

INTERACTION OF POISSON HYPERPLANE PROCESSES AND CONVEX BODIES

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Abstract

Given a stationary and isotropic Poisson hyperplane process and a convex body K in \mathbb{R}^d , we consider the random polytope defined by the intersection of all closed half-spaces containing K that are bounded by hyperplanes of the process not intersecting K . We investigate how well the expected mean width of this random polytope approximates the mean width of K if the intensity of the hyperplane process tends to infinity.

Keywords: Poisson hyperplane process; convex body; mean width; approximation

2010 Mathematics Subject Classification: Primary 60D05

Secondary 52A27

1. Introduction

Ever since the seminal papers of Rényi and Sulanke [20, 21, 22], the approximation of convex bodies by random polytopes has been a much-studied branch of stochastic geometry. A typical object of investigation is the convex hull of n independent, identically distributed random points in a given convex body in \mathbb{R}^d . A typical question concerns the asymptotic behavior of a geometric functional of this convex hull, as the number n of random points tends to infinity. Surveys at least partially devoted to this topic include [2, 3], [8], [13], [19], [24, 26], [27, Section 8.2], and [29]. The precise asymptotic formulas that have been obtained usually require that the convex body K under consideration is either sufficiently smooth (where sometimes the existence of freely rolling balls may be sufficient) or a polytope. For general convex bodies, one has a precise asymptotic formula for the volume, denoted by V . Let $K \subset \mathbb{R}^d$ be a convex body with $V(K) = 1$ (say), and let K_n denote the convex hull of n independent random points in K with uniform distribution. Then, as shown in [28],

$$\lim_{n \rightarrow \infty} n^{2/(d+1)} [1 - \mathbb{E}V(K_n)] = c(d) \int_{\partial K} \kappa^{1/(d+1)} d\mathcal{H}^{d-1}, \quad (1)$$

with an explicit constant $c(d)$, where \mathbb{E} denotes mathematical expectation. Here κ is the generalized Gauss–Kronecker curvature (which exists almost everywhere on ∂K) and \mathcal{H}^{d-1} is the $(n-1)$ -dimensional Hausdorff measure. However, for most convex bodies (in the sense of Baire category: see [30]) the right-hand side of (1) is zero, so (1) gives only partial information on the order of $V(K) - \mathbb{E}V(K_n)$. Additional information for all convex bodies is provided by a result in [4], which says that

$$n^{-1} \ln^{d-1} n \ll V(K) - \mathbb{E}V(K_n) \ll n^{-2/(d+1)}. \quad (2)$$

Received 29 December 2018; revision received 25 June 2019.

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Here the notation $f(n) \ll g(n)$ means that there exists a constant $c > 0$ such that $f(n) \leq cg(n)$ for all sufficiently large $n \in \mathbb{N}$. The constant c has to be independent of n , but it may depend on the dimension d , the convex body K , and later on the given measure φ . For the mean width W , it was shown in [23] that

$$n^{-2/(d+1)} \ll W(K) - \mathbb{E}W(K_n) \ll n^{-1/d}. \quad (3)$$

The orders are best possible; they are attained by sufficiently smooth bodies on the right side of (2) and the left side of (3), and by polytopes on the left side of (2) and the right side of (3). This change of optimality makes it difficult to conjecture how a common generalization of (2) and (3) to general intrinsic volumes might look. Although a guess has been formulated in [1, p. 675], this has remained one of the major mysteries in this area.

It should be mentioned that it follows from [11] that for most convex bodies (in the sense of Baire category) the middle terms in (2) and (3) oscillate, as $n \rightarrow \infty$, between the orders given by the left and right sides. More precise formulations are found in [4, Theorem 5] and [23, p. 305]. This shows that, for general convex bodies, two-sided inequalities of type (2, 3) with optimal orders are the best one can expect (up to the involved constants).

Vaguely ‘dual’ to the preceding are questions about the approximation of a convex body by the intersection of random closed half-spaces containing the body. Such questions have been treated in the plane in [22] and in higher dimensions in [6], [7], [10], and [15].

A common feature of these investigations is that a fixed number n of independent random objects, points, or hyperplanes is considered, and in the end this number n tends to infinity. In the case of hyperplanes, the underlying model, for example in [22] and [6], may seem a bit artificial, since the hyperplanes must be restricted so that they intersect a region close to the convex body under consideration.

Another model, which in the case of hyperplanes seems more natural, starts with a stationary Poisson process, either of points or of hyperplanes, which is then restricted, either to the points contained in the considered convex body or to the hyperplanes not intersecting the body. The intensity of the Poisson process is finally assumed to increase to infinity. For point processes, relevant investigations are [5], [9], [16, 17], and [18], and hyperplane processes are considered in [15].

The setting in this paper consists in a stationary Poisson hyperplane process X and a convex body K with interior points in \mathbb{R}^d . The K -cell of X is the random polytope defined by

$$Z_K := \bigcap_{H \in X, H \cap K = \emptyset} H^-(K), \quad (4)$$

where $H^-(K)$ denotes the closed half-space bounded by H that contains K . If the intensity of X tends to infinity, the K -cell Z_K may or may not approximate K , depending on the directional distribution of X (an even probability measure on the unit sphere) in relation to properties of K . In [14], the approximation was measured in terms of the Hausdorff metric, and various situations of good approximation were investigated. For example, Z_K converges almost surely to K in the Hausdorff metric as the intensity of X tends to infinity, if and only if the support of the directional distribution of X contains the support of the area measure of K .

