

Now we can apply Lemma 5.9 to the inside integrals, which gives

$$\int_{\mathbb{S}^d} f \, d\nu_{\mu_\nu} = \int_{[0, \infty)^{k-1}} e^{-\beta \hat{w}^T \hat{r}} \lim_{s_1 \rightarrow 0^+} \frac{1}{s_1} \int_0^\infty e^{-\beta r_1 w_1} \\ \times \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) \, d(\text{id} - e^{-\beta r_1 w_1} R_{s_1 \theta_1^T *})(\eta)(x) \, dw_1 \, d\hat{w}. \quad (5.15)$$

We now consider the function  $g$  on  $(0, \infty)$  defined by

$$g(s_1) := \frac{1}{s_1} \int_0^\infty e^{-\beta r_1 w_1} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) \, d(\text{id} - e^{-\beta r_1 w_1} R_{s_1 \theta_1^T *})(\eta)(x) \, dw_1.$$

We aim to prove that  $g(s_1) \rightarrow \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w}) \, d\eta(x)$  as  $s_1 \rightarrow 0^+$ . To this end, we compute

$$g(s_1) = \frac{1}{s_1} \int_0^\infty e^{-\beta r_1 w_1} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) \, d\eta(x) \, dw_1 \\ - \frac{1}{s_1} \int_0^\infty e^{-\beta r_1 (w_1 + s_1)} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + (w_1 + s_1) \theta_1^T) \, d\eta(x) \, dw_1.$$

Changing the variable in the second integral to get an integral over  $[s_1, \infty)$  gives

$$g(s_1) = \frac{1}{s_1} \int_0^{s_1} e^{-\beta r_1 w_1} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) \, d\eta(x) \, dw_1.$$

Now

$$g(s_1) - \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w}) \, d\eta(x) \\ = \frac{1}{s_1} \int_0^{s_1} e^{-\beta r_1 w_1} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) \, d\eta(x) \, dw_1 - \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w}) \, d\eta(x) \\ = \frac{1}{s_1} \int_0^{s_1} \int_{\mathbb{S}^d} (e^{-\beta r_1 w_1} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) - f(x + \hat{\theta}^T \hat{w})) \, d\eta(x) \, dw_1.$$

Since  $y \mapsto e^{-\beta y^T r} f(x + \hat{\theta}^T \hat{w} + \theta^T y)$  is uniformly continuous, there exists  $\delta$  such that

$$0 \leq w_1 < \delta \implies |e^{-\beta r_1 w_1} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) - f(x + \hat{\theta}^T \hat{w})| < \frac{\epsilon}{\|\eta\|}.$$

So, for  $0 \leq s_1 < \delta$ ,

$$\left| g(s_1) - \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w}) \, d\eta(x) \right| \leq \frac{1}{s_1} \int_0^{s_1} \int_{\mathbb{S}^d} \frac{\epsilon}{\|\eta\|} \, d\eta(x) \, dw_1 = \epsilon.$$

Thus  $g(s_1) \rightarrow \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w}) \, d\eta(x)$ , as we wanted.

Putting the formula for  $\lim_{s_1 \rightarrow 0^+} g(s_1)$  in (5.15) gives

$$\int_{\mathbb{S}^d} f \, d\nu_{\mu_\nu} = \int_{[0, \infty)^{k-1}} e^{-\beta \hat{w}^T \hat{r}} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w}) \, d\eta(x) \, d\hat{w},$$

which is the right-hand side of (5.13) with one  $\lim_{s \rightarrow 0^+}$  and one  $\int_0^\infty$  removed. Repeating the argument  $k - 1$  times gives

$$\int_{\mathbb{S}^d} f \, d\nu_{\mu_\nu} = \int_{\mathbb{S}^d} f \, d\nu,$$

as required. □

As described before Proposition 5.8, this completes the proof of Theorem 5.1.

6. A parametrization of the equilibrium states

We are now ready to describe the KMS states of our system. At the end of the section, we will use the following theorem to prove our main result.