The majority of investigations on random approximation deals with the asymptotic behavior of geometric functionals, such as volume, mean width, number of k -faces, of the approximating random polytopes. In the present setting, a first result of this type was proved in [15]. We assume now that the stationary Poisson hyperplane process X has intensity $n \in \mathbb{N}$, and we

denote the corresponding K -cell by $Z_K^{(n)}$. It is assumed further that the directional distribution φ of X has a positive, continuous density with respect to spherical Lebesgue measure. Under these assumptions, Kaltenbach [15] proved that

$$n^{-2/(d+1)} \ll \mathbb{E}V(Z_K^{(n)}) - V(K) \ll n^{-1/d}. \tag{5}$$

The proof can be considered as a ‘dualization’ (in a non-precise sense) of that of (3) and an extension to Poisson processes.

The purpose of this note is to obtain a similar counterpart to (2), and thus a result of type (5) with the volume replaced by the mean width W (observe that under dualization, volume and mean width interchange their roles, roughly). We have to assume now that the stationary Poisson hyperplane process X is also isotropic, that is, its distribution is invariant under rotations.

Theorem 1. *Let X be a stationary and isotropic Poisson hyperplane process in \mathbb{R}^d of intensity n . Let $K \subset \mathbb{R}^d$ be a convex body with interior points, and let $Z_K^{(n)}$ denote the K -cell of X . Then*

$$n^{-1} \ln^{d-1} n \ll \mathbb{E}W(Z_K^{(n)}) - W(K) \ll n^{-2/(d+1)}. \tag{6}$$

For random polytopes generated by finitely many independent hyperplanes with a suitable distribution, depending on K , a similar result was proved in [6]. Some ideas used there can be employed in the following. It turned out, however, that a proof for Poisson hyperplane processes is not straightforward and requires additional arguments. These will be presented in this note.

That the orders in (6) are best possible can be seen from extensions, to Poisson hyperplane processes, of precise asymptotic formulas that have been obtained for a finite number of independent random hyperplanes with a suitable distribution. These formulas are, on the one hand, Theorem 1.3 in [6], which holds for simple polytopes, and on the other hand Theorem 5.2 in [7], which yields the exact order on the right side of (6) if applied to a convex body of class C^2 . The extension, which we do not carry out here, would require a dual version of the argument sketched in [18] (proof of Lemma 1) and an estimate of the type provided by Lemma 1 below.

2. Preliminaries

The standard scalar product of \mathbb{R}^d is denoted by $\langle \cdot, \cdot \rangle$, and the induced norm by $\| \cdot \|$. The unit ball of \mathbb{R}^d is B^d , and the unit sphere is S^{d-1} . Lebesgue measure on \mathbb{R}^d is denoted by λ_d .

Hyperplanes and closed half-spaces of \mathbb{R}^d are written in the form

$$H(u, \tau) = \{x \in \mathbb{R}^d : \langle x, u \rangle = \tau\}, \quad H^-(u, \tau) = \{x \in \mathbb{R}^d : \langle x, u \rangle \leq \tau\},$$

respectively, with $u \in S^{d-1}$ and $\tau \in \mathbb{R}$. Let \mathcal{H} be the space of hyperplanes in \mathbb{R}^d with its usual topology. For a subset $M \subset \mathbb{R}^d$, we write

$$\mathcal{H}_M := \{H \in \mathcal{H} : H \cap M \neq \emptyset\}.$$

We let \mathcal{K}_d denote the space of d -dimensional convex bodies (compact, convex sets with interior points) in \mathbb{R}^d . As usual, it is equipped with the Hausdorff metric. For $K \in \mathcal{K}_d$, let $R_o(K)$ be the radius of the smallest ball with center at the origin o of \mathbb{R}^d that contains K .

For our notation concerning point processes, we refer to [27, Sections 3.1 and 3.2]. In particular, given a locally compact topological space E , we let $(\mathbf{N}_s(E), \mathcal{N}_s(E))$ denote the measurable space of simple, locally finite counting measures on E . We often identify a simple counting measure $\eta \in \mathbf{N}_s(E)$ with its support, using $\eta(\{x\}) = 1$ and $x \in \eta$ synonymously. A (simple) point process in E is a mapping $X: (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbf{N}_s(E), \mathcal{N}_s(E))$, where $(\Omega, \mathcal{A}, \mathbb{P})$ is some probability space, such that $\{X(C) = 0\}$ is measurable for all compact sets $C \subset E$. We let $\Theta = \mathbb{E} X$ denote the intensity measure of X . The point process X is a Poisson process if

$$\mathbb{P}(X(A) = k) = e^{-\Theta(A)} \frac{\Theta(A)^k}{k!}$$

for $k \in \mathbb{N}_0$ and each Borel set $A \subset E$ with $\Theta(A) < \infty$. For the independence properties of (simple) Poisson processes, we refer to [27, Theorem 3.2.2]. A stationary Poisson hyperplane process in \mathbb{R}^d is a Poisson process X in the space \mathcal{H} of hyperplanes whose intensity measure (and hence whose distribution) is invariant under translations. The intensity measure of such a process, assumed to be non-zero, is of the form

$$\Theta(A) = \gamma \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{\infty} \mathbf{1}_A(H(u, \tau)) \, d\tau \, \varphi(du)$$

for Borel sets $A \subset \mathcal{H}$. Here $\gamma > 0$ is the intensity of X and φ is an even probability measure on the sphere \mathbb{S}^{d-1} , the spherical directional distribution of X .