**THEOREM 6.1.** *Consider our standard set-up, and suppose that  $\beta > 0$ .*

(a) *Suppose that  $\mu \in P(\varprojlim(\mathbb{S}^d, E_m^T))$ . Define measures  $\mu_m \in P(\mathbb{S}^d)$  by  $\mu_m = E_{m,\infty}^T(\mu)$  and take  $\nu_{\mu_m}$  to be the subinvariant measure on  $\mathbb{S}^d$  obtained by applying Theorem 5.1 to the measure  $\mu_m$ . Then there is a  $KMS_\beta$  state  $\psi_\mu$  of  $(B_\infty, \alpha)$  such that*

$$\psi_\mu(V_{m,p}U_{m,n}V_{m,q}^*) = \delta_{p,q} e^{-\beta p^T r^m} \prod_{j=1}^k (\beta r_j^m) \int_{\mathbb{S}^d} e^{2\pi i x^T n} d\nu_{\mu_m}(x). \tag{6.1}$$

(b) *The map  $\mu \mapsto \psi_\mu$  is an affine homeomorphism of  $P(\varprojlim(\mathbb{S}^d, E_m^T))$  onto  $KMS_\beta(B_\infty, \alpha)$ .*

To prove the theorem, we first build some maps between the spaces of subinvariant measures. We will make use of Theorem 5.1, but the measures described there are not all normalized. To ensure that we are dealing with probability measures, we introduce the numbers

$$c_m := \prod_{j=1}^k (\beta r_j^m) = \beta^k \left( \prod_{j=1}^k r_j^m \right) \quad \text{and} \quad d_m := \det D_m.$$

Because  $D_m$  is diagonal, (2.4) shows that the two sets of numbers are related by  $d_m c_{m+1} = c_m$ .

With  $\Sigma_{\beta,r}$  from Proposition 3.7(c), we define  $\sigma_m : \Sigma_{\beta,r^{m+1}} \rightarrow \Sigma_{\beta,r^m}$  by

$$\sigma_m(v) = d_m^{-1} E_{m*}^T(v).$$

**LEMMA 6.2.** *Suppose that  $\mu \in P(\varprojlim(\mathbb{S}^d, E_m^T))$ , and define  $\mu_m := E_{m,\infty}^T(\mu)$  for  $m \geq 1$ . Then the measures  $\nu_{\mu_m}$  given by Theorem 5.1 satisfy  $\sigma_m(\nu_{\mu_{m+1}}) = \nu_{\mu_m}$ .*

*Proof.* We take  $f \in C(\mathbb{S}^d)$ , and using (5.1) compute

$$\begin{aligned} \int_{\mathbb{S}^d} f d\sigma_m(\nu_{\mu_{m+1}}) &= d_m^{-1} \int_{\mathbb{S}^d} f \circ E_m^T d\nu_{\mu_{m+1}} \\ &= d_m^{-1} \int_{[0,\infty)^k} e^{-\beta w^T r^{m+1}} \int_{\mathbb{S}^d} (f \circ E_m^T)(x + \theta_{m+1}^T w) d\mu_{m+1}(x) dw \\ &= d_m^{-1} \int_{[0,\infty)^k} e^{-\beta w^T r^{m+1}} \int_{\mathbb{S}^d} f(E_m^T x + E_m^T \theta_{m+1}^T w) d\mu_{m+1}(x) dw \\ &= d_m^{-1} \int_{[0,\infty)^k} e^{-\beta w^T D_m^{-1} r^m} \int_{\mathbb{S}^d} f(E_m^T x + \theta_m^T D_m^{-1} w) d\mu_{m+1}(x) dw, \end{aligned}$$

where at the last step we used both (2.4) and (2.3). Now substituting  $v = D_m^{-1} w$  in the outside integral gives

$$\int_{\mathbb{S}^d} f d\sigma_m(\nu_{\mu_{m+1}}) = \int_{[0,\infty)^k} e^{-\beta v^T r^{m+1}} \int_{\mathbb{S}^d} f(E_m^T x + \theta_m^T v) d\mu_{m+1}(x) dv. \tag{6.2}$$