We assume now that X is a stationary Poisson hyperplane process in \mathbb{R}^d of intensity $\gamma > 0$ and with a non-degenerate spherical directional distribution φ . Here, ‘non-degenerate’ means that φ is not concentrated on any great subsphere.

For $K \in \mathcal{K}_d$, we let Z_K denote the K -cell defined by X and K , as above. This is a random polytope, since it is almost surely bounded. More precisely, we show the following estimate, which later on, when the intensity tends to infinity, will allow us to restrict ourselves to K -cells contained in a sufficiently large fixed ball.

Lemma 1. *Let $K \in \mathcal{K}_d$. There are constants $a, b > 0$, depending only on φ , such that*

$$\mathbb{P}(R_o(Z_K) > b(R_o(K) + x)) \leq 2d e^{-ax} \quad \text{for } x \geq 0.$$

Proof. Since $\text{supp } \varphi$, the support of the even measure φ , is not contained in a great subsphere, we can choose vectors $\pm e_1, \dots, \pm e_d \in \text{supp } \varphi$ positively spanning \mathbb{R}^d . In the following, we write $e_{d+i} := -e_i$ for $i = 1, \dots, d$. We can choose a sufficiently large constant b and sufficiently small, pairwise disjoint neighborhoods $U_i \subset \mathbb{S}^{d-1}$ of $e_i, i = 1, \dots, 2d$, such that each intersection

$$P := \bigcap_{i=1}^{2d} H^-(u_i, 1) \quad \text{with } u_i \in U_i, i = 1, \dots, 2d,$$

is a polytope with $R_o(P) \leq b$. Let $x \geq 0$. If the numbers τ_i are such that $R_o(K) \leq \tau_i \leq R_o(K) + x$ and if $u_i \in U_i$ for $i = 1, \dots, 2d$, then

$$R_o\left(\bigcap_{i=1}^{2d} H^-(u_i, \tau_i)\right) \leq R_o((R_o(K) + x)P) = (R_o(K) + x)R_o(P) \leq b(R_o(K) + x).$$

The sets of hyperplanes

$$A_i(x) := \{H(u, \tau) : u \in U_i, R_o(K) \leq \tau \leq R_o(K) + x\}, \quad i = 1, \dots, 2d,$$

are pairwise disjoint. If $X(A_i(x)) > 0$ for $i = 1, \dots, 2d$, then $R_o(Z_K) \leq b(R_o(K) + x)$. Therefore, observing that $\Theta(A_i(x)) = \gamma x \varphi(U_i)$ and choosing $0 < a \leq \varphi(U_i)$ for $i = 1, \dots, 2d$, we get

$$\begin{aligned} &\mathbb{P}(R_o(Z_K) > b(R_o(K) + x)) \\ &\leq \mathbb{P}(X(A_i(x)) = 0 \text{ for at least one } i \in \{1, \dots, 2d\}) \\ &= 1 - \prod_{i=1}^{2d} (1 - \mathbb{P}(X(A_i(x)) = 0)) \\ &= 1 - \prod_{i=1}^{2d} (1 - e^{-\gamma \varphi(U_i)x}) \\ &\leq 1 - (1 - e^{-\gamma ax})^{2d} \\ &\leq 2d e^{-\gamma ax}, \end{aligned}$$

by Bernoulli’s inequality. This was the assertion. □

3. Proof of the upper bound

The approach to proving the right-hand estimate of (6) consists in establishing an extremal property of balls and then finding a connection to a known result on approximation of balls by convex hulls of finitely many random points. Since we are dealing with Poisson processes, this requires extra arguments in either step.

Let X be a stationary Poisson hyperplane process in \mathbb{R}^d , with a non-degenerate spherical directional distribution φ and with intensity γ . If a convex body $K \in \mathcal{K}_d$ is given, we let Z_K denote the K -cell defined by X and K , as in (4). In order to be able to compare Z_K and Z_L for different $K, L \in \mathcal{K}_d$, we use an auxiliary Poisson process. For this, we consider the product space $E := \mathbb{S}^{d-1} \times [0, \infty)$ with the product measure $\varphi \otimes \lambda_+$, where λ_+ is the Lebesgue measure on $[0, \infty)$. Let Y be the Poisson process on E with intensity measure $2\gamma \varphi \otimes \lambda_+$. (Its existence and uniqueness up to stochastic equivalence follows, for example, from [27, Theorem 3.2.1].) Let $M(E)$ denote the set of all locally finite subsets $S \subset E$ with the property that the set $\{u \in \mathbb{S}^{d-1} : (u, t) \in S \text{ for some } t \geq 0\}$ positively spans \mathbb{R}^d . For $\eta \in \mathbf{N}_s(E)$ with support in $M(E)$, we define