We write  $s := \theta_m^T v$  and consider the translation automorphism  $\tau_s$  of  $C(\mathbb{S}^d)$  defined by  $\tau_s(f)(x) = f(x + s)$ . Then the inside integral on the right of (6.2) is

$$\begin{aligned} \int_{\mathbb{S}^d} f(E_m^T x + \theta_m^T v) d\mu_{m+1}(x) &= \int_{\mathbb{S}^d} (\tau_s(f) \circ E_m^T) d\mu_{m+1} \\ &= \int_{\mathbb{S}^d} \tau_s(f) d(E_m^T)_*(\mu_{m+1}) \\ &= \int_{\mathbb{S}^d} f(x + \theta_m^T v) d\mu_m(x). \end{aligned}$$

Putting this back into the double integral in (6.2) gives the right-hand side of (5.1) for the measure  $\mu_m$ , and we deduce from (5.1) that

$$\int_{\mathbb{S}^d} f d\sigma_m(\nu_{\mu_{m+1}}) = \int_{\mathbb{S}^d} f d\nu_{\mu_m} \quad \text{for all } f \in C(\mathbb{S}^d),$$

as required. □

*Proof of Theorem 6.1(a).* Since the maps

$$E_{m,\infty}^T : \varprojlim \mathbb{S}^d \rightarrow \mathbb{S}^d$$

are surjective, each  $\mu_m$  is a probability measure on  $\mathbb{S}^d$ . Thus we deduce from Theorem 5.1 that  $\nu_m := (\prod_{j=1}^k (\beta r_j^m)) \nu_{\mu_m}$  is a probability measure satisfying the subinvariance relation (5.2). Thus Proposition 3.7(b) gives a KMS state  $\psi_m$  of  $(B_m, \alpha^m)$  such that  $\psi_m(U_{m,n}) = \int_{\mathbb{S}^d} e^{2\pi i x^T n} d\nu_m(x)$ . We now need to check that  $\psi_{m+1} \circ \pi_m = \psi_m$  so that we can deduce from [6, Proposition 3.1] that the  $\psi_m$  combine to give a  $\text{KMS}_\beta$  state of  $(B_\infty, \alpha)$ .

Since we are viewing measures as functionals on  $C(\mathbb{S}^d)$ , the map  $E_{m*}^T$  on  $M(\mathbb{S}^d)$  is induced by the continuous function  $E_m^T : x \mapsto E_m^T x$  on  $\mathbb{S}^d$ . Then, for  $f \in C(\mathbb{S}^d)$ ,

$$\int_{\mathbb{S}^d} f \circ E_m^T(x) d\nu_{m+1}(x) = \int_{\mathbb{S}^d} f(x) dE_{m*}^T(\nu_{m+1})(x). \tag{6.3}$$

For the functions  $g_n \in C(\mathbb{S}^d)$  given by  $g_n(x) = e^{2\pi i n^T x}$  (so that  $\iota_m(g_n) = U_{m,n} \in B_m$ ), we have

$$g_n \circ E_m^T(x) = e^{2\pi i (E_m^T x)^T n} = e^{2\pi i x^T E_m n} = g_{E_m n}(x).$$

Substituting this on the left-hand side of (6.3) gives

$$\int_{\mathbb{S}^d} g_{E_m n}(x) d\nu_{m+1}(x) = \int_{\mathbb{S}^d} g_n(x) dE_{m*}^T(\nu_{m+1})(x). \tag{6.4}$$

Using again  $d_m = \det D_m$  and  $c_m = \prod_{j=1}^k (\beta r_j^m)$  and the relation  $d_m c_{m+1} = c_m$ , Lemma 6.2 gives

$$\begin{aligned} E_{m*}^T(\nu_{m+1}) &= \sigma_m(d_m \nu_{m+1}) = d_m c_{m+1} \sigma_m(\nu_{\mu_{m+1}}) \\ &= d_m c_{m+1} \nu_{\mu_m} = c_m \nu_{\mu_m} = \nu_m. \end{aligned} \tag{6.5}$$

Using (6.4) at the third step and (6.5) at the fourth step, we now calculate

$$\begin{aligned} \psi_{m+1}(\pi_m(U_{n,m})) &= \psi_{m+1}(U_{m+1, E_m n}) = \int_{\mathbb{S}^d} g_{E_m n} d\nu_{m+1} \\ &= \int_{\mathbb{S}^d} g_n dE_{m*}^T(\nu_{m+1}) = \int_{\mathbb{S}^d} g_n d\nu_m = \psi_m(U_{n,m}). \end{aligned}$$