$$P(\eta, K) := \bigcap_{(u,t) \in \text{supp } \eta} H^-(u, h(K, u) + t)$$

for $K \in \mathcal{K}_d$, where $h(K, \cdot) := \max\{\langle x, \cdot \rangle : x \in K\}$ denotes the support function of K . This is a polytope containing K . We shall see below that the random polytope $P(Y, K)$ is stochastically equivalent to the K -cell Z_K defined by the hyperplane process X . We use the random polytopes $P(Y, K)$ to show that the function $K \mapsto \mathbb{E}W(Z_K)$ is concave and continuous on \mathcal{K}_d . Here a function $f : \mathcal{K}_d \rightarrow \mathbb{R}$ is called concave if

$$f((1 - \alpha)K + \alpha L) \geq (1 - \alpha)f(K) + \alpha f(L)$$

for $K, L \in \mathcal{K}_d$ and $\alpha \in [0, 1]$, where $K + L := \{x + y : x \in K, y \in L\}$ and $\beta K := \{\beta x : x \in K\}$ for $K, L \in \mathcal{K}_d$ and $\beta \geq 0$.

Lemma 2. For $K, L \in \mathcal{K}_d$ and $\alpha \in [0, 1]$,

$$\mathbb{E}W(Z_{(1-\alpha)K+\alpha L}) \geq (1 - \alpha)\mathbb{E}W(Z_K) + \alpha\mathbb{E}W(Z_L). \tag{7}$$

The functional $K \mapsto \mathbb{E}W(Z_K)$ is continuous on \mathcal{K}_d .

Proof. For $\eta \in \mathbb{N}_s(E)$ we have

$$(1 - \alpha)P(\eta, K) + \alpha P(\eta, L) \subseteq P(\eta, (1 - \alpha)K + \alpha L),$$

as follows immediately from the definition of $P(\eta, K)$ and the linearity properties of the support function. The monotonicity and linearity properties of the mean width yield

$$W(P(\eta, (1 - \alpha)K + \alpha L)) \geq (1 - \alpha)W(P(\eta, K)) + \alpha W(P(\eta, L)).$$

Here we can replace η with Y . Then linearity and monotonicity of the expectation yield

$$\mathbb{E}W(P(Y, (1 - \alpha)K + \alpha L)) \geq (1 - \alpha)\mathbb{E}W(P(Y, K)) + \alpha\mathbb{E}W(P(Y, L)). \tag{8}$$

We define the Poisson hyperplane process X_K by

$$X_K(A) := X(A \setminus \mathcal{H}_{\text{int } K})$$

for Borel sets $A \subset \mathcal{H}$. Its intensity measure is given by

$$\begin{aligned} \mathbb{E} X_K(A) &= \Theta(A \setminus \mathcal{H}_{\text{int } K}) \\ &= \gamma \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{\infty} \mathbf{1}\{H(u, \tau) \in A\} \mathbf{1}\{H(u, \tau) \cap \text{int } K = \emptyset\} \, d\tau \, \varphi(du) \\ &= \gamma \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{-h(K, -u)} \mathbf{1}\{H(u, \tau) \in A\} \, d\tau \, \varphi(du) \\ &\quad + \gamma \int_{\mathbb{S}^{d-1}} \int_{h(K, u)}^{\infty} \mathbf{1}\{H(u, \tau) \in A\} \, d\tau \, \varphi(du) \\ &= 2\gamma \int_{\mathbb{S}^{d-1}} \int_{h(K, u)}^{\infty} \mathbf{1}\{H(u, \tau) \in A\} \, d\tau \, \varphi(du), \end{aligned}$$

where we have used the fact that φ is an even measure. Next, we define a mapping $F_K: E \rightarrow \mathcal{H}$ by

$$F_K(u, t) := H(u, h(K, u) + t)$$

and denote for $\eta \in \mathbb{N}_s(E)$ by $F_K(\eta)$ the pushforward of η under F_K . Then $F_K(Y)$ is a Poisson hyperplane process. For its intensity measure we obtain, for Borel sets $A \subset \mathcal{H}$,

$$\begin{aligned} \mathbb{E}(F_K(Y))(A) &= \mathbb{E}Y(F_K^{-1}(A)) \\ &= 2\gamma \int_{\mathbb{S}^{d-1}} \int_0^{\infty} \mathbf{1}\{(u, t) \in F_K^{-1}(A)\} \, dt \, \varphi(du) \\ &= 2\gamma \int_{\mathbb{S}^{d-1}} \int_0^{\infty} \mathbf{1}\{H(u, h(K, u) + t) \in A\} \, dt \, \varphi(du) \\ &= 2\gamma \int_{\mathbb{S}^{d-1}} \int_{h(K, u)}^{\infty} \mathbf{1}\{H(u, \tau) \in A\} \, d\tau \, \varphi(du). \end{aligned}$$

Thus, X_K and $F_K(Y)$ have the same intensity measure. Since either of them is a Poisson process, they are stochastically equivalent. It follows that the zero cell Z_K is stochastically equivalent to the random polytope $P(Y, K)$. Therefore, (8) yields the assertion (7).

To prove the continuity assertion, let $K, K_i \in \mathcal{K}_d$ for $i \in \mathbb{N}$ and suppose that $K_i \rightarrow K$ in the Hausdorff metric, as $i \rightarrow \infty$. If $\varepsilon_0 > 0$ is small enough, then

$$K_\varepsilon := \bigcap_{u \in \mathbb{S}^{d-1}} H^-(u, h(K, u) + \varepsilon)$$

is a convex body for any $\varepsilon > -\varepsilon_0$. Clearly, $K_\varepsilon \rightarrow K$ as $\varepsilon \rightarrow 0$. For given $\varepsilon > 0$, let i be so large that $K_{-\varepsilon} \subset K_i \subset K_\varepsilon$. Observe also that $X(\mathcal{H}_{K_i} \Delta \mathcal{H}_K) = 0$ (where Δ denotes the symmetric difference) implies that $Z_{K_i} = Z_K$. Therefore,

$$\begin{aligned} |\mathbb{E}W(Z_{K_i}) - \mathbb{E}W(Z_K)| &\leq \mathbb{E}|W(Z_{K_i}) - W(Z_K)| \\ &\leq \mathbb{P}(X(\mathcal{H}_{K_i} \Delta \mathcal{H}_K) > 0) \mathbb{E}W(Z_{K_\varepsilon}) \\ &\leq \mathbb{P}(X(\mathcal{H}_{K_\varepsilon} \setminus \mathcal{H}_{K_{-\varepsilon}}) > 0) \mathbb{E}W(Z_{K_{\varepsilon_0}}) \\ &\rightarrow 0 \end{aligned}$$

as $\varepsilon \rightarrow 0$. The limit follows from the fact that

$$\Theta(\mathcal{H}_{K_\varepsilon} \setminus \mathcal{H}_{K_{-\varepsilon}}) = \gamma \int_{\mathbb{S}^{d-1}} [h(K_\varepsilon, u) - h(K_{-\varepsilon}, u)] \varphi(du) \rightarrow 0$$

as $\varepsilon \rightarrow 0$, by monotone convergence. This proves the continuity assertion. □

From now on, we assume that the stationary Poisson hyperplane process X is isotropic and has intensity n . Then its intensity measure is given by $\Theta = n\mu$ with

$$\mu = \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{\infty} \mathbf{1}\{H(u, \tau) \in \cdot\} d\tau \sigma(du), \tag{9}$$

where σ is the normalized spherical Lebesgue measure. For convex bodies $K, L \in \mathcal{K}_d$ with $K \subset L$, we have

$$\begin{aligned} \mu(\mathcal{H}_L \setminus \mathcal{H}_K) &= \int_{\mathcal{H} \setminus \mathcal{H}_K} \mathbf{1}\{H \cap L \neq \emptyset\} \mu(dH) \\ &= \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{\infty} \mathbf{1}\{H(u, \tau) \cap L \neq \emptyset\} \mathbf{1}\{H(u, \tau) \cap K = \emptyset\} d\tau \sigma(du) \\ &= 2 \int_{\mathbb{S}^{d-1}} [h(L, u) - h(K, u)] \sigma(du) \\ &= W(L) - W(K). \end{aligned} \tag{10}$$

Lemma 3. *If X is isotropic, then the functional*

$$K \mapsto \frac{\mathbb{E}W(Z_K)}{W(K)}, \quad K \in \mathcal{K}_d,$$

attains its maximum at balls.

Proof. If X is isotropic, then the functional $K \mapsto \mathbb{E}W(Z_K)$ is invariant under rigid motions. Since by Lemma 2 it is concave and continuous on \mathcal{K}_d , it is well known that on the set of convex bodies $K \in \mathcal{K}_d$ with given mean width $W(K)$ it attains its maximum at balls. The proof, which uses Hadwiger’s ‘Zweites Kugelungstheorem’ ([12, pp. 170–171], reproduced in [25, Theorem 3.3.5]), is carried out in [6, p. 621]. \square

To take advantage of the preceding lemma, we connect this to a known asymptotic result about convex hulls of i.i.d. random points in a ball. First we write the result of Lemma 3 in the form

$$\mathbb{E}W(Z_K) - W(K) \ll \mathbb{E}W(Z_{B^d}) - W(B^d). \tag{11}$$

We recall that here $Z_K = Z_K^{(n)}$ and $Z_{B^d} = Z_{B^d}^{(n)}$ and that we intend to let n tend to infinity. In view of this, we choose a number $R > b$, where b is the constant appearing in Lemma 1 for $K = B^d$, and state that

$$\mathbb{E}W(Z_{B^d}) - \mathbb{E}[W(Z_{B^d})\mathbf{1}\{R_o(Z_{B^d}) < R\}] = O(n^{-1}) \tag{12}$$

as $n \rightarrow \infty$ (where the constant involved in O depends on R). For the proof, we note that the left side of (12) can be estimated by

$$\begin{aligned} \mathbb{E}[W(Z_{B^d})\mathbf{1}\{R_o(Z_{B^d}) \geq R\}] &\leq \mathbb{E}[2R_o(Z_{B^d})\mathbf{1}\{R_o(Z_{B^d}) \geq R\}] \\ &= 2 \int_{\Omega} R_o(Z_{B^d})\mathbf{1}\{R_o(Z_{B^d}) \geq R\} d\mathbb{P} \\ &= 2 \int_0^\infty \mathbb{P}(R_o(Z_{B^d})\mathbf{1}\{R_o(Z_{B^d}) \geq R\} > t) dt \\ &= 2R \mathbb{P}(R_o(Z_{B^d}) \geq R) + 2 \int_R^\infty \mathbb{P}(R_o(Z_{B^d}) > t) dt. \end{aligned}$$

Lemma 1 provides an estimate for $\mathbb{P}(R_o(Z_{B^d}) \geq b(1+x))$. Using this with $b(1+x) = R$ for the first summand and with $b(1+x) = t$ for the second summand (and observing that now $\gamma = n$), we obtain (12).