Thus the states  $\psi_m$  give an element  $(\psi_m)$  of the inverse limit  $\varprojlim \text{KMS}_\beta(B_m, \alpha^{r^m})$ , and surjectivity of the isomorphism in [6, Proposition 3.1] gives a  $\text{KMS}_\beta$  state  $\psi_\mu$  of  $(B_\infty, \alpha)$  such that  $\psi_m = \psi_\mu \circ \pi_{m,\infty}$  for  $m \geq 1$ .  $\square$

*Remark 6.3.* We observe that the  $\text{KMS}_\beta$  state of Theorem 6.1(a) is given on  $B_\infty = \overline{\text{span}}\{V_{m,p}U_{m,n}V_{m,q}^*\}$  by

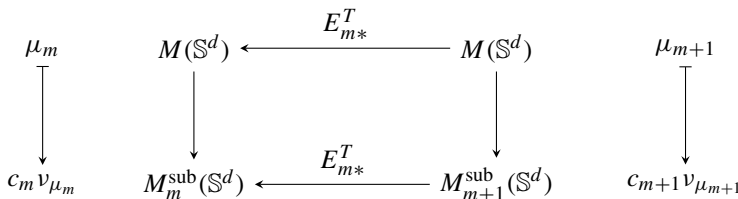
$$\psi_\mu(V_{m,p}U_{m,n}V_{m,q}^*) = \delta_{p,q} e^{-\beta p^T r^m} \int_{\mathbb{S}^d} g_n d(v_{E_{m,\infty}^T}(\mu)).$$

*Proof of Theorem 6.1(b).* We first prove that every  $\text{KMS}_\beta$  state has the form  $\psi_\mu$ . So suppose that  $\phi$  is a  $\text{KMS}_\beta$  state of  $(B_\infty, \alpha)$ . Then, for each  $m \geq 1$ ,  $\phi \circ \pi_{m,\infty}$  is a  $\text{KMS}_\beta$  state of  $(B_m, \alpha^{r^m})$ , and hence there are probability measures  $\nu_m$  such that

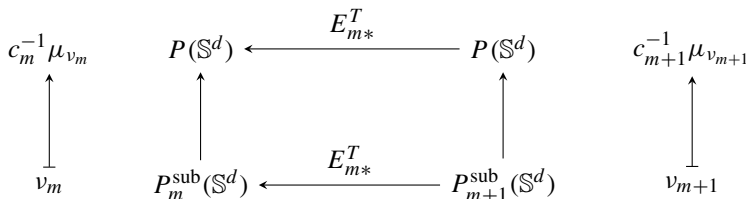
$$\phi \circ \pi_{m,\infty}(f) = \int_{\mathbb{S}^d} f d\nu_m \quad \text{for all } f \in C(\mathbb{S}^d)$$

and  $E_{m*}^T(\nu_{m+1}) = \nu_m$  for all  $m \geq 1$ . Theorem 4.1 implies that each  $\nu_m$  satisfies the corresponding subinvariance relation. More specifically, we write  $M_m^{\text{sub}}(\mathbb{S}^d)$  and  $P_m^{\text{sub}}(\mathbb{S}^d)$  for the set of measures and the set of probability measures satisfying (4.2). Then we have  $\nu_m \in P_m^{\text{sub}}(\mathbb{S}^d)$ .

Once more using  $d_m = \det D_m$  and  $c_m = \prod_{j=1}^k (\beta r_j^m)$ , the construction of Theorem 5.1(a) gives a function  $\mu \mapsto c_m \nu_\mu$  from  $M(\mathbb{S}^d)$  to the simplex  $M_m^{\text{sub}}(\mathbb{S}^d)$ . Lemma 6.2 gives commutative diagrams



and Theorem 5.1 implies that the vertical arrows are bijections. A simple set-theoretic argument then implies that we also have commutative diagrams