We use the bijective mapping

$$\xi: \mathcal{H} \setminus \mathcal{H}_{\{o\}} \rightarrow \mathbb{R}^d \setminus \{o\}, \quad \xi(H(u, \tau)) = \tau^{-1}u. \tag{13}$$

Let κ_0 be the pushforward of the measure μ , restricted to $\mathcal{H} \setminus \mathcal{H}_{B^d}$, under ξ . Then

$$\kappa_0(A) = \frac{2}{\omega_d} \int_A \|x\|^{-(d+1)} \lambda_d(dx)$$

for Borel sets $A \subset B^d \setminus \{o\}$, where ω_d is the surface area of the unit sphere. The measure κ_0 is infinite, but finite on compact subsets of $B^d \setminus \{o\}$.

Let Y_n denote the Poisson point process in \mathbb{R}^d with intensity measure $n\kappa_0$. Let Q_n be the convex hull of Y_n . Then Q_n is a random polytope, which is stochastically equivalent to the polar of Z_{B^d} . With the constant $R > b$ chosen above, we set $r = 1/R$ and $B_r = rB^d$. By (10), we have

$$W(Z_{B^d}) - W(B^d) = \int_{\mathcal{H} \setminus \mathcal{H}_{B^d}} \mathbf{1}\{H \cap Z_{B^d} \neq \emptyset\} \mu(dH) = \kappa_0(B^d \setminus Q_n),$$

hence

$$\mathbb{E}[(W(Z_{B^d}) - W(B^d))\mathbf{1}\{R_o(Z_{B^d}) < R\}] = \mathbb{E}[\kappa_0(B^d \setminus Q_n)\mathbf{1}\{B_r \subset Q_n\}].$$

Now it follows from (11) and (12) that

$$\mathbb{E}W(Z_K) - W(K) \ll \mathbb{E} [\kappa_0(B^d \setminus Q_n)\mathbf{1}\{B_r \subset Q_n\}] + O(n^{-1}). \tag{14}$$

To express the latter expectation in a suitable way, we note that Q_n is almost surely a simplicial polytope, hence each of its facets is the convex hull of d points of Y_n . For any d points $x_1, \dots, x_d \in Y_n$ (almost surely, they are affinely independent and their affine hull does not contain o), we define

$$S(x_1, \dots, x_d) := B^d \setminus H^-(x_1, \dots, x_d),$$

where $H^-(x_1, \dots, x_d)$ is the closed half-space bounded by $\text{aff}\{x_1, \dots, x_d\}$ that contains o . Further, we define

$$T(x_1, \dots, x_d) := S(x_1, \dots, x_d) \cap \text{pos}\{x_1, \dots, x_d\}.$$

Then we have

$$\begin{aligned} &\kappa_0(B^d \setminus Q_n)\mathbf{1}\{B_r \subset Q_n\} \\ &= \frac{1}{d!} \sum_{(x_1, \dots, x_d) \in \eta_n^d} \mathbf{1}\{Y_n(S(x_1, \dots, x_d)) = 0\} \kappa_0(T(x_1, \dots, x_d))\mathbf{1}\{B_r \subset Q_n\}, \end{aligned}$$

where η_n^d denotes the set of ordered d -tuples of pairwise different elements from the support of η . We note that if $B_r \subset Q_n$, then points $x_1, \dots, x_d \in Y_n$ with $Y_n(S(x_1, \dots, x_d)) = 0$ automatically satisfy $x_1, \dots, x_d \in B^d \setminus B_r$ and $\text{aff}\{x_1, \dots, x_d\} \cap B_r = \emptyset$ a.s. Therefore,

$$\begin{aligned} &\kappa_0(B^d \setminus Q_n)\mathbf{1}\{B_r \subset Q_n\} \\ &= \frac{1}{d!} \sum_{(x_1, \dots, x_d) \in \eta_n^d} \mathbf{1}\{Y_n(S(x_1, \dots, x_d)) = 0, B_r \subset Q_n\} \kappa_0(T(x_1, \dots, x_d)) \\ &\quad \times \mathbf{1}\{x_1, \dots, x_d \in B^d \setminus B_r\} \mathbf{1}\{\text{aff}\{x_1, \dots, x_d\} \cap B_r = \emptyset\}. \end{aligned}$$