Thus the sequence  $(\mu_m) := (c_m^{-1} \mu_{\nu_m})$  belongs to the inverse limit  $\varprojlim (P(\mathbb{S}^d), E_{m*}^T)$ , and hence is given by a probability measure  $\mu \in P(\varprojlim(\mathbb{S}^d, E_m^T))$ . We want to show that  $\phi = \psi_\mu$ . Since both are states, it suffices to check that they agree on elements  $V_{m,p}U_{m,n}V_{m,q}^*$ . Since  $\phi$  is a  $\text{KMS}_\beta$  state and the measure  $\nu_m$  implements  $\phi$  on  $C(\mathbb{S}^d) = \overline{\text{span}}\{U_{m,n}\}$ ,

$$\phi(V_{m,p}U_{m,n}V_{m,q}^*) = \delta_{p,q} e^{-\beta p^T r^m} \int_{\mathbb{S}^d} e^{2\pi i x^T n} d\nu_m(x).$$

Since  $\nu = \nu_{\mu_\nu}$  for all  $\nu$  and  $\mu_{\nu_m} = c_m \mu_m$ , we have  $\nu_m = c_m \nu_{\mu_m}$  and

$$\phi(V_{m,p} U_{m,n} V_{m,q}^*) = \delta_{p,q} e^{-\beta p^T r^m} c_m \int_{\mathbb{S}^d} e^{2\pi i x^T n} d\nu_{\mu_m}(x).$$

This is precisely the formula for  $\psi_\mu(V_{m,p} U_{m,n} V_{m,q}^*)$  in (6.1). Thus  $\phi = \psi_\mu$ .

Since each  $\psi_\mu$  is a state, it follows from the formula (6.1) that  $\mu \mapsto \psi_\mu$  is affine and weak\* continuous from  $M(\mathbb{S}^d) = C(\mathbb{S}^d)^*$  to the state space of  $B_\infty$ . The formula (6.1) also implies that  $\mu \mapsto \psi_\mu$  is injective, and since we have just shown that it is surjective, we deduce that it is a homeomorphism of the compact space  $P(\varprojlim(\mathbb{S}^d, E_n^T))$  onto the simplex of  $\text{KMS}_\beta$  states of  $(B_\infty, \alpha)$ .  $\square$

*Proof of Theorem 2.8.* According to (6.1) in Theorem 6.1, we have to compute

$$\int_{[0,\infty)^k} e^{2\pi i x^T n} d\nu_{\mu_m}(x),$$

which, by Theorem 5.1, is

$$\begin{aligned} & \int_{[0,\infty)^k} e^{-\beta w^T r^m} \int_{\mathbb{S}^d} e^{2\pi i(x+\theta_m^T w)^T n} d\mu_m(x) dw \\ &= \int_{[0,\infty)^k} e^{-\beta w^T r^m} e^{2\pi i w^T \theta_m n} \int_{\mathbb{S}^d} e^{2\pi i x^T n} d\mu_m(x) dw \\ &= \int_{[0,\infty)^k} e^{-\beta w^T r^m} e^{2\pi i w^T \theta_m n} M_{m,n}(\mu) dw. \end{aligned} \tag{6.6}$$

We can rewrite the integrand as

$$e^{-\beta w^T r^m} e^{2\pi i w^T \theta_m n} = e^{\sum_{j=1}^k w_j(-\beta r_j^m + 2\pi i(\theta_m n)_j)} = \prod_{j=1}^k e^{w_j(-\beta r_j^m + 2\pi i(\theta_m n)_j)}.$$

When we view  $\int_{[0,\infty)^k} dw$  as an iterated integral, we find that

$$(6.6) = \prod_{j=1}^k \left( \int_0^\infty e^{w_j(-\beta r_j^m + 2\pi i(\theta_m n)_j)} M_{m,n}(\mu) dw_j \right).$$

Since  $\beta > 0$  and each  $r_j^m > 0$ ,

$$|e^{w_j(-\beta r_j^m + 2\pi i(\theta_m n)_j)}| = e^{w_j(-\beta r_j^m)} \rightarrow 0 \quad \text{as } w_j \rightarrow \infty.$$

Thus

$$(6.6) = \prod_{j=1}^k \frac{e^{w_j(-\beta r_j^m + 2\pi i(\theta_m n)_j)}}{-\beta r_j^m + 2\pi i(\theta_m n)_j} M_{m,n}(\mu) \Big|_0^\infty = \prod_{j=1}^k \frac{1}{\beta r_j^m - 2\pi i(\theta_m n)_j} M_{m,n}(\mu),$$

and the result follows from (6.1).  $\square$

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