Using the Slivnyak–Mecke formula (see e.g. [27, Corollary 3.2.3]) and noting that $n\kappa_0$ is the intensity measure of Y_n , we obtain

$$\begin{aligned} &\mathbb{E} [\kappa_0(B^d \setminus Q_n)\mathbf{1}\{B_r \subset Q_n\}] \\ &= \frac{n^d}{d!} \int_{B^d \setminus B_r} \dots \int_{B^d \setminus B_r} \mathbb{E} \mathbf{1}\{Y_n(S(x_1, \dots, x_d)) = 0, B_r \subset \text{conv}(Y_n \cup \{x_1, \dots, x_d\})\} \\ &\quad \times \kappa_0(T(x_1, \dots, x_d))\mathbf{1}\{\text{aff}\{x_1, \dots, x_d\} \cap B_r = \emptyset\} \kappa_0(dx_1) \cdots \kappa_0(dx_d). \end{aligned}$$

Let $\lambda_0 := (2/\omega_d)\lambda_d$. For Borel sets $A \subset B^d \setminus B_r$ we have

$$\lambda_0(A) \leq \kappa_0(A) \leq r^{-(d+1)}\lambda_0(A).$$

For fixed $x_1, \dots, x_d \in B^d \setminus B_r$ with $\text{aff}\{x_1, \dots, x_d\} \cap B_r = \emptyset$, we have

$$\begin{aligned} &\mathbb{E} \mathbf{1}\{Y_n(S(x_1, \dots, x_d)) = 0, B_r \subset \text{conv}(Y_n \cup \{x_1, \dots, x_d\})\} \\ &\leq \mathbb{E} \mathbf{1}\{Y_n(S(x_1, \dots, x_d)) = 0\} \\ &= e^{-n\kappa_0(S(x_1, \dots, x_d))} \\ &\leq e^{-n\lambda_0(S(x_1, \dots, x_d))}. \end{aligned}$$

Therefore, we can estimate

$$\begin{aligned} &\mathbb{E} [\kappa_0(B^d \setminus Q_n) \mathbf{1}\{B_r \subset Q_n\}] \\ &\ll n^d \int_{B^d} \dots \int_{B^d} e^{-n\lambda_0(S(x_1, \dots, x_d))} \lambda_0(T(x_1, \dots, x_d)) \lambda_0(dx_1) \dots \lambda_0(dx_d). \end{aligned}$$

Let \tilde{Y}_n be a Poisson point process in \mathbb{R}^d with intensity measure $n\lambda_0$, and let

$$\Pi_n := \text{conv}(\tilde{Y}_n \cap B^d).$$

Using the Slivnyak–Mecke formula as above yields that

$$\begin{aligned} &\mathbb{E} \lambda_0(B^d \setminus \Pi_n) \\ &= n^d \int_{B^d} \dots \int_{B^d} e^{-n\lambda_0(S(x_1, \dots, x_d))} \lambda_0(T(x_1, \dots, x_d)) \lambda_0(dx_1) \dots \lambda_0(dx_d). \end{aligned}$$

We conclude that

$$\mathbb{E} [\kappa_0(B^d \setminus Q_n) \mathbf{1}\{B_r \subset Q_n\}] \ll \mathbb{E} \lambda_0(B^d \setminus \Pi_n).$$

It follows from Lemma 1 in [18] that

$$\mathbb{E} \lambda_0(B^d \setminus \Pi_n) \ll n^{-2/(d+1)}.$$

Together with (14), this yields the upper bound in (6).

4. Proof of the lower bound

The proof of the left-hand estimate of (6) requires only a few changes in the proof of the corresponding inequality in [6, (1.3)].

Let $K \in \mathcal{K}_d$. For $x \in \mathbb{R}^d \setminus K$, we define $K^x := \text{conv}(K \cup \{x\})$ and set

$$m(H) := \min\{W(K^x) - W(K) : x \in H\}$$

for hyperplanes $H \in \mathcal{H} \setminus \mathcal{H}_K$. For $t > 0$ we define

$$\mathcal{H}_K(t) := \{H \in \mathcal{H} \setminus \mathcal{H}_K : m(H) \leq t\}.$$

We assume now, as in Theorem 1, that X is a stationary and isotropic Poisson hyperplane process in \mathbb{R}^d of intensity $n \in \mathbb{N}$. Let $H \in \mathcal{H} \setminus \mathcal{H}_K$, and let $z \in H$ be such that $m(H) = W(K^z) - W(K)$ (clearly, such a point exists). If no hyperplane of X separates z and K , then $z \in Z_K^{(n)}$ and hence $H \cap Z_K^{(n)} \neq \emptyset$. It follows that

$$\begin{aligned} \mathbb{P}(H \cap Z_K^{(n)} \neq \emptyset) &\geq \mathbb{P}(X(\mathcal{H}_{K^z} \setminus \mathcal{H}_K) = 0) \\ &= \exp[-\Theta(\mathcal{H}_{K^z} \setminus \mathcal{H}_K)] \\ &= e^{-nm(H)}, \end{aligned}$$

where (10) was used. Therefore, we obtain (using (10) again and Fubini’s theorem)

$$\begin{aligned} \mathbb{E} W(Z_K^{(n)}) - W(K) &= \int_{\Omega} \int_{\mathcal{H} \setminus \mathcal{H}_K} \mathbf{1}\{H \cap Z_K^{(n)} \neq \emptyset\} \mu(dH) d\mathbb{P} \\ &= \int_{\mathcal{H} \setminus \mathcal{H}_K} \mathbb{P}(H \cap Z_K^{(n)} \neq \emptyset) \mu(dH) \end{aligned}$$

$$\begin{aligned} &\geq \int_{\mathcal{H} \setminus \mathcal{H}_K} e^{-nm(H)} \mu(dH) \\ &\geq \int_{\mathcal{H} \setminus \mathcal{H}_K} \mathbf{1}\{m(H) \leq t\} e^{-nt} \mu(dH) \\ &= e^{-nt} \mu(\mathcal{H}_K(t)), \end{aligned}$$

where $t > 0$ can be any number. The choice $t = 1/n$ gives

$$\mathbb{E}W(Z_K^{(n)}) - W(K) \geq e^{-1} \mu(\mathcal{H}_K(1/n)). \tag{15}$$

The remainder of the proof is similar to that in [6, p. 619]. For the reader’s convenience, we adapt the argument to the present case. The idea is to use a result from Bárány and Larman [4], together with polarity.

We translate K so that (say) its centroid is at the origin. Let K° be the polar body of K . For $x \in K^\circ$, let $v(x)$ denote the minimal volume that a closed half-space containing x cuts off from K° , and define

$$K^\circ(t) := \{x \in K^\circ : v(x) \leq t\}$$

for $t > 0$. We can choose $\rho, t_0 > 0$ such that $\rho B^d \subset K^\circ(t)$ for $0 < t \leq t_0$.

Let $\psi : \mathbb{R}^d \setminus \{o\} \rightarrow \mathcal{H} \setminus \mathcal{H}_{\{o\}}$ be the inverse of the map ξ defined by (13); thus

$$\psi(ru) = H(u, r^{-1}) \quad \text{for } u \in \mathbb{S}^{d-1}, r > 0.$$

Let ν denote the pushforward of the Lebesgue measure λ_d under ψ ; thus

$$\nu(A) = \omega_d \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}\{H(u, \tau) \in A\} \tau^{-(d+1)} d\tau \sigma(du) \tag{16}$$

for Borel sets $A \subset \mathcal{H} \setminus \mathcal{H}_{\{o\}}$.

Let $H(u, \tau)$ be a hyperplane contained in $\mathcal{H}_{\rho^{-1}B^d} \setminus \mathcal{H}_K$. Since $H(u, \tau) \cap K = \emptyset$, we have $\tau \geq h(K, u)$, which is bounded from below by a positive constant depending only on K . Since $H(u, \tau) \cap \rho^{-1}B^d \neq \emptyset$, we have $\tau \leq \rho^{-1}$. Now comparison of (16) and (9) yields the existence of constants $c_1, c_2 > 0$, depending only on d and K , such that

$$c_1 \nu(A) \leq \mu(A) \leq c_2 \nu(A) \quad \text{if } A \subset \mathcal{H}_{\rho^{-1}B^d} \setminus \mathcal{H}_K.$$

Let $0 < t \leq t_0$ and $x \in K^\circ(t) \setminus \partial K^\circ$. There is a hyperplane E through x that bounds a closed half-space E^+ not containing o , such that $\lambda_d(K^\circ \cap E^+) \leq t$. If $H := \psi(x)$ and $y := \psi^{-1}(E)$, then $y \in H \in \mathcal{H}_{\rho^{-1}B^d} \setminus \mathcal{H}_K$. The mapping ψ maps the cap $K^\circ \cap E^+$ bijectively onto the set of hyperplanes (weakly) separating y and K . We denote this set by \mathcal{H}_K^y . It follows that

$$m(H) \leq W(K^y) - W(K) = \mu(\mathcal{H}_K^y) \leq c_2 \nu(\mathcal{H}_K^y) = c_2 \lambda_d(K^\circ \cap E^+) \leq c_2 t,$$

hence $H \in \mathcal{H}_K(c_2 t)$. Since $x \in K^\circ(t) \setminus \partial K^\circ$ was arbitrary, this shows that $\psi(K^\circ(t)) \subset \mathcal{H}_K(c_2 t)$. Therefore,

$$\lambda_d(K^\circ(t)) = \nu(\psi(K^\circ(t))) \leq \nu(\mathcal{H}_K(c_2 t)) \leq c_1^{-1} \mu(\mathcal{H}_K(c_2 t)).$$

If we choose $t = 1/(c_2 n)$, then this inequality together with (15) shows that

$$\mathbb{E}W(Z_K^{(n)}) - W(K) \geq e^{-1} \mu(\mathcal{H}_K(1/n)) \geq e^{-1} c_1 \lambda_d(K^\circ((c_2 n)^{-1})).$$

Theorem 2 of [4] says that

$$\lambda_d(K^\circ(\varepsilon)) \geq c\varepsilon \log^{d-1}(1/\varepsilon)$$

for sufficiently small $\varepsilon > 0$, with some constant c depending only on K . It follows that

$$\mathbb{E}W(Z_K^{(n)}) - W(K) \geq c_3 n^{-1} \log^{d-1} n$$

for all sufficiently large n , where c_3 is a constant depending only on d and K . By adapting the constant, we can assume that this holds for all $n \in \mathbb{N}$. This completes the proof of the lower bound in (6).

Acknowledgement

Thanks are due to the referees, whose comments have led to corrections, clarifications, and simplifications.

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